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**CLAIM PROBLEMS AND EGALITARIAN CRITERIA**

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# Claim problems and egalitarian criteria\*

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## Abstract

The paper presents a unified framework where claim and cost sharing problems are jointly analyzed. Both problems have the following common characteristic: given a proposal the agents evaluate the suitability of the proposal in two ways, checking how much they lose and how much they gain. Taking this into account, we construct the vector of awards and losses for any proposal and we use different egalitarian criteria to select among these vectors. We use the Lorenz, the Least Square and the lexicographic criteria and we analyze the solutions arising from the application of these criteria in the sets of vectors of awards-losses. In particular, we characterize the members of two families of solutions: the family of Weighted Least Square Solutions and the family of Imputation Selector Weighted Least Square Solutions. The second family includes between its members well-known solutions as Constrained Equal Awards and Constrained Equal Losses solutions.

## 1 Introduction

A claim problem consists of a set of claimants who must divide between themselves an infinitely divisible good, the endowment, that is not enough to satisfy their claims entirely. The question of how to divide the endowment fairly between the claimants has been widely analyzed and many rules have been defined and characterized to provide axiomatic support. Following this approach, fairness is identified with a list of axioms and any solution satisfying the list of axioms is considered as a potential solution to the problem. These rules and their characterizations play a central role in the literature of fair allocation.

A similar approach has been used in a different problem that shares an almost identical mathematical formulation with claim problems. For example,

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in a cost sharing problem where agents are ordered according to their needs for a public project, if the need of the last agent (the one with the greatest need) is met then all agents with lesser needs are also satisfied. A solution to any of these problems consists of dividing the cost of a common project between the agents involved in it. Again, an important aspect of the problem is to define rules that provide fair division of the total cost between the agents.

The existence of many different rules claiming to be valid for providing fair allocation indicates how difficult is to define fairness. On page 3 of his book “*Equity in Theory and Practice*” [19] Young says the following:

These sharing rules, which in some cases are elaborately defined, express a notion of equity in the division of jointly produced goods. By equitable I do not necessarily mean ethical or moral, but that which a given society considers to be appropriate to the need, status and contributions of its various members.

Under the same ideal of equity or fairness different societies could give rise to different sharing rules for the same sharing problems.

When considering the fairness of an allocation, in both types of problem, agents realize that any allocation can be analyzed in two ways: assuming that we are dealing with a claim problem, given a distribution of the endowment agents perceive that they gain something (the amount they receive) but they also lose something (the amount they do not receive: the difference between their claim and the amount received). The terms gain and loss are reversed if we consider cost sharing problems.

This paper also deals with this approach -albeit indirectly- but the initial viewpoint is slightly different. Instead of dealing directly with axioms, in our approach the fairness of a proposal is associated with the selection of egalitarian proposals that take into account both aspects of any distribution of the endowment: awards and losses.

Our paper seeks to give a more unified view of some normative solutions. We consider that the idea of fairness should be explicated by answering the following two questions:

1. What egalitarian criterion is used to make egalitarian comparisons between elements?
2. From what set are those elements taken?

The answers may help us to understand the similarities and differences between solutions and this is the first major contribution of this paper.

The first question asks what tool is used to make comparisons (of fairness) between different elements (vectors). Using arguments of fairness when selecting a specific element implicitly imposes the idea of comparison between different elements. This is needed to argue that the elements chosen are *better* than the elements not chosen. In other words, the selected vector *dominates*<sup>1</sup> the vectors that are not selected.

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<sup>1</sup>The terms *better* and *dominates* require precise definitions.

Probably the most relevant criteria in literature are the Lorenz criterion, the Least Square criterion and the Lexicographic criterion.

In most of the literature on egalitarianism it is agreed that an allocation should be maximal according to the Lorenz criterion as a least requirement for calling it egalitarian. The Lorenz order is not a complete order and therefore, in general, does not select a unique element. The criterion selects a set of Lorenz maximal elements that are not comparable one with another (from the point of view of the Lorenz criterion). The other two criteria are complete and select Lorenz maximal elements.

The second question is about the set from which these maximal egalitarian allocations must be selected. In our work we consider sets of weighted vectors of awards and losses. The weights seek to reflect the importance of these two aspects of the allocations. If only awards matter the vector of losses is irrelevant and therefore weighted with parameter 0. In the symmetric case awards and losses are treated equally.

The most relevant, most widely, analyzed solutions for claim problems are Constrained Equal Awards (CEA), Constrained Equal Losses (CEL), the Talmud Rule and the Proportional Rule. In this study we examine what answers are given by these solutions to these two questions. They are all solutions that select maximal egalitarian vectors in a given set. They can be seen as selectors of lexicographical maximal elements in a determined set. We also provide an alternative definition of the CEA, CEL and Proportional Rule as selectors of Least Square maximal elements in a determined set. Clearly, they are Lorenz maximal element selectors.

A second major contribution of the paper is the axiomatic analysis provided for the solutions arising from the application of the Lorenz and Least Square criteria. This aspect can be seen as the answer to a third question that is also central in the literature of fair allocation.

### 3. What set of axioms supports each solution?

With the Lorenz criterion we show that any Lorenz maximal allocation should be order preserving for gains and losses (the agent with a lower demand gains and loses no more than the agent with a higher demand) and any allocation that is order preserving (in gains and losses) is a Lorenz maximal allocation.

With the Least Square criterion we identify the family of solutions that satisfy additivity, order preservation and anonymity. These solutions are new in the literature and each of them is characterized by adding an axiom to the previous three axioms. If we require the solution to select imputations (asking for a property that we call the Imputation Selector Property) we are dealing with another family of solutions that are characterized using, among other axioms, a weaker version of the classical axiom of additivity. This family includes the CEA, CEL and the Reverse Talmud Rule. We also provide a characterization for each member of the family.

The two families arise because additivity and the Imputation Selector Property are incompatible. It is because, in general, only solutions that select impu-

tations have been considered<sup>2</sup> that the existence of Least Square Solutions has remained hidden. The requirement of Imputation Selection implies that other axioms (in particular additivity) cannot be satisfied.

The Lexicographic criterion defines a family of solutions that includes the well-known Talmud Rule, characterized by Aumann and Maschler [2]. We also analyze Weighted Talmud Rules by weighting awards and losses differently (the Talmud Rule weights awards and losses equally). These solutions have been studied by Moreno-Tertero and Villar [12].

In our paper, many relevant solutions in the literature of claim problems are presented as optimal elements in a given set. A similar analysis has been performed in the context of TU games where the most relevant single-valued solution concepts (the Shapley value and the nucleolus among them) are identified with an egalitarian criterion applied in a given set.

The rest of the paper is organized as follows: Section 2 introduces the model, the egalitarian criteria and the set of vectors of awards and losses. Section 3 studies the Lorenz criterion. Section 4 and 5 deal with the Least Square criterion in two different sets. Section 6 is for the lexicographical criterion and Section 7 concludes the paper.

## 2 Preliminaries

### 2.1 Claim problems

The tuple  $(N, d, E)$  is a claim *problem* if:

- a)  $N$  is a finite nonempty set.
- b)  $\sum_{i \in N} d_i > E$ .

$N$  represents the set of *agents or claimants*. Then  $i \preceq j$  means that we assume  $d_i \leq d_j$  and  $d_1 \geq 0$ . We denote by  $\Gamma$  the class of claim problems.

An allocation to the claimants is represented by a real valued vector  $x \in \mathbb{R}^N$ . The  $i$ -th coordinate of the vector  $x$  denotes the allocation given to claimant  $i$ . The vector  $x$  is called *efficient* if  $x(N) = \sum_{i \in N} x_i = E$  and the set of all efficient vectors is called the *preimputation set* and is denoted by  $PI(N, d, E)$ .

A subset of the preimputation set is the *imputation set* denoted by  $I(N, d, E)$ . An imputation is an efficient vector where  $d_i \geq x_i \geq 0$  for all  $i \in N$ .

#### SOLUTIONS AND PROPERTIES

A solution  $\phi$  on a set of problems  $\Gamma$  is a mapping that associates a vector  $\phi(N, d, E) \in PI(N, d, E)$  with every problem  $(N, d, E)$  in  $\Gamma$ . Unlike many other authors we do not require a solution concept to select only imputations. This is

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<sup>2</sup>Note that in TU games this requirement implies that the Shapley value and the prenucleolus are not solutions (at least in the class of all TU games).

also why we prefer the term **solution** to the term **rule** which is used assuming that a rule selects imputations.

Some well-known solutions are<sup>3</sup>:

**Constrained equal awards (CEA)**. This solution divides the endowment equally among the agents under the constraint that no claimant receives more than his claim.

**Constrained equal losses (CEL)**. This solution divides the total loss ( $\sum_{i \in N} d_i - E$ ) equally among the agents under the constraint that no claimant receives a negative amount.

The **Proportional Solution (PS)**. This solution divides the endowment among the claimants proportionally to their claims.

These three solutions always select imputations. The no constrained versions of the first two solutions (EA and EL) do not necessarily select imputations.

Some convenient, well-known properties of a solution  $\phi$  in  $\Gamma$  are the following.

P1)  $\phi$  satisfies **scale invariance** (SCIN) if for each  $(N, d, E)$  in  $\Gamma$  and  $\lambda > 0$  we have that  $\phi(N, \lambda d, \lambda E) = \lambda \phi(N, d, E)$ .

P2)  $\phi$  satisfies **anonymity** (AN) if for each  $(N, d, E)$  in  $\Gamma$  and each bijective mapping  $\tau : N \rightarrow N'$  such that  $(N', \tau d, E) \in \Gamma$  we have that  $\phi(N', \tau d, E) = \tau(\phi(N, d, E))$ . In this case  $(N, d, E)$  and  $(N', \tau d, E)$  are equivalent claim problems.

P3)  $\phi$  satisfies the **equal treatment property** (ETP) if for each  $(N, d, E)$  in  $\Gamma$  equal claimants  $i, j$  are treated equally, i.e.,  $\phi_i(N, d, E) = \phi_j(N, d, E)$ . Here,  $i$  and  $j$  are equal if  $d_i = d_j$ .

P4)  $\phi$  satisfies **order preservation for awards and losses** (ORDPRE) if for each  $(N, d, E)$  in  $\Gamma$  we have that  $\phi(N, d, E)$  is order preserving for awards and losses. An allocation  $x$  is order preserving for awards and losses if  $d_i \leq d_j$  implies that  $x_i \leq x_j$  and  $d_i - x_i \leq d_j - x_j$ .

P5)  $\phi$  satisfies **resource monotonicity** (MON) if given  $(N, d, E), (N, d, E^*) \in \Gamma$  with  $E^* > E$  then  $\phi_i(N, d, E) \leq \phi_i(N, d, E^*)$  for all  $i \in N$ .

P6)  $\phi$  satisfies **additivity** (ADD) if  $(N, d, E), (N, d', E') \in \Gamma$  with  $(N, d + d', E + E') \in \Gamma$  then  $\phi(N, d + d', E + E') = \phi(N, d, E) + \phi(N, d', E')$ .

The first five properties are satisfied for the most important solutions. Property 4 implies property 3. Auman and Maschler [2] introduced order preservation. In the survey by Thomson [16] Scale Invariance is called Homogeneity.

## 2.2 Egalitarian Criteria

For any vector  $z \in \mathbb{R}^d$  we denote by  $\theta(z)$  the vector that results from  $z$  by permuting the coordinates in such a way that  $\theta_1(z) \leq \theta_2(z) \leq \dots \leq \theta_d(z)$ . Let  $x, y \in \mathbb{R}^d$ .

