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Application of the L-fuzzy concept analysis in the morphological image and signal processing

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Abstract In this work we are going to set up a new relationship between the L-fuzzy Concept Analysis and the Fuzzy Mathematical Morphology. Specifically we prove that the problem of finding fuzzy images or signals that remain invariant under a fuzzy morphological opening or under a fuzzy morphological closing, is equal to the problem of finding the L-fuzzy concepts of some L-fuzzy context. Moreover, since the Formal Concept Analysis and the Mathematical Morphology are the particular cases of the fuzzy ones, the showed result has also an interpretation for binary images or signals.

Keywords L-fuzzy concept analysis · Fuzzy mathematical morphology · Morphological image processing

Q1 **Mathematics Subject Classifications (2010)**

1 Introduction

The L-fuzzy Concept Analysis [15, 16], as an extension of the Formal Concept Analysis [36], and the Fuzzy Mathematical Morphology [5, 10, 11] were developed in different contexts but both use the lattice theory as algebraic framework.

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In the case of the L-fuzzy Concept Analysis, the L-fuzzy concepts were defined using a fuzzy implication. In the Fuzzy Mathematical Morphology, a fuzzy implication is also used to define the erosion but a t-norm also appears to introduce the dilation.

In the literature, there are papers that propose the use of Formal Concept Analysis for image processing [25] but no others to establish a explicit relation between the L-fuzzy Concept Analysis and the Fuzzy Mathematical Morphology. This will be the main target of this work.

On the other hand, both theories have been used in knowledge extraction processes in data bases [18–20].

The paper is organized as follows: Section 2 provides a background about L-fuzzy Concept Analysis, Mathematical Morphology and Fuzzy Mathematical Morphology. Section 3 sets up the relation between L-fuzzy Concept Analysis and Fuzzy Mathematical Morphology showing also two interesting examples. Finally, the conclusions and future work are detailed in Section 4.

2 Antecedents

2.1 L-fuzzy concept analysis

The Formal Concept Analysis of R. Wille [21, 36] extracts information from a binary table that represents a formal context (X, Y, R) with X and Y sets of objects and attributes respectively and $R \subseteq X \times Y$. The information is obtained by means of the formal concepts that are pairs (A, B) with $A \subseteq X$, $B \subseteq Y$ verifying $A^* = B$ and $B^* = A$, where $*$ is the derivation operator that associates the attributes related to the elements of A to every object set A, and the objects related to the attributes of B to every attribute set B.

In previous works [14–16] we have defined the L-fuzzy contexts (L, X, Y, R), with L a complete lattice, X and Y sets of objects and attributes respectively and $R \in L^{X\times Y}$ a fuzzy relation [22, 26] between the objects and the attributes. This is an extension of the Wille's formal contexts to the fuzzy case when we want to study the relationship between the objects and the attributes with values in a complete lattice L, instead of binary values. The use of a non residuated implication operator in our first works supposed a difficulty in the development of the algorithms.

Latter, R. Belohlavek [7–9] and S. Polland [33] have published very important papers that generalize the Formal Concepts Analysis using residuated implication operators. In these papers, the mathematical and computational results are more important than in the first ones. The results in the present paper are based on using these residuated implications.

In our original papers, we have defined the derivation operators $(\cdot)_1$ and $(\cdot)_2$ in order to work with these L-fuzzy contexts:

$$\forall A \in L^X, \forall B \in L^Y, \quad A_1(y) = \inf_{x\in X}\{I(A(x), R(x, y))\},$$
$$B_2(x) = \inf_{y\in Y}\{I(B(y), R(x, y))\},$$

with I a fuzzy implication [4] defined in the lattice $(L, \leq)$ and where A_1 represents the attributes related to the objects of A in a fuzzy way, and B_2, the objects related to all the attributes of B.

We will follow this notation and we will use residuated implications I as R. Belohlavek [7–9] and S. Polland [33] do.

The information stored in the context is visualized by means of the L-fuzzy concepts that are some pairs $(A, A_1) \in L^X \times L^Y$ with $A_{12} = A$. These pairs, whose first and second components are said to be the fuzzy extension and intension respectively, represent a set of objects that share a set of attributes in a fuzzy way.

The set $\mathcal{L} = \{(A, A_1)/A = A_{12}\}$ with the order relation $\leq$ defined as:

$$\forall (A, A_1), (C, C_1) \in \mathcal{L},\ (A, A_1) \leq (C, C_1) \text{ if } A \leq C \text{ (or } A_1 \geq C_1),$$

that is, $A(x) \leq C(x)$ (or $A_1(x) \geq C_1(x)$), $\forall x \in X$, is a complete lattice that is said to be the L-fuzzy concept lattice [15, 16].

