

Evolution in time of L-Fuzzy context sequences

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Abstract

In this work, we consider a complete lattice L and we study L -fuzzy context sequences which represent the evolution in time of an L -fuzzy context. To carry out this study, in the first part of the paper, we consider n -ary OWA operators in complete lattices, which enable us to make a general analysis and a temporal analysis at any moment in time of L -fuzzy context sequences. After that, evolution in time of the relationship between the objects and the attributes is considered. In particular, we analyze the concepts of Trend and Persistent formal contexts. Finally, we illustrate our results with an example where we consider the particular lattice $L = \mathcal{J}([0, 1])$.

Keywords: L-fuzzy context, L-fuzzy concept, L-fuzzy context sequences, n -ary OWA operators.

1. Introduction

L -Fuzzy Concept Analysis [2, 7, 10, 11, 16, 25, 30] is a mathematical tool for analyzing data and representing conceptual knowledge in a formal way. This theory makes use of L -fuzzy concepts to extract information from an L -fuzzy context. Recall that an L -fuzzy context consists of a tuple (L, X, Y, R) , where L is a complete lattice [19], X and Y are sets of objects and attributes, and $R \in L^{X \times Y}$ is an L -fuzzy relation between the objects and the attributes. L -fuzzy contexts can be seen as an extension of Formal Concept Analysis [23, 36].

In some situations, we have several relations between the object set X and the attribute set Y . Such relations lead to the notion of L -fuzzy context sequence. If this sequence recovers the evolution in time of an L -fuzzy context,

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we try to predict future trends from the analysis of past behaviors.

In Artificial Intelligence, there is a great need to represent temporal knowledge and to modelize change over time. In many domains, as science, medicine, treasury, population and weather patterns, change is noticeable from one moment to another. In this sense, the aim of this paper is to introduce an interesting tool for the study of temporal phenomena. Specifically, the main goal of the present work is the study of L -fuzzy context sequences for a complete lattice L .

We specially focus on searching trends in the evolution of the relationship between objects and attributes. Several works which consider temporal evolution in a Formal context can be found in the literature as, for instance, [33, 37, 38].

In particular, in [37, 38], Wolff introduces a Conceptual Time System to define the Temporal Concept Analysis. In this Conceptual Time System, the state and phase spaces are defined as concept lattices which represent the meaning of the states with respect to the chosen time description. Besides, authors define the hidden evolution trends in [33, 35], using temporal matching in the case of Formal Concept Analysis.

The existence of Triadic contexts [27] gives us the possibility of using ternary relations for representing time. However, this approach is very demanding for our interest and, for this reason, it is only developed for formal contexts.

Trend analysis is usually referred to techniques for extracting an underlying pattern of behavior in statistics. In this paper, we show a new and different method for L -fuzzy Contexts with quantitative data that allows one to detect some regularities. This method will establish trends that can be used as a basis for making decisions.

In the paper, the behaviour of the observed data is described by the model and some statements about tendencies are made. These are the other main contributions of this work.

We apply our results to an illustrative example that shows the monthly sales of sports items in certain shops throughout a period of time.

The paper is organized as follows: Section 2 provides a background about L -fuzzy Concept Analysis and n -ary OWA operators. Section 3 sets up a general study of L -fuzzy context sequences and Section 4 tackles a temporal study using n -ary OWA operators. Section 5 analyzes temporal trends in the L -fuzzy context sequence defining Trend and Persistent formal concepts. Finally, conclusions and future work are drawn in Section 6.

2. Preliminaries

2.1. L -fuzzy contexts

The Formal Concept Analysis of R. Wille [36] extracts information from a binary table that represents a formal context (X, Y, R) , where $R \subseteq X \times$

X and Y are finite sets of objects and attributes, respectively. The hidden information consists of pairs (A, B) with $A \subseteq X$ and $B \subseteq Y$, called formal concepts. A and B are such that $A^* = B$ and $B^* = A$, where $(\cdot)^*$ is a derivation operator that associates attributes and objects: A^* is the set of attributes common to the objects in A and B^* the set of objects which have all attributes in B . A formal concept can be interpreted as a group of objects A sharing the attributes of B .

In previous works [10, 11], we have defined L -fuzzy contexts (L, X, Y, R) , where L is a complete lattice, $R \in L^{X \times Y}$ is a fuzzy relation between the set of objects X and the set of attributes Y . This definition is an extension of Wille's Formal contexts to the fuzzy setting which allows us to study the relations between objects and attributes with values in a complete lattice L , instead of binary values.

To work with these L -fuzzy contexts, we have defined the derivation operators 1 and 2 by means of the expressions:

For all $A \in L^X$, for all $B \in L^Y$

$$A_1(y) = \inf_{x \in X} \{I(A(x), R(x, y))\}$$

$$B_2(x) = \inf_{y \in Y} \{I(B(y), R(x, y))\}$$

with I being a fuzzy implication operator defined in the lattice (L, \leq) .

Although any fuzzy implication operator can be used to define the derivation operators, in this paper we use residuated implications. Other authors have also used residuated implication operators for defining derivation operators [9, 31, 32].

The information stored in the context is visualized by means of the L -fuzzy concepts, which represent a group of objects that share a group of attributes in a fuzzy way. These are pairs $(M, M_1) \in L^X \times L^Y$, where $M \in \text{fix}(\varphi)$ is the set of fixed points of the operator φ , which is defined from the derivation operators 1 and 2 as $\varphi(M) = (M_1)_2 = M_{12}$. The first and the second components of an L -fuzzy concept are called fuzzy extension and intension, respectively.

Using the usual order relation between fuzzy sets, that is,

$$\text{for all } M, N \in L^X, \quad M \leq N \iff M(x) \leq N(x) \quad \text{for all } x \in X,$$

we define the set $\mathcal{L} = \{(M, M_1) / M \in \text{fix}(\varphi)\}$ with the order relation \preceq given by:

$$\text{for all } (M, M_1), (N, N_1) \in \mathcal{L}, \quad (M, M_1) \preceq (N, N_1) \text{ if } M \leq N \text{ (or } N_1 \leq M_1).$$

As φ is an order preserving operator, by Tarski's theorem [34], the set $\text{fix}(\varphi)$ is a complete lattice and (\mathcal{L}, \preceq) is also a complete lattice, called [10, 11] the L -fuzzy concept lattice.

Moreover, given $A \in L^X$ (or $B \in L^Y$), we can obtain the associated L -fuzzy concept applying the derivation operators twice. If we use a residuated implication, as it is the case in this work, the associated L -fuzzy concept is (A_{12}, A_1)

(or (B_2, B_{21})).

Other important results about this theory can be found in [6, 8, 9, 12]. Extensions of Formal Concept Analysis to the interval-valued case are in [3, 20, 21] and to the fuzzy property-oriented concept lattice framework, in [24, 26, 29].

