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# NON-ANONYMOUS BALLOT AGGREGATION: AN AXIOMATIC GENERALIZATION OF APPROVAL VOTING

Jorge Alcalde–Unzu
Marc Vorsat

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Departamento de Economía

Ekonomia Saila

Universidad Pública de Navarra

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# Non-anonymous ballot aggregation: an axiomatic generalization of Approval Voting

Jorge Alcalde–Unzu* and Marc Vorsatz†

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**Abstract**

We study axiomatically situations in which the society agrees to treat voters with different characteristics distinctly. In this setting, we propose a set of six intuitive axioms and show that they jointly characterize a new class of voting procedures, called Personalized Approval Voting. According to this family, each voter has a strictly positive and finite weight (the weight is necessarily the same for all voters with the same characteristics) and the alternative with the highest number of weighted votes is elected. Hence, the implemented voting procedure reduces to Approval Voting in case all voters are identical or the procedure assigns the same weight to all types.

*Keywords:* Approval Voting, Characterization, Anonymity.

*JEL-Number:* D71.

*Department of Economics, Public University of Navarre, Campus Arrosadia, 31006 Pamplona, Spain. `Email: jorge.alcalde@unavarra.es`. Financial support from the Spanish Ministry of Education and Science, through the projects ECO2009–11213 and ECO2009–12836, is gratefully acknowledged.

†Corresponding author. Fundación de Estudios de Economía Aplicada (FEDEA), Calle Jorge Juan 46, 28001 Madrid, Spain. `Email: mvorsatz@fedea.es`. Financial support from the Spanish Ministry of Education and Science, through the Ramón y Cajal program and the project ECO2009–07530, is gratefully acknowledged.

# 1 Introduction

**Motivation** There are many instances in which the members of a society or an institution vote in order to take a decision and each voter's impact on the outcome depends on her/his underlying characteristics (type). Examples include the EU Member Council or the IMF Board of Directors, where the weight of a country is determined by its population size or its stake, respectively (see, Figures 1 and 2 in the Appendix); management boards, where the vote of the CEO tends to count double in case of a tie; or hiring decisions in academic institutions, where the opinion of senior members is usually given more weight. From a theoretical point of view, this implies that voters are not treated equally and that existing axiomatic results on the question of which voting procedure to implement do not directly apply. It is consequently the aim of this study to complement the existing literature on axiomatic voting theory by suggesting a general class of voting procedures that is able to cover these kind of situations.

*Plurality Voting* is the most widely used voting procedure. According to it, every individual is allowed to cast one vote and the alternative with most votes is elected. One common critique of Plurality Voting is that it may actually result in the election of the worst alternative for a majority of individuals even in single–winner elections. As a simple example, consider the case when there are three alternatives, two of which are very similar. Then, if the votes for the two similar alternatives are distributed equally, the third alternative may be elected even though a majority of the voters would prefer either of the other two alternatives.

*Approval Voting*, introduced by Brams and Fishburn (1978), has been explicitly designed to overcome this drawback of Plurality Voting by allowing individuals to vote for (or approve of) as many alternatives as they wish to.

As usual, the alternative with most votes wins. Recent evidence from field experiments by Laslier and van der Straten (2008) in France and Alós–Ferrer and Granić (2011) in Germany has shown that Approval Voting modifies the overall ranking of the candidates and that it tends to elect the candidate that is most widely *accepted* in the population. This is the main reason why we deviate from using Plurality Voting as a benchmark and frame our analysis in the (more general and more complex) context when individuals can approve any number of alternatives.

**Characterization** We are interested in general voting procedures that are operable in different voting environments in which the set of voters and the set of alternatives might vary. In particular, given a population of potential voters and a conceivable set of alternatives, a voting procedure should specify an outcome (a non-empty subset of the set of feasible alternatives) for every electorate (the individuals that indeed vote) and every set of feasible alternatives (the alternatives actually standing for election). We also assume that voters are classified by types according to some exogenous characteristic. In the examples of indirect democracy mentioned earlier, one can think of a type as the number of people or the stake the voter represents. In problems of decision making in small groups, the voter's type could be associated with her personal characteristics such as seniority or age.

In this setting, we consider a set of six intuitive properties. *Consistency in Alternatives*, which is the analogue of Arrow's Choice Axiom, states that if the set of feasible alternatives is reduced but some of the originally elected alternatives remain feasible, then exactly those alternatives have to be elected in the new situation; *Consistency in Voters*, which requires that if two disjoint electorates select a common set out of two feasible alternatives, then exactly this set has to be elected when the two electorates are assembled;

*Weak Anonymity*, which means that voters with the same type have to be treated equally; *Neutrality*, which is symmetry across alternatives; *Faithfulness*, which asks that if there is a single voter who approves $x$ but not $y$, then $x$ has to be elected whenever $x$ and $y$ are the only two feasible alternatives; and *Continuity* which states that no electorate is able to change completely the result of the election when joint with a sufficiently large sequence of electorates that all elect the same set of alternatives.

We show that these six properties fully characterize a general class of voting procedures that we will call *Personalized Approval Voting*. Each voting procedure of this family is associated with a vector of strictly positive and finite weights, one for each type of voter, and the winning alternative is the one with the highest number of weighted votes. So, if the weight is the same for all types, the voting procedure reduces to Approval Voting.

**Related Literature** Our work contributes to the existing literature on axiomatic voting theory. Roberts (1991) was the first to characterize Plurality Voting. Richelson (1978), Ching (1996), and Yeh (2008) also characterize the Plurality Rule, but as a social choice correspondence and not as a voting procedure; that is, in these studies, the domain is the Cartesian product of all linear orders on the set of alternatives. Fishburn (1978, 1979), Sertel (1988), Baigent and Xu (1991), Goodin and List (2006) and Vorsatz (2007) provide different characterizations of Approval Voting. Alós–Ferrer (2006) shows that the properties in one of Fishburn's characterizations are not independent. Finally, Massó and Vorsatz (2008) and Alcalde–Unzu and Vorsatz (2009) introduce classes of voting procedures that generalize Approval Voting in natural ways. In Massó and Vorsatz (2008), the neutrality property is relaxed; in Alcalde–Unzu and Vorsatz (2009), the weight of a vote is a decreasing function in the number of approved alternatives.

**Remainder** In Section 2, we present basic notation and definitions. Section 3 introduces the axioms and presents the characterization result. In Section 4, we discuss some aspects of the procedures that we characterize. Finally, the Appendix contains the proof of the characterization and the independence of the axioms.

# 2 Notation and Definitions

We consider a setting with a variable set of voters and alternatives. Formally, let $X$ be a finite set of *conceivable alternatives*. Generic alternatives will be denoted by $x$, $y$, and $z$; subsets of $X$ by $S$ and $T$. The cardinality of $X$, $|X|$, is greater than or equal to 3.[1] The set of *feasible alternatives* $K$, the alternatives that are actually standing for election, is a non–empty subset of $X$. Our analysis focuses on the idea that the individuals participating in the election may differ in their characteristics. To model this, we assume that there is a finite set of *types* $\Theta = \{1, 2, \ldots, \theta\}$ and that for each type $t \in \Theta$, there is an infinite number of *potential voters* $I_t$. Hence, $I \equiv \bigcup_{t\in\Theta} I_t$ is the *population* of all potential voters. The individuals actually participating in the election, the *electorate* $N$, is a non–empty and finite subset of the population $I$. We will also make frequent use of the capital letters $A$ and $B$ to denote electorates.