As we are using a residuated implication operator I, the composition of the derivation operators $(\cdot)_1$ and $(\cdot)_2$ (or $(\cdot)_2$ and $(\cdot)_1$) is a closure operator with important properties studied by R. Belohlavek [6]. On the other hand, given $A \in L^X$, (or $B \in L^Y$) we can obtain the associated L-fuzzy concept (A_{12}, A_1) (or (B_2, B_{21})).

Other extensions of Formal Concept Analysis to the interval-valued case are in [1, 17] and to the fuzzy property-oriented concept lattice framework in [31, 32].

A very interesting particular case of L-fuzzy contexts appears trying to analyze situations where the objects and the attribute sets are coincident [2, 3], that is, L-fuzzy contexts (L, X, X, R) with $R \in L^{X \times X}$, (this relation can be reflexive, symmetrical …). In these situations, the L-fuzzy concepts are pairs (A, B) such that $A, B \in L^X$.

These are the L-fuzzy contexts that we are going to use to obtain the main results of this work. Specifically, we are going to take a complete chain $(L, \leq)$ as the valuation set, and L-fuzzy contexts as $(L, \mathbb{R}^n, \mathbb{R}^n, R)$ or $(L, \mathbb{Z}^n, \mathbb{Z}^n, R)$. In the first case, the L-fuzzy concepts (A, B) are interpreted as signal or image pairs related by means of R. In the second case, A and B are digital versions of these signals or images.

2.2 Mathematical morphology

The Mathematical Morphology is a theory concerned with the processing and analysis of images or signals using filters and other operators that modify them. The fundamentals of this theory (initiated by G. Matheron [29, 30] and J. Serra [34]), are in the set theory, the integral geometry and the lattice algebra. Actually this methodology is used in general contexts related to activities as the information extraction in digital images, the noise elimination, the pattern recognition and others.

2.2.1 Mathematical morphology in binary images and grey levels images

In this theory images A from $X = \mathbb{R}^n$ or $X = \mathbb{Z}^n$ (digital images or signals when $n = 1$) are analyzed.

The *morphological filters* are defined as operators $F : \wp(X) \to \wp(X)$ that transform, simplify, clean or extract relevant information from these images $A \subseteq X$, information that is encapsulated by the filtered image $F(A) \subseteq X$.

These morphological filters are obtained by means of two basic operators, the *dilation* δ_S and the *erosion* ε_S, that are defined in the case of binary images with the *sum* and *difference of Minkowski* [34], respectively.

$$\delta_S(A) = A \oplus S = \bigcup_{s \in S} A_s, \quad \varepsilon_S(A) = A \ominus \check{S} = \bigcap_{s \in \check{S}} A_s,$$

where A is an image that is treated with another $S \subseteq X$, that is said to be *structuring element*, or with its opposite $\check{S} = \{-x/x \in S\}$ and where A_s represents a translation of A: $A_s = \{a + s/a \in A\}$.

The structuring image S represents the effect that we want to produce over the initial image A.

These operators are not independent since they are dual transformations with respect to the complementation [35], that is, if A^c represents the complementary set of A, then:

$$\varepsilon_S(A) = (\delta_S(A^c))^c, \forall A, S \in \wp(X).$$

We can compose these operators dilation and erosion associated with the structuring element S and obtain the basic filters *morphological opening* $\gamma_S : \wp(X) \to \wp(X)$ and *morphological closing* $\phi_S : \wp(X) \to \wp(X)$ defined by:

$$\gamma_S = \delta_S \circ \varepsilon_S, \quad \phi_S = \varepsilon_S \circ \delta_S.$$

The opening γ_S and the closing ϕ_S over these binary images verifies the two conditions that characterize the morphological filters: They are isotone and idempotent operators, and moreover it is verified, for all $A, S \in \wp(X)$:

a) $\gamma_S(A) \subseteq A \subseteq \phi_S(A)$.
b) $\gamma_S(A) = (\phi_S(A^c))^c$.

These operators will characterize some special images (the S-open and the S-closed ones) that will play an important role in this work.

This theory is generalized introducing some tools to treat images with grey levels [34]. The images and the structuring elements are now maps defined in $X = \mathbb{R}^n$ and with values in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ or defined in $X = \mathbb{Z}^n$ and with values in finite chains as, for instance, $\{0, 1, \ldots, 255\}$. The erosion and dilation can be defined as follows:

$$\varepsilon_S(A)(x) = \inf\{A(y) - S(y - x)/y \in X\},$$
$$\delta_S(A)(x) = \sup\{A(y) + S(x - y)/y \in X\}.$$

The previous definitions can be immersed in a more general framework that considers each image as a point $x \in L$ of a partially ordered structure $(L, \leq)$ (complete lattice), and the filters as operators $F : L \to L$ with properties related to the order in these lattices [24, 34].