Next section summarizes the main results about n -ary OWA Operators that will be of interest in the study of L -fuzzy context sequences.

2.2. n -ary OWA Operators

Families of OWA operators were introduced by Yager [39] as a new aggregation technique based on the ordered weighted averaging. The definition of these operators is as follows:

Definition 1. A mapping $F : [0, 1]^n \rightarrow [0, 1]$ is an OWA operator of dimension n if it exists a weighting n -tuple $W = (w_1, w_2, \dots, w_n)$ such that $w_i \in [0, 1]$ and $\sum_{1 \leq i \leq n} w_i = 1$, such that $F(a_1, a_2, \dots, a_n) = w_1.b_1 + w_2.b_2 + \dots + w_n.b_n$, where b_i is the i th largest element in a_1, a_2, \dots, a_n .

To study fuzzy context sequences, we are interested in the use of operators which are close to the *or* operator. To measure this closeness, we can use the concept of orness degree [39].

The extension of Yager's OWA operators to a complete lattice L is not an easy task. The main difficulty is that Yager's construction is based on a previous arrangement of the real values to be aggregated and such an arrangement is not always possible in a partially ordered set. To overcome this problem, Lizasoain and Moreno [28] built an ordered vector for each given vector of elements in the lattice. This construction allowed them to define an n -ary OWA operator on any complete lattice, in such a way that Yager's OWA operator is recovered as a particular case.

The construction, for each vector $(a_1, \dots, a_n) \in L^n$, of a totally ordered vector (b_1, \dots, b_n) is done as shown in the following proposition:

Proposition 1. Let (L, \leq_L) be a complete lattice. For any $(a_1, a_2, \dots, a_n) \in L^n$, consider the values

- $b_1 = a_1 \vee \dots \vee a_n \in L$
- $b_2 = [(a_1 \wedge a_2) \vee \dots \vee (a_1 \wedge a_n)] \vee [(a_2 \wedge a_3) \vee \dots \vee (a_2 \wedge a_n)] \vee \dots \vee [a_{n-1} \wedge a_n] \in L$
- \vdots
- $b_k = \bigvee \{a_{j_1} \wedge \dots \wedge a_{j_k} \mid \{j_1, \dots, j_k\} \subseteq \{1, \dots, n\}\} \in L$
- \vdots
- $b_n = a_1 \wedge \dots \wedge a_n \in L$

Then $a_1 \wedge \cdots \wedge a_n = b_n \leq_L b_{n-1} \leq \cdots \leq_L b_1 = a_1 \vee \cdots \vee a_n$.

Moreover, if the set $\{a_1, \dots, a_n\}$ is totally ordered, then the vector (b_1, \dots, b_n) is the same as $(a_{\sigma(1)}, \dots, a_{\sigma(n)})$ for some permutation σ of $\{1, \dots, n\}$.

Besides, it is very easy to see that if $\{a_1, \dots, a_n\}$ is a chain, b_k is the k th order statistic.

Proposition 1 allows one to generalize Yager's n -ary OWA operators from $[0, 1]$ to any complete lattice. To do so, Lizasoain and Moreno gave the following:

Definition 2. Let (L, \leq_L, T, S) be a complete lattice endowed with a t -norm T and a t -conorm S . We will say that $(\alpha_1, \alpha_2, \dots, \alpha_n) \in L^n$ is

(i) a weighting vector in (L, \leq_L, T, S) if $S(\alpha_1, \dots, \alpha_n) = 1_L$, and

(ii) a distributive weighting vector in (L, \leq_L, T, S) if it is a weighting vector such that $a = T(a, S(\alpha_1, \dots, \alpha_n)) = S(T(a, \alpha_1), \dots, T(a, \alpha_n))$ for any $a \in L$.

Definition 3. Let $(\alpha_1, \dots, \alpha_n) \in L^n$ be a distributive weighting vector in (L, \leq_L, T, S) . For each $(a_1, \dots, a_n) \in L^n$, let (b_1, \dots, b_n) be the totally ordered vector constructed in Proposition 1. The function $F_\alpha : L^n \rightarrow L$ given by

$$F_\alpha(a_1, \dots, a_n) = S(T(\alpha_1, b_1), \dots, T(\alpha_n, b_n)),$$

$(a_1, \dots, a_n) \in L^n$, is called n -ary OWA operator.

We will use these n -ary OWA operators in the following sections.

3. General study of L -fuzzy context sequences

A first study of L -fuzzy context sequences when $L = [0, 1]$ is done in [5]. We begin by recalling the main definition:

Definition 4. An L -fuzzy context sequence is a sequence of tuples (L, X, Y, R_i) , $i \in \{1, \dots, n\}$, $n \in \mathbb{N}$, with L a complete lattice, X and Y sets of objects and attributes, respectively, and $R_i \in L^{X \times Y}$ for all $i \in \{1, \dots, n\}$, a family of L -fuzzy relations between X and Y .

In [1] we have developed a general study of these L -fuzzy context sequences using n -ary OWA operators.

For summarizing the information stored in the L -fuzzy context sequence, we define:

Definition 5. Let (L, \leq_L, T, S) be a complete lattice endowed with a t -norm T and a t -conorm S . Let (L, X, Y, R_i) , $i \in \{1, \dots, n\}$, be an L -fuzzy context sequence, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ a distributive weighting vector and F_α the n -ary OWA operator associated with α . We can define an L -fuzzy relation R_{F_α} that aggregates the information of the different L -fuzzy contexts by means of the

following expression:
For all $x \in X, y \in Y$,

$$\begin{aligned} R_{F_\alpha}(x, y) &= F_\alpha(R_1(x, y), R_2(x, y), \dots, R_n(x, y)) = \\ &= S(T(\alpha_1, b_1(x, y)), T(\alpha_2, b_2(x, y)), \dots, T(\alpha_n, b_n(x, y))), \end{aligned}$$

with $(b_1(x, y), b_2(x, y), \dots, b_n(x, y))$ the totally ordered vector constructed in Proposition 1 for $(R_1(x, y), R_2(x, y), \dots, R_n(x, y))$.

Given a certain $k \in \mathbb{N}, k \leq n$, we define the relation $R_{F_{\alpha^k}}$ using an n -ary OWA operator F_{α^k} with the distributive weighting vector $\alpha^k = (\alpha_1, \alpha_2, \dots, \alpha_n)$ such that $\alpha_k = 1_L$ and $\alpha_i = 0_L$, for all $i \neq k$:

$$\begin{aligned} R_{F_{\alpha^k}}(x, y) &= F_{\alpha^k}(R_1(x, y), R_2(x, y), \dots, R_n(x, y)) = \\ &= S(T(0_L, b_1(x, y)), T(0_L, b_2(x, y)), \dots, T(1_L, b_k(x, y)), \dots, T(0_L, b_n(x, y))), \end{aligned}$$

for all $x \in X, y \in Y$.