For any individual $i \in I$, let $M_i \in 2^X$ be the set of alternatives she votes for. A *profile* $M = (M_i)_{i\in I} \in (2^X)^I$ is a list of all votes. Given a profile $M$ and an electorate $N$, a *response profile* $M_N = (M_i)_{i\in N} \in (2^X)^N$ is the $n$–tuple of votes coming from the electorate $N$ at profile $M$. Given

[1] If there are only two conceivable alternatives, all the results of the paper hold true. The unique difference is that, when $|X| = 2$, one of the axioms, Consistency in Alternatives, is superfluous. This will become evident from the proof.

the response profile $M_N$, the number of votes $x$ receives from the individuals of type $t$ who belong to the electorate $N$ is denoted by $G_x^t(M_N)$. Thus, $G_x(M_N) = \sum_{t\in\Theta} G_x^t(M_N)$ is the total number of votes $x$ gets at $M_N$. Also, let $d_{x,y}^t(M_N) = G_x^t(M_N) - G_y^t(M_N)$ be the difference of the number of votes of type $t$ between alternatives $x$ and $y$ at $M_N$. Finally, given two disjoint electorates $A$ and $B$ and two response profiles $M_A$ and $M_B$, denote the response profile $(M_i)_{i\in A\cup B} \in (2^X)^{A\cup B}$ by $M_A + M_B$.

Given a set of feasible alternatives $K$ and an electorate $N$, a *voting rule* $v^{K,N} : (2^X)^I \to (2^K \setminus \emptyset)$ selects for all profiles $M$ a non–empty set of feasible alternatives $v^{K,N}(M)$ with the property that for all $M, \bar{M} \in (2^X)^I$ such that $M_N^K = \bar{M}_N^K$, $v^{K,N}(M) = v^{K,N}(\bar{M})$. We write $v^K(M_N)$ instead of $v^{K,N}(M)$. A *voting procedure* $\{v^{K,N} : (2^X)^I \to (2^K \setminus \emptyset)\}_{K\subseteq X, N\subset I}$ is a family of voting rules, one for every set of feasible alternatives $K$ and every electorate $N$. It is denoted by $v$. Given the voting procedure $v$ and a set of feasible alternatives $K$, the subfamily $\{v^{K,N} : (2^X)^I \to (2^K \setminus \emptyset)\}_{N\subset I}$ is denoted by $v^K$.

As we have already outlined in the Introduction, the main objective of our study is to relax the anonymity assumption underlying Approval Voting. One natural way to achieve this goal is to treat individuals with the same type equally but to possibly discriminate between votes coming from individuals of distinct types. The family we introduce next conceptualizes this idea by assigning an exogenous weight to each type.

**Definition 1** *The voting procedure $v$ is a Personalized Approval Voting if there exists a vector of weights $\mathbf{p} = (p_1, p_2, \ldots, p_\theta) \in \mathbb{R}_{++}^\theta$ such that for all sets of feasible alternatives $K \subseteq X$ and all electorates $N \subset I$,*

$$x \in v^K(M_N) \text{ if and only if } \sum_{t\in\Theta} p_t \cdot G_x^t(M_N) \geq \sum_{t\in\Theta} p_t \cdot G_y^t(M_N) \text{ for all } y \in K.$$

We denote the Personalized Approval Voting associated with the vector of weights $\mathbf{p} = (p_1, p_2, \ldots, p_\theta)$ by $v_{\mathbf{p}}$. If $\mathbf{p}$ is such that $p_s = p_t$ for all $s, t \in \Theta$, then *all* voters are treated equally and $v_{\mathbf{p}}$ coincides with Approval Voting.

# 3 Axioms and Characterization

In this section, we are going to present a characterization of all Personalized Approval Voting procedures. Since we allow in our analysis for a variable set of feasible alternatives and voters, we necessarily need two consistency conditions that establish how the selected set of alternatives adapts as either of these changes. The first property states that if the set of feasible alternatives is reduced and some of the alternatives that originally were selected remain feasible, then exactly those have to be selected in the new situation.

Consistency in alternatives: The voting procedure $v$ is *consistent in alternatives* if for all feasible sets of alternatives $S \subset T \subseteq X$, all profiles $M \in (2^X)^I$, and all electorates $N \subset I$ such that $v^T(M_N) \cap S \neq \emptyset$,

$$v^S(M_N) = v^T(M_N) \cap S.$$

The property of consistency in alternatives is important because it allows us to reformulate the question of which alternatives to choose from each subset of alternatives to the question of how to order all alternatives of the universal set $X$. To say it differently, the problems of constructing a social choice function and a social welfare function become equivalent (see, Arrow 1959). This is the reason why we can restrict our attention in the remaining five axioms to sets of feasible alternatives that only contain two alternatives.

The second consistency property, consistency in voters, says that if two disjoint groups of voters elect some common alternatives from the set $\{x, y\}$,

then exactly those alternatives have to be elected if the two groups are joined. This kind of property has been used in many characterizations of voting procedures and social choice functions; examples include, Smith (1973), Young (1974), Hansson and Sahlquist (1976), Fishburn (1978), Richelson (1978), Sertel (1988), Alós–Ferrer (2006), Massó and Vorsatz (2008), and Alcalde-Unzu and Vorsatz (2009).

Consistency in voters: The voting procedure $v$ is *consistent in voters* if for all alternatives $x, y \in X$, all profiles $M \in (2^X)^I$, and all disjoint electorates $A, B \subset I$ such that $v^{\{x,y\}}(M_A) \cap v^{\{x,y\}}(M_B) \neq \emptyset$,

$$v^{\{x,y\}}(M_A + M_B) = v^{\{x,y\}}(M_A) \cap v^{\{x,y\}}(M_B).$$

The third condition, weak anonymity, relaxes the classical symmetry condition according to which the result of the election should be invariant to permutations of voters. Here, we only require this symmetry condition to hold true if voters of the same type are permuted. Formally, two response profiles $M_A$ and $M'_B$ are *isomorphic relative to* $\{x, y\}$ if for all $t \in \Theta$, there is a permutation $\pi_t : I_t \cap A \to I_t \cap B$ such that $M_{\pi_t(i)} \cap \{x, y\} = M_i \cap \{x, y\}$.

Weak Anonymity: The voting procedure $v$ is *weakly anonymous* if for all alternatives $x, y \in X$ and all response profiles $M_A$ and $M'_B$ that are isomorphic relative to $\{x, y\}$,

$$v^{\{x,y\}}(M_A) = v^{\{x,y\}}(M'_B).$$

The next property, neutrality, is also a standard condition. It states that if alternatives are permuted, then the set of elected alternatives has to be permuted accordingly. Formally, given a permutation $\mu : X \to X$ and a pair of alternatives $\{x, y\}$, let $\mu(M_N)$ and $\mu(v^{\{x,y\}}(M_N))$ be the response profile

and the set of elected alternatives permuted according to $\mu$.