We say that the vector  $x$  *Lorenz dominates* the vector  $y$  (denoted by  $x \succ_L y$ ) if  $\sum_{i=1}^k \theta_i(x) \geq \sum_{i=1}^k \theta_i(y)$  for all  $k \in \{1, 2, \dots, d\}$  and if at least one of these inequal-

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<sup>3</sup>A long list of solutions can be found in a survey by Thomson [16].

ities is strict. The vector  $x$  *weakly Lorenz dominates* the vector  $y$  (denoted by  $x \succeq_L y$ ) if  $\sum_{i=1}^k \theta_i(x) \geq \sum_{i=1}^k \theta_i(y)$  for all  $k \in \{1, 2, \dots, d\}$ .

The vector  $x$  *lexicographically dominates* the vector  $y$  (denoted by  $x \succ_{lex} y$ ) if there exists  $k$  such that  $\theta_i(x) = \theta_i(y)$  for all  $i \in \{1, 2, \dots, k-1\}$  and  $\theta_k(x) > \theta_k(y)$ . The vector  $x$  *weakly lexicographically dominates* the vector  $y$  (denoted by  $x \succeq_{lex} y$ ) if either there exists  $k$  such that  $\theta_i(x) = \theta_i(y)$  for all  $i \in \{1, 2, \dots, k-1\}$  and  $\theta_k(x) > \theta_k(y)$  or  $\theta_i(x) = \theta_i(y)$  for all  $i$ .

The vector  $x$  *least square dominates* the vector  $y$  (denoted by  $x \succ_{LS} y$ ) if  $\sum_{i=1}^d x_i^2 < \sum_{i=1}^d y_i^2$ . The vector  $x$  *weakly least square dominates* the vector  $y$  (denoted by  $x \succeq_{LS} y$ ) if  $\sum_{i=1}^d x_i^2 \leq \sum_{i=1}^d y_i^2$ .

The vector  $x$  *lexicographically dominates* the vector  $y$  (denoted by  $x \succ_{lex} y$ ) if there exists  $k$  such that  $\theta_i(x) = \theta_i(y)$  for all  $i \in \{1, 2, \dots, k-1\}$  and  $\theta_k(x) > \theta_k(y)$ . The vector  $x$  *weakly lexicographically dominates* the vector  $y$  (denoted by  $x \succeq_{lex} y$ ) if either there exists  $k$  such that  $\theta_i(x) = \theta_i(y)$  for all  $i \in \{1, 2, \dots, k-1\}$  and  $\theta_k(x) > \theta_k(y)$  or  $\theta_i(x) = \theta_i(y)$  for all  $i$ .

These last two criteria provide maximal Lorenz allocations.

### 2.3 The set of awards-losses vectors

Let  $(N, d, E)$  be a problem and let  $x$  be an efficient allocation. Each agent measures  $x_i$  in two ways. In one sense  $x_i$  measures how much he\she receives. In the other sense,  $d_i - x_i$  measures how much he\she does not receive. We consider a unified framework where different models can be jointly analyzed. For cost problems the vector  $x$  measures the cost that each agent is going to receive and can be seen as a loss while the vector  $d - x$  measures the gain that he\she obtains from  $x$ . If we think in terms of bankruptcy problems the arguments are reversed<sup>4</sup>. Given the allocation  $x$  we define its associated ordered<sup>5</sup> vector of awards-losses as follows:

$$x^{AL} = (x_1, \dots, x_n, x_1 - d_1, \dots, x_n - d_n).$$

We also use the following notation;

$$x^A = (x_1, \dots, x_n) \text{ and}$$

$$x^L = (x_1 - d_1, \dots, x_n - d_n).$$

<sup>4</sup>In fact, it is only in Section 3 that this consideration matters. Given that we use the Least Square criterion or the set of vectors in absolute terms, the issue of which elements of the vector are losses and which are awards plays no role in the rest of the sections.

<sup>5</sup>Fixing the order of coalitions is a technical trick to obtain a homeomorphism between the two topological vector spaces: the space of preimputations and the space of ordered vectors of awards-losses.

In this vector, awards and losses are equally weighted and equally treated. We also consider vectors where awards and losses are not equally treated. Given the allocation  $x$  we define its associated weighted vector of awards-losses as follows:

$$\lambda\text{-}x^{AL} = ((1 - \lambda)x_1, \dots, (1 - \lambda)x_n, \lambda(x_1 - d_1), \dots, \lambda(x_n - d_n))$$

where  $\lambda \in [0, 1]$ . Note that  $\lambda\text{-}x^{AL}$  with  $\lambda = \frac{1}{2}$  is the vector of equal weights, which in our study is equivalent to considering  $x^{AL}$  or  $\lambda\text{-}x^{AL}$  with  $\lambda = \frac{1}{2}$ .

We also use the following notation:

$$AL^\lambda(PI(N, d, E)) = \{\lambda\text{-}x^{AL}; x \in PI(N, d, E)\} \text{ and } AL^\lambda(I(N, d, E)) = \{\lambda\text{-}x^{AL}; x \in I(N, d, E)\}.$$

The following notation  $|AL^\lambda(PI(N, d, E))| = \{|\lambda\text{-}x^{AL}|; x \in PI(N, d, E)\}$  and  $|AL^\lambda(I(N, d, E))| = \{|\lambda\text{-}x^{AL}|; x \in I(N, d, E)\}$  is used to denote the set of vectors of awards-losses taken in absolute terms.

Note that we use the notation  $\lambda\text{-}x^{AL}$  instead of  $\lambda\text{-}x^{AL}(N, d, E)$ . We consider there is no confusion, so we prefer the notation  $\lambda\text{-}x^{AL}$  for the sake of simplicity.

### 3 The Lorenz criterion

The first egalitarian criterion we consider is the Lorenz criterion. The Lorenz order is not complete and therefore by applying this criterion we do not, in general, obtain uniqueness. In this sense, the set of Lorenz maximal allocations (the set of Lorenz undominated allocations) can be seen as the maximal set of fair allocations. A Lorenz dominated allocation is not a candidate for selection when looking for fair allocations.

We define the **Lorenz maximal set as the set of Lorenz undominated allocations, that is,**

$$\mathcal{L}(PI(N, d, E)) = \left\{ x \in PI(N, d, E); \text{there is no } y \in PI(N, d, E) \text{ such that } y^{AL} \succ_L x^{AL} \right\}.$$

The Lorenz maximal set coincides with the set of allocations that satisfy order preservation in both ways, awards and losses. Therefore, order preservation emerges as a minimal requirement for a fair allocation.

The proof of this result relies on the following fact. For two elements  $k$  and  $l$ , a vector  $x$ , and a real number  $\alpha > 0$ , we say that  $(k, l, x, \alpha)$  is an *equalizing bilateral transfer* (of size  $\alpha$  from  $k$  to  $l$  with respect to  $x$ ) if

$$x_k - \alpha \geq x_l + \alpha.$$

Now, Lemma 2 of Hardy, Littlewood and Polya [6] implies that an allocation  $y$  Lorenz dominates another allocation  $x$  only if  $y$  can be obtained from  $x$  by a finite sequence of equalizing bilateral transfers.

**Theorem 1** *The Lorenz maximal set coincides with the set of all allocations that satisfy order preservation in both ways: awards and losses.*

**Proof.** Let  $x \in PI(N, d, E)$  be such that  $x$  is not order preserving for awards. Therefore, there are claimants  $i, j$  such that  $d_i \geq d_j$  and  $x_i < x_j$ . Then it also holds that  $d_i - x_i > d_j - x_j$ . Consider the following allocation  $z$  :

$$z_l = \begin{cases} x_l + \varepsilon & \text{if } l = i \\ x_l - \varepsilon & \text{if } l = j \\ x_l & \text{otherwise} \end{cases}$$

where  $\varepsilon = \min(\frac{x_j - x_i}{2}, \frac{(d_i - x_i) - (d_j - x_j)}{2})$ .

It is not difficult to check that  $z \succeq_L x$  since it still holds that  $z_i \leq z_j$  and  $d_i - z_i \geq d_j - z_j$ . The proof is similar in the case where  $x$  violates order preservation for losses.

Let  $x$  be an allocation satisfying order preservation for awards and losses. Then

$$\sum_{1 \leq i \leq n} \theta_i(x^{AL}) = E - \sum_{1 \leq i \leq n} d_i$$

since the first  $n$  elements of the vector  $\theta(x^{AL})$  are the ordered losses  $(x_n - d_n, \dots, x_1 - d_1)$ <sup>6</sup>. Note also that

$$\sum_{i=n+1}^{2n} \theta_i(x^{AL}) = \sum_{1 \leq i \leq n} x_i = E$$

since the last  $n$  elements of the vector  $\theta(x^{AL})$  are the ordered awards  $(x_1, \dots, x_n)$ . Therefore, if there is an allocation  $z$  such that  $z^{AL} \succ_L x^{AL}$  should be the case that  $z^L \succ_L x^L$  and  $z^A \succeq_L x^A$  or  $z^L \succeq_L x^L$  and  $z^A \succ_L x^A$ . If  $z^A \succ_L x^A$  then  $z^A$  can be obtained from  $x^A$  by a finite sequence of equalizing bilateral transfers.

Now consider a vector  $y^A$  resulting from  $x^A$  after a bilateral equalizing transfer. Let  $i, j$  two claimants such that  $x_i < x_j$

$$y_l = \begin{cases} x_l + \varepsilon & \text{if } l = i \\ x_l - \varepsilon & \text{if } l = j \\ x_l & \text{otherwise} \end{cases}$$

where  $0 < \varepsilon \leq \frac{x_j - x_i}{2}$ .

It is clear that  $y^A \succ_L x^A$  implies that  $x^L \succ_L y^L$  and therefore  $y^{AL}$  does not Lorenz dominate  $x^{AL}$ .

A similar consideration follows for the case where we consider Lorenz domination with respect to the vector  $x^L$ . That is, if there exists an allocation  $y$  such that  $y^L \succ_L x^L$  then  $x^A \succ_L y^A$  and therefore  $y^{AL}$  does not Lorenz dominate  $x^{AL}$ . ■

The following corollary arises immediately since a convex combination of order preserving allocations is also order preserving.

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<sup>6</sup>If there is any  $x_1 < (x_i - d_i)$  we have the following contradiction:  
 $x_1 < (x_i - d_i) \leq (x_1 - d_1) < x_1$ .

**Corollary 2** *The Lorenz maximal set is convex.*

Similar results are obtainable if we restrict the search of allocations to the set of imputations. In this way, the set

$$\mathcal{L}(I(N, d, E)) = \left\{ \begin{array}{l} x \in I(N, d, E); \text{ there is no } y \in PI(N, d, E) \\ \text{such that } y^{AL} \succ_L x^{AL} \end{array} \right\}$$

coincides with the set of all imputations that satisfy order preservation in both ways.

However, the theorem is not true if we consider weighted vectors of awards-losses. For example, it is immediately apparent that if we take  $\lambda = 0$

$$\mathcal{L}^0(PI(N, d, E)) = \left\{ \begin{array}{l} x \in PI(N, d, E); \text{ there is no } y \in PI(N, d, E) \\ \text{such that } y^A \succ_L x^A \end{array} \right\}$$

coincides with  $\{(\frac{E}{n}, \dots, \frac{E}{n})\}$  and if we take  $\lambda = 1$  the set

$$\mathcal{L}^1(PI(N, d, E)) = \left\{ \begin{array}{l} x \in PI(N, d, E); \text{ there is no } y \in PI(N, d, E) \\ \text{such that } y^L \succ_L x^L \end{array} \right\}$$

coincides with  $\left\{ \left( d_i - \frac{\sum_{i \in N} d_i - E}{n} \right)_{i \in N} \right\}$ .