For every pair (x, y) , this relation represents the minimum of the k largest observations as long as there are k observations which are greater than the other ones, and will be used in next section.

Moreover, in [1] we studied different results in order to establish comparisons between the L -fuzzy concepts associated with the different relations $R_{F_\alpha^k}$.

4. Temporal analysis of L -fuzzy context sequences

4.1. Temporal analysis in any complete lattice L

The general study of L -fuzzy context sequences as defined in the previous section uses different n -ary OWA operators to aggregate values. However, such an study may not allow one to make an analysis of their evolution in time.

To accomplish this analysis, the following definition tries to provide a value to estimate the relation between each object and each attribute at an instant h .

Definition 6. In the complete lattice (L, \leq_L, T, S) endowed with the t -norm T and the t -conorm S , let us consider the L -fuzzy context sequence (L, X, Y, R_i) , $i \in \{1, \dots, n\}$. Fixed $h \in \mathbb{N}, h \leq n$, let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ be a distributive weighting vector with $k = n - h + 1$ and F_α the k -ary OWA operator associated with α . We can define an L -fuzzy relation $R_{F_\alpha^h}^h$ that aggregates the information of the different L -fuzzy contexts by means of the following expression:

$$\begin{aligned} R_{F_\alpha^h}^h(x, y) &= F_\alpha(R_h(x, y), R_{h+1}(x, y), \dots, R_n(x, y)) = \\ &= S(T(\alpha_1, b_1^h(x, y)), T(\alpha_2, b_2^h(x, y)), \dots, T(\alpha_k, b_k^h(x, y))), \end{aligned}$$

for all $x \in X, y \in Y$,

where $(b_1^h(x, y), b_2^h(x, y), \dots, b_k^h(x, y))$ is the totally ordered vector constructed in Proposition 1 for $(R_h(x, y), R_{h+1}(x, y), \dots, R_n(x, y))$.

Remark 1. Note that the chain

$$b_1^h(x, y) \geq_L \cdots \geq_L b_k^h(x, y)$$

where $k = n - h + 1$, obtained in this case is not necessarily contained in the chain $b_1(x, y) \geq_L \cdots \geq_L b_n(x, y)$ obtained starting from the whole family $\{R_i(x, y) \mid i \in \{1, \dots, n\}\}$. For instance,

$$b_k^h(x, y) = R_h(x, y) \wedge \cdots \wedge R_n(x, y),$$

whereas

$$b_k(x, y) = (R_1(x, y) \wedge \cdots \wedge R_k(x, y)) \vee \cdots \vee (R_h(x, y) \wedge \cdots \wedge R_n(x, y)),$$

with all the possible intersections $R_{j_1}(x, y) \wedge \cdots \wedge R_{j_k}(x, y)$ inside the dots. In particular, in this case we see that $b_k(x, y) \geq_L b_k^h(x, y)$ for any $1 \leq h \leq n$.

A particular case of special interest is obtained when we take distributive weighting vectors with a single non null value:

Relevant case 1. Let $(L, X, Y, R_i), i \in \{1, \dots, n\}$, be an L -fuzzy context sequence with (L, \leq_L, T, S) a complete lattice, $R_i \in L^{X \times Y}$ and X and Y are sets of objects and attributes, respectively, and consider $h, k \in \mathbb{N}, h \leq n$ and $k = n - h + 1$. We define the relation $R_{F_\alpha}^h$ using a k -ary OWA operator F_α with $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ being such that $\alpha_k = 1_L$ and $\alpha_i = 0_L$, for all $i \neq k$, as:

$$\begin{aligned} R_{F_\alpha}^h(x, y) &= F_\alpha(R_h(x, y), R_{h+1}(x, y), \dots, R_n(x, y)) = \\ &= S(T(0_L, b_1^h(x, y)), T(0_L, b_2^h(x, y)), \dots, T(1_L, b_k^h(x, y))) \\ &\text{for all } x \in X, y \in Y. \end{aligned}$$

with $(b_1^h(x, y), b_2^h(x, y), \dots, b_k^h(x, y))$ the totally ordered vector constructed in Proposition 1 for $(R_h(x, y), R_{h+1}(x, y), \dots, R_n(x, y))$.

$R_{F_\alpha}^h$ is said to be the h -minimum L -fuzzy relation associated with α .

Observe that α is a distributive weighting vector. Moreover, notice that if $L = [0, 1]$, then we are using step-OWA operators [40].

These h -minimum L -fuzzy relations have interesting properties as shown below.

Proposition 2. For any t -norm T and t -conorm S , it holds that $R_{F_\alpha}^h(x, y) = b_k^h(x, y)$, for all $x \in X, y \in Y$, with $k = n - h + 1$.

Proof: The proof is straightforward taking into account the basic properties of a t -norm T and a t -conorm S , and the definition of a distributive weighting vector. \square

Remark 2. As L is a complete lattice, then

$$R_{F_\alpha}^h(x, y) = \bigwedge_{i \geq h} R_i(x, y), \text{ for all } x \in X, y \in Y.$$

In particular, if we take $L = [0, 1]$, α is the distributive weighting vector of Yager and Filev [41] when $\lambda = 1$.

Note that in this case, and after Proposition 2 and Remark 2, it is not necessary to build the corresponding chain in order to get the h -minimum L -fuzzy relation.

Another interesting case is the following one:

Relevant case 2. If $L = [0, 1]$, $T(a, b) = ab$ and $S(a, b) = \min\{a+b, 1\}$, for all $a, b \in [0, 1]$, and $k = n-h+1$, using a distributive weighting vector $\hat{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_k)$ such that $\alpha_i = 1/k$, for all i , the resulting relation $R_{F_{\hat{\alpha}}}^h$ is given by:

$$R_{F_{\hat{\alpha}}}^h(x, y) = \sum_{i=1}^k \frac{b_i^h(x, y)}{k}, \text{ for all } (x, y) \in X \times Y.$$

$R_{F_{\hat{\alpha}}}^h$ is said to be the h -average L -fuzzy relation associated with $\hat{\alpha}$.

Remark 3. Note that in this case, as any rearrangement of the values $\{R_h(x, y), \dots, R_n(x, y)\}$ would provide the same result, it is not necessary to build the corresponding chain and the resulting relation can be written as:

$$R_{F_{\hat{\alpha}}}^h(x, y) = \sum_{i=h}^n \frac{R_i(x, y)}{k}, \text{ for all } (x, y) \in X \times Y$$

From this point on, we will use the definition of h -minimum relation in order to obtain the following results.

First, we can compare the different L -fuzzy concepts obtained from the different relations $R_{F_\alpha}^h$, for all $h \leq n$.

Proposition 3. Consider $A \in L^X$. Let $h, l \leq n$ and let (A^h, B^h) and (A^l, B^l) be the L -fuzzy concepts associated with A in the L -fuzzy contexts $(L, X, Y, R_{F_\alpha}^h)$ and $(L, X, Y, R_{F_\alpha}^l)$. If $h \leq l$ then $B^h \leq B^l$.

