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ON THE INTUIOTIONISTIC FUZZY RELATIONS

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ABSTRACT:

Intuitionistic fuzzy relations which generalize the already existing such relations are introduced and their basic properties are shown.

The concept of the Intuitionistic Fuzzy Relation (IFR) is based on the definition of the Intuitionistic Fuzzy Sets (IFSs) [1]. It is introduced in different forms and different ways, and practically independently, in [2-9]. We must note that the approaches in the various IFR definitions are different in researches by different authors. The approach from [7-9] was in some sense the most general. In the present form it includes Buhaescu's [4], Stoyanova's [6], Burillo and Bustince's [8,9] results as particular cases.

Let everywhere below X, Y and Z be ordinary finite non empty sets (universes).

An IFS A^* in X is an object with the form:

$$A^* = \{<x, \mu_A(x), \gamma_A(x)>/x \in E\},$$

where the functions $\mu_A : E \to [0, 1]$ and $\gamma_A : E \to [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in E$, respectively, and for every $x \in E$:

$$0 \le \mu(x) + \gamma(x) \le 1.$$

Below we shall write A instead of A^*. We shall denote by IFS the set (or class, in the sense of the NBG set theory) of all IFSs and by FS - of all fuzzy sets.

We shall call Intuitionistic Fuzzy Norm (IFN) in [0, 1] x [0, 1] every couple $\langle S, T\rangle$ of two mappings $S, T: [0, 1] \times [0, 1] \to [0, 1]$ satisfying the properties (cf. [7-10])

i) Boundary conditions, $T(x, 1) = x$, $T(x\ 0) = 0$, $S(x, 1) = 1$ and $S(x, 0) = x$ for every $x \in [0, 1]$;

ii) Monotony, $T(x, y) \le T(z, t)$ and $S(x, y) \le S(z, t)$ iff $x \le z$ and $y \le t$, where $x, y, z, t \in [0, 1]$;

iii) Commutative, $T(x, y) = T(y, x)$ and $S(x, y) = S(y, x)$ for every $x, y \in [0, 1]$;

iv) Associative, $T(T(x, y), z) = T(x, T(y, z))$ and $S(S(x, y), z) = S(x, S(y, z))$ for every $x, y, z \in [0, 1]$;

v) Connection, $S(x, y) + T(z, t) \le 1$ for every $x, y, z, t \in [0, 1]$, such that $x + z \le 1$ and $y + t \le 1$.

We shall note that when $S(x, y) + T(x, y) = 1$ we obtain the definition from [8] with some corrections, where T is called a norm and S - a conorm.

An IFR is called every IFS $R \subset X \times Y$ with the form (cf. [8, 9]):

$$R = \{\langle\langle x, y\rangle, \mu_R(x, y), \gamma_R(x, y)\rangle / x \in X \ \& \ y \in Y\},$$

where $\mu_R : X \times Y \to [0, 1]$, $\gamma_R : X \times Y \to [0, 1]$ are degrees of membership and non-membership as the ordinary IFSs (see [1]) or degrees of validity and non-validity of the relation R; and for every $\langle x, y\rangle \in X \times Y$:

$$0 \le \mu_R(x, y), \gamma_R(x, y) \le 1.$$

We shall denote with IFR(X, Y) the set of all IFRs over $X \times Y$.

This new relation, which obviously is an IFR, we shall call inverse relation to R.

Let $P, R \in IFR(X, Y)$ and let below $(x, y) \in X \times Y$. We shall define the following relations and operations over IFRs (ch. [1, 8]).