These results were noted by Bosmans et al. [4] when studying Lorenz comparisons between vectors of  $n$  elements (being  $n$  the number of claimants). Many other authors have considered Lorenz comparisons of vectors of  $n$  elements in their works. For example, this type of analysis can be found in Kasajima et al. [9] and Thomson [17].

## 4 The Least Square criterion

### 4.1 Least Square Solutions

The second egalitarian criterion that we consider is the Least Square criterion. The Least Square order is complete and by applying this criterion in a convex set we obtain uniqueness. The criterion selects Lorenz maximal allocations.

We define the **Least Square Solution** as follows:

$$\mathcal{LS}(PI(N, d, E)) = \{x \in PI(N, d, E); x^{AL} \succ_{LS} y^{AL}, \text{ for all } y \in PI(N, d, E)\}$$

and the **Weighted Least Square Solutions associated with the weights  $\lambda$** .

$$\lambda\text{-}\mathcal{LS}(PI(N, d, E)) = \left\{ \begin{array}{l} x \in PI(N, d, E); \lambda \cdot x^{AL} \succ_{LS} \lambda \cdot y^{AL}, \\ \text{for all } y \in PI(N, d, E) \end{array} \right\}.$$

The following theorem gives a simple formula with which the  $LS$  can be easily computed.

**Theorem 3** *Let  $(N, d, E)$  be a problem and consider the allocation  $z$  defined as follows:*

$$z_1 = \frac{E - \sum_{1 \leq i \leq n} \frac{d_i - d_1}{2}}{n} \text{ and } z_i = z_1 + \frac{d_i - d_1}{2} \text{ for all } i \neq 1.$$

Then  $LS(PI(N, d, E)) = z$ .

**Proof.** Let  $z = LS(PI(N, d, E))$ . Then for all  $i, j \in N$  we have that  $z_j - z_i = (d_j - z_j) - (d_i - z_i)$ . Assume on the contrary that  $z_j - z_i < (d_j - z_j) - (d_i - z_i)$ . Consider the allocation  $y$  where

$$y_l = \begin{cases} z_l - \varepsilon & \text{if } l = i \\ z_l + \varepsilon & \text{if } l = j \\ x_l & \text{otherwise} \end{cases}$$

where  $\varepsilon$  is such that  $y_j - y_i = (d_j - y_j) - (d_i - y_i)$ . It is clear that  $y \succ_{LS} z$ .

Now assume that  $z_j - z_i > (d_j - z_j) - (d_i - z_i)$ . Consider the allocation  $y$  where

$$y_l = \begin{cases} z_l + \varepsilon & \text{if } l = i \\ z_l - \varepsilon & \text{if } l = j \\ x_l & \text{otherwise} \end{cases}$$

where  $\varepsilon$  is such that  $y_j - y_i = (d_j - y_j) - (d_i - y_i)$ . It is clear that  $y \succ_{LS} z$ .

Therefore,  $z_j - z_i = (d_j - z_j) - (d_i - z_i)$  for all  $i, j \in N$  and consequently for all  $j \in N$  it results  $z_j = z_1 + \frac{1}{2}(d_j - d_1)$ . The final step is to apply efficiency, that is,  $\sum_{1 \leq i \leq n} z_i = E$ . ■

Similarly, it can be proved that the Weighted Least Square Solution can be computed by the following formula:

**Theorem 4** *Let  $(N, d, E)$  be a claim problem,  $\lambda \in [0, 1]$  and consider the allocation  $z$  defined as follows:*

$$z_1 = \frac{E - \lambda \sum_{1 \leq i \leq n} (d_i - d_1)}{n} \text{ and } z_i = z_1 + \lambda(d_i - d_1) \text{ for all } i \neq 1.$$

Then  $\lambda$ - $LS(PI(N, d, E)) = z$ .

**Proof.** Let  $z = \lambda$ - $LS(PI(N, d, E))$ . Similarly to the previous theorem it can be proved that for all  $i, j \in N$  we have that  $(1 - \lambda)(z_j - z_i) = \lambda((d_j - z_j) - (d_i - z_i))$ . Consequently for all  $j \in N$  it results  $z_j = z_1 + \lambda(d_j - d_1)$ . The final step is to apply efficiency, that is,  $\sum_{1 \leq i \leq n} z_i = E$ . ■

Note that if  $\lambda = 0$ ,  $\lambda$ - $LS(PI(N, d, E)) = (\frac{E}{n}, \dots, \frac{E}{n}) = EA(N, d, E)$  and if  $\lambda = 1$ ,  $\lambda$ - $LS(PI(N, d, E)) = (d_i - \frac{\sum_{1 \leq i \leq n} d_i - E}{n})_{i \in N} = EL(N, d, E)$ .

If  $\lambda \notin [0, 1]$  the allocation selected by the solution is not order preserving.

Figure 1 shows how these solutions perform in two-claimant problems when  $E$  moves from 0 to  $d_1 + d_2$ .

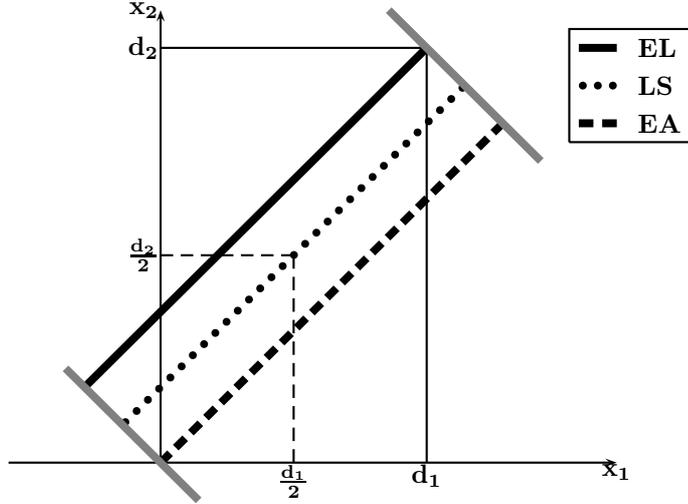


Figure 1: *Illustration of the **Weighted Least Square Solution** for different  $\lambda$  when  $E$  moves from 0 to  $d_1 + d_2$ . The lines of slope 1 are paths of awards. Also we draw two lines of slope  $-1$  passing through the origin and the claims vector which correspond with the lowest budget set and the highest budget set respectively.*

## 4.2 Characterizations

Not surprisingly Least Square Solutions are additive solutions and additive solutions satisfying other reasonable properties must be Least Square Solutions. The similarities with the Shapley value (Shapley [15]) and other Least Square Solutions (the  $\lambda$ -prenucleolus<sup>7</sup> among them) for TU games are clear. In TU games, the Shapley value is a Least Square Solution (see Keane, [10]) that has several characterizations. One of them has the additivity axiom as one of its main axioms.

In Littlechild [11], a restricted version of additivity is used to characterize the Shapley value of airport problems. In this version the order in the set of claimants when considering the new airport problem must be the same as the sum of two airport problems. In our analysis we impose only the necessary requirement that the set of claimants must coincide in the two problems.

The proof of this characterization uses the results of the following two lemmas.

We say that  $(N, d, E) \in \Gamma^n$  if  $|N| = n$ .

**Lemma 5** *Let  $\phi$  be a solution that satisfies AN, ETP and ADD. Let  $A = (N, d^1, E^1) \in \Gamma^n$  and  $B = (N, d^2, E^2) \in \Gamma^n$  be two claim problems such that for claimants  $i, j$  we have that  $d_j^1 - d_i^1 = d_j^2 - d_i^2$ . Then  $\phi_j(A) - \phi_i(A) = \phi_j(B) - \phi_i(B)$ .*

<sup>7</sup>See Ruiz et al. [13].

**Proof.** Assume that the lemma is not true. Therefore, there exist two claim problems,  $A = (N, d^1, E^1)$  and  $B = (N, d^2, E^2)$ , such that for claimants  $i, j$  we have that  $d_j^1 - d_i^1 = d_j^2 - d_i^2$  and  $\phi_j(A) - \phi_i(A) \neq \phi_j(B) - \phi_i(B)$ .

Let  $\phi_j(A) - \phi_i(A) = \lambda_1$  and  $\phi_j(B) - \phi_i(B) = \lambda_2$  where by assumption we know that  $\lambda_1 \neq \lambda_2$ . Consider the problem  $C = (N, d^*, E^1)$  where  $d^*$  is defined as follows:

$$d_l^* = \begin{cases} d_i^1 & \text{if } l = j \\ d_j^1 & \text{if } l = i \\ d_l^1 & \text{otherwise.} \end{cases}$$

By AN we know that  $\phi_i(C) = \phi_j(A)$  and  $\phi_j(C) = \phi_i(A)$ . By ADD we know that  $\phi_i(B + C) = \phi_i(B) + \phi_i(C)$ .

And  $\phi_j(B + C) = \phi_j(B) + \phi_j(C) = \phi_j(B) + \phi_i(A) = \phi_j(B) + \phi_j(A) - \lambda_1 = \phi_j(B) + \phi_i(C) - \lambda_1 = \phi_i(B) + \lambda_2 + \phi_i(C) - \lambda_1$ .

Note that in the problem  $(B + C)$  claimants  $i$  and  $j$  are symmetric and therefore by ETP  $\phi_i(B + C) = \phi_j(B + C)$ .

Consequently  $\lambda_1 = \lambda_2$ , contradicting our initial assumption. ■

**Lemma 6** *Let  $\phi$  be a solution that satisfies AN, ETP, SCIN and ADD. Then, there exists  $\lambda_n \in \mathbb{R}$  such that for any problem  $(N, d, E) \in \Gamma^n$  we have that  $\phi_j(N, d, E) - \phi_i(N, d, E) = \lambda(d_j - d_i)$  for all  $i, j \in N$ .*

**Proof.** Assume that the lemma is not true. We consider two cases:

a) There exists a claim problem,  $A = (N, d, E)$ , such that for claimants  $i, j, m, p \in N^8$  we have that  $\phi_j(A) - \phi_i(A) = \lambda_1(d_j - d_i)$  and  $\phi_m(A) - \phi_p(A) = \lambda_2(d_m - d_p)$ . By the Lemma 5 it should be the case that  $d_j - d_i \neq d_m - d_p$ . Now let  $k$  be a constant such that  $d_j - d_i = k(d_m - d_p)$ . By SCIN we know that

$$\phi_m(kA) - \phi_p(kA) = k\phi_m(A) - k\phi_p(A) = k\lambda_2(d_m - d_p).$$

Therefore

$$\phi_m(kA) - \phi_p(kA) = k\lambda_2(d_m - d_p) = \lambda_2(d_j - d_i).$$

By Lemma 5 it must hold that

$$\phi_m(kA) - \phi_p(kA) = \lambda_1(d_j - d_i)$$

and therefore,

$$\lambda_2(d_j - d_i) = \lambda_1(d_j - d_i)$$

contradicting the initial assumption.

b) There exist claim problems  $A$  and  $B$  with the same set of claimants  $N$  such that there exist claimants  $i, j \in N$  for which  $\phi_j(A) - \phi_i(A) = \lambda_1(d_j - d_i)$  and  $\phi_i(B) - \phi_j(B) = \lambda_2(d_i - d_j)$ . In this case the proof of the contradiction is similar to case a). ■

The main theorem of this section results from this:

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<sup>8</sup>If  $|N| = 2$  there is no contradiction and if  $|N| = 3$  the proof below is valid assuming  $j = m$ .