Moreover, if we use a residuated implication operator I and a crisp singleton $A = \{x_0\}$, then

$$A^h(x_0) = A^l(x_0) = 1_L$$

A similar result is obtained taking as starting point an L -fuzzy set of attributes $B \in L^Y$.

Proof:

Consider $A \in L^X$ and $R_{F_\alpha}^h$ and $R_{F_\alpha}^l$. If $h \leq l$,

$$R_{F_\alpha}^h(x, y) = R_h(x, y) \wedge \dots \wedge R_l(x, y) \wedge \dots \wedge R_n(x, y) \leq R_l(x, y) \wedge \dots \wedge R_n(x, y) = R_{F_\alpha}^l(x, y)$$

Hence, unfolding the fuzzy extensions of both L -fuzzy concepts and taking into account that a fuzzy implication operator is increasing on its second argument:

$$B^h(y) = \inf_{x \in X} \{I(A(x), R_{F_\alpha}^h(x, y))\} \leq \inf_{x \in X} \{I(A(x), R_{F_\alpha}^l(x, y))\} = B^l(y)$$

This result holds for every L -fuzzy set A and for every implication operator.

Besides, if we take a crisp singleton $\{x_0\}$ and a residuated implication, then the membership degree of x_0 in the fuzzy extension of the L -fuzzy concepts is equal to 1_L :

$$B^h(y) = \inf_{x \in X} \{I(A(x), R_{F_\alpha}^h(x, y))\} = R_{F_\alpha}^h(x_0, y)$$

$$A^h(x) = \inf_{y \in Y} \{I(B^h(y), R_{F_\alpha}^h(x, y))\} = \inf_{y \in Y} \{I(R_{F_\alpha}^h(x_0, y), R_{F_\alpha}^h(x, y))\}.$$

Therefore, $A^h(x_0) = 1_L$. In the same way, we can prove that $A^l(x_0) = 1_L$.

The result for the other L -fuzzy set $B \in L^Y$ can be proved analogously. \square

This result can be interpreted highlighting that the fuzzy intensions obtained for the different L -fuzzy contexts of the sequence represent a non-decreasing chain for all $y \in Y$.

Remark 4. Observe that if we take the h -average relation $R_{F_\alpha}^h$, this result is not true since if $h \leq l$ then $R_{F_\alpha}^h \leq R_{F_\alpha}^l$ does not necessarily hold.

Furthermore, if an object and an attribute are related from instant h , then they are related at least $k = n - h + 1$ times, hence the following result can be proved:

Consider $k \in \mathbb{N}$, $k \leq n$. Let $R_{F_{\alpha^k}}$ be defined as in Section 3, using an n -ary OWA operator F_{α^k} with the distributive weighting vector $\alpha^k = (\alpha_1, \alpha_2, \dots, \alpha_n)$ such that $\alpha_k = 1_L$ and $\alpha_i = 0_L$, for all $i \neq k$. Then, we have:

Proposition 4. Consider $A \in L^X$, and $h \leq n$. The fuzzy intension B^h of the L -fuzzy concept (A^h, B^h) obtained in $(L, X, Y, R_{F_\alpha}^h)$ is included in the fuzzy intension \bar{B}^k of the L -fuzzy concept (\bar{A}^k, \bar{B}^k) obtained in $(L, X, Y, R_{F_{\alpha^k}})$ with $k = n - h + 1$. That is,

$$B^h(y) \leq \bar{B}^k(y), \quad \text{for all } y \in Y$$

We also have a similar result for $B \in L^Y$.

Proof: It is straightforward taking into account the proof of the previous proposition and the fact that, when $k = n - h + 1$, the following inequality holds:

$$\begin{aligned} R_{F_\alpha}^h(x, y) &= R_h(x, y) \wedge \dots \wedge R_n(x, y) \leq \\ &\leq \bigvee \{R_{j_1}(x, y) \wedge \dots \wedge R_{j_k}(x, y) \mid \{j_1, \dots, j_k\} \subseteq \{1, \dots, n\}\} = R_{F_{\alpha^k}}(x, y) \end{aligned}$$

□

To conclude this subsection, we present the following important result which allows us to study the attributes associated with some elements of X from an instant h in two different ways:

Theorem 1. *Let (L, \leq_L) be a complete lattice, A a crisp subset of X , and I a residuated implication. Let $h \leq n$ and $R_{F_\alpha}^h$ the h -minimum L relation associated with a weighting vector as defined in relevant case 1 with respect to the weighing vector $\alpha = (\alpha_1, \dots, \alpha_k)$ with $\alpha_k = 1_L$ and $\alpha_i = 0_L$ for every $i \neq k$.*

The fuzzy intension $B^h \in L^Y$ of the L -fuzzy concept derived from A in $(L, X, Y, R_{F_\alpha}^h)$ is equal to the intersection of the fuzzy intensions B_i of the L -fuzzy concepts obtained in the L -fuzzy contexts (L, X, Y, R_i) with $i \geq h$. That is,

$$B^h(y) = \bigwedge_{i \geq h} B_i(y), \quad \text{for all } y \in Y$$

Proof:

If we use a residuated implication operator I , then for all $y \in Y$ we have

$$B^h(y) = \inf_{x \in X} \{I(A(x), R_{F_\alpha}^h(x, y))\} = \bigwedge_{x \in X/A(x)=1_L} R_{F_\alpha}^h(x, y)$$

Therefore, by the definition of $R_{F_\alpha}^h(x, y)$, we can say that:

$$B^h(y) = \bigwedge_{x \in X/A(x)=1_L} \{ \bigwedge_{i \geq h} \{R_i(x, y)\} \} = \bigwedge_{i \geq h} \{ \bigwedge_{x \in X/A(x)=1_L} \{R_i(x, y)\} \} = \bigwedge_{i \geq h} B_i(y).$$

□

Remark 5. *This result justifies the utility of the defined relations $R_{F_\alpha}^h$ since they allow one to study the attributes associated with some objects from an instant h looking only at the L -fuzzy context $(L, X, Y, R_{F_\alpha}^h)$ instead of at all the L -fuzzy contexts of the sequence.*

4.2. Temporal analysis of $\mathcal{J}([0, 1])$ -fuzzy context sequences

The use of interval-valued L -fuzzy contexts gives us the opportunity to represent some real problems. We have previously published some papers [3, 14] in which the considered lattice is $L = \mathcal{J}([0, 1])$.

