Type-2 Fuzzy Entropy-Sets

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***Abstract*—The final goal of this study is to adapt the concept of fuzzy entropy of De Luca and Termini to deal with Type-2 Fuzzy Sets. We denote this concept Type-2 Fuzzy Entropy-Set. However, the construction of the notion of entropy measure on an infinite set, such us $[0,1]$, is not effortless. For this reason, we first introduce the concept of quasi-entropy of a Fuzzy Set on the universe $[0,1]$. Furthermore, whenever the membership function of the considered Fuzzy Set in the universe $[0,1]$ is continuous, we prove that the quasi-entropy of that set is a fuzzy entropy in the sense of De Luca y Termini. Finally, we present an illustrative example where we use Type-2 Fuzzy Entropy-Sets instead of fuzzy entropies in a classical fuzzy algorithm.**

***Index Terms*—Type-2 Fuzzy Sets; Quasi-entropy measure; Entropy measure.**

I. Introduction

The concept of fuzzy entropy measure was introduced by De Luca and Termini in [1] in order to measure how far a Fuzzy Set is from a crisp one. Since then, this concept has been adapted to the different extensions of Fuzzy Sets [2] and with different interpretations. All of them measure how far the considered extension is from a set of reference (which may be that of crisp sets, of Fuzzy Sets, etc).

In this sense, it is worth mentioning the following concepts: the Atanassov intuitionistic fuzzy entropy measure, given by Szmidt et al. [3] to measure how far an Atanassov Intuitionistic Fuzzy Set (AIFS) is from a crisp set; the entropy for Interval-Valued Fuzzy Sets (IVFS) defined by Burillo et al. [4], which measures how far an IVFS or AIFS is from a Fuzzy Set; and finally, the idea given by Pal et al. [5] which combines two concepts similar to those given by Szmidt et al. and Burillo et al. in one single bi-valuated measure. (We should recall that AIFSs, IVFSs and Fuzzy Sets are particular instances of Type-2 Fuzzy Sets (T2FS) (see Fig. 1) [2]).

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Furthermore, we know that, for a Fuzzy Set on the finite universe $U = \{u_1, \cdots, u_n\}$, the value $\mathcal{A}(u_i) \in [0,1]$ is a number which represents the membership degree of u_i to $\mathcal{A}$. From the beginning of fuzzy theory in 1965, many authors were very critical with it: if Fuzzy Sets are used to represent uncertainty associated to a fact, how can the membership values be an exact number $\mathcal{A}(u_i)$ without taking into account the uncertainty associated to the way these numbers are built? This fact led to the introduction in 1971 [6] of T2FS in the following sense: for a Type-2 Fuzzy Set A_2 defined on the finite universe U, the membership degree of each element to the set, i.e., $A_2(u_i)$, is a Fuzzy Set on the infinite universe $[0,1]$. With Zadeh's interpretation, in this paper we consider that the Fuzzy Set $A_2(u_i)$ represents the uncertainty associated to the building of $\mathcal{A}(u_i) \in [0,1]$.

In this setting, we understand De Luca and Termini fuzzy entropy E of the set $A_2(u_i)$, $E(A_2(u_i))$, as a measure of the *doubt (uncertainty)* associated to the value $\mathcal{A}(u_i) \in [0,1]$ given by the expert. In this way, if $E(A_2(u_i)) = 0$, we assume that there is no doubt associated with the value $\mathcal{A}(u_i)$; that is, there is no doubt associated with the numerical value given to represent the membership degree of u_i to the Fuzzy Set $\mathcal{A}$. However, if $E(A_2(u_i)) = 1$, then the doubt with respect to the value $\mathcal{A}(u_i)$ is maximal.

Taking into account the definition of fuzzy entropy, if the Fuzzy Set $A_2(u_i)$ on $[0,1]$ is

$$A_2(u_i)(x) = \begin{cases} 1 \text{ if } x = \mathcal{A}(u_i) \\ 0 \text{ otherwise} \end{cases}$$

then $E(A_2(u_i)) = 0$.

Similarly if $A_2(u_i)(x) = 0.5$ for all $x \in [0,1]$ then $E(A_2(u_i)) = 1$.

From these considerations, in this work we aim at the following objectives:

(A) To extend the concept of fuzzy entropy in the sense of De Luca and Termini to T2FSs.
(B) To provide a construction method of such entropies.
(C) To introduce an illustrative example where the notion of entropy that we propose for T2FSs is used in an algorithm that was originally developed using the concept of fuzzy entropy for Fuzzy Sets or for some extensions.