**Theorem 7** *Let  $\phi$  be a solution that satisfies AN, SCIN, ORDPRE and ADD. Then, there exists  $\lambda_n \in [0, 1]$  such that for any problem  $(N, d, E) \in \Gamma^n$  we have that  $\phi(N, d, E) = \lambda_n$ -LS( $N, d, E$ ).*

**Proof.** Lemma 6 implies that  $\phi$  is a Weighted Least Square Solution in  $\Gamma^n$ . By ORDPRE it must be true that  $\lambda_n \in [0, 1]$ . Therefore it only remains to prove that Weighted Least Square Solutions with  $\lambda_n \in [0, 1]$  satisfy the properties. This is immediate, so we omit the details. ■

Note that because additivity requires that the claim problems have the same set of claimants the theorem applies in particular domains with fixed set of claimants. That is, a solution  $\phi$  satisfying the axioms and defined in the class of all claim problems can be the following. Let  $\phi$  be the Least Square if the cardinality of the set of claimants is two and let  $\phi$  be the Equal Awards otherwise. Note that this type of solutions does not satisfy consistency. A solution  $\phi$  satisfies consistency if for any problem  $(N, d, E)$  and any  $S \subset N$  it holds that  $\phi_i(S, (d_i)_{i \in S}, \sum_{i \in S} \phi_i(N, d, E)) = \phi_i(N, d, E)$  for all  $i \in S$ .

The family of Least Square Solutions contains different solutions. Each solution is characterized adding a new property to the set of properties.

P7) We say that  $\phi$  satisfies **half claim boundedness (HCM)** if for any  $(N, d, E) \in \Gamma$  we have that either  $\phi_i(N, d, E) \geq \frac{d_i}{2}$  for all  $i \in N$  or  $\phi_i(N, d, E) \leq \frac{d_i}{2}$  for all  $i \in N$ .

The idea that no one must receive more than half of his\her claim when someone else receives less than his\hers half is extensively discussed in Aumann and Maschler [2] and is presented as one of the foundations of the Talmud Rule.

**Theorem 8** *Let  $\phi$  be a solution that satisfies AN, SCIN, ETP, ADD and HCB. Then,  $\phi$  is the Least Square Solution.*

**Proof.** By the previous theorem we know that any solution  $\phi$  satisfying AN,ETP, SCIN and ADD is a Weighted Least Square Solution (with  $\lambda \in \mathbb{R}$  since ORDPRE is not required). Let  $\lambda$ -LS be a weighted Least Square Solution such that  $\lambda \neq \frac{1}{2}$ . Consider the following claim problem:

$$A = (\{1, 2, \dots, n-1, n\}, (2, 2, \dots, 2, 4), (n-1) + 2).$$

It is immediate that  $\lambda$ -LS( $A$ )  $\neq (1, 1, \dots, 1, 2)$ . For any solution  $\phi$  satisfying HCB  $\phi(A) = (1, 1, \dots, 1, 2)$ . Therefore it only remains to prove that the Least Square Solution satisfies HCB. Let  $(N, d, E)$  be a problem and let  $LS(N, d, E) = z$ . Note that for any  $i, j \in N$  we have that  $z_j = z_i + \frac{1}{2}(d_j - d_i)$ .

Assume that  $z_i < \frac{d_i}{2}$ . Then  $z_j = z_i + \frac{1}{2}(d_j - d_i) < \frac{d_i}{2} + \frac{1}{2}(d_j - d_i) = \frac{d_j}{2}$ .

Assume that  $z_i > \frac{d_i}{2}$ . Then  $z_j = z_i + \frac{1}{2}(d_j - d_i) > \frac{d_i}{2} + \frac{1}{2}(d_j - d_i) = \frac{d_j}{2}$ .

Therefore there is no pair of claimants  $i, j$  such that

$z_i < \frac{d_i}{2}$  and  $z_j > \frac{d_j}{2}$ . ■

Note that unlike the case of Theorem 7 this theorem applies in the domain of all claim problems.

The HCB property inspires the following property.

P8) We say that  $\phi$  satisfies  **$\lambda$ -claim boundedness ( $\lambda$ -CB)** if for any  $(N, d, E) \in \Gamma$  we have that  $\phi_l(N, d, E) \geq \lambda d_i$  for all  $i \in N$  or  $\phi_l(N, d, E) \leq \lambda d_i$  for all  $i \in N$ .

The next theorem follows immediately.

**Theorem 9** *Let  $\phi$  be a solution that satisfies AN, ETP, SCIN, ADD and  $\lambda$ -CB. Then,  $\phi$  is the  $\lambda$ -Least Square Solution.*

In this way, each member of the family is individually characterized.

Finally, we relate the Proportional Solution to the Least Square family. Since the Proportional Solution satisfies  $\lambda$ -CB for any  $\lambda$  the corollary below follows:

**Corollary 10** *Let  $A = (N, d, E)$  be a problem where  $E = \lambda \sum_{n \geq l \geq 1} d_l$  and let  $\phi$  be a solution that satisfies  $\lambda$ -CB. Then,  $\phi(A) = \lambda$ -LS( $A$ ) = PS( $\bar{A}$ ).*

## 5 Imputations and Weighted Least Square Solutions

### 5.1 A family of solutions

Most papers on claim problems assume that a solution must select an imputation. Least Square Solutions, in general, do not select imputations. This issue can be easily solved if we restrict the domain where egalitarian criteria are applied. Therefore if the choice of the Least Square maximal allocation is restricted to the Imputation Set the resulting allocation must be an imputation. From a computational point of view it is not difficult to find a new formula for the new LS Solutions. However it is not so easy to characterize the new solutions. It is clear that a solution that always selects imputations and also satisfies the minimal desirable properties (ETP, AN and SCIN) cannot satisfy additivity.

The following example<sup>9</sup> shows that additive solutions satisfying ETP do not, in general, select imputations. Alternatively, the result follows from the fact that Weighted Least Square Solutions do not necessarily select imputations. Bergantiños et al. [3] proves this result for two claimant problems. Therefore if we require a solution to be an *imputation selector* we cannot ask for additive solutions.

**Example 11** *Consider the following three claim problems:  $A = (0, 2, 2 : 2)$ ,  $B = (2, 0, 2 : 2)$ ,  $C = (0, 0, 2 : 2)$  and  $D = (2, 2, 2 : 2)$ .*

Let  $\phi$  be a solution that selects imputations satisfying additivity and ETP. It is immediate that  $\phi(A) = (0, 1, 1)$ ,  $\phi(B) = (1, 0, 1)$ ,  $\phi(C) = (0, 0, 2)$  and  $\phi(D) = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ .

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<sup>9</sup>The example can be seen as an airport problem and it also illustrates that in airport problems no solution satisfies additivity and ETP.

By additivity  $\phi(A + B) = \phi(A) + \phi(B) = (1, 1, 2)$  and  $\phi(C + D) = \phi(C) + \phi(D) = (\frac{2}{3}, \frac{2}{3}, \frac{8}{3})$ .

But  $\phi(C + D)$  should be identical to  $\phi(A + B)$  since they are the same claim problem.

We focus on solutions that select imputations. Weighted Least Square Solutions arise when the Least Square criterion is applied in the set  $AL^\lambda(I(N, d, E))$ . Consequently,

$$\lambda\text{-LS}(I(N, d, E)) = \{x \in I(N, d, E); \lambda\text{-}x^{AL} \succ_{LS} \lambda\text{-}y^{AL}, \text{ for all } y \in I(N, d, E)\}.$$

We use the term Imputation Selector Weighted Least Square Solution ( $\lambda\text{-ISLS}$ ) for these solutions. The following theorem can be proved following the arguments of the proof of Theorem 4 and provides the formula for the new solutions.

**Theorem 12** *Let  $(N, d, E)$  be a problem and  $\lambda \in [0, 1]$ . Then:*

- a) *If  $\frac{E - \lambda \sum_{n > i > 1} (d_i - d_1)}{n - k + 1} \in [0, d_1]$  then  $\lambda\text{-ISLS}(N, d, E) = \lambda\text{-LS}(N, d, E)$ .*  
b) *If  $E - \lambda \sum_{n \geq i \geq 1}^n (d_i - d_1) < 0$  then  $\lambda\text{-ISLS}(N, d, E) = z$  where*

$$z_i = \begin{cases} 0 & \text{if } i < k \\ \frac{E - \lambda \sum_{n > i > k} (d_i - d_k)}{n - k + 1} & \text{if } i = k \\ \lambda(d_i - d_k) & \text{if } i > k \end{cases}$$

*and  $k$  is the first index such that  $E - \lambda \sum_{n > i > k} (d_i - d_k) \geq 0$ .*

- c) *If  $E - \lambda \sum_{n \geq i \geq 1} d_i - d_1 > nd_1$  then  $\lambda\text{-ISLS}(N, d, E) = z$  where*

$$z_i = \begin{cases} d_i & \text{if } i < k \\ \frac{E - \sum_{k-1 > l \geq 1} d_l - \lambda \sum_{n > i > k} (d_i - d_k)}{n - k + 1} & \text{if } i = k \\ \lambda(d_i - d_k) & \text{if } i > k \end{cases}$$

*and  $k$  is the first index such that  $E - \sum_{k-1 \geq l \geq 1} d_l - \lambda \sum_{n \geq i \geq k} (d_i - d_k) \leq (n - k + 1)d_k$ .*

Figure 2 illustrates how these solutions perform in two-claimant problems when  $E$  moves from 0 to  $d_1 + d_2$ . Figure 2(a) shows the CEA and CEL solutions, and the shaded area between them corresponds to the ISLS solutions for different  $\lambda$ . Similarly, figure 2(b) shows the solutions ISLS when  $\lambda = \frac{1}{2}$ .

In this setting CEA and CEL can be seen as members of a family of solutions. A similar result is observed by Thomson [17]. In his paper, Thomson proves that CEA and CEL are members of the CIC family. He also mentions that the Reverse Talmud Rule is a member of the same family. In Thomson [18] a subfamily of the CIC family is introduced. This subfamily coincides with the family of Weighted ISLS and similar pictures to Figure 2 can also be seen in this paper.

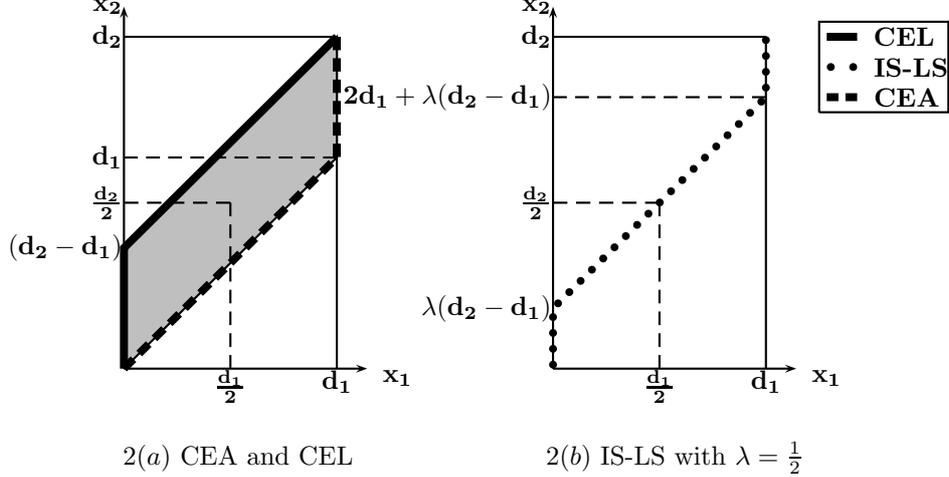


Figure 2: Illustration of the **Imputation Selector Weighted Least Square Solution** for different  $\lambda$  when  $E$  moves from 0 to  $d_1 + d_2$ .