Note that if we take $(\mathcal{J}([0, 1]), \leq)$ with the usual order $([a_1, c_1] \leq [a_2, c_2] \iff a_1 \leq a_2 \text{ and } c_1 \leq c_2)$, this is a complete but not totally ordered lattice. In this setting, for an interval-valued L -fuzzy context sequence $(\mathcal{J}([0, 1]), X, Y, R_i), i \in \{1, \dots, n\}$, Definition 5 translates into the following:

$$R_{F_{[\alpha, \beta]}}(x, y) = F_{[\alpha, \beta]}(R_1(x, y), R_2(x, y), \dots, R_n(x, y)) = \mathbf{S}(\mathbf{T}([\alpha_1, \beta_1], [b_1(x, y), d_1(x, y)]), \dots, \mathbf{T}([\alpha_n, \beta_n], [b_n(x, y), d_n(x, y)]))$$

where $([b_1(x, y), d_1(x, y)], [b_2(x, y), d_2(x, y)], \dots, [b_n(x, y), d_n(x, y)])$ is the totally ordered vector constructed from $(R_1(x, y), R_2(x, y), \dots, R_n(x, y))$ and \mathbf{T}, \mathbf{S} are an interval-valued t-norm and t-conorm, respectively.

In this setting, relevant case 1 becomes the following:

Relevant case 3. Consider $h, k \in \mathbb{N}$ such that $h \leq n$ and $k = n - h + 1$. If we represent by $\llbracket \alpha \rrbracket$ the distributive weighting vector $\llbracket \alpha \rrbracket = ([\alpha_1, \beta_1], [\alpha_2, \beta_2], \dots, [\alpha_k, \beta_k])$ such that $[\alpha_k, \beta_k] = [1, 1]$, and $[\alpha_i, \beta_i] = [0, 0]$, for all $i \neq k$, we can define the relation $R_{F_{\llbracket \alpha \rrbracket}}^h$ as:

$$R_{F_{\llbracket \alpha \rrbracket}}^h(x, y) = [b_k^h(x, y), d_k^h(x, y)]$$

for all $(x, y) \in X \times Y$ where $([b_1^h(x, y), d_1^h(x, y)], [b_2^h(x, y), d_2^h(x, y)], \dots, [b_k^h(x, y), d_k^h(x, y)])$ is the totally ordered vector constructed from $(R_h(x, y), R_{h+1}(x, y), \dots, R_n(x, y))$.

$R_{F_{\llbracket \alpha \rrbracket}}^h$ is said to be the h -minimum L -fuzzy relation associated with $\llbracket \alpha \rrbracket$.

Remark 6. Observe that, if for all $(x, y) \in X \times Y$ we denote $R_i(x, y) = [\underline{R}_i(x, y), \overline{R}_i(x, y)]$, the previous h -minimum L -fuzzy relation can be written as:

$$R_{F_{\llbracket \alpha \rrbracket}}^h(x, y) = \left[\bigwedge_{i \geq h} \underline{R}_i(x, y), \bigwedge_{i \geq h} \overline{R}_i(x, y) \right]$$

Besides, relevant case 2 can also be translated into this interval-valued setting, as follows.

Relevant case 4. In the complete lattice $(\mathcal{J}([0, 1]), \leq)$,

considering $h \leq n$, $k = n - h + 1$ and the weighting vector $\llbracket \hat{\alpha} \rrbracket = ([\alpha_1, \beta_1], \dots, [\alpha_k, \beta_k]) \in \mathcal{J}([0, 1])^n$ verifying $[\alpha_i, \beta_i] = [\frac{1}{k}, \frac{1}{k}]$, for all i , we can define the relation $R_{F_{\llbracket \hat{\alpha} \rrbracket}}^h$, for all $(x, y) \in X \times Y$, as follows:

$$R_{F_{\llbracket \hat{\alpha} \rrbracket}}^h(x, y) = \left[\sum_{i=1}^k \frac{b_i^h(x, y)}{k}, \sum_{i=1}^k \frac{d_i^h(x, y)}{k} \right]$$

where $([b_1^h(x, y), d_1^h(x, y)], [b_2^h(x, y), d_2^h(x, y)], \dots, [b_k^h(x, y), d_k^h(x, y)])$ is the totally ordered vector constructed from $(R_h(x, y), R_{h+1}(x, y), \dots, R_n(x, y))$.

$R_{F_{\llbracket \hat{\alpha} \rrbracket}}^h$ is called the h -average L -fuzzy relation associated with $\llbracket \hat{\alpha} \rrbracket$.

Remark 7. Note that, also in this case, any rearrangement of the values $R_i(x, y)$, with $i \geq h$, would provide the same result:

$$R_{F_{\llbracket \hat{\alpha} \rrbracket}}^h(x, y) = \left[\sum_{i=h}^n \frac{\underline{R}_i(x, y)}{k}, \sum_{i=h}^n \frac{\overline{R}_i(x, y)}{k} \right]$$

In order to clarify our study in this specific interval-valued case, let us consider the following example.

Example 1. Consider the L -fuzzy context sequence (L, X, Y, R_i) , $i \in \{1, \dots, 5\}$, that represents the sports items $X = \{x_1, x_2, x_3\}$ sales in some establishment $Y = \{y_1, y_2, y_3\}$ during 5 months. Every interval-valued observation of the relations $R_i \in \mathcal{J}([0, 1])^{X \times Y}$, represents the variation of the percentage of the daily product sales in each establishment along a month.

R_1	y_1	y_2	y_3
x_1	[0.7, 0.8]	[1, 1]	[0.8, 1]
x_2	[0, 0]	[0.1, 0.4]	[0.1, 0.3]
x_3	[0, 0]	[0.1, 0.3]	[0, 0.4]

R_2	y_1	y_2	y_3
x_1	[1, 1]	[0.8, 1]	[1, 1]
x_2	[0.8, 0.9]	[0.4, 0.5]	[0.1, 0.3]
x_3	[0, 0]	[0, 0.2]	[0.2, 0.4]

R_3	y_1	y_2	y_3
x_1	[1, 1]	[1, 1]	[1, 1]
x_2	[0.6, 0.8]	[0.5, 0.5]	[0.7, 0.8]
x_3	[0, 0]	[0.1, 0.2]	[0.2, 0.4]

R_4	y_1	y_2	y_3
x_1	[0.5, 0.5]	[0.4, 0.6]	[0.6, 0.8]
x_2	[0.1, 0.3]	[0.5, 0.6]	[0.3, 0.5]
x_3	[0.6, 0.6]	[0.8, 0.9]	[0.8, 1]

R_5	y_1	y_2	y_3
x_1	[0.1, 0.4]	[0, 0.2]	[0, 0.2]
x_2	[0, 0]	[0.6, 0.8]	[0, 0.2]
x_3	[0.8, 1]	[1, 1]	[0.9, 1]

If we want to study the evolution of the sequence, we can take a value h and analyze the corresponding L -fuzzy concepts.