NEUTRALITY: The voting procedure $v$ is *neutral* if for all alternatives $x, y \in X$, all profiles $M \in (2^X)^I$, all electorates $N \subset I$, and all permutations $\mu : X \to X$,

$$\mu(v^{\{x,y\}}(M_N)) = v^{\mu(\{x,y\})}(\mu(M_N)).$$

So far, we have not introduced any kind of monotonicity condition which insures that getting more votes is desirable. The following property is a weak condition inspired by Fishburn (1978).

FAITHFULNESS: The voting procedure $v$ is *faithful* if for all individuals $i \in I$ and all alternatives $x, y \in X$,

$$M_i = \{x\} \Rightarrow v^{\{x,y\}}(M_i) = \{x\}.$$

To introduce the last property, Continuity, consider an infinite number of disjoint electorates such that all of them only select the same alternative $x$ from the set $\{x, y\}$. Suppose also that there is another electorate $A$, disjoint from the other electorates, for which $y$ is the unique alternative elected from the set $\{x, y\}$. The idea of continuity is that if a sufficient number of electorates that elect $x$ are joined together with $A$, then alternative $x$ should be elected (but not necessarily excluding $y$). In the literature, similar conditions are found under the names of *Archimedean Property* or *Overwhelming Majority*; see, Smith (1973), Young (1974, 1975), Richelson (1978), Myerson (1996), or Alcalde-Unzu and Vorsatz (2009). This condition eliminates, for example, dictatorship–like procedures that give an infinite weight to some type $t \in \Theta$ of voters or procedures that break ties in a lexicographic way.

CONTINUITY: The voting procedure $v$ is *continuous* if for all alternatives $x, y \in X$, all profiles $M \in (2^X)^I$, all successions of disjoint electorates $\{N_p\}$

such that $v^{\{x,y\}}(N_p) = \{x\}$ for all $p \in \mathbb{N}$, and any other electorate $A$ for which $A \cap N_p = \emptyset$ for all $p \in \mathbb{N}$ and $v^{\{x,y\}}(M_A) = \{y\}$, there exists $k \in \mathbb{N}$ such that

$$x \in v^{\{x,y\}}(M_{N_1} + M_{N_2} + \cdots + M_{N_k} + M_A).$$

Our main result state that these six properties fully characterize the set of all Personalized Approval Voting.

**Theorem 1** *The voting procedure $v$ is consistent in alternatives, consistent in voters, weakly anonymous, neutral, faithful, and continuous if and only if it is a Personalized Approval Voting.*

Proof: See the Appendix. □

We also show that the six properties are independent.

**Proposition 1** *The properties in Theorem 1 are independent.*

Proof: See the Appendix. □

The proof that the mentioned properties imply $v$ to be a Personalized Approval Voting is constructive and divided into several steps. We now shortly explain the structure of the proof (that is included in the Appendix) in order to facilitate its reading.

1. It is shown in Lemma 1 that if one individual either approves both $x$ and $y$ or neither of the two alternatives, then eliminating this individual from the electorate does not affect the result of the election in case $x$ and $y$ are the only two feasible alternatives.

2. Lemma 2 shows that that if $x$ and $y$ receive the same number of votes from each type $t \in \Theta$, then both alternatives have to be elected if they are the only two feasible alternatives.

3. Lemma 3 establishes that if alternatives $z$ and $w$ receive the same number of votes from each type $t \in \Theta$ under the response profile $M'_B$ as alternatives $x$ and $y$, respectively, under the response profile $M_A$, then (i) $z$ is elected from the set $\{z, w\}$ if and only if $x$ is elected from the set $\{x, y\}$ and (ii) $w$ is elected from the set $\{z, w\}$ if and only if $y$ is elected from the set $\{x, y\}$.

4. We construct a binary relation $\succsim$ over vectors $(x_1, x_2, \ldots, x_\theta) \in \mathbb{N}^\theta$, interpreting each of the vectors as a possible combination of numbers of votes from each type. The binary relation represents which combinations of votes are better for an alternative to be selected by the voting procedure.

5. With the help of Lemma 3 and step 4, we establish that the conditions of Theorem 1 in Krantz *et al.* (1971) are satisfied. This allows us to show that the subfamily $v^{\{x,y\}}$ is a Personalized Approval Voting.

6. Finally, we apply consistency in alternatives to show that, independently of the number of alternatives, the voting procedure $v$ is a Personalized Approval Voting.

# 4 Conclusion

We have characterized a voting procedure –a family of voting rules– that generalizes Approval Voting for the cases when the society agrees not to treat all voters equally. In particular, we have shown that a voting procedure is consistent in alternatives, consistent in voters, weakly anonymous (*i.e.*, only voters with the same characteristics are necessarily treated in the same way), neutral, faithful, and continuous if and only if there exists a strictly positive

and finite weight for each type of voter and the alternatives with the maximal sum of weighted votes are selected.

Since we characterize a large class of voting rules and therefore allow a priori for a wide variety of discrimination between voters, there is the need to discuss how the weights are ultimately determined. In particular, one needs to identify first the relevant characteristics of the voters. Only afterwards one has to decide how to weigh different voters.

In some cases of indirect democracy in which each voter represents a set of citizens, the first step is not controversial: the classification should be done in function of the number of people each voter represents. However, there is a vast literature that discusses which weight each representative should have as a function of her type. At a first sight, one would think that the weights should be proportional to the number of people each voter represents. Yet, it has been shown that this is probably not the best voting rule. Barbera and Jackson (2006) characterize the efficient weights –the ones that maximize the total expected utility–, which turn out to be different from the proportional ones. Other authors have proposed also structures of weights in basis of other criteria such as the equality of the probability of each person to be pivotal in the election or the equality of the expected satisfaction of each person with the outcome. See, for example, Laruelle and Valenciano (2008) for this strand of the literature.

In many other cases the voters only represent themselves, yet it might still be desirable to implement a discriminative voting rule. This is the point where the discussion about which characteristics should determine the weight clearly emerges. Even though it is impossible to provide a definite guideline, one can look, for example, at elections in universities. In some of them, voters are classified depending on their type of affiliation (students, administrative

staff, professors, *etc.* ); in others, the classification is more detailed and also considers other aspects such as seniority.

All in all, and independently of the more or less difficulty to define the appropriate criteria to classify voters, our axiomatic study provides a theoretical background for the use of a Personalized Approval Voting procedure.

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## Proof of the Theorem

It is easy to check that all Personalized Approval Voting procedures satisfy consistency in voters, consistency in alternatives, weak anonymity, neutrality, faithfulness, and continuity. The proof that these properties imply $v$ to be a Personalized Approval Voting follows the before-mentioned steps.

**Lemma 1** *If the voting procedure $v$ is consistent in voters and neutral, then for all alternatives $x, y \in X$, all profiles $M \in (2^X)^I$, all electorates $N \subset I$ and all voters $i \in N$ such that $M_i \cap \{x, y\} \in \{\emptyset, \{x, y\}\}$,*

$$v^{\{x,y\}}(M_N) = v^{\{x,y\}}(M_{N\setminus\{i\}}).$$

<u>Proof</u>: Take any two alternatives $x, y \in X$, any profile $M \in (2^X)^I$, any electorate $N \subset I$ and any voter $i \in N$ such that $M_i \cap \{x, y\} \in \{\emptyset, \{x, y\}\}$. We are going to show first by contradiction that $v^{\{x,y\}}(M_i) = \{x, y\}$.