Regarding objective (A), it is important to remark the following: In the same spirit as in the work by Pal et al. [5], we consider that the entropy of a T2FS A_2 on a finite universe U must not be a number, but a Fuzzy Set (Type-1) $E_{T2}(A_2)$ on the same universe U. We call this Fuzzy Set, Type-2 Fuzzy Entropy-Set. The values $E_{T2}(A_2)(u_i) \in [0,1]$ are given by the fuzzy entropies of the Fuzzy Sets on the universe $[0,1]$ used to represent the membership of u_i to the set A_2. With our interpretation we have that each value of $E_{T2}(A_2)(u_i)$ represents the doubt associated to the membership degree of the element u_i to the Fuzzy Set $\mathcal{A}$ on the considered finite universe U.

We also introduce a measure call pointwise measure which assigns to each T2FS a numerical value obtained through an appropriate aggregation of the elements in the Fuzzy Set $E_{T2}(A_2)$. We see that this measure has properties similar to those of De Luca and Termini's fuzzy entropy.

Regarding objective (B): In order to build the Type-2 Fuzzy Entropy-Set the following problem arises: we should calculate the fuzzy entropy of Fuzzy Sets which are defined on non-finite universes (the interval $[0, 1]$). This problem leads us to introduce the concept of quasi-entropy. The latter does not exactly match fuzzy entropy as defined by De Luca and Termini. However, if we consider Fuzzy Sets defined on the universe $[0, 1]$ with a continuous membership function, then the concept of quasi-entropy and the concept of fuzzy entropy defined by De Luca and Termini are the same. We build Type-2 Fuzzy Entropy-Sets from the quasi-entropies.

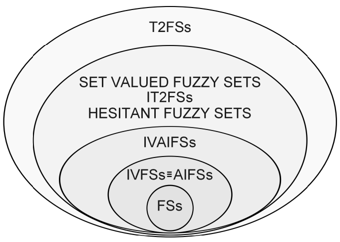

Fig. 1. Inclusion relationships of extensions of Fuzzy Sets in [2]

Regarding objective (C): As an illustrative example of the utility of our theoretical developments, we rewrite the algorithm for image segmentation which uses fuzzy techniques, i.e., Huang and Wang's algorithm [7], [8]. We consider an image as a Type-2 Fuzzy Set and we replace fuzzy entropy by our concept of Type-2 Fuzzy Entropy-Set. It is worth to note that *the purpose of this example is not to provide a new method*, but just to show how our theoretical developments can be used to understand an image as a Type-2 Fuzzy Set (over the universe of the intensity levels) and hence how a well-known algorithm can be extended to this setting, as it has already been extended to some other settings such as IVFSs or AIFSs [9], [10].

This paper is organized as follows. In the following section we recall some definitions and properties which will be used in the subsequent of this work. Then, in Section III we introduce the concept of fuzzy quasi-entropy measure for an infinite universe $[0, 1]$ analyzing the particular case of continuous membership degrees. Sections IV and V present the Type-2 Fuzzy Entropy-Set together with some specific cases of these sets and the definition of pointwise measure. Section VI presents an illustrative example in image thresholding. Finally, in Section VII we include some conclusions and references.

II. Preliminary notions

In this paper, we denote by X a non-empty universe in a Fuzzy Set that can be either finite or infinite. When the universe is finite, it is denoted by U.

Definition 2.1: [11] A Fuzzy Set (FS) (or Type-1 Fuzzy Set) A is a mapping $A : X \mapsto [0, 1]$ where the value $A(x)$ is referred to as the membership degree of the element x to the Fuzzy Set A.

The set of all FSs on X is denoted by $FS(X)$.

From the notions given by Zadeh in [12], a Type-2 Fuzzy Set (T2FS) can be defined as follows.

Definition 2.2: A Type-2 Fuzzy Set (T2FS) A_2 on X is a mapping $A_2 : X \mapsto FS([0, 1])$ where the membership degree of an element of the universe X is a Fuzzy Set on the infinite universe $[0, 1]$.

From Definition 2.2, it can be seen that, mathematically, a T2FS is a mapping $A_2 : X \mapsto [0, 1]^{[0,1]}$, where

$$[0, 1]^{[0,1]} = \{f \mid f : [0, 1] \mapsto [0, 1]\}.$$

We denote by $T2FS(X)$ the class of all T2FSs on the universe X.

Fuzzy entropy measure was formalized in terms of axiom construction by De Luca and Termini in [1] in order to assess the amount of vagueness within a FS. However, depending on the properties demanded, we can find in the literature different axiomatic definitions of the concept of fuzzy entropy measure, such as [13], [14], [15]. In particular, we base our definition on [14].

Definition 2.3: A function $E : FS(X) \mapsto [0, 1]$ is called an entropy measure on $FS(X)$ if it satisfies the following properties:

(E1) $E(A) = 0$ if and only if A is crisp.

(E2) $E(A) = 1$ if and only if $A(x) = \frac{1}{2}$ for all $x \in X$.

(E3) If $A, B \in FS(X)$, and for all $x \in X$

$$\left.\begin{array}{l} A(x) \leq B(x) \leq \frac{1}{2} \\ \text{or} \\ A(x) \geq B(x) \geq \frac{1}{2} \end{array}\right\} \text{ then } E(A) \leq E(B)$$

(E4) $E(A) = E(N(A))$ for all $A \in FS(X)$, where $N(A) = \{(x, 1 - A(x))\}$ for all $x \in X$.