For instance, if $h = 4$ (fourth month), using the h -minimum L -fuzzy relation associated with $\llbracket \alpha \rrbracket$, we have:

$R_{F_{\llbracket \alpha \rrbracket}}^4$	y_1	y_2	y_3
x_1	[0.1, 0.4]	[0, 0.2]	[0, 0.2]
x_2	[0, 0]	[0.5, 0.6]	[0, 0.2]
x_3	[0.6, 0.6]	[0.8, 0.9]	[0.8, 1]

and, taking as L -fuzzy context $(L, X, Y, R_{F_{\llbracket \alpha \rrbracket}}^4)$, we can obtain the L -fuzzy concepts starting from the crisp singletons:

$$\begin{aligned}
\{x_1\} &\longrightarrow (\{x_1/[1, 1], x_2/[0.6, 0.6], x_3/[1, 1]\}, \\
&\quad \{y_1/[0.1, 0.4], y_2/[0, 0.2], y_3/[0, 0.2]\}) \\
\{x_2\} &\longrightarrow (\{x_1/[0.5, 0.6], x_2/[1, 1], x_3/[1, 1]\}, \\
&\quad \{y_1/[0, 0], y_2/[0.5, 0.6], y_3/[0, 0.2]\}) \\
\{x_3\} &\longrightarrow (\{x_1/[0.2, 0.2], x_2/[0.2, 0.2], x_3/[1, 1]\}, \\
&\quad \{y_1/[0.6, 0.6], y_2/[0.8, 0.9], y_3/[0.8, 1]\})
\end{aligned}$$

We can say that the future tendency from the fourth month on is that item x_3 is the only one which will achieve good sales in all the establishments. The other articles, x_1 and x_2 , will have poor sales, and all of them associated with x_3 . Item x_1 will be sold essentially in establishment y_1 , while x_2 will be sold in y_2 .

Now, using the h – average L -fuzzy relation associated with $[\hat{\alpha}]$, we obtain:

$R_{F_{[a]}}^4$	y_1	y_2	y_3
x_1	[0.3, 0.45]	[0.2, 0.4]	[0.3, 0.5]
x_2	[0.05, 0.15]	[0.55, 0.7]	[0.15, 0.35]
x_3	[0.7, 0.8]	[0.9, 0.95]	[0.85, 1]

And, in this case:

$$\begin{aligned}
\{x_1\} &\longrightarrow (\{x_1/[1, 1], x_2/[0.7, 0.7], x_3/[1, 1]\}, \\
&\quad \{y_1/[0.3, 0.45], y_2/[0.2, 0.4], y_3/[0.3, 0.5]\}) \\
\{x_2\} &\longrightarrow (\{x_1/[0.65, 0.7], x_2/[1, 1], x_3/[1, 1]\}, \\
&\quad \{y_1/[0.05, 0.15], y_2/[0.55, 0.7], y_3/[0.15, 0.35]\}) \\
\{x_3\} &\longrightarrow (\{x_1/[0.3, 0.45], x_2/[0.3, 0.35], x_3/[1, 1]\}, \\
&\quad \{y_1/[0.7, 0.8], y_2/[0.9, 0.95], y_3/[0.85, 1]\})
\end{aligned}$$

We can see that the sales tendency for all the items and establishments is better since the demand for this new relation is lower.

Obviously, the smaller the value of h the more certain the prediction that we do.

5. Temporal trends

In this section, we want to study temporal trends to identify the evolution in time of the L -fuzzy context sequence $(L, X, Y, R_i), i \in \{1, \dots, n\}$, when L is a complete lattice.

Our interest is focused on the study of the evolution of the relationship between the objects (or attributes) with respect to one or several attributes (or objects). In the first part of the paper, we have done different approximations considering different instants h . Now, we want to conduct a general study of the complete sequence by means of new tools.

For this purpose, we also use residuated implication operators in the calculation of the L -fuzzy concepts associated with certain objects or attributes.

In [5], we conducted a preliminary study in $[0, 1]$ and we want to extend and deepen those results to any complete lattice L . In this case, we have to take into account that, except for a complete chain, the elements of the lattice L are not necessarily comparable.

5.1. Trend and persistent objects and attributes

The best way to study the evolution in time of an object or attribute is to study its associated L -fuzzy concepts in the different L -fuzzy contexts of the sequence. This is the idea for the following:

Definition 7. Consider $x_0 \in X, y_0 \in Y$. Let $(A_{i\{x_0\}}, B_{i\{x_0\}})$ and $(A_{i\{y_0\}}, B_{i\{y_0\}})$ be the L -fuzzy concepts associated with the crisp singletons $\{x_0\}$ and $\{y_0\}$ in the L -fuzzy context sequence (L, X, Y, R_i) with $i \leq n$.

Then:

- (i) $Trend(x_0) = \{y \in Y / B_{i\{x_0\}}(y) \leq B_{i+1\{x_0\}}(y), \text{ for all } i < n\}$
is the attribute set whose membership degrees in the different intensions of the L -fuzzy concepts $(A_{i\{x_0\}}, B_{i\{x_0\}})$ are non decreasing.
- (ii) $Trend(y_0) = \{x \in X / A_{i\{y_0\}}(x) \leq A_{i+1\{y_0\}}(x), \text{ for all } i < n\}$
is the object set whose membership degrees in the different extensions of the L -fuzzy concepts $(A_{i\{y_0\}}, B_{i\{y_0\}})$ are non decreasing.

We can say that they are the attributes that are more and more related to object x_0 and the objects more and more related to attribute y_0 .

Example 2. In our example, $L = \mathcal{J}([0, 1])$ and we obtain that $Trend(x_3) = \{y_1, y_3\}$ and $Trend(y_1) = Trend(y_3) = \{x_3\}$.

This is a very demanding definition but it allows us to establish trends with a high degree of fulfillment.

Moreover, it is easy to prove the following result:

Proposition 5. Consider $x \in X, y \in Y$. Then:

$$y \in Trend(x) \iff x \in Trend(y)$$

Proof: Consider the crisp singleton $\{x\}$. Using a residuated implication operator, $B_{i\{x\}}(y) \leq B_{(i+1)\{x\}}(y)$, for all $i < n \iff R_i(x, y) \leq R_{i+1}(x, y)$, for all $i < n \iff A_{i\{y\}}(x) \leq A_{(i+1)\{y\}}(x)$, for all $i < n \iff x \in Trend(y)$. \square

We can extend this definition to the case of more than one object or attribute:

Definition 8. For every $Z, T \neq \emptyset, Z \subseteq X$ and $T \subseteq Y$:

- (i) We define $Trend(Z)$ as:

$$Trend(Z) = \{y \in Y / B_{i\{x\}}(y) \leq B_{i+1\{x\}}(y), \text{ for all } i < n, \text{ for all } x \in Z\}$$

- (ii) And $Trend(T)$ as:

$$Trend(T) = \{x \in X / A_{i\{y\}}(x) \leq A_{i+1\{y\}}(x), \text{ for all } i < n, \text{ for all } y \in T\}$$

In this case, the following can also be proven:

Proposition 6. For every $Z, T \neq \emptyset, Z \subseteq X$ and $T \subseteq Y$:

- (i) If $Trend(Z) = T$, then $Z \subseteq Trend(T)$
- (ii) If $Trend(T) = Z$, then $T \subseteq Trend(Z)$

Proof:

- (i) If $y \in Trend(Z) = T$ then $B_{i\{x\}}(y) \leq B_{(i+1)\{x\}}(y)$, for all $i < n$, for all $x \in Z \iff R_i(x, y) \leq R_{i+1}(x, y)$, for all $i < n$, for all $x \in Z$, for all $y \in Trend(Z) = T \iff A_{i\{y\}}(x) \leq A_{(i+1)\{y\}}(x)$, for all $i < n$, for all $x \in Z$, for all $y \in Trend(Z) = T \iff Z \subseteq Trend(T)$.