Suppose that $v^{\{x,y\}}(M_i) = \{x\}$. Consider the permutation $\mu : X \to X$ such that $\mu(x) = y$, $\mu(y) = x$ and $\mu(z) = z$ for all $z \in X \setminus \{x, y\}$. Then, by neutrality, $\mu(v^{\{x,y\}}(M_i)) = v^{\mu(\{x,y\})}(\mu(M_i))$. Given that $\mu(v^{\{x,y\}}(M_i)) = \{y\}$, that $\mu(\{x, y\}) = \{x, y\}$ by definition of $\mu$ and that $\mu(M_i) = M_i$, we have that $v^{\{x,y\}}(M_i) = \{y\}$. This is a contradiction. Since $v^{\{x,y\}}(M_i) = \{y\}$ can be excluded using a similar argument and since $v^{\{x,y\}}(M_i) \neq \emptyset$ by definition, we can conclude that $v^{\{x,y\}}(M_i) = \{x, y\}$. Finally, $v^{\{x,y\}}(M_{N\setminus\{i\}}) \cap \{x, y\} \neq \emptyset$ implies that we can apply consistency in voters to obtain that $v^{\{x,y\}}(M_N) = v^{\{x,y\}}(M_{N\setminus\{i\}})$. This concludes the proof of the lemma. $\square$

The successive applications of Lemma 1 implies that given an electorate $N$ and any two alternatives $x$ and $y$ standing for election, it can be assumed that all individuals belonging to $N$ vote for one and only one of these two alternatives (voters who do not declare a strict preference between $x$ and $y$

can simply be discarded). Also note if $M_i \in \{\emptyset, \{x, y\}\}$ for all $i \in N$, then both alternatives have to be elected by neutrality.

**Lemma 2** *If the voting procedure $v$ is consistent in voters, weakly anonymous, and neutral, then for all alternatives $x, y \in X$, all profiles $M \in (2^X)^I$, and all electorates $N \subset I$ such that $G_x^t(M_N) = G_y^t(M_N)$ for all $t \in \Theta$,*

$$v^{\{x,y\}}(M_N) = \{x, y\}.$$

<u>Proof</u>: Take any two alternatives $x, y \in X$, any profile $M \in (2^X)^I$, and any electorate $N \subset I$ such that $G_x^t(M_N) = G_y^t(M_N)$ for all $t \in \Theta$. By Lemma 1, we can assume that $M_i \cap \{x, y\} \in \{\{x\}, \{y\}\}$ for all $i \in N$. Partition the electorate $N$ into $\theta$ sub–electorates $N_1, \ldots, N_\theta$ in such a way that $i \in N_t$ if and only if $i \in N \cap I_t$.

Consider any type $t \in \Theta$ for which $|N_t| > 0$. We are going to show by contradiction that $v^{\{x,y\}}(M_{N_t}) = \{x, y\}$. Suppose that $v^{\{x,y\}}(M_{N_t}) = \{x\}$. Take the permutation $\mu : X \to X$ such that $\mu(x) = y, \mu(y) = x$, and $\mu(z) = z$ for all $z \in X \setminus \{x, y\}$. Then, by neutrality, $v^{\mu(\{x,y\})}(\mu(M_{N_t})) = \mu(v^{\{x,y\}}(M_{N_t})) = \{y\}$. Since $\mu(\{x, y\}) = \{x, y\}$ by the definition of the permutation, the former equation can be rewritten as $v^{\{x,y\}}(\mu(M_{N_t})) = \{y\}$. Now observe that $\mu(M_{N_t})$ is an isomorphic copy of $M_{N_t}$ relative to $\{x, y\}$ because $G_x^t(M_N) = G_y^t(M_N)$ by assumption. By weak anonymity, $v^{\{x,y\}}(M_{N_t}) = v^{\{x,y\}}(\mu(M_{N_t})) = \{y\}$, which contradicts our initial assumption that $v^{\{x,y\}}(M_{N_t}) = \{x\}$. A symmetric argument proves that $v^{\{x,y\}}(M_{N_t}) \neq \{y\}$ and, therefore, we are able to conclude that $v^{\{x,y\}}(M_{N_t}) = \{x, y\}$.

Finally, using that $\bigcap_{t \in \Theta : |N_t| > 0} v^{\{x,y\}}(M_{N_t}) \neq \emptyset$, the iterative application of consistency in voters implies that

$$v^{\{x,y\}}(M_N) = v^{\{x,y\}}\left(\sum_{t \in \Theta : |N_t| > 0} M_{N_t}\right) = \bigcap_{t \in \Theta : |N_t| > 0} v^{\{x,y\}}(M_{N_t}) = \{x, y\}.$$

This concludes the proof of the lemma. $\square$

**Lemma 3** *If the voting procedure $v$ is consistent in voters, weakly anonymous, and neutral, then for all alternatives $x, y, z, w \in X$, all profiles $M, M' \in (2^X)^I$, and all electorates $A, B \subset I$ such that $G^t_x(M_A) = G^t_z(M'_B)$ and $G^t_y(M_A) = G^t_w(M'_B)$ for all $t \in \Theta$,*

$$x \in v^{\{x,y\}}(M_A) \Leftrightarrow z \in v^{\{z,w\}}(M'_B) \text{ and } y \in v^{\{x,y\}}(M_A) \Leftrightarrow w \in v^{\{z,w\}}(M'_B).$$

<u>Proof</u>: Take any four alternatives $x, y, z, w \in X$, any two profiles $M, M' \in (2^X)^I$, and any two electorates $A, B \subset I$ such that $G^t_x(M_A) = G^t_z(M'_B)$ and $G^t_y(M_A) = G^t_w(M'_B)$ for all $t \in \Theta$. By Lemma 1, we can assume that $M_i \cap \{x, y\} \in \{\{x\}, \{y\}\}$ for all $i \in N$. Partition the electorate $A$ into $\theta$ sub–electorates $A_1, \ldots, A_\theta$ in such a way that $i \in A_t$ if and only if $i \in A \cap I_t$. Construct the electorates $B_1, \ldots, B_\theta$ in an identical manner.

For each type $t \in \Theta$, partition the electorate $A_t$ into two sub–electorates, $A_{t_1}$ and $A_{t_2}$, in such a way that exactly $|d^t_{x,y}(M_A)|$ individuals belong to $A_{t_1}$ and all these individuals only vote for the alternative that receives more votes at $M_A$; that is, for all $i \in A_{t_1}$, $M_i \cap \{x, y\} = \{x\}$ whenever $G^t_x(M_A) > G^t_y(M_A)$ and $M_i \cap \{x, y\} = \{y\}$ whenever $G^t_x(M_A) < G^t_y(M_A)$ and, obviously, $A_{t_1} = \emptyset$ in case $G^t_x(M_A) = G^t_y(M_A)$. Then, $A_{t_2} = A_t \setminus A_{t_1}$. The electorates $B_{t_1}$ and $B_{t_2}$ are derived from $B_t$ in a similar fashion.

