Appendix B

In this Appendix we explain in detail how the proofs of the results of the paper can be derived for the extended cost allocation problems defined in the subsection entitled "Extension on the structure of the transfer rate" of Section 4. Consider then the cost allocation problems $(N_t, N_u, C, \underline{t}, \overline{t}, \underline{u}, \overline{u})$ where $N_t = \{1,2,3\}$ and $N_u = \{3,4,5\}$. Then, we can establish some limits for the transfer rate in a similar way as in Proposition 3 of the paper.

Remark 1 (Extension of Proposition 3) *The values $\hat{t}$ and $\hat{u}$ for the transfer rates are compatible with a problem $(N_t, N_u, C, \underline{t}, \overline{t}, \underline{u}, \overline{u})$ if and only if $\hat{t} \in [\underline{t}, \overline{t}^*(\overline{t}, \overline{u}, C)]$ and $\hat{u} \in [\underline{u}^*(\underline{u}, \underline{t}, C), \overline{u}^*(\overline{u}, C)]$, where*

$$\overline{t}^*(\overline{u}, \overline{t}, C) = \min\{\frac{c_2}{c_1}, \overline{t}, h(\overline{u}^*(\overline{u}, C))\},$$

$$\overline{u}^*(\overline{u}, C) = \min\{\frac{c_4}{c_3}, \frac{c_5}{c_4 + c_5}, \overline{u}\}, \textit{ and}$$

$$\underline{u}^*(\underline{u}, \underline{t}, C) = \max\{h^{-1}(\underline{t}), \underline{u}\},$$

being $h(u) = \frac{c_3}{c_3 + c_2 \cdot (1-u)}$.

Proof. Let $(N_t, N_u, C, \underline{t}, \overline{t}, \underline{u}, \overline{u})$ be a problem where $N_t = \{1,2,3\}$ and $N_u = \{3,4,5\}$. If the actual transfer rate t is 1, we have that $c_i = 0$ for all $i \in \{1,2\}$ and $\overline{t} = 1$. Therefore, $\overline{t}^*(\overline{u}, \overline{t}, C) = 1$. Similarly, if the actual transfer rate u is 1, we have that $c_i = 0$ for all $i \in \{3,4\}$ and $\overline{u} = 1$. Therefore, $\overline{u}^*(\overline{u}, C) = 1$.

Let us assume now that t and/or u are smaller than 1. Then, the cost that we observe, c_i, is the difference between all the waste entering the segment, denoted as V_i^*, and the amount transferred to the next segments, given by tV_i^* if $i \in \{1,2\}$, by uV_i^* if $i \in \{3,4\}$ or by 0 if $i = 5$. Then, we have that

$$V_i^* = \begin{cases} \frac{c_i}{1-t} & \text{if } i \in \{1,2\} \\ \frac{c_i}{1-u} & \text{if } i \in \{3,4\} \\ c_i & \text{if } i = 5. \end{cases} \tag{1}$$

Similarly, the amount thrown into the water by region i, denoted as V_i, is the difference between the total amount entered segment i, V_i^*, and the amount transferred from its immediate upstream segment, given by 0 if $i = 1$, by tV_{i-1}^* if $i \in \{2,3\}$ and by uV_{i-1}^* if $i \in \{4,5\}$. Then, we can deduce using expression (1) that:

$$V_i(t,u,C) = \begin{cases} \frac{c_i}{1-t} & \text{if } i=1 \\ \frac{c_i}{1-t} - \frac{c_{i-1}}{1-t}t & \text{if } i=2 \\ \frac{c_i}{1-u} - \frac{c_{i-1}}{1-t}t & \text{if } i=3 \\ \frac{c_i}{1-u} - \frac{c_{i-1}}{1-u}u & \text{if } i=4 \\ c_i - \frac{c_{i-1}}{1-u}u & \text{if } i=5. \end{cases} \quad (2)$$

Given that $V_i(t,u,C) \geq 0$ by definition and taking into account the expressions obtained in (2), we have that