- (ii) Analogous to the previous one. □

As a particular case, we have the sets $Trend(X)$ and $Trend(Y)$ for which the following remark holds:

Remark 8. $Trend(X) = Y \iff Trend(Y) = X$.

Moreover, it is verified:

Proposition 7. $Trend(X) = Y$ (equiv. $Trend(Y) = X$) if and only if the L -fuzzy relations $R_{F_{\alpha^k}}$ and $R_{F_{\alpha^h}}$ defined in Sections 3 and 4, and R_h given in the L -fuzzy context sequence definition, are the same for $k = n - h + 1$.

Proof: If for all $y \in Y$, it is verified that $y \in Trend(x)$ for all $x \in X$, then this means that for all $y \in Y$, for all $x \in X, B_{i\{x\}}(y) \leq B_{i+1\{x\}}(y)$, for all $i < n$ where $B_{i\{x\}}$ is the Fuzzy intension of the L -fuzzy concept derived from the basic point associated with $x \in X$ in the L -fuzzy context (L, X, Y, R_i) .

As we are using a residuated implication operator, this is equivalent to: $R_i(x, y) \leq R_{i+1}(x, y)$, for all $x \in X$, for all $y \in Y$, for all $i < n \iff R_{F_{\alpha^k}}(x, y) = R_{n-k+1}(x, y) = \min_{i \geq h} \{R_i(x, y)\} = R_{F_{\alpha^h}}(x, y)$. □

This is a particular but very interesting case for some practical situations. In our example, as the L -fuzzy contexts store the sales, then we are saying that these sales are always increasing.

Next, we prove that if the objects and attributes are trends, then a relationship exists between the L -fuzzy concepts obtained using the context relations $R_i, i \leq n$ and the relation $R_{F_{\alpha^i}}^i$ defined from instant i .

Proposition 8. For all $x_0 \in X$ and for all $y_0 \in Y$, it is provided:

- (i) Let $(A_{i\{x_0\}}, B_{i\{x_0\}})$ and $(A_{\{x_0\}}^i, B_{\{x_0\}}^i)$, for all $i \leq n$, be the L -fuzzy concepts associated with $\{x_0\}$ in the L -fuzzy contexts (L, X, Y, R_i) and $(L, X, Y, R_{F_{\alpha^i}}^i)$, respectively.
The attribute $y \in Trend(x_0)$ if and only if $B_{i\{x_0\}}(y) = B_{\{x_0\}}^i(y)$, for all $i < n$.

- (ii) Let $(A_{i\{y_0\}}, B_{i\{y_0\}})$ and $(A_{i\{x_0\}}, B_{i\{x_0\}})$, for all $i \leq n$, be the L -fuzzy concepts associated with $\{y_0\}$ in the L -fuzzy contexts (L, X, Y, R_i) and $(L, X, Y, R_{F_\alpha}^i)$, respectively.
The object $x \in Trend(y_0)$ if and only if $A_{i\{y_0\}}(x) = A_{i\{x_0\}}(x)$, for all $i < n$.

Proof:

- (i) \implies] If $y \in Trend(x_0)$, then for all $i < n$, $B_{i\{x_0\}}(y) \leq B_{i+1\{x_0\}}(y)$. Moreover $B_{i\{x_0\}}(y) = R_i(x_0, y)$, for all $i \leq n$ then, we can say that $R_i(x_0, y) \leq R_{i+1}(x_0, y)$, for all $i < n$.
So, for all $i \leq n$, $B_{i\{x_0\}}(y) = R_{F_\alpha}^i(x_0, y) = \min_{k \geq i} \{R_k(x_0, y)\} = R_i(x_0, y) = B_{i\{x_0\}}(y)$.
 \Leftarrow] If $B_{i\{x_0\}}(y) = B^i(y)$, for all $i < n$, as by definition $B_{i\{x_0\}}(y) = R_{F_\alpha}^i(x_0, y)$ and the L -fuzzy relations $R_{F_\alpha}^i(x_0, y) \leq R_{F_\alpha}^{i+1}(x_0, y)$, for all $i \leq n$, we can prove that $B_{i\{x_0\}}(y) \leq B_{i+1\{x_0\}}(y)$, for all $i < n$. Therefore, $y \in Trend(x_0)$.

- (ii) The proof of this statement is similar to the previous one. □

This result does not hold if the objects and attributes are not trends, therefore, the study of the L -fuzzy relations $R_{F_\alpha}^i$ is very useful when we want to study future tendencies with trend objects or attributes.

As the definition of *Trend* is very demanding, in some cases, it is interesting to study *Persistent* objects and attributes in order to relax the demand level.

Definition 9. Given $x_0 \in X, y_0 \in Y$. Let $(A_{i\{x_0\}}, B_{i\{x_0\}})$ and $(A_{i\{y_0\}}, B_{i\{y_0\}})$ be the L -fuzzy concepts associated with the crisp singletons $\{x_0\}$ and $\{y_0\}$, in the L -fuzzy context sequence (L, X, Y, R_i) with $i \leq n$:

- (i) $Persistent(x_0) = \{y \in Y / B_{i\{x_0\}}(y) \geq B_{1\{x_0\}}(y), \text{ for all } i, 1 < i \leq n\}$ is the set of attributes whose membership degrees in the fuzzy intensions of the L -fuzzy concepts $(A_{i\{x_0\}}, B_{i\{x_0\}})$ are bigger than or equal to the values of the L -fuzzy concept $(A_{1\{x_0\}}, B_{1\{x_0\}})$.
- (ii) $Persistent(y_0) = \{x \in X / A_{i\{y_0\}}(x) \geq A_{1\{y_0\}}(x), \text{ for all } i, 1 < i \leq n\}$ is the set of objects whose membership degrees in the fuzzy extensions of the L -fuzzy concepts $(A_{i\{y_0\}}, B_{i\{y_0\}})$ are bigger than or equal to the values of the L -fuzzy concept $(A_{1\{y_0\}}, B_{1\{y_0\}})$.