It should be pointed out that $(E1) - (E3)$ generate De Luca and Termini axiomatic definition and $(E4)$ is a property frequently demanded in image processing.

Definition 2.3 is based on the standard negation $N(x) = 1 - x$. In the case of another strong negation being considered, property $(E2)$ would be

$$E(A) = 1 \text{ if and only if } A(x) = e \text{ for all } x \in X,$$

where e is the equilibrium point of the strong negation considered.

Finally, in Definition 2.3 it does not matter whether the universe X is finite or infinite, but dealing with infinite universes requires a more complicated mathematical formalism. Thus, most of the works in the literature take into account only the finite case (universe U).

A construction method of entropies was given in [14], using aggregation functions and the concept of E_N function, which we recall now.

Definition 2.4: A function $E_N : [0, 1] \to [0, 1]$ is called a normal E_N-function associated with the strong negation N, if it satisfies the following conditions:

1) $E_N(x) = 1$ if and only if $x = e$ (where e is the equilibrium point of N; that is, $N(e) = e$.)
2) $E_N(x) = 0$ if and only if $x = 0$ or $x = 1$.
3) If $y \geq x \geq e$ or $y \leq x \leq e$, then $E_N(x) \geq E_N(y)$.
4) $E_N(x) = E_N(N(x))$ for all $x \in [0,1]$.

In particular, entropies of FSs on finite universes can be built from E_N-functions as follows.

Theorem 2.1: Let $M : [0,1]^n \to [0,1]$ be such that it fulfills

(M1) $M(x_1, \cdots, x_n) = 0$ if and only if $x_1 = \cdots = x_n = 0$;
(M2) $M(x_1, \cdots, x_n) = 1$ if and only if $x_1 = \cdots = x_n = 1$;
(M3) For any pair $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ of n-tuples such that $x_i, y_i \in [0,1]$ for all $i \in \{1, \cdots, n\}$, if $x_i \leq y_i$ for all $i \in \{1, \cdots, n\}$, then $M(x_1, \cdots, x_n) \leq M(y_1, \cdots, y_n)$;
(M4) M is a symmetric function in all its arguments.

Then $E(\mathcal{A}) = M_{i=1}^n E_N(\mathcal{A}(u_i))$ for all $\mathcal{A} \in FS(U)$ satisfies $(E1) - (E4)$ of Definition 2.4.

Example 2.2: If we take $E_N(x) = 1 - |2x - 1|$ and $M(x_1, \cdots, x_n) = \frac{1}{n}\sum\limits_{i=1}^{n} x_i$, then

$$E(\mathcal{A}) = \frac{1}{n}\sum_{i=1}^{n} 1 - |2\mathcal{A}(u_i) - 1|$$

is Yager's measure of fuzziness [16].

Restricted Equivalence Functions R are functions which satisfy frequently demanded properties for the comparison of images. They were introduced by Bustince et al. in [8], [14], [17].

Definition 2.5: A function $R : [0,1]^2 \to [0,1]$ is called a restricted equivalence function if it satisfies the following conditions:

(R1) $R(x, y) = R(y, x)$ for all $x, y \in [0,1]$;
(R2) $R(x, y) = 1$ if and only if $x = y$;
(R3) $R(x, y) = 0$ if and only if $\{x, y\} = \{0, 1\}$;
(R4) $R(x, y) = R(N(x), N(y))$ for all $x, y \in [0,1]$, being N a strong negation on $[0,1]$;
(R5) For all $x, y, z \in [0,1]$ such that $x \leq y \leq z$ then $R(x, z) \leq R(x, y)$ and $R(x, z) \leq R(y, z)$.

III. Fuzzy quasi-entropy measure for an infinite universe

In order to develop our notion of entropy measure on T2FSs, we study some results about entropy measures on FSs whose universe X is infinite. In particular, we focus on the notion of an entropy measure on $FS([0,1])$. When the universe X is infinite some mathematical operations, such as the integration operation, yield the same value for different sets $A_1, A_1^{'}$ such that $A_1 = A_1^{'}$ a.e. (almost everywhere). [1] To handle this situation in a suitable way, we adapt the concept of entropy measure given by De Luca and Termini [1].

As we have seen in Theorem 2.1, in the case of finite universes, entropy can be built aggregating appropriate functions (E_N-functions); in particular, the arithmetic mean can be used for the aggregation. If we try to extend this procedure to the universe $[0,1]$, it is natural to use an integral instead of the arithmetic mean. A problem arises, however, with axioms $(E1)$ and $(E2)$. For instance, consider the functions $f_1(t) = 0$ for all $t \in [0,1]$, $f_2(t) = 0.3$ if $t = 0.3$ or $t = 0.8$ and $f_2(t) = 0$ otherwise. These functions are different, but the integral of both on $[0,1]$ equals 0, since they differ in a zero-measure set (a finite set of points).