Consider the permutation $\mu : X \to X$ such that $\mu(z) = x$, $\mu(w) = y$, and that $\mu(s) = s$ for all $s \in X \setminus \{z, w\}$. Then, $\mu(M'_{B_{t_j}})$ and $M_{A_{t_j}}$ are isomorphic relative to $\{x, y\}$ for all $t \in \Theta$ and all $j \in \{1, 2\}$. Summing up over all types we can see that the response profiles $M_{A_1} = \sum_{t \in \Theta : |A_{t_1}| > 0} M_{A_{t_1}}$ and $\mu(M'_{B_1}) = \sum_{t \in \Theta : |B_{t_1}| > 0} \mu(M'_{B_{t_1}})$ are isomorphic relative to $\{x, y\}$. By weak anonymity,

$$v^{\{x,y\}}(M_{A_1}) = v^{\{x,y\}}(\mu(M'_{B_1})). \tag{1}$$

Also, we have that $d^t_{x,y}(M_{A_{t_2}}) = d^t_{z,w}(M'_{B_{t_2}}) = 0$ for all $t \in \Theta$ by the way we partitioned the electorates. So, define $M_{A_2} = \sum_{t\in\Theta:|A_{t_2}|>0} M_{A_{t_2}}$ and $\mu(M'_{B_2}) = \sum_{t\in\Theta:|B_{t_2}|>0} \mu(M'_{B_{t_2}})$ and apply Lemma 2 to see that

$$v^{\{x,y\}}(M_{A_2}) = v^{\{x,y\}}(\mu(M'_{B_2})) = \{x, y\}. \tag{2}$$

Now, apply consistency in voters together with Equations (1) and (2) to see that $v^{\{x,y\}}(M_A) = v^{\{x,y\}}(M_{A_1}+M_{A_2}) = v^{\{x,y\}}(M_{A_1})$ and that $v^{\{x,y\}}(\mu(M'_B)) = v^{\{x,y\}}(\mu(M'_{B_1})+\mu(M'_{B_2})) = v^{\{x,y\}}(\mu(M'_{B_1}))$. This, together with Equation (1), implies that

$$v^{\{x,y\}}(M_A) = v^{\{x,y\}}(\mu(M'_B)). \tag{3}$$

Finally, consider the permutation $\mu^{-1}$. By neutrality, $\mu^{-1}(v^{\{x,y\}}(\mu(M'_B))) = v^{\mu^{-1}(\{x,y\})}(\mu^{-1}(\mu(M'_B))) = v^{\{z,w\}}(M'_B)$. This, together with Equation (3), implies that $\mu^{-1}(v^{\{x,y\}}(M_A)) = v^{\{z,w\}}(M'_B)$. Hence, $x \in v^{\{x,y\}}(M_A)$ if and only if $z \in v^{\{z,w\}}(M'_B)$ and $y \in v^{\{x,y\}}(M_A)$ if and only if $w \in v^{\{z,w\}}(M'_B)$. $\square$

Next, consider the following binary relation $\succsim$ defined over $\mathbb{N}^\theta \times \mathbb{N}^\theta$: for all $(x_1, \ldots, x_\theta), (y_1, \ldots, y_\theta) \in \mathbb{N}^\theta$, $(x_1, \ldots, x_\theta) \succsim (y_1, \ldots, y_\theta)$ if there exists a response profile $M_N$ and two alternatives $x, y \in X$ such that $x \in v^{\{x,y\}}(M_N)$ and for all $t \in \Theta$, $G^t_x(M_N) = x_t$ and $G^t_y(M_N) = y_t$. Our objective is to show that the triple $(\mathbb{N}^\theta, \succsim, +)$ is a *closed extensive structure*; that is, this triple satisfies the following properties (see Krantz *et al.* 1971):

1. Complete preorder: $\succsim$ is a complete preorder over $\mathbb{N}^\theta \times \mathbb{N}^\theta$.

2. Associativity: For all $(x_1, \ldots, x_\theta), (y_1, \ldots, y_\theta), (z_1, \ldots, z_\theta) \in \mathbb{N}^\theta$, we have that $(x_1, \ldots, x_\theta) + ((y_1, \ldots, y_\theta) + (z_1, \ldots, z_\theta)) \sim ((x_1, \ldots, x_\theta) + (y_1, \ldots, y_\theta)) + (z_1, \ldots, z_\theta)$.

3. Independence: For all $(x_1, \ldots, x_\theta), (y_1, \ldots, y_\theta), (z_1, \ldots, z_\theta) \in \mathbb{N}^\theta$, we have that $(x_1, \ldots, x_\theta) \succsim (y_1, \ldots, y_\theta) \Leftrightarrow ((x_1, \ldots, x_\theta) + (z_1, \ldots, z_\theta)) \succsim$

$((y_1, \ldots, y_\theta)+(z_1, \ldots, z_\theta)) \Leftrightarrow ((z_1, \ldots, z_\theta)+(x_1, \ldots, x_\theta)) \succsim ((z_1, \ldots, z_\theta)+(y_1, \ldots, y_\theta))$.

4. Archimedean: For all four $(x_1, \ldots, x_\theta), (y_1, \ldots, y_\theta), (z_1, \ldots, z_\theta)$ and $(w_1, \ldots, w_\theta) \in \mathbb{N}^\theta$, if $(x_1, \ldots, x_\theta) \succ (y_1, \ldots, y_\theta)$, then there exists a positive integer $t$ such that $(t \cdot (x_1, \ldots, x_\theta)+(z_1, \ldots, z_\theta)) \succsim (t \cdot (y_1, \ldots, y_\theta)+(w_1, \ldots, w_\theta))$.

**Lemma 4** *The triple* $(\mathbb{N}^\theta, \succsim, +)$ *is a closed extensive structure.*

Proof: We show that the triple $(\mathbb{N}^\theta, \succsim, +)$ satisfies the conditions of Complete Preorder, Associativity, Independence, and Archimedean.

Complete Preorder: To see that the binary relation $\succsim$ is well-defined, take any $(x_1, \ldots, x_\theta), (y_1, \ldots, y_\theta) \in \mathbb{N}^\theta$ and consider the response profiles $M_A, M'_B$ together with the alternatives $x, y, z, w \in X$ such that $G^t_x(M_A) = G^t_z(M'_B) = x_t$ and $G^t_y(M_A) = G^t_w(M'_B) = y_t$ for all $t \in \Theta$. We have to establish that $x \in v^{\{x,y\}}(M_A) \Leftrightarrow z \in v^{\{z,w\}}(M'_B)$ and that $y \in v^{\{x,y\}}(M_A) \Leftrightarrow w \in v^{\{z,w\}}(M'_B)$. But this is exactly what we have shown in Lemma 3. Hence, $\succsim$ is well-defined.

To show that the binary relation $\succsim$ is complete, note first that for any $(x_1, \ldots, x_\theta), (y_1, \ldots, y_\theta) \in \mathbb{N}^\theta$, we can consider a response profile $M_N$ and two alternatives $x, y \in X$ such that for all $t \in \Theta$, $N \cap I_t$ consists of $x_t$ individuals voting only for alternative $x$ and $y_t$ individuals voting only for alternative $y$. This is always possible because $I_t$ is an infinite set for all $t \in \Theta$. By definition of $v$, we have that $v^{\{x,y\}}(M_N) \in \{\{x\}, \{y\}, \{x, y\}\}$. Then, it follows from the definition of $\succsim$ that $(x_1, \ldots, x_\theta) \succsim (y_1, \ldots, y_\theta)$ and/or $(y_1, \ldots, y_\theta) \succsim (x_1, \ldots, x_\theta)$. Hence, $\succsim$ is complete.