- $\frac{c_2}{1-t} - \frac{c_1}{1-t}t \geq 0$. If $c_1 = c_2 = 0$, the condition is satisfied. Otherwise, we deduce that $t \leq \frac{c_2}{c_1}$.
- $\frac{c_3}{1-u} - \frac{c_2}{1-t}t \geq 0$. If $c_2 = c_3 = 0$, the condition is satisfied. Otherwise, we deduce that $t \leq \frac{c_3}{c_2(1-u)+c_3} = h(u)$ or, what is the same, that $u \geq \frac{t(c_2+c_3)-c_3}{c_2 t} = h^{-1}(t)$.
- $\frac{c_4}{1-u} - \frac{c_3}{1-u}u \geq 0$. If $c_3 = c_4 = 0$, the condition is satisfied. Otherwise, we deduce that $u \leq \frac{c_4}{c_3}$.
- $c_5 - \frac{c_4}{1-u}u \geq 0$. If $c_4 = c_5 = 0$, the condition is satisfied. Otherwise, we deduce that $u \leq \frac{c_5}{c_4+c_5}$.

Then, we have obtained that $u \leq \overline{u}^*(\overline{u},C) = \min\{\frac{c_4}{c_3}, \frac{c_5}{c_4+c_5}, \overline{u}\}$. Given that the function $h(u)$ is increasing, we obtain a new upper limit for t: $t \leq h(\overline{u}^*(\overline{u},C))$. Therefore, we have that $t \leq \overline{t}^*(\overline{t},\overline{u},C) = \min\{\frac{c_2}{c_1}, \overline{t}, h(\overline{u}^*(\overline{u},C))\}$. Moreover, using the inverse function of h, we have a new lower limit for u: $u \geq \underline{u}^*(\underline{u},\underline{t},C) = \max\{h^{-1}(\underline{t}), \underline{u}\}$.

Additionally, it is easy to see from the previous reasoning that any value of $\hat{t}$ between $\underline{t}$ and $\overline{t}^*(\overline{t},\overline{u},C)$ and of $\hat{u}$ between $\underline{u}^*(\underline{u},\underline{t},C)$ and $\overline{u}^*(\overline{u},C)$ are compatible with $(N_t, N_u, C, \underline{t}, \overline{t}, \underline{u}, \overline{u})$. Then we have arrived at the desired result. ■

Remark 2 (Extension of Proposition 4) *Let* $(N_t, N_u, C, \underline{t}, \overline{t}, \underline{u}, \overline{u})$ *be a problem where* $N_t = \{1,2,3\}$ *and* $N_u = \{3,4,5\}$. *Then,*

$$\underline{l}_i^i(\cdot) = \begin{cases} c_i & \textit{if } i = 1 \\ c_i - c_{i-1} \cdot \overline{t}^*(\overline{t}, \overline{u}, C) & \textit{if } i = 2 \\ c_i - c_{i-1} \cdot \frac{\overline{t}^*(\overline{t},\overline{u},C)}{1-\overline{t}^*(\overline{t},\overline{u},C)} \cdot (1 - \underline{u}^*(\underline{u}, \underline{t}, C)) & \textit{if } i = 3 \textit{ and } \overline{t}^*(\overline{t}, \overline{u}, C) < 1 \\ 0 & \textit{if } i = 3 \textit{ and } \overline{t}^*(\overline{t}, \overline{u}, C) = 1 \\ c_i - c_{i-1} \cdot \overline{u}^*(\overline{u}, C) & \textit{if } i = 4 \\ c_i - \frac{c_{i-1} \cdot \overline{u}^*(\overline{u},C)}{1-\overline{u}^*(\overline{u},C)} & \textit{if } i = 5 \textit{ and } \overline{u}^*(\overline{u}, C) < 1 \\ 0 & \textit{if } i = 5 \textit{ and } \overline{u}^*(\overline{u}, C) = 1. \end{cases}$$