So we should modify axioms $(E1)$ and $(E2)$. This can be done in two different ways.

1) They can be kept as they stand in Definition 2.3. In this case, the value of the function in one single point would determine that the entropy was not zero or one, even if the function equals 0 or 0.5, respectively, in any other point. This would be too harsh.
2) We can rewrite axioms $(E1)$ and $(E2)$ considering that functions which are equal almost everywhere must have the same entropy. This is something which is usually done for many applications, and it is the approach that we choose in this work.

Taking into account these considerations, we propose the following definition (note axioms $E1^*$ and $E2^*$). We take the name of quasi-entropy because an exact copy of De Luca and Termini's definition of entropy would correspond to approach 1) above, which we have not followed.

Definition 3.1: Let $A \in FS([0,1])$, we define the set $H_A = \{x \mid A(x) \in]0,1[\}$.

Definition 3.2: A function $E^* : FS([0,1]) \mapsto [0,1]$ is called a quasi-entropy measure on $FS([0,1])$ if it satisfies the following properties:

$(E1^*)$ $E^*(A) = 0$ if and only if the Lebesgue measure of H_A is null, i.e., $m(H_A) = 0$, where m denotes the Lebesgue measure in $\mathbb{R}$.
$(E2^*)$ $E^*(A) = 1$ if and only if $A(x) = \frac{1}{2}$ a.e. in $[0,1]$.
$(E3^*)$ If $A, B \in FS([0,1])$, and for all $x \in [0,1]$

$$\left.\begin{array}{l} A(x) \leq B(x) \leq \frac{1}{2} \\ \text{ or} \\ A(x) \geq B(x) \geq \frac{1}{2} \end{array}\right\} \text{ then } E^*(A) \leq E^*(B).$$

$(E4^*)$ $E^*(A) = E^*(N(A))$ for all $A \in FS([0,1])$ where $N(A) = \{(x, 1 - A(x))\}$ for all $x \in [0,1]$.

Remark 1: Notice that properties $(E3^*)$ and $(E4^*)$ are exactly equal to the properties $(E3)$ and $(E4)$ of entropy measure in FSs given in Definition 2.3.

From here on, we only consider FSs in the universe $X = [0,1]$ and such that the function $A : X \mapsto [0,1]$ is a Lebesgue integrable function. Observe that since Lebesgue integrable functions are a large class of functions, even restricting to them is not a major concern.

In order to construct a quasi-entropy measure we start by defining a function Γ and we study under which conditions it fulfills properties $(E1^*) - (E4^*)$ individually.

Let $g :]0,1[\mapsto [0,1]$ be a Lebesgue integrable function. We define function $\Gamma : FS([0,1]) \mapsto [0,1]$ as

$$\Gamma(A) = \int_{H_A} g(A(y))dy. \tag{1}$$

[1]Given two functions f_1, f_2, we say $f_1 = f_2$ a.e. if $f_1(x) = f_2(x)$ for all x in the domain except for a set of null measure. Particularly, $f_1 = c$ a.e. where c is a constant if $f_1(x) = c$ except for a set of null measure.

Example 3.1: Let $g(x) = 2\min(x, 1-x)$ and consider the following FS on $[0,1]$: $A(x) = 1$ for all $x \in [0,1]$. Then, by Eq. (1) we have

$$\Gamma(A) = \int_{H_A} g(A(y))dy = \int_{H_A} 2\min(1,0) = 0.$$

In Theorem 3.2, we study those sets which have minimum entropy measure, i.e., property $(E1^*)$.

Theorem 3.2: Let $\Gamma : FS([0,1]) \mapsto [0,1]$ be a function given by Eq. (1). Then

Γ satisfies $(E1^*)$ if and only if $g(z) \neq 0$ for all $z \in \,]0,1[$.

Proof. See Appendix.

Example 3.3: Figure 2 shows $g_1(z) = 1 - z$, $g_2(z) = z^2$ and $g_3(z) = 0.3$ for $z \in]0,1[$ which satisfy the property of Theorem 3.2.

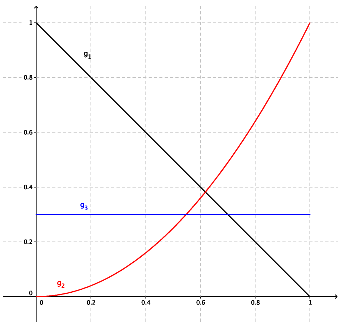

Fig. 2. Functions g_1, g_2, g_3 satisfying $E1^*$.