Now, the erosions $\varepsilon : L \to L$ are operators that preserve the infimum $\varepsilon(\inf M) = \inf \varepsilon(M), \forall M \subseteq L$ and the dilations $\delta : L \to L$, the supremum: $\delta(\sup M) = \sup \delta(M), \forall M \subseteq L$. The opening $\gamma : L \longrightarrow L$ and the closing $\phi : L \longrightarrow L$ are isotone and idempotent operators verifying $\gamma(x) \leq x \leq \phi(x), \forall x \in L$.

2.2.2 Fuzzy mathematical morphology

In this new framework and associated with lattices, a new *fuzzy morphological image processing* has been developed [5, 10–13, 27, 28] using L-fuzzy sets [22, 26] A and S (with $X = \mathbb{R}^2$ or $X = \mathbb{Z}^2$) as images and structuring elements.

In this interpretation, the filters are operators $F_S : L^X \to L^X$, where L is the chain $[0, 1]$ or a finite chain $L_k = \{0 = \alpha_1, \alpha_2, ..., \alpha_{k-1}, \alpha_k = 1\}$ with $0 < \alpha_1 < ... < \alpha_{k-1} < 1$.

In all these cases, *fuzzy morphological dilations* $\delta_S : L^X \to L^X$ and *fuzzy morphological erosions* $\varepsilon_S : L^X \to L^X$ are defined using some operators of the fuzzy logic [5, 10, 13, 28].

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In general, there are two types of relevant operators in the Fuzzy Mathematical Morphology. One of them is formed by those obtained by using some pairs $(*, I)$ of adjunct operators related by:

$$(\alpha * \beta \leq \psi) \Longleftrightarrow (\beta \leq I(\alpha, \psi)).$$

The other type are the morphological operators obtained by pairs $(*, I)$ related by a strong negation $' : L \to L$:

$$\alpha * \beta = (I(\alpha, \beta'))', \forall(\alpha, \beta) \in L \times L.$$

An example of one of these pairs that belongs to both types is the formed by the t-norm and the implication of Lukasiewicz.

In this paper, we work taking as $(X, +)$ the commutative group $(\mathbb{R}^n, +)$ or the commutative group $(\mathbb{Z}^n, +)$, and as $(L, \leq, ', I, *)$, the complete chain $L = [0, 1]$ or a finite chain as $L = L_k = \{0 = \alpha_1, \alpha_2, ..., \alpha_{k-1}, \alpha_k = 1\}$ with the Zadeh negation and $(*, I)$ the Lukasiewicz t-norm and implication.

We interpret the L-fuzzy sets $A : X \to L$ and $S : X \to L$ as n-dimensional images in the space $X = \mathbb{R}^n$ (or n-dimensional digital images in the case of $X = \mathbb{Z}^n$).

In the literature, (see [5, 10, 23, 28]), erosion and dilation operators are introduced associated with the residuated pair $(*, I)$ as follows:

If $S : X \to L$ is an image that we take as *structuring element*, then we consider the following definitions associated with (L, X, S).

Definition 1 [10] The fuzzy erosion of the image $A \in L^X$ by the structuring element S is the L-fuzzy set $\varepsilon_S(A) \in L^X$ defined as:

$$\varepsilon_S(A)(x) = \inf\{I(S(y - x), A(y))/y \in X\}, \quad \forall x \in X.$$

The fuzzy dilation of the image A by the structuring element S is the L-fuzzy set $\delta_S(A)$ defined as:

$$\delta_S(A)(x) = \sup\{S(x - y) * A(y)/y \in X\}, \quad \forall x \in X.$$

Then we obtain fuzzy erosion and dilation operators $\varepsilon_S, \delta_S : L^X \to L^X$.

The following result, that also appears in a more general context [6], is verified:

Proposition 1 *(1) If $\leq$ represents now the usual order in L^X obtained by the order extension in the chain L, then the pair $(\varepsilon_S, \delta_S)$ is an adjunction in the lattice $(L^X, \leq)$, that is:*

$$\delta_s(A_1) \leq A_2 \Longleftrightarrow A_1 \leq \varepsilon_s(A_2).$$

(2) If A' is the negation of A defined by $A'(x) = (A(x))', \forall x \in X$ and if $\breve{S}$ represents the image associated with S such that $\breve{S}(x) = S(-x), \forall x \in X$, then it is verified:

$$\varepsilon_S(A') = (\delta_{\breve{S}}(A))', \quad \delta_S(A') = (\varepsilon_{\breve{S}}(A))', \quad \forall A, S \in L^X.$$

This proposition can be proved as a consequence of the properties of the closure operators [6].