To see that $\succsim$ is transitive, take any $(x_1, \ldots, x_\theta), (y_1, \ldots, y_\theta), (z_1, \ldots, z_\theta) \in \mathbb{N}^\theta$ such that $(x_1, \ldots, x_\theta) \succsim (y_1, \ldots, y_\theta)$ and $(y_1, \ldots, y_\theta) \succsim (z_1, \ldots, z_\theta)$. Con-

sider any response profile $M_N$ and any three alternatives $x, y, z \in X$ such that for all $t \in \Theta$, $N \cap I_t$ consists of $x_t$ individuals voting only for alternative $x$, $y_t$ individuals voting only for alternative $y$, and $z_t$ individuals voting only for alternative $z$. Since $(x_1, \ldots, x_\theta) \succsim (y_1, \ldots, y_\theta)$ and $(y_1, \ldots, y_\theta) \succsim (z_1, \ldots, z_\theta)$ by assumption, the definition of $\succsim$ implies that $x \in v^{\{x,y\}}(M_N)$ and $y \in v^{\{y,z\}}(M_N)$. Suppose that transitivity is violated; that is, $v^{\{x,z\}}(M_N) = \{z\}$. Then,

(a) $x \notin v^{\{x,y,z\}}(M_N)$. If it was the case that $x \in v^{\{x,y,z\}}(M_N)$, then, by consistency in alternatives, we would have $x \in v^{\{x,z\}}(M_N)$. This contradicts $v^{\{x,z\}}(M_N) = \{z\}$.

(b) $y \notin v^{\{x,y,z\}}(M_N)$. If it was the case that $y \in v^{\{x,y,z\}}(M_N)$, then, by consistency in alternatives, we would have $y \in v^{\{x,y\}}(M_N)$. This, together with the assumption $x \in v^{\{x,y\}}(M_N)$, would imply that $v^{\{x,y\}}(M_N) = \{x, y\}$. Hence, by consistency in alternatives, $x \in v^{\{x,y,z\}}(M_N)$, which contradicts case (a).

(c) $z \notin v^{\{x,y,z\}}(M_N)$. If it was the case that $z \in v^{\{x,y,z\}}(M_N)$, then, by consistency in alternatives, we would have $z \in v^{\{y,z\}}(M_N)$. This, together with the assumption $y \in v^{\{y,z\}}(M_N)$, would imply that $v^{\{y,z\}}(M_N) = \{y, z\}$. Hence, by consistency in alternatives, $y \in v^{\{x,y,z\}}(M_N)$, which contradicts case (b).

The three cases together imply that $v^{\{x,y,z\}}(M_N) = \emptyset$. This is not possible by definition and, therefore, we have reached a contradiction. Consequently, $\succsim$ is transitive. Since the binary relation $\succsim$ is well-defined, complete, and transitive, it is a complete preorder.

Associativity: The property holds because $+$, the usual addition operator on vectors, is associative.

Independence: Consider any triple $(x_1, \ldots, x_\theta), (y_1, \ldots, y_\theta), (z_1, \ldots, z_\theta) \in \mathbb{N}^\theta$ such that $(x_1, \ldots, x_\theta) \succsim (y_1, \ldots, y_\theta)$. Take any profile $M$, any two disjoint electorates $A, B \subset I$, and any two alternatives $x, y \in X$ such that for all $t \in \Theta$, $G_x^t(M_A) = x_t$, $G_y^t(M_A) = y_t$, and $G_x^t(M_B) = G_y^t(M_B) = z_t$. Since $(x_1, \ldots, x_\theta) \succsim (y_1, \ldots, y_\theta)$ by assumption, the definition of $\succsim$ implies that $x \in v^{\{x,y\}}(M_A)$. Also, $v^{\{x,y\}}(M_B) = \{x, y\}$ by Lemma 2. By consistency in voters, $x \in v^{\{x,y\}}(M_A + M_B)$. Given that $G_x^t(M_A + M_B) = x_t + z_t$ and $G_y^t(M_A + M_B) = y_t + z_t$ for all $t \in \Theta$, it follows from the definition of $\succsim$ that $(x_1 + z_1, \ldots, x_\theta + z_\theta) \succsim (y_1 + z_1, \ldots, y_\theta + z_\theta)$. Hence, as desired, $((x_1, \ldots, x_\theta) + (z_1, \ldots, z_\theta)) \succsim ((y_1, \ldots, y_\theta) + (z_1, \ldots, z_\theta))$ and $((z_1, \ldots, z_\theta) + (x_1, \ldots, x_\theta)) \succsim ((z_1, \ldots, z_\theta) + (y_1, \ldots, y_\theta))$.

Archimedean: Take any $(x_1, \ldots, x_\theta), (y_1, \ldots, y_\theta), (z_1, \ldots, z_\theta), (w_1, \ldots, w_\theta)$ belonging to $\mathbb{N}^\theta$ such that $(x_1, \ldots, x_\theta) \succ (y_1, \ldots, y_\theta)$. Consider any profile $M$, any two alternatives $x, y \in X$, any electorate $A \subset I$, and any 0succession of disjoint electorates $\{N_p\}_{p \in \mathbb{N}}$ such that $G_x^t(M_A) = z_t$, $G_y^t(M_A) = w_t$, $v^{\{x,y\}}(N_p) = \{x\}$, $A \cap N_p = \emptyset$, $G_x^t(M_{N_p}) = x_t$ and $G_y^t(M_{N_p}) = y_t$ for all $p \in \mathbb{N}$. By continuity, there exists $k \in \mathbb{N}$ such that $x \in v^{\{x,y\}}(M_{N_1} + \ldots + M_{N_k} + M_A)$. Since $G_x^t(M_{N_1} + \ldots + M_{N_k} + M_A) = k \cdot x_t + z_t$ and $G_y^t(M_{N_1} + \ldots + M_{N_k} + M_A) = k \cdot y_t + w_t$ for all $t \in \Theta$, we have, as desired, that $(t \cdot (x_1, \ldots, x_\theta) + (z_1, \ldots, z_\theta)) \succsim (t \cdot (y_1, \ldots, y_\theta) + (w_1, \ldots, w_\theta))$. $\square$

Since the triple $(\mathbb{N}^\theta, \succsim, +)$ is a closed extensive structure, we can apply Theorem 1 in Krantz *et al.* (1971) which guarantees that there exists a real-valued function $f$ over $\mathbb{N}^\theta$ such that for all $(x_1, \ldots, x_\theta), (y_1, \ldots, y_\theta) \in \mathbb{N}^\theta$:

(i) $(x_1, \ldots, x_\theta) \succsim (y_1, \ldots, y_\theta) \Leftrightarrow f(x_1, \ldots, x_\theta) \geq f(y_1, \ldots, y_\theta)$ and

(ii) $f((x_1, \ldots, x_\theta) + (y_1, \ldots, y_\theta)) = f(x_1, \ldots, x_\theta) + f(y_1, \ldots, y_\theta)$.

Additionally, any other function $g$ satisfies conditions $(i)$ and $(ii)$ if and only if there exists $t \in \mathbb{R}_{++}$ such that $g = t \cdot f$.