$P \subset R$ iff $(\forall (x, y) \in X \times Y)(\mu_P(x, y) \le \mu_R(x, y) \ \& \ \gamma_P(x, y) \ge \gamma_R(x, y))$;

$P = R$ iff $P \subset R \ \& \ R \subset P$;

$\bar{P} = \{\langle x, \gamma_P(x, y), \mu_P(x, y)\rangle / (x, y) \in X \times Y\}$

$P \cap R = \{\langle\langle x, y\rangle, \min(\mu_P(x, y), \mu_R(x, y)), \max(\gamma_P(x, y), \gamma_R(x, y))\rangle / (x, y) \in X \times Y\}$

$P \cup R = \{\langle\langle x, y\rangle, \max(\mu_P(x, y), \mu_R(x, y)), \min(\gamma_P(x, y), \gamma_R(x, y))\rangle / (x, y) \in X \times Y\}$

$P + R = \{\langle\langle x, y\rangle, \mu_P(x, y)+\mu_R(x, y)-\mu_P(x, y).\mu_R(x, y), \gamma_P(x, y).\gamma_R(x, y)\rangle /$

$(x, y)\in X \times Y\}$

$P . R = \{\langle\langle x, y\rangle, \mu_P(x, y).\mu_R(x, y), \gamma_P(x, y)+\gamma_R(x, y)-\gamma_P(x, y).\gamma_R(x, y)\rangle /$

$(x, y)\in X \times Y\}$.

It is seen easily that the above constructed sets (IFRs) are IFSs. Therefore, all assertions for IFSs (see [1]) are valid for them, too.

Let $P \in IFR(X, Y)$, $R \in IFR(Y, Z)$. Then the object

$P\ (S, T)\ R = \{\langle\langle x, y\rangle, S(\mu_P(x, y), \mu_R(x, y)), T(\gamma_P(x, y), \gamma_R(x, y))\rangle /$

$(x, y)\in X \times Y\}$

is called a composed (S, T)-relation.

From the definition of S and T it follows that P (S, T) R is an IFR. The definitions from [6, 8] are particular cases of the above ones.

THEOREM 1: For the IFRs P, P_1, P_2, R, R_1, R_2:

(a) if $P_1 \subset P_2$, then $P_1\ (S, T)\ R \subset P_2\ (S, T)\ R$;

(b) if $R_1 \subset R_2$, then $P\ (S, T)\ R_1 \subset P\ (S, T)\ R_2$;

Proof: (a) Let $P_1 \subset P_2$. Then for

$P_i\ (S, T)\ R = \{\langle\langle x, y\rangle, S(\mu_{P_i}(x, y), \mu_R(x, y)), T(\gamma_{P_i}(x, y), \gamma_R(x, y))\rangle /$

$(x, y)\in X \times Y\}$

$(i = 1, 2)$ it follows from ii) that

$$S(\mu_{P_1}(x, y), \mu_R(x, y)) \le S(\mu_{P_2}(x, y), \mu_R(x, y))$$

and

$$T(\gamma_{P_1}(x, y), \gamma_R(x, y)) \ge T(\gamma_{P_2}(x, y), \gamma_R(x, y)),$$

from where it follows the validity of the assertion.

(b) is proved analogically.

This assertion is a generalization of Theorem 5 [8].

THEOREM 2: For the IFRs P, Q, R:

(a) $(P \cup Q)\ (S, T)\ R \supset (P\ (S, T)\ R) \cup (Q\ (S, T)\ R)$

(b) $(P \cap Q)\ (S, T)\ R \subset (P\ (S, T)\ R) \cap (Q\ (S, T)\ R)$

(c) $(P\ (S, T)\ Q)\ (S, T)\ R = P\ (S, T)\ (Q\ (S, T)\ R)$

Proof: (a) From $P \cup Q \supset P$ and $P \cup Q \supset Q$ it follows that

$(P \cup Q)\ (S, T)\ R \supset P\ (S, T)\ R$ and $(P \cup Q)\ (S, T)\ R \supset Q\ (S, T)\ R$ from where $(P \cup Q)\ (S, T)\ R \supset (P\ (S, T)\ R) \cup (Q\ (S, T)\ R)$.

(b) and (c) are proved analogically.