Next, we prove an interesting result from a practical point of view that shows how to obtain the fuzzy erosions and dilations with a complex structuring element reducing the problem to the application of those operators using more simple structuring elements.

Proposition 2 *Let S_1 and S_2 be two structuring elements and $\delta_{S_j} : L^X \to L^X$ and $\varepsilon_{S_j} : L^X \to L^X$ $(j = 1, 2)$ the fuzzy dilation and erosion operators in L associated with them.*

If $\circ$ *represents the usual composition, it is verified*

$$(\delta_{S_2} \circ \delta_{S_1}) = \delta_{\delta_{S_2}(S_1)}, \quad (\varepsilon_{S_2} \circ \varepsilon_{S_1}) = \varepsilon_{\delta_{S_1}(S_2)}.$$

Proof Let us prove the first equality: $\forall A \in L^X$ $(\delta_{S_2} \circ \delta_{S_1})(A) = \delta_{S_2}(\delta_{S_1}(A))$, therefore $\forall x \in X$:

$$\begin{aligned} ((\delta_{S_2} \circ \delta_{S_1})(A))(x) &= (\delta_{S_2}(\delta_{S_1}(A)))(x) = \sup_{y \in X}\{S_2(x-y) * \delta_{S_1}(A)(y)\} \\ &= \sup_{y \in X}\{S_2(x-y) * \sup_{\omega \in X}\{S_1(y-\omega) * A(\omega)\}\}, \end{aligned}$$

Due to the left continuity, this expression is equal to:

$$\sup_{y \in X}\{\sup_{\omega \in X}\{S_2(x-y) * (S_1(y-\omega) * A(\omega))\}\}.$$

Now, we can interchange the calculus of the suprema and apply the left continuity, associativity and commutativity properties:

$$\sup_{\omega \in X}\{\sup_{y \in X}\{S_2(x-y) * (S_1(y-\omega) * A(\omega))\}\} = \sup_{\omega \in X}\{A(\omega) * \sup_{y \in X}\{S_2(x-y) * S_1(y-\omega)\}\}.$$

If we take $y - \omega = z$, then the last expression is equal to:

$$\sup_{\omega \in X}\{A(\omega) * \sup_{z \in X}\{S_2(x-\omega-z) * S_1(z)\}\} = \sup_{\omega \in X}\{A(\omega) * \delta_{S_2}(S_1)(x-\omega)\},$$

and applying the commutativity of $*$, we have:

$$\sup_{\omega \in X}\{\delta_{S_2}(S_1)(x-\omega) * A(\omega)\} = (\delta_{\delta_{S_2}(S_1)}(A))(x),$$

that proves the equality $(\delta_{S_2} \circ \delta_{S_1}) = \delta_{\delta_{S_2}(S_1)}$.

Let us prove now that $(\varepsilon_{S_2} \circ \varepsilon_{S_1}) = \varepsilon_{\delta_{S_1}(S_2)}$.

$(\varepsilon_{S_2} \circ \varepsilon_{S_1})(A) \quad = \varepsilon_{S_2}(\varepsilon_{S_1}(A))$ $\forall A \in L^X$, then $\forall x \in X$:

$$\begin{aligned} ((\varepsilon_{S_2} \circ \varepsilon_{S_1})(A))(x) &= (\varepsilon_{S_2}(\varepsilon_{S_1}(A)))(x) = \inf_{y \in X}\{I(S_2(y-x), \varepsilon_{S_1}(A)(y))\} \\ &= \inf_{y \in X}\{I(S_2(y-x), \inf_{\omega \in X}\{I(S_1(\omega-y), A(\omega))\})\} \\ &= \inf_{y \in X}\{\inf_{\omega \in X}\{I(S_2(y-x), I(S_1(\omega-y), A(\omega)))\}\} \\ &= \inf_{\omega \in X}\{\inf_{y \in X}\{I(S_2(y-x), I(S_1(\omega-y), A(\omega)))\}\} \\ &= \inf_{\omega \in X}\{\inf_{y \in X}\{I(S_2(y-x) * S_1(\omega-y), A(\omega))\}\} \\ &= \inf_{\omega \in X}\{\inf_{y \in X}\{(S_2(y-x) * S_1(\omega-y)) * A'(\omega))'\}\} \\ &= \inf_{\omega \in X}\{(\sup_{y \in X}(S_2(y-x) * S_1(\omega-y)) * A'(\omega))'\} \\ &= \inf_{\omega \in X}\{(\sup_{y \in X}(S_1(\omega-y) * S_2(y-x)) * A'(\omega))'\} \\ &= \inf_{\omega \in X}\{(\delta_{S_1}(S_2)(\omega-x) * A'(\omega))'\} = \inf_{\omega \in X}\{I(\delta_{S_1}(S_2)(\omega-x), A(\omega))\} \\ &= (\varepsilon_{\delta_{S_1}(S_2)}(A))(x). \end{aligned}$$

□

3 Relation between both theories

The erosion and dilation operators given in Definition 1 are used to construct the basic morphological filters: the opening and the closing (see [5, 10, 23, 28]).