Example 3.4: Let $g(x) = 2\min(x, 1-x)$ and consider the following FS on $[0,1]$: $A(x) = 0.5$ for all $x \in [0,1]$. Then, by Eq.(1) we have

$$\Gamma(A) = \int_{H_A} g(A(y))dy = \int_{H_A} 2\min(0.5, 0.5) = 1$$

In Theorem 3.5 we focus on the sets with maximum entropy measure, namely, property $(E2^*)$.

Theorem 3.5: Let $\Gamma : FS([0,1]) \mapsto [0,1]$ be a function given by Eq. (1). Then,

$$\Gamma \text{ satisfies } (E2^*) \text{ if and only if } g^{-1}(1) = \left\{\frac{1}{2}\right\}$$

Proof. See Appendix.

Example 3.6: Figure 3 shows three functions which satisfy the property of Theorem 3.5.

$$g_1(z) = -\left(z - \frac{1}{2}\right)^2 + 1 \qquad \text{for } z \in]0,1[$$

$$g_2(z) = \begin{cases} 0 & \text{if } 0 < z \leq 0.1, \\ 2.5z - 0.25 & \text{if } 0.1 < z \leq 0.5, \\ 1.5 - z & \text{if } 0.5 < z < 1. \end{cases}$$

$$g_3(z) = \begin{cases} z & \text{if } 0 < z < 0.5, \\ -2z + 2 & \text{if } 0.5 \leq z < 1. \end{cases}$$

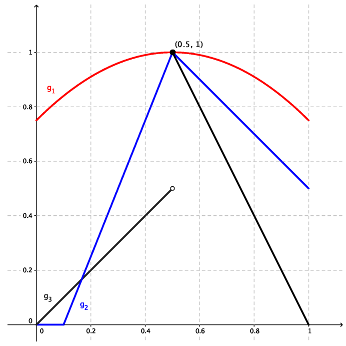

Fig. 3. Functions g_1, g_2, g_3 satisfying $E2^*$.

In Theorem 3.7, the monotonicity of quasi-entropy measure, property $(E3^*)$, is analyzed.

Theorem 3.7: Let $\Gamma : FS([0,1]) \mapsto [0,1]$ be a function given by Eq. (1). Then, Γ satisfies $(E3^*)$ if and only if g is increasing on $\left]0, \frac{1}{2}\right]$ and decreasing on $\left[\frac{1}{2}, 1\right[$.

Proof. See Appendix.

Example 3.8: Figure 4 shows functions which satisfy the property of Theorem 3.7.

$$g_1(z) = \begin{cases} 5z & \text{if } 0 < z < 0.2, \\ 1 & \text{if } 0.2 \leq z < 1, \end{cases}$$

$$g_2(z) = \begin{cases} z & \text{if } 0 < z < 0.5, \\ 1 - z & \text{if } 0.5 \leq z < 1. \end{cases}$$

$$g_3(z) = \begin{cases} z + 0.3 & \text{if } 0 < z \leq 0.5, \\ 1.4 - 1.4z & \text{if } 0.5 < z < 1. \end{cases}$$

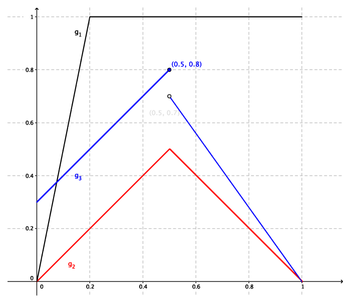

Fig. 4. Functions g_1, g_2, g_3 satisfying $E3^*$.

Finally, in Theorem 3.9 we study property $(E4^*)$, analyzing the symmetry of entropy measures.

Theorem 3.9: Let $\Gamma : FS([0,1]) \mapsto [0,1]$ be a function given by Eq. (1). Then,

Γ satisfies $(E4^*)$ if and only if g is a symmetric function with respect to $z = \frac{1}{2}$, i.e., $g(z) = g(1-z)$ for all $z \in \,]0,1[$.

Proof. See Appendix.

Example 3.10: Figure 5 shows functions g_1, g_2, g_3 which satisfy property of Theorem 3.9.

$$g_1(z) = 4\,(z - 0.5)^2 \qquad \text{for } z \in]0,1[$$

$$g_2(z) = \begin{cases} 0 & \text{if } 0 < z \leq 0.2, \\ z - 0.2 & \text{if } 0.2 < z \leq 0.5, \\ -z + 0.8 & \text{if } 0.5 < z \leq 0.8, \\ 0 & \text{if } 0.8 < z < 1. \end{cases}$$

$$g_3(z) = \min\{8z^3, 8(1-z)^3\} \qquad \text{for } z \in]0,1[$$

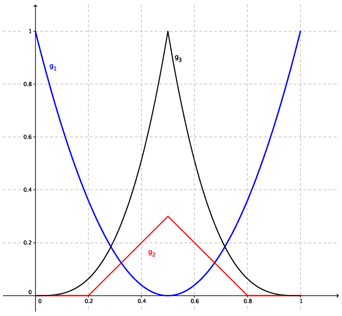

Fig. 5. Functions g_1, g_2, g_3 satisfying $E4^*$.