The relation $\Delta \in IFR(X, X)$ is called relation of total identity if for every $(x, y) \in X \times X$:

$$\mu_{\Delta}(x, y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \gamma_{\Delta}(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{otherwise} \end{cases}$$

Let its complementary relation be marked by ∇, i.e. $\nabla = \bar{\Delta}$. Therefore:

$$\mu_{\nabla}(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{otherwise} \end{cases} \quad \text{and} \quad \gamma_{\nabla}(x, y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise} \end{cases}$$

The relation $I\Delta \in IFR(X, X)$ is called a relation of intuitionistic fuzzy identity if for every $x \in X$:

$$\mu_{I\Delta}(x, x) \begin{cases} \geq \gamma_{I\Delta}(x, x), & \text{if } x = y \\ < \gamma_{I\Delta}(x, x), & \text{if } x \neq y \end{cases}$$

Let its complementary relation be marked by $I\nabla$, i.e., $I\nabla = \overline{I\Delta}$. Therefore:

$$\mu_{I\nabla}(x, x) \begin{cases} > \gamma_{I\nabla}(x, x), & \text{if } x = y \\ \geq \gamma_{I\nabla}(x, x), & \text{if } x \neq y \end{cases}$$

Obviously, every total identity relation is an intuitionistic fuzzy identity relation.

The relation $R \in IFR(X, Y)$ is called totally reflexive if for every $x \in X$: $\mu_R(x, x) = 1$ (and therefore, $\gamma_R(x, x) = 0$); and totally antireflexive if for every $x \in X$: $\gamma_R(x, x) = 1$ (and therefore, $\mu_R(x, x) = 0$).

Obviously, in the second case we cannot define the condition to be for every $x \in X$: $\mu_R(x, x) = 0$, because in this case the validity of $\gamma_R(x, x) = 1$ is not obligatory.

The relation $R \in IFR(X, Y)$ is called intuitionistic fuzzy reflexive, if for every $x \in X$: $\mu_R(x, x) \geq \gamma_R(x, x)$; an intuitionistic fuzzy antireflexive if for every $x \in X$: $\gamma_R(x, x) \leq \mu_R(x, x)$; an intuitionistic fuzzy irreflexive if for every $x \in X$: $\gamma_R(x, x) < \mu_R(x, x)$; an intuitionistic fuzzy symmetrical if for every $x \in X$

and for every $y \in Y$: $\mu_R(x, y) = \mu_R(y, x)$ and $\gamma_R(x, y) = \gamma_R(y, x)$.

Obviously, every total reflexive and total antireflexive relations are intuitionistic fuzzy reflexive and intuitionistic fuzzy antireflexive relations, respectively.

The vallidity of the next assertions follows easily from the definitions.

THEOREM 3: For every $R \in IFR(X, X)$:

(a) if R is a reflexive, then $\Delta \subset R$

(b) if R is an antireflexive, then $\nabla \supset R$.

(c) if R is an intuitionistic fuzzy reflexive, then $I\Delta \subset R$

(d) if R is an intuitionistic fuzzy antireflexive, then $I\nabla \supset R$.

THEOREM 4: For every $R \in IFR(X, X)$:

(a) if R is a reflexive, then R (S, T) R is a reflexive

(b) if R is an antireflexive, then R (S, T) R is an antireflexive relation.

(c) if R is an intuitionistic fuzzy reflexive, then R (S, T) R is an intuitionistic fuzzy reflexive

(d) if R is an intuitionistic fuzzy antireflexive, then R (S, T) R is an intuitionistic fuzzy antireflexive relation.

THEOREM 5: Let $R \in IFR(X, X)$ be an intuitionistic fuzzy reflexive relation. Then $R \cup P$ is an intuitionistic fuzzy reflexive relation for every $P \in IFR(X, X)$.

Proof: Let $R \in IFR(X, X)$ be an intuitionistic fuzzy reflexive relation. Therefore for every $x \in X$: $\mu_R(x, x) \geq \gamma_R(x, x)$. Let $P \in IFR(X, X)$. Then

$$\mu_{R\cup P}(x, x) = \max(\mu_R(x, x), \mu_P(x, x)) \geq \mu_R(x, x) \geq \gamma_R(x, x)$$

$$\geq \min(\gamma_R(x, x), \gamma_P(x, x)) = \gamma_{R\cup P}(x, x)$$

and hence $R \cup P$ is an intuitionistic fuzzy reflexive relation.

Finally, we shall note that the above definitions related to the IFR can be extended in another direction, too. For the ordinary case of the IFR this is done in [11] and all results from there can be transformed into the above defined form of the IFR, too.

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