The Reverse Talmud Rule is defined as follows. Let  $(N, d, E)$  be a claim problem. Then

$$T_i^r(N, d, E) = \begin{cases} \max\{\frac{d_i}{2} - \alpha, 0\} & \text{if } E \leq \frac{\sum_{n>i>1} d_l}{2} \\ \frac{d_i}{2} + \min\{\frac{d_i}{2}, \alpha\} & \text{otherwise} \end{cases}$$

where  $\alpha$  is chosen such that  $\sum_{n \geq i \geq 1} T_i^r(N, d, E) = E$ .

This solution is also a member of the family of IS Weighted LS Solutions. In fact, the Imputation Selector Least Square Solution and the Reverse Talmud Rule are the same solution.

**Theorem 13** Let  $(N, d, E)$  be a claim problem. Then  $T^r(N, d, E) = ISLS(N, d, E)$ .

**Proof.** Let  $z = T^r(N, d, E)$ . We distinguish 4 cases:

a)  $E \leq \frac{\sum_{n>i>1} d_i}{2}$  and  $z_i \in (0, d_i)$  for all  $i \in N$ .

In this case  $(z_i - z_j) = \frac{d_i}{2} - \alpha - (\frac{d_j}{2} - \alpha) = \frac{1}{2}(d_i - d_j)$  for all  $i, j \in N$ . Therefore  $z = ISLS(N, d, E)$ .

b)  $E \leq \frac{\sum_{n>i>1} d_i}{2}$  and  $z_l = 0$  for all  $l \in \{1, \dots, k\}$ .

In this case  $(z_i - z_j) = \frac{d_i}{2} - \alpha - (\frac{d_j}{2} - \alpha) = \frac{1}{2}(d_i - d_j)$  for all  $i, j \in \{k+1, \dots, n\}$ . We need to prove that

$$E - \frac{1}{2} \sum_{n \geq i \geq k} (d_i - d_k) \leq 0$$

or equivalently,

$$E - \frac{1}{2} \sum_{k \leq i \leq n} d_i + (n - k + 1) \frac{1}{2} d_k \leq 0.$$

And this is so because we know that

$$\frac{1}{2} \sum_{n \geq i \geq k} d_i - E \geq (n - k + 1) \frac{1}{2} d_k$$

since otherwise  $z_k > 0$ .

c)  $E > \frac{\sum_{n \geq i \geq 1} d_i}{2}$  and  $z_i \in (0, d_i)$  for all  $i \in N$ .

The proof of this case is the proof of case a).

d)  $E > \frac{\sum_{n \geq i \geq 1} d_i}{2}$  and  $z_l = d_l$  for all  $l \in \{1, \dots, k\}$ .

In this case  $(z_i - z_j) = \frac{d_i}{2} - \alpha - (\frac{d_j}{2} - \alpha) = \frac{1}{2}(d_i - d_j)$  for all  $i, j \in \{k + 1, \dots, n\}$ .

We need to prove that

$$\frac{E - \sum_{k-1 \geq i \geq 1} d_i - \frac{1}{2} \sum_{n \geq i \geq k} (d_i - d_k)}{n - k + 1} \geq d_k$$

or equivalently,

$$E - \sum_{k-1 \geq l \geq 1} d_l - \frac{1}{2} \sum_{n \geq i \geq k} d_i + (n - k + 1) \frac{1}{2} d_k \geq (n - k + 1) d_k$$

$$E - \sum_{k-1 \geq i \geq 1} d_i - \frac{1}{2} \sum_{n \geq i \geq k} d_i \geq (n - k + 1) \frac{1}{2} d_k.$$

And this is so because we know that

$$E - \sum_{k-1 \geq i \geq 1} d_i - \frac{1}{2} \sum_{n \geq i \geq k} d_i \geq (n - k + 1) \frac{1}{2} d_k$$

since otherwise  $z_k < d_k$ . ■

A Imputation Selector  $\lambda$ -Least Square can be formulated as a Reverse  $\lambda$ -Talmud Rule in the following terms:

Let  $(N, d, E)$  be a claim problem. Then

$$\lambda\text{-}T_i^r(N, d, E) = \begin{cases} \max\{\lambda d_i - \alpha, 0\} & \text{if } E \leq \lambda \sum_{n \geq l \geq 1} d_l \\ \lambda d_i + \min\{(1 - \lambda)d_i, \alpha\} & \text{otherwise} \end{cases}$$

where  $\alpha$  is chosen such that  $\sum_{n \geq i \geq 1} \lambda\text{-}T_i^r(N, d, E) = E$ .

## 5.2 Characterizations

In the following we axiomatize this family of solutions and each of its members.

Example 11 implies that these solutions do not satisfy additivity. We now introduce a new property called Restricted Additivity. We shall see that this weaker property of additivity is compatible with imputation selector and, in fact, we use the property to characterize the new family of solutions, all of them imputation selectors.

Let  $A = (N, d, E)$  be a problem and let  $\phi$  be a solution. Let  $T(A, \phi) = \{i \in N : 0 < \phi_i(A) < d_i\}$ .

P9) We say that  $\phi$  satisfies **Restricted Additivity** (RADD) if, given  $A = (N, d, E)$  and  $B = (N, d', E')$  with  $T(A, \phi) = T(B, \phi)$  and  $\phi_l(A) = \phi_l(B) = 0$  for all  $l \notin T(A, \phi)$  (or  $\phi_l(A) = d_l$  and  $\phi_l(B) = d_l$  for all  $l \notin T(A, \phi)$ ) then  $\phi(A + B) = \phi(A) + \phi(B)$ .

For this section, we also introduce the following notation. A problem  $A \in C^{n, \phi, k, 0}$  if  $[N] = n$ ,  $[T(A, \phi)] = k$  and  $\phi_i(A) = 0$  for all  $i \notin T(A, \phi)$ . Similarly, a problem  $B \in C^{n, \phi, k, d}$  if  $[N] = n$ ,  $[T(B, \phi)] = k$  and  $\phi_i(B) = d_i$  for all  $i \notin T(B, \phi)$ .

Given a problem  $A = (N, d, E)$  by  $d_i(A)$  we denote the claim of  $i$  in the problem  $A$ .

The following lemma is immediate.

**Lemma 14** *Let  $\phi$  be a solution that satisfies ORDPRE. Let  $(N, d, E)$  be a claim problem. If  $\phi_j(N, d, E) = d_j$  then  $\phi_i(N, d, E) = d_i$  for all  $i$  such that  $d_i \leq d_j$ . If  $\phi_j(N, d, E) = 0$  then  $\phi_i(N, d, E) = 0$  for all  $i$  such that  $d_i \leq d_j$ .*

Note that if a solution satisfies ADD then it satisfies RADD. The issue is that additive solutions are not Imputation Selectors, that is, they violate the following property:

P10) We say that a solution satisfies **Imputation Selection** (IS) if, given any claim problem, it selects an imputation.

The following lemma presents IS Weighted LS Solutions as candidates for satisfying the new properties (P9 and P10).

**Lemma 15** *Let  $\phi$  be a solution that satisfies AN, SCIN, ORDPRE and RADD. Then:*

a) *There exists  $\lambda \in [0, 1]$  such that for any  $A = (N, d, E) \in C^{n, \phi, k, 0}$  we have that  $\phi_j(A) - \phi_i(A) = \lambda(d_j - d_i)$  for all claimants  $i, j \in T(A, \phi)$ .*

b) *There exists  $\mu \in [0, 1]$  such that for any  $A = (N, d, E) \in C^{n, \phi, k, d}$  we have that  $\phi_j(A) - \phi_i(A) = \mu(d_j - d_i)$  for all claimants  $i, j \in T(A, \phi)$ .*

**Proof.** Similar to Lemma 6. ■

The lemma suggests that a solution satisfying these properties seems to share some similarities with the family of solutions introduced. In order to prove that

only these solutions satisfy the properties we need to show the following two facts;

1.- The solution  $\phi$  must be a Imputation Selector Weighted Least Square Solution for a problem  $A$  where  $[T(A, \phi)] = k$ . In other words:

Let  $\phi$  be a solution satisfying the properties, let  $A = (N, d, E)$  be a problem and let  $k$  be the last claimant for whom  $\phi_k(A) = 0$ . Then

$$E - \lambda \sum_{k \leq i \leq n} (d_i - d_k) \leq 0 \text{ and } E - \lambda \sum_{k+1 \leq i \leq n} (d_i - d_{k+1}) > 0$$

where  $\lambda$  is such that  $\phi_l(A) - \phi_i(A) = \lambda(d_l - d_i)$  for claimants  $i, l \in T(A, \phi)$ .

Let  $\phi$  be a solution satisfying the properties. Let  $A = (N, d, E)$  be a problem and let  $k-1$  be the last claimant for whom  $\phi_{k-1}(A) = d_{k-1}$ . Then

$$\begin{aligned} \frac{E - \sum_{1 \leq i < k-1} d_i - \lambda \sum_{k \leq i \leq n} (d_i - d_{k-1})}{n - k + 2} &\geq d_{k-1} \text{ and} \\ \frac{E - \sum_{1 \leq i < k} d_i - \lambda \sum_{k \leq i \leq n} (d_i - d_k)}{n - k + 1} &< d_k \end{aligned}$$

where  $\lambda$  is such that  $\phi_l(A) - \phi_i(A) = \lambda(d_l - d_i)$  for claimants  $i, l \in T(A, \phi)$ .

2.- The solution  $\phi$  is the same  $\lambda$ -ISLS for any type of claim problems. That is, let  $A$  and  $B$  be two claim problems such that  $[T(A, \phi)] \neq [T(B, \phi)]$ . Then  $\phi(A) = \lambda_1$ -ISLS and  $\phi(B) = \lambda_2$ -ISLS and  $\lambda_1 = \lambda_2$ .

The following two lemmas investigate the first fact. In the proofs the property of resource monotonicity is also used. Therefore in the new characterizations ADD is replaced by RADD, IS and MON.

**Lemma 16** *Let  $\phi$  be a solution that satisfies AN, SCIN, ORDPRE, IS, MON and RADD. Then there exists  $\lambda \in [0, 1]$  such that for any problem  $A = (N, d, E) \in C^{n, \phi, k, 0}$  we have that  $\phi(A) = \lambda$ -ISLS(A).*

**Proof.** By Lemma 15 we only need to prove that for any problem  $A = (N, d, E) \in C^{n, \phi, k, 0}$  with  $\phi_j(A) = 0$  and  $\phi_{j+1}(A) > 0$  we have that

$$E - \lambda \sum_{j \leq i \leq n} (d_i - d_j) \leq 0.$$

where  $\lambda$  is such that  $\phi_l(A) - \phi_i(A) = \lambda(d_l - d_i)$  for claimants  $i, l \in T(A, \phi)$ . The case  $k = 1$  (that is  $|T(A, \phi)| = 1$ ) is immediate and we omit the details. We only consider when  $k > 1$ .

Assume on the contrary that

$$\frac{E - \lambda \sum_{j \leq i \leq n} (d_i - d_j)}{n - j + 1} = k > 0.$$

Note that it must be true that

$$\phi_{j+1}(A) - \phi_j(A) > \lambda(d_{j+1} - d_j). \quad (1)$$

Let  $C = (N, d, E^*)$  be a problem such that the set of claimants and their claims are identical to problem A and  $E^*$  is the maximal amount for which  $\phi_j(C) = 0$ . Clearly,  $E^* \geq E$ .