After studying each property separately, the following corollary holds true.

Corollary 3.11: Let Γ be given by Eq. (1). Then Γ is a quasi-entropy measure if and only if g satisfies the conditions demanded in Theorems 3.2, 3.5, 3.7 and 3.9.

Proposition 3.12: Let g be an E_N-function associated with the strong negation N given by $N(x) = 1-x$ for all $x \in [0,1]$. Then the function Γ given by Eq. (1) in terms of g is a quasi-entropy.

Proof. It follows from the Corollary 3.11 and properties of E_N-functions (see [14]).

In [14], it is proved that, from a restricted equivalence function R, we can build an E_N-function as follows: $E_N(x) = R(x, 1-x)$. So the following corollary is straight.

Corollary 3.13: Let R be a restricted equivalence function and let $g(x) = R(x, 1-x)$. Then, Γ given by Eq. (1) in terms of g is a quasi-entropy.

Example 3.14: Fig. 6 shows three functions g_1, g_2, g_3 which satisfy all the conditions of Theorems 3.2, 3.5, 3.7 and 3.9, so from Corollary 3.11 they generate quasi-entropy measures:

$$g_1(z) = -4z^2 + 4z \qquad \text{for } z \in]0,1[$$

$$g_2(z) = \min\{2z, 2-2z\} \qquad \text{for } z \in]0,1[$$

$$g_3(z) = \min\{8z^3, 8(1-z)^3\} \qquad \text{for } z \in]0,1[$$

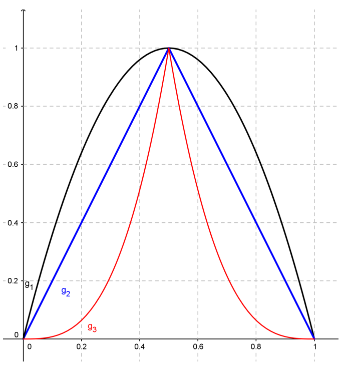

Fig. 6. Functions g_1, g_2, g_3 which generate a quasi-entropy measure.

In the following we compute an example of the calculation of a quasi-entropy.

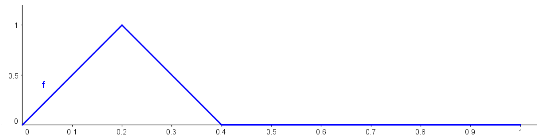

Fig. 7. Graph of function f of Example 3.15.

Example 3.15: Let $f \in FS([0,1])$ be given by

$$f(x) = \begin{cases} 5x & \text{if } 0 \le x \le 0.2, \\ 2-5x & \text{if } 0.2 < x \le 0.4, \\ 0 & \text{otherwise,} \end{cases}$$

displayed in Figure 3.15. Consider the quasi-entropy measure E^* generated as in Eq. (1) by $g(z) = \min\{2z, 2-2z\}$. Then:

$$E^*(f) = \int_0^{0.1} 10y dy + \int_{0.1}^{0.2} (2-10y)dy + \int_{0.2}^{0.3} (10y-2)dy + \int_{0.3}^{0.4} (4-10y)dy = 0.2$$

A. Quasi-entropy measure on Continuous functions

As we have said before, when we use integrals sets of zero measure are ignored. This has led us to modify in the previous section the first and second axioms of the definition of entropy 2.3 by De Luca and Termini. But in the case of continuous functions, if a function is constant almost everywhere, then it is constant everywhere, and this kind of technical problems may be ignored. That is, if we consider just those FSs on the universe $[0,1]$ with a continuous membership function, then our definition of entropy can be written as the one which was introduced by De Luca and Termini; i.e., Definition 2.3. For this reason in this section we study quasi-entropy measures restricted to the class of $FS([0,1])$ whose membership degree is a continuous function.

Definition 3.3: Let $FS_C([0,1])$ be the set of all FSs on the universe $X = [0,1]$ whose membership degree $A : X \mapsto [0,1]$ leads to a continuous function.

In the following theorem we introduce a method to build entropies in the sense of De Luca and Termini as long as the membership function of the considered FS on $[0,1]$ is continuous.

Theorem 3.16: Let be $g\ :]0,1[\mapsto [0,1]$ satisfying the properties of the Theorems 3.2, 3.5, 3.7 and 3.9 and let Γ be given as in Eq. (1). If we restrict to FS_C then $\Gamma|_{FS_C}$ is an entropy measure in the sense of De Luca and Termini [1]. Namely, the function Γ on $FS_C([0,1])$ satisfies:

(E1) $\Gamma(A) = 0$ if and only if A is crisp.
(E2) $\Gamma(A) = 1$ if and only if $A(x) = \frac{1}{2}$ in $[0,1]$.
(E3) If $A, B \in FS_C([0,1])$, and for all $x \in [0,1]$

$$\left.\begin{array}{l} A(x) \le B(x) \le \frac{1}{2} \\ \text{or} \\ A(x) \ge B(x) \ge \frac{1}{2} \end{array}\right\} \text{ then } \Gamma(A) \le \Gamma(B)$$

(E4) $\Gamma(A) = \Gamma(N(A))$ for all $A \in FS([0,1])$, where $N(A) = \{(x, 1-A(x))\}$ for all $x \in X$.