Using this result we construct the vector of weights $\mathbf{p} = (p_1, \ldots, p_\theta)$ by setting $f(1, 0, \ldots, 0)$ equal to $p_1$, $f(0, 1, 0, \ldots, 0)$ equal to $p_2$, and so forth. Since we know from condition (ii) that $f(x_1, \ldots, x_\theta) = f(x_1, 0, \ldots, 0) + f(0, x_2, 0, \ldots, 0) + \ldots + f(0, 0, \ldots, x_\theta)$, we have that

$$f(x_1, \ldots, x_\theta) \geq f(y_1, \ldots, y_\theta) \Leftrightarrow \sum_{t=1}^{\theta} p_t \cdot x_t \geq \sum_{t=1}^{\theta} p_t \cdot y_t.$$

Then, it follows from condition (i) and the definition of $\succsim$ that for all response profiles $M_N$ and all alternatives $x, y \in X$,

$$x \in v^{\{x,y\}}(M_N) \Leftrightarrow \sum_{t=1}^{\theta} p_t \cdot G_x^t(M_N) \geq \sum_{t=1}^{\theta} p_t \cdot G_y^t(M_N).$$

We also know from Faithfulness that $p_t > 0$ for all $t \in \Theta$ and, therefore, we have shown that the subfamily $\{v^K\}_{|K|=2}$ is a Personalized Approval Voting with respect to the vector of weights $\mathbf{p} = (p_1, p_2, \ldots, p_\theta)$. Consequently, it remains to be shown that the vector of weights $\mathbf{p} = (p_1, p_2, \ldots, p_\theta)$ is such that for all sets of feasible alternatives $K \subseteq X$, independently of its size, all response profiles $M_N$,

$$x \in v^K(M_N) \text{ if and only if } \sum_{t \in \Theta} p_t \cdot G_x^t(M_N) \geq \sum_{t \in \Theta} p_t \cdot G_y^t(M_N) \text{ for all } y \in K.$$

Suppose first that $x \in v^K(M_N)$. Then, by consistency in alternatives, $x \in v^{\{x,y\}}(M_N)$ for all $y \in K \setminus \{x\}$. Since we already know that $v^{\{x,y\}}$ is the Personalized Approval Voting with respect to $\mathbf{p} = (p_1, p_2, \ldots, p_\theta)$, it has to be the case that $\sum_{t \in \Theta} p_t \cdot G_x^t(M_N) \geq \sum_{t \in \Theta} p_t \cdot G_y^t(M_N)$ for all $y \in K$.

Suppose now that $\sum_{t \in \Theta} p_t \cdot G_x^t(M_N) \geq \sum_{t \in \Theta} p_t \cdot G_y^t(M_N)$ for all $y \in K$. Then, $x \in v^{\{x,y\}}(M_N)$ for $y \in K \setminus \{x\}$ because $v^{\{x,y\}}(M_N)$ is the Personalized Approval Voting with respect to $\mathbf{p} = (p_1, p_2, \ldots, p_\theta)$. If there is some $z \neq x$

such that $z \in v^K(M_N)$, then $v^K(M_N) \cap \{x, z\} \neq \emptyset$ and it follows from consistency in alternatives that $v^{\{x,z\}}(M_N) = v^K(M_N) \cap \{x, z\}$. Since we have already seen that $x \in v^{\{x,z\}}(M_N)$ it also has to be that $x \in v^K(M_N)$. Finally, if there is no $z \neq x$ such that $z \in v^K(M_N)$, then $v^K(M_N) = \{x\}$ because $v^K(M_N) \neq \emptyset$. This concludes the proof of the theorem.

# Independence of the Axioms

We will finally establish the independence of the properties.

**Consistency in Alternatives:** Take any type $t \in \Theta$. Let the voting procedure $v$ be equal to Approval Voting whenever the set of feasible alternatives $K$ contains exactly two alternatives; otherwise, apply the Personalized Approval Voting with weights $p_t = 1$ and $p_s = 2$ for all types $s \neq t$. This procedure is consistent in voters, weakly anonymous, neutral, faithful, and continuous. The following example shows that it is not consistent in alternatives.

Consider $X = \{x, y, z\}$ and suppose that $i \in I_s$ and $j \in I_t$. If $M_i = \{x\}$ and $M_j = \{y\}$, then $v^X(M_i + M_j) = \{x\}$ and $v^{\{x,y\}}(M_i + M_j) = \{x, y\}$. Since $v^X(M_i + M_j) \cap \{x, y\} \neq \emptyset$, consistency in alternatives implies that $v^{\{x,y\}}(M_i + M_j) = v^X(M_i + M_j) \cap \{x, y\} = \{x\}$. This contradicts that $v^{\{x,y\}}(M_i + M_j) = \{x, y\}$.

**Consistency in Voters:** Let the voting procedure $v$ be equal to Approval Voting whenever all individuals belonging to the electorate $N$ are of the same type; otherwise select all feasible alternatives. This procedure is consistent in alternatives, weakly anonymous, neutral, faithful, and continuous. The following example shows that it is not consistent in voters.

Consider $X = \{x, y\}$ and suppose that $1 \in I_s$ and $2 \in I_t$. If $M_1 = M_2 = \{x\}$, then $v^{\{x,y\}}(M_1) = v^{\{x,y\}}(M_2) = \{x\}$ and $v^{\{x,y\}}(M_1 + M_2) = \{x, y\}$. Since $v^{\{x,y\}}(M_1) \cap v^{\{x,y\}}(M_2) \neq \emptyset$, consistency in voters implies that $v^{\{x,y\}}(M_1 + M_2) = \{x\}$. This contradicts that $v^{\{x,y\}}(M_1 + M_2) = \{x, y\}$.

**Weak Anonymity:** Assign to each individual $i \in I$ a weight $p_i$ greater than a strictly positive number $\epsilon$. Also assume that $p_i > p_j$ for some pair $i, j \in I_t$ for some $t \in \Theta$. Let the voting procedure $v$ be such that for all sets of feasible alternatives $K \subseteq X$, all profiles $M \in (2^X)^I$, and all electorates $N \subset I$, $x \in v^K(M_N)$ if and only if $\sum_{i \in N: x \in M_i} p_i \geq \sum_{i \in N: y \in M_i} p_i$ for all $y \in K$. This procedure is consistent in alternatives, consistent in voters, neutral, faithful, and continuous. The following example shows that it is not weakly anonymous.

Consider $X = \{x, y\}$ and $i, j \in I_t$ for some $t \in \Theta$ such that $p_i > p_j$. If $M_i = \{x\}$ and $M_j = \{y\}$, then $v^{\{x,y\}}(M_i + M_j) = \{x\}$. Now, take any permutation $\pi_t : I_t \to I_t$ such that $\pi_t(i) = j$ and $\pi_t(j) = i$. Then, $v^{\{x,y\}}(M_{\pi_t(i)} + M_{\pi_t(j)}) = \{y\}$. Since weak anonymity implies that $v^{\{x,y\}}(M_{\pi_t(i)} + M_{\pi_t(j)}) = v^{\{x,y\}}(M_i + M_j)$, this is a contradiction.