$$\overline{l}_i^i(\cdot) = \begin{cases} c_i & \textit{if } i = 1 \\ c_i - c_{i-1} \cdot \underline{t} & \textit{if } i = 2 \\ c_i - c_{i-1} \cdot \frac{\underline{t}}{1-\underline{t}} \cdot (1 - \overline{u}^*(\overline{u}, C)) & \textit{if } i = 3 \\ c_i - c_{i-1} \cdot \underline{u}^*(\underline{u}, \underline{t}, C) & \textit{if } i = 4 \\ c_i - \frac{c_{i-1} \cdot \underline{u}^*(\underline{u},\underline{t},C)}{1-\underline{u}^*(\underline{u},\underline{t},C)} & \textit{if } i = 5 \end{cases}$$

Proof. The limits for regions 1, 2, 4 and 5 are derived exactly in the same way as in the proof of Proposition 4, taking into account the corresponding transfer rate in each particular segment (t for regions 1 and 2, and u for 4 and 5). Consider then region 3. If $t \in (0, 1)$ we have that $\frac{c_2}{1-t}$ units of waste entered region 2. Then $\frac{c_2}{1-t} \cdot t$ units of waste entered region 3 from region 2 and $\frac{c_2}{1-t} \cdot t \cdot u$ of these units left region 3 to region 4. Therefore, $\frac{c_2 \cdot t}{1-t} \cdot (1 - u)$ units of waste present in region 3 are responsibility of the regions situated upstream from 3. Then we have that $l_3^3(\cdot) = c_3 - \frac{c_2 \cdot t}{1-t} \cdot (1 - u)$. If $t = 1$, we have that c_2 equals 0 and, in this case, there is no information at all about how much of the waste present in region 3 is the responsibility of region 3. Then, $l_3^3(\cdot) \in [0, c_3]$. Finally, if $t = 0$ we have that all the waste present in region 3 is of its own responsibility and thus $l_3^3(\cdot) = c_3$. In situations in which there is uncertainty over t and/or u, $t \in [\underline{t}, \overline{t}^*(\overline{t}, \overline{u}, C)]$ and $u \in [\underline{u}^*(\underline{u}, \underline{t}, C), \overline{u}^*(\overline{u}, C)]$, we can summarize these expressions as stated in the remark. ■

Remark 3 (Extension of Theorem 1) *Let $(N_t, N_u, C, \underline{t}, \overline{t}, \underline{u}, \overline{u})$ be the cost allocation problems where $N_t = \{1, 2, 3\}$ and $N_u = \{3, 4, 5\}$. A rule satisfies LR, NDR, CR and MIT if and only if it is the Upstream Responsibility rule (whose formal definition is in the paper).*

Proof. It is easy to see that the Upstream Responsibility rule γ satisfies LR, NDR, CR and MIT. To prove the other implication, consider a problem

$(N_t, N_u, C, \underline{t}, \overline{t}, \underline{u}, \overline{u})$ where $N_t = \{1, 2, 3\}$ and $N_u = \{3, 4, 5\}$ and their corresponding $\overline{t}^*(\overline{t}, \overline{u}, C)$, $\overline{u}^*(\overline{u}, C)$ and $\underline{u}^*(\underline{u}, \underline{t}, C)$ inferred from Remark 1. Let x be a rule satisfying LR, NDR, CR and MIT. We are going to show that x has to correspond to γ. We will calculate the assignment given by x in 5 steps. In the j-th step, we calculate the values of $x_i^j(\cdot)$ for all $i \in \{1, \ldots, 5\}$:

- Steps 1 and 2: The distribution of the costs c_1 and c_2 follow the same arguments as in the proof of Theorem 1. By NDR we have that $x_i^1(\cdot) = 0$ for all $i > 1$ and $x_i^2(\cdot) = 0$ for all $i > 2$. By LR and MIT we can conclude that $x_1^1(\cdot) = c_1$ and $x_2^2(\cdot) = c_2 - c_1 s$, where $s = \frac{\underline{t} + \overline{t}^*(\overline{t}, \overline{u}, C)}{2}$. Then, we obtain by definition that $x_1^2(\cdot) = c_1 s$.