Note that imposing continuity is not a too hard restriction, since, for instance, in many applications, in order to build linguistic labels, these are defined through continuous membership functions (triangular, trapezoidal, etc. [18]).

Corollary 3.17: Let g be an E_N-function associated with the strong negation N given by $N(x) = 1 - x$ for all $x \in [0,1]$. Then

$$\Gamma(A) = \int_{H_A} g(A(y))dy$$

is a fuzzy entropy in the sense of De Luca and Termini on $FS_C([0,1])$. In particular, if R is a restricted equivalence function, then

$$\Gamma(A) = \int_{H_A} R(A(x), 1 - A(x))dx$$

is also an entropy in the sense of De Luca and Termini.

IV. Type-2 Fuzzy Entropy-Set

De Luca and Termini introduced the notion of entropy measure as a function whose domain and codomain are a FS and $[0,1]$, respectively, i.e. a function $E : FS(X) \mapsto [0,1]$. In this way, the codomain of the entropy function and the codomain of the FS coincide. Due to the introduction of the concept of T2FS (by Zadeh [12]) as a function whose image is a FS, the proposal of this work is to define the entropy measure of a T2FS by means of a function whose domain is a T2FS and the codomain is a FS.

Given a T2FS (with universe X), each element $x \in X$ is associated with a $FS([0,1])$ where its quasi-entropy measure can be calculated. Observe that since the universe is infinite, most of the entropy measure constructions on the literature cannot be applied. By calculating the quasi-entropy measure, for each $x \in X$ we obtain a value in $[0,1]$, i.e., each element of the universe X is associated with a value in $[0,1]$. A reasonable way of expressing the entropy measure of a T2FS is by means of a function $E_{T2} : T2FS(X) \mapsto FS(X)$.

Definition 4.1: Let X be the universe of a T2FS A_2 and let $E^* : FS([0,1]) \mapsto [0,1]$ be a quasi-entropy measure. A Type-2 Fuzzy Entropy-Set is a function $E_{T2} : T2FS(X) \mapsto FS(X)$ given by

$$E_{T2}(A_2) = \{(x, E^*(A_2(x))) | \, x \in X\}. \quad (2)$$

The given construction of Type-2 Fuzzy Entropy-Set on Definition 4.1 measures the lack of knowledge or uncertainty about the membership degrees. Thereby, any set with "crisp" membership degrees such as FSs or IVFSs has entropy measure 0.

Next, we present an example where the Type-2 Fuzzy Entropy-Set is calculated.

Example 4.1: Let $U = \{u_1, u_2, u_3, u_4\}$ be the universe and $A_2 : T2FS(U) \mapsto FS(U)$ be the T2FS given by $A_2 = \{(u_i, A_2(u_i) = f_i) \mid i \in \{1,2,3,4\}\}$ where

$$f_1(x) = \begin{cases} 0.5 & \text{if } x = 0.3, \\ 0.25 & \text{if } x = 0.5, \\ 1 & \text{if } x = 0.8, \\ 0 & \text{otherwise.} \end{cases}$$

$$f_2(x) = \begin{cases} 1 & \text{if } x \in [0.2; 0.4] \cup [0.7; 1[, \\ 0 & \text{otherwise.} \end{cases}$$

$$f_3(x) = \begin{cases} 0 & \text{if } 0 < x \leq 0.5, \\ 2.5x - 1.25 & \text{if } 0.5 < x \leq 0.9, \\ 1 & \text{if } 0.9 < x \leq 1 \end{cases}$$

$$f_4(x) = \begin{cases} 5x & \text{if } 0 < x \leq 0.2, \\ -1.25x + 1.25 & \text{if } 0.2 < x \leq 1. \end{cases}$$

as in Figure 8. Consider the quasi-entropy measure E^* generated as in Eq. (1) by $g(z) = -4z^2 + 4z$. Then:

$E^*(f_1) = 0$, $E^*(f_2) = 0$, $E^*(f_3) = \frac{4}{15}$ and $E^*(f_4) = \frac{2}{3}$, and consequently the Type-2 Fuzzy Entropy-Set is given by

$$E_{T2}(A_2) = \left\{(u_1, 0), (u_2, 0), \left(u_3, \frac{4}{15}\right), \left(u_4, \frac{2}{3}\right)\right\}$$

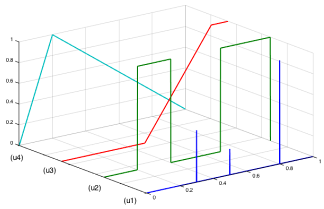

Fig. 8. Graph of the Type-2 Fuzzy Set A_2.