Therefore,  $\phi_j(N, d, E^* + \varepsilon) > 0$  for any positive  $\varepsilon$ . Let  $D = (N, d, E^* + \varepsilon)$  be a problem where  $\varepsilon$  is as small as needed. Then  $j, j+1, j+2 \in T(D, \phi)$  and by Lemma 15 there exists  $\mu \in [0, 1]$  such that

$$\phi_{j+2}(D) - \phi_{j+1}(D) = \mu(d_{j+2} - d_{j+1}). \quad (2)$$

We consider two cases;

a) Assume that  $d_{j+2} > d_{j+1}$ .<sup>10</sup>

Note that also by Lemma 15 we have that

$$\phi_{j+2}(C) - \phi_{j+1}(C) = \lambda(d_{j+2} - d_{j+1}). \quad (3)$$

By monotonicity of  $\phi$  it must be true that

$$\phi_{j+2}(D) - \phi_{j+2}(C) \leq \varepsilon \text{ and } \phi_{j+1}(D) - \phi_{j+1}(C) \leq \varepsilon. \quad (4)$$

By (4) we know that

$$\phi_{j+1}(D) - \phi_{j+1}(C) - (\phi_{j+2}(D) - \phi_{j+2}(C)) \leq \varepsilon \quad (5)$$

and

$$\phi_{j+2}(D) - \phi_{j+2}(C) - (\phi_{j+1}(D) - \phi_{j+1}(C)) \leq \varepsilon. \quad (6)$$

Now assume that  $\lambda > \mu$ .

Subtracting (3) from (2) we conclude that

$$\phi_{j+2}(C) - \phi_{j+2}(D) - (\phi_{j+1}(C) - \phi_{j+1}(D)) = (\lambda - \mu)(d_{j+2} - d_{j+1}) > 0.$$

Since  $\varepsilon$  is as small as needed this inequality contradicts (5).

Now assume that  $\lambda < \mu$ .

Subtracting (2) from (3) we conclude that

$$\phi_{j+2}(C) - \phi_{j+2}(D) - (\phi_{j+1}(C) - \phi_{j+1}(D)) = (\mu - \lambda)(d_{j+2} - d_{j+1}) > 0.$$

Since  $\varepsilon$  is as small as needed this inequality contradicts (6).

Therefore we conclude that  $\mu = \lambda$ .

Remember that as a consequence of assuming that the lemma is not true we have that  $\phi_{j+1}(A) - \phi_j(A) > \lambda(d_{j+1} - d_j) = \mu(d_{j+1} - d_j)$ .

But it is immediate that if  $\phi_{j+1}(D) - \phi_j(D) = \mu(d_{j+1} - d_j)$  then  $\phi_{j+1}(A) - \phi_j(A) \leq \mu(d_{j+1} - d_j)$  contradicting (1).

Therefore  $\phi(N, d, E) = \lambda\text{-ISLS}(N, d, E)$ .

b)  $d_{j+1} = d_{j+2} = \dots = d_n$ . Let  $F \in C^{n, \phi, k, 0}$  be a problem where

$$E - \lambda \sum_{j \leq i \leq n} (d_i - d_j) = 0 \text{ and } d_{j+1} < d_{j+2}.$$

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<sup>10</sup>The proof is identical if being  $d_{j+1} = d_{j+2}$  there exists claimant  $l$  such that  $d_l > d_{j+1}$ . In this case the role played by claimant  $j+2$  is played by claimant  $l$ .

By RADD of  $\phi$  we have that  $(A+F) \in C^{n,\phi,k,0}$  and therefore  $\phi(A+F) = \lambda$ -ISLS( $A+F$ ). It is clear that  $d_{j+1}(A+F) < d_{j+2}(A+F)$  and that implies that

$$E(A) - \lambda \sum_{j \leq i \leq n} (d_i(A) - d_j(A)) \leq 0.$$

■

**Lemma 17** *Let  $\phi$  be a solution that satisfies AN, SCIN, ORDPRE, MON and RADD. Then there exists  $\lambda \in [0, 1]$  such that for any problem  $A = (N, d, E) \in C^{n,\phi,k,d}$  we have that  $\phi(A) = \lambda$ -ISLS( $A$ ).*

**Proof.** By Lemma 15 we only need to prove that for any problem  $A = (N, d, E) \in C^{n,\phi,k,d}$  with  $\phi_j(A) = d_j$  and  $\phi_{j+1}(A) < d_{j+1}$  we have that

$$\frac{E - \sum_{1 \leq l \leq j-1} d_l - \lambda \sum_{j \leq i \leq n} (d_i - d_j)}{n - j + 1} \geq d_j.$$

where  $\lambda$  is such that  $\phi_l(A) - \phi_i(A) = \lambda(d_l - d_i)$  for claimants  $i, l \in T(A, \phi)$ . The case  $k = 1$  (that is  $|T(A, \phi)| = 1$ ) is immediate and we omit the details. We only consider when  $k > 1$ .

Assume on the contrary that

$$\frac{E - \sum_{1 \leq l \leq j-1} d_l - \lambda \sum_{j \leq i \leq n} (d_i - d_j)}{n - j + 1} < d_j.$$

Note that it must be true that

$$\phi_{j+1}(A) - \phi_j(A) < \lambda(d_{j+1} - d_j). \quad (7)$$

Let  $C = (N, d, E^*)$  be a problem such that the set of claimants and their claims are identical to problem A and  $E^*$  is the minimal amount for which  $\phi_j(C) = d_j$  (since  $\phi$  is ORDPRE it also holds that  $\phi_l(C) = d_l$  for any  $l < j$ ). Clearly,  $E^* \leq E$ .

Therefore,  $\phi_j(N, d, E^* - \varepsilon) < d_j$  for any positive  $\varepsilon$ . Let  $D = (N, d, E^* - \varepsilon)$  be a problem where  $\varepsilon$  is as small as needed. Then  $j, j+1, j+2 \in T(D, \phi)$  and by Lemma 15 there exists  $\mu \in [0, 1]$  such that

$$\phi_{j+2}(D) - \phi_{j+1}(D) = \mu(d_{j+2} - d_{j+1}). \quad (8)$$

We consider two cases;

a) Assume that  $d_{j+2} > d_{j+1}$ .

Note that also by Lemma 15 we have that

$$\phi_{j+2}(C) - \phi_{j+1}(C) = \lambda(d_{j+2} - d_{j+1}). \quad (9)$$

By monotonicity of  $\phi$  it must be true that

$$\phi_{j+2}(C) - \phi_{j+2}(D) \leq \varepsilon \text{ and } \phi_{j+1}(C) - \phi_{j+1}(D) \leq \varepsilon. \quad (10)$$

By (10) we know that

$$\phi_{j+2}(C) - \phi_{j+2}(D) - (\phi_{j+1}(C) - \phi_{j+1}(D)) \leq \varepsilon \quad (11)$$

and

$$\phi_{j+1}(C) - \phi_{j+1}(D) - (\phi_{j+2}(C) - \phi_{j+2}(D)) \leq \varepsilon. \quad (12)$$

Now assume that  $\lambda > \mu$ .

Subtracting (9) from (8) we conclude that

$$\phi_{j+2}(C) - \phi_{j+2}(D) - (\phi_{j+1}(C) - \phi_{j+1}(D)) = (\lambda - \mu)(d_{j+2} - d_{j+1}) > 0.$$

Since  $\varepsilon$  is as small as needed this inequality contradicts (11).

Now assume that  $\lambda < \mu$ .

Subtracting (8) from (9) we conclude that

$$\phi_{j+2}(C) - \phi_{j+2}(D) - (\phi_{j+1}(C) - \phi_{j+1}(D)) = (\mu - \lambda)(d_{j+2} - d_{j+1}) > 0.$$

Since  $\varepsilon$  is as small as needed this inequality contradicts (12).

Therefore we conclude that  $\mu = \lambda$ .

Remember that as a consequence of assuming that the lemma is not true we have that  $\phi_{j+1}(A) - \phi_j(A) < \lambda(d_{j+1} - d_j) = \mu(d_{j+1} - d_j)$ .

But it is immediate that if  $\phi_{j+1}(D) - \phi_j(D) = \mu(d_{j+1} - d_j)$  then  $\phi_{j+1}(A) - \phi_j(A) \geq \mu(d_{j+1} - d_j)$  contradicting (7).

Therefore  $\phi(N, d, E) = \lambda\text{-ISLS}(N, d, E)$ .

b)  $d_{j+2} = d_{j+1}$ . This case is similar to the proof of case b) of Lemma 16 and is therefore omitted. ■

The next two lemmas investigate the second aspect of the problem.

**Lemma 18** *Let  $\phi$  be a solution that satisfies AN, SCIN, ORDPRE, MON, IS and RADD. Then there exists  $\lambda \in [0, 1]$  such that for any problem  $A \in C^{n, \phi, k, 0}$  (with  $k < |N|$ ) and any problem  $B \in C^{n, \phi, k+1, 0}$  we have that  $\phi(A) = \lambda\text{-ISLS}(A)$  and  $\phi(B) = \lambda\text{-ISLS}(B)$ .*

**Proof.** We consider 2 cases:

a) Let  $A \in C^{n, \phi, k, 0}$  with  $1 < k < |N|$  such that  $j$  is the last claimant for whom  $\phi_j(A) = 0$  and  $d_j > d_{j-1}$  in case that  $d_j \neq d_1$  (that is,  $k < n - 1$ ).

Let  $C = (N, d, E^*)$  be a problem such that the set of claimants and their claims are identical to problem A and  $E^*$  is the maximal amount for which  $C \in C^{n, \phi, k, 0}$ . Clearly,  $E^* \geq E$ .

Let  $D = (N, d, E^* + \varepsilon)$  be a problem where  $\varepsilon$  is as small as needed. Therefore,  $D \in C^{n, \phi, m, 0}$  where  $m > k$ . Then  $j, j+1, j+2 \in T(D, \phi)$  and by Lemma 15 there exists  $\mu \in [0, 1]$  such that

$$\phi_{j+2}(D) - \phi_{j+1}(D) = \mu(d_{j+2} - d_{j+1}). \quad (13)$$

Note that also by Lemma 15 we have that

$$\phi_{j+2}(C) - \phi_{j+1}(C) = \lambda(d_{j+2} - d_{j+1}). \quad (14)$$

By monotonicity of  $\phi$  it must be true that

$$\phi_{j+2}(C) - \phi_{j+2}(D) \leq \varepsilon \text{ and } \phi_{j+1}(C) - \phi_{j+1}(D) \leq \varepsilon. \quad (15)$$

By (15) we know that

$$\phi_{j+1}(D) - \phi_{j+1}(C) - (\phi_{j+2}(D) - \phi_{j+2}(C)) \leq \varepsilon \quad (16)$$

and

$$\phi_{j+2}(D) - \phi_{j+2}(C) - (\phi_{j+1}(D) - \phi_{j+1}(C)) \leq \varepsilon. \quad (17)$$

Now assume that  $\lambda > \mu$ .

Subtracting (14) from (13) we conclude that

$$\phi_{j+2}(C) - \phi_{j+2}(D) - (\phi_{j+1}(C) - \phi_{j+1}(D)) = (\lambda - \mu)(d_{j+2} - d_{j+1}) > 0.$$

Since  $\varepsilon$  is as small as needed this inequality contradicts (16).

Now assume that  $\lambda < \mu$ .

Subtracting (13) from (14) we conclude that

$$\phi_{j+2}(C) - \phi_{j+2}(D) - (\phi_{j+1}(C) - \phi_{j+1}(D)) = (\mu - \lambda)(d_{j+2} - d_{j+1}) > 0.$$

Since  $\varepsilon$  is as small as needed this inequality contradicts (17).