Following the usual process in Mathematical Morphology, we extend these operators to the fuzzy case:

Definition 2 The fuzzy opening of the image $A \in L^X$ by the structuring element $S \in L^X$ is the fuzzy subset $\gamma_S(A)$ that results from the composition of the erosion $\varepsilon_S(A)$ of A by S followed by its dilation:

$$\gamma_s(A) = \delta_s(\varepsilon_s(A)) = (\delta_s \circ \varepsilon_s)(A).$$

The fuzzy closing of the image $A \in L^X$ by the structuring element $S \in L^X$ is the fuzzy subset $\phi_S(A)$ that results from the composition of the dilation $\delta_S(A)$ of A by S followed by its erosion:

$$\phi_s(A) = \varepsilon_s(\delta_s(A)) = (\varepsilon_s \circ \delta_s)(A).$$

It can be proved that the operators γ_S and ϕ_S are morphological filters, that is, they preserve the order and they are idempotent:

$$A_1 \leq A_2 \Longrightarrow (\gamma_S(A_1) \leq \gamma_S(A_2)) \text{ and } (\phi_S(A_1) \leq \phi_S(A_2)),$$

$$\gamma_S(\gamma_S(A)) = \gamma_S(A), \phi_S(\phi_S(A)) = \phi_S(A), \ \forall A \in L^X, \forall S \in L^X.$$

Moreover, these filters verify that:

$$\gamma_S(A) \leq A \leq \phi_S(A), \ \forall A \in L^X, \forall S \in L^X.$$

Analogous results that those obtained for the erosion and dilation operators can be proved for the opening and closing:

Proposition 3 *If A' is the negation of A defined by $A'(x) = (A(x))' \quad \forall x \in X$, then:*

$$\gamma_S(A') = (\phi_{\breve{S}}(A))', \ \phi_S(A') = (\gamma_{\breve{S}}(A))', \ \forall A, S \in L^X$$

Proof

$$\gamma_S(A') = \delta_S(\varepsilon_S(A')) = \delta_S((\delta_{\breve{S}}(A))') = (\varepsilon_{\breve{S}}(\delta_{\breve{S}}(A)))' = (\phi_{\breve{S}}(A))'.$$

The other equality can be proved in an analogous way. □

Since the operators γ_S and ϕ_S are increasing in the complete lattice $(L^X, \leq)$, by Tarski's theorem, the respective fixed points sets are not empty. These fixed points will be used in the following definition:

Definition 3 An image $A \in L^X$ is said to be S-open if $\gamma_S(A) = A$ and it is said to be S-closed if $\phi_S(A) = A$.

These S-open and S-closed sets provide a connection between the Fuzzy Mathematical Morphology and the Fuzzy Concept Theory, as we will see next.

For that purpose, given the complete chain L that we are using, and a commutative group $(X, +)$, we will associate with any fuzzy image $S \in L^X$, the fuzzy relation $R_S \in L^{X \times X}$ such that:

$$R_S(x, y) = S(x - y), \ \forall (x, y) \in X \times X.$$

It is evident that $R_{S'} = R_S^{'}$ and, if R_S^{op} represents the opposite relation of R_S, then $R_S^{op} = R_{\breve{S}}$.

The aim is to interpret a structuring image as a relation of a context.

In agreement with this last point, we can redefine the erosion and dilation as follows:

$$\begin{aligned}
\varepsilon_S(A)(x) &= \inf\{I(R_S(y, x), A(y))/y \in X\} \\
&= \inf\{I(R_S^{op}(x, y), A(y))/y \in X\}, \ \forall x \in X, \\
\delta_S(A)(x) &= \sup\{R_S(x, y) * A(y)/y \in X\}, \ \forall x \in X.
\end{aligned}$$

With this rewriting, given the structuring element $S \in L^X$, we can interpret the triple (L, X, S) as an L-fuzzy context (L, X, X, R_S') where the sets of objects and attributes are coincident. The incidence relation $R_S' \in L^{X \times X}$ is at the same time the negation of an interpretation of the fuzzy image by the structuring element S.