Fixed $j \leq n$, an alternative definition of $Persistent(x_0)$ and $Persistent(y_0)$ can be given as follows:

$$Persistent_j(x_0) = \{y \in Y / B_{i\{x_0\}}(y) \geq B_{j\{x_0\}}(y), \text{ for all } i, j < i \leq n\}$$

$$Persistent_j(y_0) = \{x \in X / A_{i\{y_0\}}(x) \geq A_{j\{y_0\}}(x), \text{ for all } i, j < i \leq n\}$$

However, in the rest of the paper we will use Definition 9.

With this definition, results similar to those of Proposition 5 and 6 hold:

Proposition 9. *Consider $x \in X, y \in Y$.
The following result is verified:*

$$y \in \text{Persistent}(x) \iff x \in \text{Persistent}(y)$$

We can extend this definition to the case of more than one object or attribute:

Definition 10. *For every $Z, T \neq \emptyset, Z \subseteq X$ and $T \subseteq Y$:*

(i) *We define $\text{Persistent}(Z)$ as:*

$$\text{Persistent}(Z) = \{y \in Y / B_{i\{x\}}(y) \geq B_{1\{x\}}(y), \text{ for all } i < n, \text{ for all } x \in Z\}$$

(ii) *And $\text{Persistent}(T)$ as:*

$$\text{Persistent}(T) = \{x \in X / A_{i\{y\}}(x) \geq A_{1\{y\}}(x), \text{ for all } i < n, \text{ for all } y \in T\}$$

Proposition 10. *For all $Z, T \neq \emptyset, Z \subseteq X$ and $T \subseteq Y$:*

(i) *If $\text{Persistent}(Z) = T$, then $Z \subseteq \text{Persistent}(T)$*

(ii) *If $\text{Persistent}(T) = Z$, then $T \subseteq \text{Persistent}(Z)$*

However, results similar to those in Propositions 7 and 8 are not necessarily verified.

5.2. Trend and Persistent Formal contexts

The Trend and Persistent definitions set up pairs of objects and attributes that can be used for a more complete analysis of the evolution of the L -fuzzy sequence $(L, X, Y, R_i), i \in \{1, \dots, n\}$.

Following this idea and Definition 7, the study of tendencies of the L -fuzzy context sequence can be completed with the construction of binary Trend matrices.

Definition 11. *The Trend matrix $TM \subseteq X \times Y$ is defined as:*

$$TM(x, y) = \begin{cases} 1 & \text{if } y \in \text{Trend}(x) \text{ (equiv. } x \in \text{Trend}(y)) \\ 0 & \text{in other case} \end{cases}$$

By Proposition 5, to obtain the Trend matrix it is only necessary the calculation of $\text{Trend}(x)$, for all $x \in X$ or similarly $\text{Trend}(y)$, for all $y \in Y$.

Example 3. In our example, the Trend matrix is:

TM	y_1	y_2	y_3
x_1	0	0	0
x_2	0	1	0
x_3	1	0	1

We can consider now the formal context (X, Y, TM) and obtain its formal concepts to have a general idea of the trends between the objects X and the attributes Y .

Definition 12. Consider the formal context (X, Y, TM) with X the set of objects, Y the set of attributes and $TM \subseteq X \times Y$. The formal concepts of (X, Y, TM) are called Trend formal concepts.

Example 4. In our case, the Trend formal concepts of the L-fuzzy context sequence are:

$$(\{x_1\}, \emptyset), (\{x_2\}, \{y_2\}), (\{x_3\}, \{y_1, y_3\}), (\emptyset, \{y_1, y_2, y_3\})$$

We can say that the main trends are that sport item $\{x_2\}$ is sold in establishment $\{y_2\}$ and sport item $\{x_3\}$ in $\{y_1\}$ and $\{y_3\}$.

Remark 9. If we consider the mappings $Trend^1 : 2^X \rightarrow 2^Y$ and $Trend^2 : 2^Y \rightarrow 2^X$, that are decreasing, by Proposition 6 they form a Galois connection and then the Trend formal concept lattice is straightforwardly obtained.

It is also possible to conduct a different study using Persistent definition, following Definition 9:

Definition 13. The matrix $PM \subseteq X \times Y$ such that

$$PM(x, y) = \begin{cases} 1 & \text{if } y \in Persistent(x) (x \in Persistent(y)) \\ 0 & \text{in other case} \end{cases}$$

is called Persistent Matrix.

Example 5. In our example, the Persistent Matrix is:

PM	y_1	y_2	y_3
x_1	0	0	0
x_2	1	1	0
x_3	1	0	1

We can now consider (X, Y, PM) and calculate their formal concepts to obtain information about the tendencies between the objects of X and the attributes of Y .

Definition 14. Consider the formal context (X, Y, PM) . The formal concepts of (X, Y, PM) are called *Persistent formal concepts*.

Example 6. In our case, the Persistent formal concepts of the L -fuzzy context sequence are:

$(\{x_1\}, \emptyset), (\{x_2\}, \{y_1, y_2\}), (\{x_3\}, \{y_1, y_3\}), (\{x_2, x_3\}, \{y_1\}), (\emptyset, \{y_1, y_2, y_3\})$

The differences with the Trend formal concepts are that the tendency is that $\{x_2\}$ is also sold in $\{y_1\}$ and that a new formal concept $(\{x_2, x_3\}, \{y_1\})$ appears.

Remark 10. Also in this case, the mappings $Persistent^1 : 2^X \rightarrow 2^Y$ and $Persistent^2 : 2^Y \rightarrow 2^X$ form a Galois connection and the Persistent formal concept lattice is straightforwardly obtained.

Moreover, as the definition of Persistent is less demanding than the one of Trend, we can easily prove the following:

Proposition 11. If TM and PM are the Trend and Persistent matrices obtained from an L -fuzzy context sequence, then $TM \subseteq PM$ holds.

If we denote by $\mathcal{L}(X, Y, TM)$ and $\mathcal{L}(X, Y, PM)$ the concept lattices of the formal contexts (X, Y, TM) and (X, Y, PM) , respectively, then [23]:

Proposition 12. If $(A, B) \in \mathcal{L}(X, Y, TM)$ then there exists $(C, D) \in \mathcal{L}(X, Y, PM)$ such that $A \subseteq C$ and $B \subseteq D$.

6. Conclusions and future work

In this work, we have used n -ary OWA operators to study the evolution in time of L -fuzzy context sequences considering different departure instants h and by means of the L -fuzzy concepts. After that, we have studied tendencies that we find when we consider all the sequence using Trend and Persistent matrices. The study of these matrices gives us a general summary of the evolution of the relationship between the objects and the attributes using formal concepts.

In the future, we want to use these trends to analyze contexts with absent values. Furthermore, we will study the particular case $X = Y$ that allows the study of preference L -fuzzy contexts. Moreover, our intention is to extend Trend and Persistent definitions to the fuzzy field defining tendency and persistence degrees.

Finally, these L -fuzzy contexts that evolve in time can be generalized if we study L -fuzzy contexts where the observations are red given in terms of other L -fuzzy contexts.

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