**Neutrality:** Assign to each alternative $x \in X$ a strictly positive weight $p_x$. Assume also that $p_x > p_y$ for some $x, y \in X$. Let the voting procedure $v$ be such that for all sets of feasible alternatives $K \subseteq X$, all profiles $M \in (2^X)^I$, and all electorates $N \subset I$, $x \in v^K(M_N)$ if and only if $p_x \cdot G_x(M_N) \geq p_y \cdot G_y(M_N)$ for all $y \in K$. This procedure is consistent in alternatives, consistent in voters, weakly anonymous, faithful and continuous. The following example shows that it is not neutral.

Consider $X = \{x, y\}$ and $N = \{i, j\}$ and suppose that $p_x < p_y$. If $M_i = \{x\}$ and $M_j = \{y\}$, then $v^{\{x,y\}}(M_N) = \{y\}$. Now let the per-

mutation $\mu : X \to X$ be such that $\mu(x) = y$ and $\mu(y) = x$. Then, $\mu(v^{\{x,y\}}(M_N)) = \{x\}$ and $v^{\mu(\{x,y\})}(\mu(M_N)) = \{y\}$. Since neutrality implies that $\mu(v^{\{x,y\}}(M_N)) = v^{\mu(\{x,y\})}(\mu(M_N))$, this is a contradiction.

**Faithfulness:** Let the voting procedure $v$ be such that for all sets of feasible alternatives $K \subseteq X$, all profiles $M \in (2^X)^I$, and all electorates $N \subset I$, $v^K(M_N) = K$. This procedure is consistent in alternatives, consistent in voters, weakly anonymous, neutral, and continuous. The following example shows that it is not faithful.

Consider $X = \{x, y\}$ and $N = \{i\}$. Suppose that $M_i = \{x\}$. Then, $v^{\{x,y\}}(M_i) = \{x, y\}$. However, faithfulness implies that $v^{\{x,y\}}(M_i) = M_i \cap \{x, y\} = \{x\}$, which is a contradiction.

**Continuity:** Take any vector $\mathbf{q} = (q_1, q_2, \ldots, q_\theta)$ of strictly positive weights such that $q_i \neq q_j$ for some $i, j \in \Theta$. Let the voting procedure $v$ be such that for all sets of feasible alternatives $K \subseteq X$, all profiles $M \in (2^X)^I$, and all electorates $N \subset I$, $x \in v^K(M_N)$ if and only if (a) $G_x(M_N) \geq G_y(M_N)$ for all $y \in K$ and (b) $\sum_{t\in\Theta} q_t \cdot G^t_x(M_N) \geq \sum_{t\in\Theta} q_t \cdot G^t_y(M_N)$ for all $y \in K$ such that $G_x(M_N) = G_y(M_N)$. This procedure is consistent in alternatives, consistent in voters, weakly anonymous, neutral, and faithful. The following example shows that it is not continuous.

Consider $X = \{x, y\}$, $I_s = \{j_i\}_{i\in\mathbb{N}}$, $I_t = \{k_i\}_{i\in\mathbb{N}}$ such that $q_s > q_t$. Suppose that $M_j = \{x\}$ for all $j \in I_s$ and $M_k = \{y\}$ for all $k \in I_t$. Suppose additionally that $M_l = \{y\}$ for some $l \in I_v$, with $v \notin \{s, t\}$. Consider the electorates $N_i = \{j_i, k_i\}$ for all $i \in \mathbb{N}$. By the definition of $v$, $v^{\{x,y\}}(M_l) = \{y\}$ and $v^{\{x,y\}}(M_{N_i}) = \{x\}$ for all $i \in \mathbb{N}$. Consequently, continuity implies that there is some $b \in \mathbb{N}$ such that $x \in v^{\{x,y\}}(M_{N_1} + M_{N_2} + \ldots + M_{N_b} + M_l)$. However, since $G_y(M_{N_1} + M_{N_2} + \ldots + M_{N_b} + M_l) > G_x(M_{N_1} + M_{N_2} + \ldots +$

$M_{N_b} + M_l)$ for all $b \in \mathbb{N}$, $v^{\{x,y\}}(M_{N_1} + M_{N_2} + \ldots + M_{N_b} + M_l) = \{y\}$ for all $b \in \mathbb{N}$. This is a contradiction.

| EU Member State | Population | Weight |
|---|---|---|
| Germany | 81,757,595 | 29 |
| France | 64,709,480 | 29 |
| United Kingdom | 62,041,708 | 29 |
| Italy | 60,397,353 | 29 |
| Spain | 46,087,170 | 27 |
| Poland | 38,163,895 | 27 |
| Romania | 21,466,174 | 14 |
| Netherlands | 16,576,800 | 13 |
| Greece | 11,125,179 | 12 |
| Belgium | 10,827,519 | 12 |
| Portugal | 10,636,888 | 12 |
| Czech Republic | 10,512,397 | 12 |
| Hungary | 10,013,628 | 12 |
| Sweden | 9,372,899 | 10 |
| Austria | 8,372,930 | 10 |
| Bulgaria | 7,576,751 | 10 |
| Denmark | 5,547,088 | 7 |
| Slovakia | 5,424,057 | 7 |
| Finland | 5,350,475 | 7 |
| Ireland | 4,450,878 | 7 |
| Lithuania | 3,329,227 | 7 |
| Latvia | 2,248,961 | 4 |
| Slovenia | 2,054,119 | 4 |
| Estonia | 1,340,274 | 4 |
| Cyprus | 801,851 | 4 |
| Luxembourg | 502,207 | 4 |
| Malta | 416,333 | 3 |

Table 1: Voting Weights in the EU Member State Council as of July 2011.

| IMF Board of Directors | Percentage of Fund | Weight |
|---|---|---|
| United States | 16.77% | 421,964 |
| Japan | 6.24% | 157,025 |
| Germany | 5.82% | 146,395 |
| France | 4.30% | 108,125 |
| United Kingdom | 4.30% | 108,125 |
| Belgium (Austria) | 4.98% | 125,221 |
| Mexico (Venezuela) | 4.65% | 117,053 |
| Netherlands (Ukraine) | 4.52% | 113,822 |
| Italy (Greece) | 4.26% | 107,077 |
| Singapore (Indonesia) | 3.94% | 99,062 |
| China | 3.82% | 95,999 |
| Australia (Korea) | 3.63% | 91,347 |
| Canada (Ireland) | 3.61% | 90,708 |
| Denmark (Norway) | 3.39% | 85,352 |
| Lesotho (Gambia) | 3.22% | 81,085 |
| Egypt (Lebanon) | 3.13% | 78,692 |
| India (Sri Lanka) | 2.81% | 70,705 |
| Saudi Arabia | 2.81% | 70,595 |
| Brazil (Colombia) | 2.79% | 70,188 |
| Switzerland (Poland) | 2.78% | 69,842 |
| Russian Federation | 2.36% | 60,194 |
| Iran (Morocco) | 2.27% | 57,092 |
| Argentina (Chile) | 1.84% | 46,335 |
| Togo (Chad) | 1.55% | 39,039 |

Table 2: Voting Weights in the Board of Directors of the IMF as of July 2011. One voter tends to represent several countries, the only exceptions are the United States, Japan, Germany, France, United Kingdom, China, and the Russian Federation. For example, Argentina is grouped with Bolivia, Chile, Paraguay, Peru, and Uruguay. In case the Argentinian representant is absent, the Chilean representant replaces her/him. The exact categorization can be consulted at http://www.imf.org/external/np/sec/memdir/eds.aspx.