We will use this representation as L-fuzzy context to prove the most important results that connect both theories:

Theorem 1 *Let (L, X, S) be the triple associated with the structuring element $S \in L^X$. Let $(L, X, X, R_S^{'})$ be the L-fuzzy context whose incidence relation $R_S^{'} \in L^{X \times X}$ is the negation of the relation R_S associated with S. Then the operators erosion ε_S and dilation δ_S en (L, X, S) are related to the derivation operators $(\cdot)_1$ and $(\cdot)_2$ in the L-Fuzzy context $(L, X, X, R_S^{'})$ by:*

$$\begin{aligned}
\varepsilon_S(A) &= (A')_1, \ \forall A \in L^X, \\
\delta_S(A) &= (A_2)', \ \forall A \in L^X.
\end{aligned}$$

Proof Taking into account the properties of the Lukasiewicz implication, for any $x \in X$, it is verified that:

$$\begin{aligned}
\varepsilon_S(A)(x) &= \inf\{I(R_S(y, x), A(y))/y \in X\} \\
&= \inf\{I(A'(y), R_S^{'}(y, x))/y \in X\} = (A')_1.
\end{aligned}$$

Analogously,

$$\begin{aligned}
\delta_S(A)(x) &= \sup\{R_S(x, y) * A(y)/y \in X\} = \sup\{(I(R_S(x, y), A'(y)))'/y \in X\} \\
&= (\inf\{I(R_S(x, y), A'(y))/y \in X\})' = (\inf\{I(A(y), R_S^{'}(x, y))/y \in X\})' \\
&= (A_2)'.
\end{aligned}$$

□

As a consequence, we obtain the following result which proves the connection between the outstanding morphological elements and the L-fuzzy concepts:

Theorem 2 *Let $S \in L^X$ and $R_S \in L^{X \times X}$ be its associated relation. The following propositions are equivalent:*

1. *The pair $(A, B) \in L^X \times L^X$ is an L-fuzzy concept of the context $(L, X, X, R_S^{'})$, where $R_S^{'}(x, y) = S'(x - y) \ \forall (x, y) \in X \times X$.*

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2. *The pair* $(A, B) \in L^X \times L^X$ *is such that the negation* A' *of* A *is* S*-open* ($\gamma_S(A') = A'$) *and* B *is the* S*-erosion of* A' *(that is,* $B = \varepsilon_S(A')$*).*
3. *The pair* $(A, B) \in L^X \times L^X$ *is such that* B *is* S*-closed* ($\phi_S(B) = B$) *and* A *is the negation of the* S*-dilation of* B *(that is,* $A = (\delta_S(B))'$*).*

Proof

1 $\Longrightarrow$ 2) Let $S \in L^X$ and $R_S \in L^{X \times X}$ be its associated relation. Let us consider an L-fuzzy concept (A, B) of the L-fuzzy context $(L, X, X, R_S^{'})$ in which $R_S^{'}$ is the negation of R_S.
Then, it is verified that $B = A_1$ and $A = B_2$, and, by the previous theorem,

$$\varepsilon_S(A') = A_1 = B.$$

Moreover, it is fulfilled that

$$\gamma_S(A') = \delta_S(\varepsilon_S(A')) = \delta_S(B) = (B_2)' = A'.$$

which proves that A' is S-open.

2 $\Longrightarrow$ 3) Let us suppose that the hypothesis of *2* are fulfilled. Then,

$$\phi_S(B) = \varepsilon_S(\delta_S(B)) = \varepsilon_S(\delta_S(\varepsilon_S(A'))) = \varepsilon_S(\gamma_S(A')) = \varepsilon_S(A') = B,$$

which proves that B is S-closed.
On the other hand, from the hypothesis $B = \varepsilon_S(A')$ can be deduced that

$$\delta_S(B) = \delta_S(\varepsilon_S(A')) = \gamma_S(A'),$$

and consequently, taking into account that A' is S-open, we obtain that $\delta_S(B) = A'$, and finally, $A = (\delta_S(B))'$.

3 $\Longrightarrow$ 1) Let (A, B) be a pair fulfilling the hypothesis of *3*. Let us consider the L-fuzzy context $(L, X, X, R_S^{'})$.
Then, by the previous theorem we can deduce that

$$B_2 = (\delta_S(B))' = A.$$

Fig. 1 Pixelated binary image D

Fig. 2 B is the fuzzy closing of the image D

On the other hand, applying the previous theorem and the hypothesis,

$$A_1 = \varepsilon_S(A') = \varepsilon_S(\delta_S(B)) = \phi_S(B) = B,$$

which finishes the proof.

□

Let us see now some examples.

Example 1 Interpretation of some binary images as formal concepts.

In the referential set $X = \mathbb{Z}^2$, the subsets of X can be interpreted as pixelated binary images, where a point belonging to the subset is represented by a white pixel, and the pixel is black otherwise.