- Step 3: We distribute the cost c_3. By the application of NDR, $x_i^3(\cdot) = 0$ for all $i > 3$. Consider other problem $(N_s, N_v, C, s, s, v, v)$ where $N_s = N_t$, $N_v = N_u$, $s = \frac{\underline{t} + \overline{t}^*(\overline{t}, \overline{u}, C)}{2}$ and $v = \frac{\underline{u}^*(\underline{u}, \underline{t}, C) + \overline{u}^*(\overline{u}, C)}{2}$. We have by LR that $x_3^3(N_s, N_v, C, s, s, v, v) = c_3 - \frac{c_2 s(1-v)}{1-s}$. By MIT, using a similar argument as in the proof of Theorem 1, we have that $x_3^3(N_t, N_u, C, \underline{t}, \overline{t}, \underline{u}, \overline{u}) = c_3 - \frac{c_2 s(1-v)}{1-s}$. If $s = 0$, we have that $x_3^3(N_t, N_u, C, \underline{t}, \overline{t}, \underline{u}, \overline{u}) = c_3$ and the proof of this step is finished. If $s \in (0, 1)$, we can deduce by CR as in the proof of Theorem 1 that

  $$x_1^3(N_t, N_u, C, \underline{t}, \overline{t}, \underline{u}, \overline{u}) \cdot (x_1^2(N_t, N_u, C, \underline{t}, \overline{t}, \underline{u}, \overline{u}) + x_2^2(N_t, N_u, C, \underline{t}, \overline{t}, \underline{u}, \overline{u})) = x_1^2(N_t, N_u, C, \underline{t}, \overline{t}, \underline{u}, \overline{u}) \cdot (x_1^3(N_t, N_u, C, \underline{t}, \overline{t}, \underline{u}, \overline{u}) + x_2^3(N_t, N_u, C, \underline{t}, \overline{t}, \underline{u}, \overline{u})).$$

  Given that $x_1^2(\cdot) + x_2^2(\cdot) = c_2$, $x_1^2(\cdot) = c_1 \cdot s$ and $x_1^3(\cdot) + x_2^3(\cdot) = c_2 \cdot \frac{s(1-v)}{1-s}$ we have that $x_1^3(N_t, N_u, C, \underline{t}, \overline{t}, \underline{u}, \overline{u}) = c_1 \frac{s^2(1-v)}{1-s}$ and $x_2^3(N_t, N_u, C, \underline{t}, \overline{t}, \underline{u}, \overline{u}) = c_2 \frac{s(1-v)}{1-s} - c_1 \frac{s^2(1-v)}{1-s}$.

- Step 4: We distribute the cost c_4. By similar arguments as before, we can conclude by NDR that $x_5^4(\cdot) = 0$ and by LR and MIT that $x_4^4(\cdot) = c_4 - c_3 \cdot v$. Finally, applying CR as before, we obtain that $x_3^4(\cdot) = c_3 \cdot v - c_2 \frac{s(1-v)v}{1-s}$, $x_2^4(\cdot) = c_2 \frac{s(1-v)v}{1-s} - c_1 \frac{s^2(1-v)v}{1-s}$ and $x_1^4(\cdot) = c_1 \frac{s^2(1-v)v}{1-s}$.

- Step 5: We distribute the cost c_5. By similar arguments as before, we can conclude by LR and MIT that $x_5^5(\cdot) = c_5 - c_4 \frac{v}{1-v}$. Finally, applying CR as before we obtain that $x_4^5(\cdot) = \frac{c_4 v - c_3 v^2}{1-v}$, $x_3^5(\cdot) = \frac{c_3 v^2}{1-v} - \frac{c_2 s v^2}{1-s}$, $x_2^5(\cdot) = \frac{c_2 s v^2 - c_1 s^2 v^2}{1-s}$ and $x_1^5(\cdot) = \frac{c_1 s^2 v^2}{1-s}$.

■