Therefore we conclude that  $\mu = \lambda$ .

Note that if  $d_j > d_{j-1}$  when selecting the problem  $D = (N, d, E^* + \varepsilon)$  we can guarantee that

$$E^* + \varepsilon - \lambda \sum_{j \leq i \leq n} (d_i - d_{j-1}) \leq 0$$

and in this way  $D \in C^{n, \phi, k+1, 0}$ . If  $k = n - 1$  we also can guarantee (by choosing  $\varepsilon$ ) that  $D \in C^{n, \phi, k+1, 0}$ .

Therefore  $D \in C^{n, \phi, k+1, 0}$  and  $\phi(D) = \lambda\text{-ISLS}(D)$ . By Lemma 15 we know that for any  $F \in C^{n, \phi, k+1, 0}$  it also must be true that  $\phi(F) = \lambda\text{-ISLS}(F)$ .

b) Let  $A \in C^{n, \phi, 1, 0}$  with  $d_{n-1} > d_{n-2}$ . Let  $C = (N, d, E^*)$  be a problem such that the set of claimants and their claims are identical to problem  $A$  and  $E^*$  is the maximal amount for which  $C \in C^{n, \phi, 1, 0}$ . Clearly,  $E^* \geq E$ .

Let  $D = (N, d, E^* + \varepsilon)$  be a problem where  $\varepsilon$  is as small as needed. Therefore,  $D \in C^{n, \phi, m, 0}$  where  $m > 1$ . By the previous case we know that there exists  $\lambda \in [0, 1]$  such that  $\phi(D) = \lambda\text{-ISLS}(D)$ . We need to prove that  $E^* - \lambda(d_n - d_{n-1}) \leq 0$ . Note that

$$\phi_{n-1}(D) = \frac{E^* + \varepsilon - \lambda(d_n - d_{n-1})}{2}.$$

If  $E^* - \lambda(d_n - d_{n-1}) > 0$  we conclude that for  $\varepsilon$  small enough it must be true that  $\phi_{n-1}(D) - \phi_{n-1}(C) = \phi_{n-1}(D) > \varepsilon$  and that contradicts that  $\phi$  satisfies MON. If  $d_{n-1} > d_{n-2}$  we can guarantee (by choosing  $\varepsilon$ ) that  $D \in C^{n, \phi, 2, 0}$ .

Therefore  $D \in C^{n, \phi, 2, 0}$  and  $\phi(D) = \lambda\text{-ISLS}(D)$ . By Lemma 15 we know that for any  $F \in C^{n, \phi, 2, 0}$  it also must be true that  $\phi(F) = \lambda\text{-ISLS}(F)$ . ■

**Lemma 19** *Let  $\phi$  be a solution that satisfies AN, SCIN, ORDPRE, MON, IS and RADD. Then there exists  $\lambda \in [0, 1]$  such that for any problem  $A \in C^{n,\phi,k,d}$  (with  $k < |N|$ ) and any problem  $B \in C^{n,\phi,k+1,d}$  we have that  $\phi(A) = \lambda\text{-ISLS}(A)$  and  $\phi(B) = \lambda\text{-ISLS}(B)$ .*

**Proof.** We consider 2 cases:

a) Let  $A \in C^{n,\phi,k,d}$  with  $1 < k < |N|$  such that  $j$  is the last claimant for which  $\phi_j(A) = d_j$  and  $d_j > d_{j-1}$  in case  $k < n - 1$

Let  $C = (N, d, E^*)$  be a problem such that the set of claimants and their claims are identical to problem A and  $E^*$  is the minimal amount for which  $C \in C^{n,\phi,k,d}$ . Clearly,  $E^* \leq E$ .

Let  $D = (N, d, E^* - \varepsilon)$  be a problem where  $\varepsilon$  is as small as needed. Therefore,  $D \in C^{n,\phi,m,0}$  where  $m < k$ . Then  $j, j + 1, j + 2 \in T(D, \phi)$  and by Lemma 15 there exists  $\mu \in [0, 1]$  such that

$$\phi_{j+2}(D) - \phi_{j+1}(D) = \mu(d_{j+2} - d_{j+1}). \quad (18)$$

Note that also by Lemma 15 we have that

$$\phi_{j+2}(C) - \phi_{j+1}(C) = \lambda(d_{j+2} - d_{j+1}). \quad (19)$$

By monotonicity of  $\phi$  it must be true that

$$\phi_{j+2}(C) - \phi_{j+2}(D) \leq \varepsilon \text{ and } \phi_{j+1}(C) - \phi_{j+1}(D) \leq \varepsilon. \quad (20)$$

By (20) we know that

$$\phi_{j+2}(C) - \phi_{j+2}(D) - (\phi_{j+1}(C) - \phi_{j+1}(D)) \leq \varepsilon \quad (21)$$

and

$$\phi_{j+1}(C) - \phi_{j+1}(D) - (\phi_{j+2}(C) - \phi_{j+2}(D)) \leq \varepsilon. \quad (22)$$

Now assume that  $\lambda > \mu$ .

Subtracting (19) from (18) we conclude that

$$\phi_{j+2}(C) - \phi_{j+2}(D) - (\phi_{j+1}(C) - \phi_{j+1}(D)) = (\lambda - \mu)(d_{j+2} - d_{j+1}) > 0.$$

Since  $\varepsilon$  is as small as needed this inequality contradicts (21).

Now assume that  $\lambda < \mu$ .

Subtracting (18) from (19) we conclude that

$$\phi_{j+2}(C) - \phi_{j+2}(D) - (\phi_{j+1}(C) - \phi_{j+1}(D)) = (\mu - \lambda)(d_{j+2} - d_{j+1}) > 0.$$

Since  $\varepsilon$  is as small as needed this inequality contradicts (22).

Therefore we conclude that  $\mu = \lambda$ .

Note that if  $d_j > d_{j-1}$  when selecting the problem  $D = (N, d, E^* - \varepsilon)$  we can guarantee that

$$\frac{E^* - \varepsilon - \sum_{1 \leq l \leq j-2} d_l - \lambda \sum_{j-1 \leq i \leq n} (d_i - d_{j-1})}{n - j + 2} \geq d_{j-1}$$

and in this way  $D \in C^{n,\phi,k+1,d}$ . If  $k = n - 1$  we also can guarantee (by choosing  $\varepsilon$ ) that  $D \in C^{n,\phi,k+1,0}$ .

Therefore  $D \in C^{n,\phi,k+1,d}$  and  $\phi(D) = \lambda\text{-ISLS}(D)$ . By Lemma 15 we know that for any  $F \in C^{n,\phi,k+1,d}$  it also must be true that  $\phi(F) = \lambda\text{-ISLS}(F)$ .

b) Let  $A \in C^{n,\phi,1,d}$  with  $d_{n-2} < d_{n-1}$ . Let  $C = (N, d, E^*)$  be a problem such that the set of claimants and their claims are identical to problem  $A$  and  $E^*$  is the minimal amount for which  $C \in C^{n,\phi,1,d}$ . Clearly,  $E^* \leq E$ . Let  $D = (N, d, E^* - \varepsilon)$  be a problem where  $\varepsilon$  is as small as needed. Therefore,  $D \in C^{n,\phi,2,d}$ . By the previous case we know that there exists  $\lambda \in [0, 1]$  such that  $\phi(D) = \lambda\text{-ISLS}(D)$ . We need to prove that

$$\frac{E^* - \sum_{1 \leq l \leq n-2} d_l - \lambda(d_n - d_{n-1})}{2} \geq d_{n-1}. \quad (23)$$

Assume on the contrary that

$$d_{n-1} - \frac{E^* - \sum_{1 \leq l \leq n-2} d_l - \lambda(d_n - d_{n-1})}{2} = k > 0.$$

Note that since  $D \in C^{n,\phi,2,d}$  we have that

$$\phi_{n-1}(D) = \frac{E^* - \varepsilon - \sum_{1 \leq l \leq n-2} d_l - \lambda(d_n - d_{n-1})}{2}.$$

If (23) does not hold we conclude that for  $\varepsilon$  small enough it must be true that

$$\begin{aligned} \phi_{n-1}(C) - \phi_{n-1}(D) &= d_{n-1} - \phi_{n-1}(D) = \\ &= \frac{E^* - \sum_{1 \leq l \leq n-2} d_l - \lambda(d_n - d_{n-1})}{2} + k - \phi_{n-1}(D) = \\ &= k + \frac{\varepsilon}{2}. \end{aligned}$$

This contradicts that  $\phi$  satisfies MON since it must be true that  $\phi_{n-1}(C) - \phi_{n-1}(D) \leq \varepsilon$  and that implies that for any  $\varepsilon$  it must be true that  $k \leq \frac{\varepsilon}{2}$  and that implies  $k \leq 0$ .

Note that if  $d_{n-1} > d_{n-2}$  we can guarantee (by choosing  $\varepsilon$ ) that  $D \in C^{n,\phi,2,d}$ .

Therefore  $D \in C^{n,\phi,2,d}$  and  $\phi(D) = \lambda\text{-ISLS}(D)$ . By Lemma 15 we know that for any  $F \in C^{n,\phi,2,d}$  it also must be true that  $\phi(F) = \lambda\text{-ISLS}(F)$ . ■

With the above lemmas the two questions are answered positively and the main theorem results.

**Theorem 20** *Let  $\phi$  be a solution that satisfies AN, SCIN, ORDPRE, MON, IS and RADD. Then, there exists  $\lambda_n \in [0, 1]$  such that for any problem  $(N, d, E) \in \Gamma^n$  we have that  $\phi(N, d, E) = \lambda_n\text{-ISLS}(N, d, E)$ .*

**Proof.** With the above lemmas we know that if  $\phi$  satisfies AN, SCIN, ORDPRE, MON, IS and RADD on  $\Gamma^n$  then  $\phi$  must be an  $\lambda_n\text{-ISLS}$  with  $\lambda_n \in [0, 1]$  in  $\Gamma^n$ . It only remains to prove that  $\lambda_n\text{-ISLS}$  satisfies AN, SCIN, ORDPRE,

MON, IS and RADD in  $\Gamma^n$ . The fact that  $\lambda_n$ -ISLS satisfies RADD is a consequence of the fact that if given two claim problems,  $A$  and  $B$ , with the same set of claimants we have that  $A, B \in C^{n, \lambda_n\text{-ISLS}, p, 0}$  (or  $A, B \in C^{n, \lambda_n\text{-ISLS}, p, d}$ ) then  $A + B \in C^{n, \lambda_n\text{-ISLS}, p, 0}$  (or  $A + B \in C^{n, \lambda_n\text{-ISLS}, p, d}$ ). It is immediate that  $\lambda_n$ -ISLS satisfies the rest of the properties. ■

Note that because restricted additivity requires problems to have the same set of claimants the theorem applies in particular domains with fixed set of claimants. That is, a solution  $\phi$  defined in the class of all claim problems satisfying the axioms of Theorem 20 is the following. Let  $\phi$  be the Imputation Selector Least Square if the cardinality of the set of claimants is two and let  $\phi$  be the Constrained Equal Awards otherwise.

In order to single out the Imputation Selector Least Square Solution from the family of  $\lambda$ -ISLS we again use the axiom of HCB.

**Theorem 21** *Let  $\phi$  be a solution that satisfies AN, SCIN, ETP, MON, IS, RADD and HCB. Then,  $\phi$  is the Imputation Selector Least Square Solution.*

**Proof.** By Theorem 8 it only remains to prove that the Imputation Selector Least Square Solution satisfies HCB. This is immediate since  $T^r(N, d, E) = ISLS(N, d, E)$ . ■

Note that unlike the case of Theorem 20 this theorem applies in the domain of all claim problems. Thomson [18] also characterizes the reverse Talmud rule by showing that it is the unique member of the CIC family that satisfies self-duality (Proposition 1).