If we consider the structuring binary image S given by,

$$S = \{(x_1, x_2) \in \mathbb{Z}^2 / x_1^2 + x_2^2 \leq 1\},$$

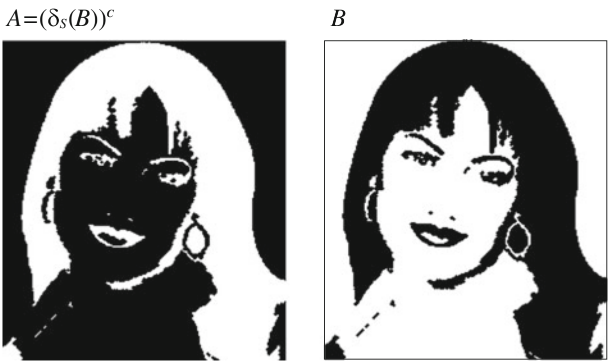


Fig. 3 The pair (A, B) is a formal concept of the context $(\mathbb{Z}^2, \mathbb{Z}^2, R_S^c)$

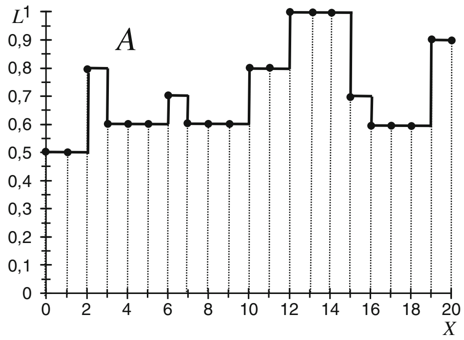


Fig. 4 Discrete signal A as an L-Fuzzy set

then the associated incidence relation $R_S^c \subset \mathbb{Z}^2 \times \mathbb{Z}^2$ is such that:

$$(x_1, x_2) R_S^c (y_1, y_2) \Longleftrightarrow ((x_1 - y_1)^2 + (x_2 - y_2)^2 > 1),$$

which is irreflexive and transitive. Using this relation we can define the formal context $(\mathbb{Z}^2, \mathbb{Z}^2, R_S^c)$.

As we have seen in the previous theorem, the problem of obtaining the formal concepts of this context $(\mathbb{Z}^2, \mathbb{Z}^2, R_S^c)$ is reduced to obtaining the S-closed sets in $\mathbb{Z}^2$ and vice versa. Let us take, for example, the binary image D in Fig. 1.

If we calculate the fuzzy closing of the image D, we can see that it is not a S-closed set because the obtained image B (see Fig. 2) is not equal to D. Therefore, by Theorem 2, there is not any formal concept of the context $(\mathbb{Z}^2, \mathbb{Z}^2, R_S^c)$ which intension is the image D.

However, the pair (A, B) showed in Fig. 3 verifies that B is S-closed ($\phi_S(B) = B$) and A is the complementary of the S-dilation of B, therefore, it is a formal concept of the context $(\mathbb{Z}^2, \mathbb{Z}^2, R_S^c)$.

Example 2 Interpretation of some open digital signals as fuzzy concepts.

If the referential set $X \subseteq \mathbb{Z}$ and the lattice $L = \{0, 0.1, 0.2, ..., 0.9, 1\}$ then, the maps $A : X \rightarrow L$ can be interpret as 1-D discrete signals.

Let us consider the referential set $X = \{0, 1, 2, ..., 20\}$ and the discrete signal represented by the L-fuzzy set A (See Fig. 4).

$$A = \{0/0.5, 1/0.5, 2/0.8, 3/0.6, 4/0.6, 5/0.6, 6/0.7, 7/0.6, 8/0.6, 9/0.6, 10/0.8$$
$$11/0.8, 12/1, 13/1, 14/1, 15/0.7, 16/0.6, 17/0.6, 18/0.6, 19/0.9, 20/0.9\}.$$

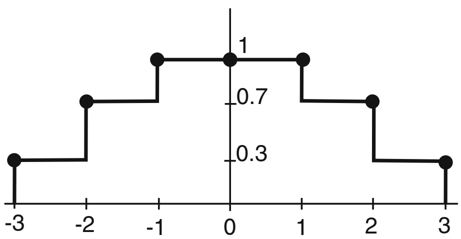
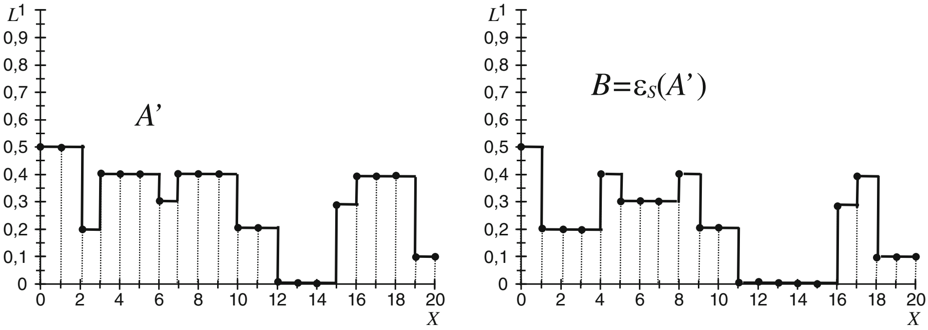


Fig. 5 Structuring element S



(a) Negation of the discrete signal A (b) Discrete signal $B = \varepsilon_S(A')$

Fig. 6 The signal A' and its fuzzy erosion B

Let us consider the structuring element given by the L-fuzzy set S:

$$S(x) = \begin{cases} 0 & \text{if } |x| > 3 \\ 0.3 & \text{if } 2 < |x| \leq 3 \\ 0.7 & \text{if } 1 < |x| \leq 2 \\ 1 & \text{if } |x| \leq 1 \end{cases}$$

represented in Fig. 5.

The associated context is (L, X, X, R'_S) where $R_S(x, y) = S(x - y)$. Then, the relation R'_S of the context is given by:

$$R'_S(x, y) = \begin{cases} 0 & \text{if } |x - y| \leq 1 \\ 0.3 & \text{if } 1 < |x - y| \leq 2 \\ 0.7 & \text{if } 2 < |x - y| \leq 3 \\ 1 & \text{if } |x| > 3 \end{cases}$$

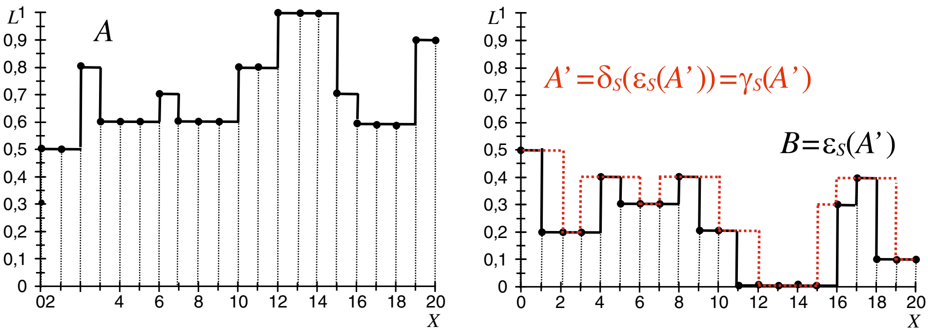


Fig. 7 The pair (A, B) is an L-Fuzzy concept of the L-Fuzzy context (L, X, X, R'_S)

If we calculate the fuzzy erosion of the negation of the signal represented by the L-fuzzy set A, we obtain the signals showed in Fig. 6:

$$\begin{aligned} A' = \{&0/0.5, 1/0.5, 2/0.2, 3/0.4, 4/0.4, 5/0.4, 6/0.3, 7/0.4, 8/0.4, 9/0.4, 10/0.2\\ &11/0.2, 12/0, 13/0, 14/0, 15/0.3, 16/0.4, 17/0.4, 18/0.4, 19/0.1, 20/0.1\},\\ \varepsilon_S(A') = \{&0/0.5, 1/0.2, 2/0.2, 3/0.2, 4/0.4, 5/0.3, 6/0.3, 7/0.3, 8/0.4, 9/0.2, 10/0.2\\ &11/0, 12/0, 13/0, 14/0, 15/0, 16/0.3, 17/0.4, 18/0.1, 19/0.1, 20/0.1\}. \end{aligned}$$

If we take now this last signal and calculate its fuzzy dilation, we can see that $\gamma_S(A') = \delta_S(\varepsilon_S(A')) = A'$. Then, the set A' is an S-open set.

Therefore, the pair $(A, \varepsilon_S(A'))$ represented in Fig. 7 is an L-fuzzy concept of the associated L-fuzzy context (L, X, X, R'_S).

4 Conclusions and future work

The main results of this work show an interesting relation between the L-fuzzy Concept Analysis and the Fuzzy Mathematical Morphology that we want to develop in future works. So, we can apply the algorithms of the L-fuzzy concept theory in Fuzzy Mathematical Morphology and vice versa. Specifically, it will be interesting to study the Morphological Gradient, and Top-Hat and Hit-or-Miss transforms and their interpretation in the L-fuzzy Concept Analysis.

On the other hand, we want to extend these results to other type of operators $(', I, *)$ (negation, fuzzy implication and conjunction) associated with certain properties in complete lattices $(L, \leq)$ and to some L-fuzzy contexts where the objects and the attributes are not related to signal or images.

Moreover, we will study the relation between both theories when we are using structuring relations $R \in L^{X\times X}$ which represent different effects that we want to produce over an initial fuzzy set $A \in L^X$.

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