In order to single out the Imputation Selector Weighted Least Square Solutions from the family of  $\lambda$ -ISLS we use again the axiom of  $\lambda$ -CB.

**Theorem 22** *Let  $\phi$  be a solution that satisfies AN, SCIN, ETP, MON, IS, RADD and  $\lambda$ -CB with  $\lambda \in (0, 1)$ . Then,  $\phi$  is the  $\lambda$ -Imputation Selector Least Square Solution.*

For  $\lambda = 0$  and  $\lambda = 1$  the definition of  $\lambda$ -CB should be modified slightly. Otherwise CEA and CEL cannot be characterized with the axioms of the Theorem above since by definition any solution satisfying IS must satisfy 0-CB and 1-CB.

P11) We say that  $\phi$  satisfies **strict 0-claim boundedness (strict 0-CB)** if for any claim problem,  $(N, d, E) \in \Gamma$ ,  $E > 0$  and  $d_1 > 0$ , we have that  $\phi_l(N, d, E) > 0$  for any  $l \in N$ .

P12) We say that  $\phi$  satisfies **strict 1-claim boundedness (strict 1-CB)** for any claim problem,  $(N, d, E) \in \Gamma$ ,  $E > 0$  and  $d_1 > 0$ , we have that  $\phi_l(N, d, E) < d_l$  for any  $l \in N$ .

CEA is the only solution that satisfies AN, SCIN, ETP, MON, IS, RADD and strict 0-CB and CEL is characterized with AN, SCIN, ETP, MON, IS, RADD and strict 1-CB

## 6 The Lexicographic criterion

### 6.1 A family of solutions

Another central solution in the literature of claim problems is the Talmud Rule introduced by Aumann and Maschler [2]. The solution is presented as the rule that explains the resolution of three numerical examples that can be found in the Talmud. For many years was an open problem what solution was behind these examples. Aumann and Maschler prove that their rule prescribes the proposals of the examples in the Talmud. They also prove that the solution coincides with the nucleolus of a TU game associated with the claim problem.

The nucleolus (Schmeidler [14]) selects lexicographical maximal elements in the set of vectors of satisfactions of the coalitions. It can be seen that the Talmud Rule is also a Lexicographic solution. First we introduce the definition of the Talmud rule.

Let  $(N, d, E)$  be a claim problem. Then

$$T_i(N, d, E) = \begin{cases} \min \left\{ \frac{d_i}{2}, \alpha \right\} & \text{if } E \leq \frac{\sum_{n \geq i \geq 1} d_i}{2} \\ \frac{d_i}{2} + \max \left\{ \frac{d_i}{2} - \alpha, 0 \right\} & \text{otherwise} \end{cases}$$

where  $\alpha$  is chosen such that  $\sum_{n \geq i \geq 1} T_i(N, d, E) = E$ .

This solution provides the allocation whose vector of awards-losses is the lexicographically maximal vector in the set  $|AL(I(N, d, E))|$ . That is,

**Theorem 23** *Let  $(N, d, E)$  be a claim problem. Then*

$$T(N, d, E) = \{x \in I(N, d, E); |x^{AL}| \succeq_{Lex} |y^{AL}|, \text{ for all } y \in I(N, d, E)\}.$$

**Proof.** Let  $z = T(N, d, E)$ . We distinguish 4 cases:

a)  $E \leq \frac{\sum_{n \geq l \geq 1} d_l}{2}$  and  $z_i < \frac{d_i}{2}$  for all  $i \in N$ .

Then the first  $n$  elements of the vector  $\theta(|z^{AL}|)$  are  $(\frac{E}{n}, \dots, \frac{E}{n})$  and clearly  $|z^{AL}|$  lexicographically dominates any other vector  $|y^{AL}|$  where  $y$  is a imputation.

b)  $E \leq \frac{\sum_{n \geq l \geq 1} d_l}{2}$  and  $z_l = \frac{d_l}{2}$  for all  $l \in \{1, \dots, k\}$ . Then the first  $2k$  elements of the vector  $\theta(|z^{AL}|)$  are  $(\frac{d_1}{2}, \frac{d_1}{2}, \dots, \frac{d_k}{2}, \frac{d_k}{2})$  and the next  $(n - k)$  elements are  $(\frac{E - \frac{1}{2} \sum_{k \geq l \geq 1} d_l}{n - k}, \dots, \frac{E - \frac{1}{2} \sum_{k \geq l \geq 1} d_l}{n - k})$ . Clearly  $|z^{AL}|$  lexicographically dominates any other vector  $|y^{AL}|$  where  $y$  is a imputation.

c)  $E > \frac{\sum_{n \geq l \geq 1} d_l}{2}$  and  $d_i > z_i > \frac{d_i}{2}$  for all  $i \in N$ .

Then the first  $n$  elements of the vector  $\theta(|z^{AL}|)$  are  $(\frac{\sum_{n \geq l \geq 1} d_l - E}{n}, \dots, \frac{\sum_{n \geq l \geq 1} d_l - E}{n})$  and clearly  $|z^{AL}|$  lexicographically dominates any other vector  $|y^{AL}|$  where  $y$  is a imputation.

d)  $E > \frac{\sum_{n \geq l \geq 1} d_l}{2}$  and  $z_l = \frac{d_l}{2}$  for all  $l \in \{1, \dots, k\}$ . Then the first  $2k$  elements of the vector  $\theta(|z^{AL}|)$  are  $(\frac{d_1}{2}, \frac{d_1}{2}, \dots, \frac{d_k}{2}, \frac{d_k}{2})$  and the next  $(n - k)$  elements are

$$\left( \frac{\frac{1}{2} \sum_{n \geq l \geq k+1} d_l - \frac{1}{2} E}{n - k}, \dots, \frac{\frac{1}{2} \sum_{n \geq l \geq k+1} d_l - \frac{1}{2} E}{n - k} \right).$$

Clearly  $|z^{AL}|$  lexicographically dominates any other vector  $|y^{AL}|$  where  $y$  is a imputation. ■

Weighted Talmud Rules<sup>11</sup> are introduced and studied by Moreno-Tertero and Villar [12]. They call this family of solutions the TAL-family.

Let  $(N, d, E)$  be a problem. Then

$$\lambda\text{-}T_i(N, d, E) = \begin{cases} \min \{\lambda d_i, \alpha\} & \text{if } E \leq \lambda \sum_{n \geq i \geq 1} d_l \\ \lambda d_i + \max \{(1 - \lambda)d_i - \alpha, 0\} & \text{otherwise} \end{cases}$$

where  $\alpha$  is chosen such that  $\sum_{n \geq i \geq 1} \lambda\text{-}T_i(N, d, E) = E$ .

It is not difficult to check that this solution provides the allocation whose vector of awards-losses is maximal in the set  $|AL^\lambda(I(N, d, E))|$ . That is for  $\lambda \in (0, 1)$  we have that

$$\lambda\text{-}T(N, d, E) = \{x \in I(N, d, E); |x^{AL}| \succeq_{Lex} |y^{AL}|, \text{ for all } y \in I(N, d, E)\}.$$

Weighted Talmud Rules always select imputations and satisfy their associated  $\lambda$ -CB.

Since the Proportional Solution satisfies  $\lambda$ -CB for any  $\lambda$  the following corollary is immediate and relates the Proportional Solution with the family of Weighted Talmud Rules.

**Corollary 24** *Let  $A = (N, d, E)$  be a problem where  $E = \lambda \sum_{n \geq i \geq 1} d_l$  and let  $\phi$  be a solution that satisfies  $\lambda$ -CB. Then,  $\phi(A) = \lambda\text{-}T(A) = PS(A)$ .*

## 6.2 Characterizations

Auman and Maschler [2] characterize the Talmud rule as the unique consistent solution for bankruptcy problems (Theorem A). In their work consistency is also called CG-consistency and explained as follows:

Intuitively, a solution is consistent if any two claimants  $i, j$  use the *contested garment principle* to divide between them the total amount  $x_i + x_j$  awarded to them by the solution.

The *contested garment principle* is a solution used to solve two-claimant problems. The solution coincides with the Talmud Rule and the theorem can be interpreted as follows; the Talmud rule is the unique solution that consistently extends to  $n$  claimant problems the *contested garment principle*.

Replacing the *contested garment principle* by the solution prescribed by a Weighted Talmud Rule in two claimant problems we can characterize this

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<sup>11</sup>Hokari and Thomson [7] use this term for Talmud rules that do not satisfy ETP. That is, the weights refer to the claimants and not to awards-losses. We keep the term since we think there is no confusion and it is more consistent with the rest of the paper.

Weighted Talmud Rule as the unique solution that consistently extends to  $n$  claimant problems this solution prescribed for two claimant problems.

Thomson [18] also characterizes the Talmud rule by showing that is the unique member of the ICI family that satisfies self-duality (Proposition 1).

## 7 Conclusions

### 7.1 A summary

A reminder of the two questions that motivated this research

1. What egalitarian criterion is used to make egalitarian comparisons between elements?
2. From what set are those elements taken?

The table below is a summary of the answers.

Solution	Criterion	Domain	Weight: $\lambda$
<b>CEA</b>	LS and Lex	$AL^\lambda(I(N, d, E))$	0
<b>CEL</b>	LS and Lex	$AL^\lambda(I(N, d, E))$	1
ISLS or $T^r$	LS	$AL^\lambda(I(N, d, E))$	$\frac{1}{2}$
$\lambda$ -ISLS or $\lambda$ - $T^r$	LS	$AL^\lambda(I(N, d, E))$	$\lambda$
<b>T</b>	Lex	$ AL^\lambda(I(N, d, E)) $	$\frac{1}{2}$
$\lambda$ - <b>T</b>	Lex	$ AL^\lambda(I(N, d, E)) $	$\lambda$
<b>PS</b>	LS and Lex	$ AL^\lambda(I(N, d, E)) $	$\frac{E}{\sum_{n \geq i \geq 1} d_i}$
<b>LS</b>	LS	$AL^\lambda(PI(N, d, E))$	$\frac{1}{2}$
$\lambda$ - <b>LS</b>	LS	$AL^\lambda(PI(N, d, E))$	$\lambda$

The table gives a unified framework to place many different solutions that have been introduced and analyzed by several authors.

Arin [1] presents a similar summary concerning solution concepts defined for TU games.

## 7.2 Extensions

The table also indicates how to extend this type of solutions to other different settings. In particular, in airport problems (Littlechild, [11]) it is generally accepted that solutions must select core allocations and not merely imputations. Therefore, the search for egalitarian maximal elements should be restricted to the core of the airport problem and Core Selector Weighted Least Square Solutions are immediately defined. In other settings, other constraints may exist and solutions are required to satisfy them. This is a restriction of the set where egalitarian maximal elements are sought. Also in claim problems different constraints could be considered. Chun et al. [5] define an egalitarian solution with respect to awards that satisfy the Half Claim Boundedness property. In his case the constraint is not only to be an Imputation Selector but also to select allocations that do not violate HCB.

An identical approach can be taken to analyze surplus sharing problems where agents must divide a surplus jointly generated but with different *participation rights*. Agents can evaluate the fairness of any proposal by checking how much they have received and how much they have gained with respect to their rights.

In fact, Ju et al. [8] also follow a general approach to deal with different economic problems that share similarities in their mathematical modelling. They use the term "entity" instead of claimant or even agent in order to ensure that their analysis is valid for different settings.

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