V. Specific cases. Pointwise measure

A. Some specific cases

In this section we show how we can recover Fuzzy Sets and extensions from T2FSs such that its Type-2 Fuzzy Entropy-Set is null.

Let $A_2 \in T2FS(U)$ such that

$$E_{T2}(A_2) = \{(u_i, 0) | u_i \in U\} \, ;$$

that is,

$$E^*(A_2(u_i)) = 0 \text{ for every } u_i \in U$$

where E^* is the quasi-entropy associated to E_{T2}.

Then:

- If the Fuzzy Sets $A_2(u_i)$ on the universe $[0,1]$, (built to represent the doubt associated to the membership degrees of the elements u_i to the Fuzzy Set $\mathcal{A}$ on the universe U), are crisp sets as the following:

$$A_2(u_i)(x) = \begin{cases} 1 \text{ if } x = a_{0_i} \\ 0 \text{ otherwise,} \end{cases}$$

  then, taking into account the interpretation discussed in the introduction, we do not have any doubt about the membership degrees of the elements to the Fuzzy Set $\mathcal{A} \in FS(U)$ and it is the ideal case. In this setting, we can take as Fuzzy Set $\mathcal{A}$:

$$\mathcal{A} = \{(u_i, \mathcal{A}(u_i) = a_{0_i}) | i \in \{1, \cdots, n\}\}$$

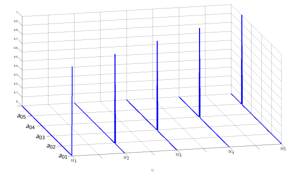

Fig. 9. Example of a Fuzzy Set.

- If the Fuzzy Sets $A_2(u_i)$ on the universe $[0,1]$ are crisp sets as follows:

$$A_2(u_i)(x) = \begin{cases} 1 \text{ if } x = a_{0_i^1} \text{ or } x = a_{0_i^2} \text{ or } x = a_{0_i^{m_i}} \\ 0 \text{ otherwise,} \end{cases}$$

then we can take as set $\mathcal{A}$ the following Typical Fuzzy Multiset $\mathcal{A}$ (on the universe U) [2] for which there is no doubt on the numerical values taken for representing the membership degrees:

$$\mathcal{A} = \{(u_i, a_{0_i^1}, a_{0_i^2}, a_{0_i^{m_i}}) | i \in \{1, \cdots, n\}\}$$

where m_1 denotes the cardinal of the Fuzzy Multiset associated with u_i.

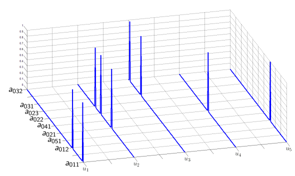

Fig. 10. Example of a Typical Fuzzy Multiset.

- If the Fuzzy Sets $A_2(u_i)$ on the universe $[0,1]$ are crisp sets as follows:

$$A_2(u_i)(x) = \begin{cases} 1 \text{ if } x \in [\underline{a}_{0_i}, \overline{a}_{0_i}] \\ 0 \text{ otherwise} \end{cases}$$

then we can take as $\mathcal{A}$ the following Interval-Valued Fuzzy Set:

$$\mathcal{A} = \{(u_i, [\underline{a}_{0_i}, \overline{a}_{0_i}]) | i \in \{1, \cdots, n\}\}$$

Notice that with our interpretation, it comes out that we have no doubt about the values for the intervals given in order to represent the membership values of the elements to the set.

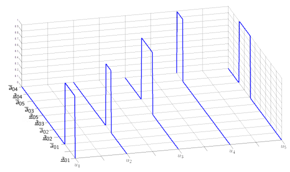

Fig. 11. Example of an Interval-valued Type-2 Fuzzy Set.

In the three considered cases, we recover Fuzzy Sets (in the first case) or well-known extensions of Fuzzy Sets (in the other two cases) whose Type-2 Fuzzy Entropy-Set is always a null Fuzzy Set. In any case, for each of the considered cases (Fuzzy Sets, Fuzzy Multisets and Interval-Valued Fuzzy Sets) there exist ad hoc definitions to calculate their entropy. For instance, De Luca and Termini's for Fuzzy Sets, Szmidt et al.'s or Burillo et al.'s for Interval-Valued Fuzzy Sets, etc.

Although we are not recovering a fuzzy extension, it is worth to mention that if

$$A_2(u_i) = \{(x, A_2(u_i)(x) = 0.5) | x \in [0,1]\} \text{ for all } u_i \in U$$

then $E_{T2}(A_2) = \{(u_i, 1) | u_i \in U\}$.

B. Pointwise measure

In this section, we introduce the concept of pointwise measure. With this measure we assign to each $A_2 \in T2FS(U)$ a numerical value which is obtained aggregating the values in the corresponding Type-2 Fuzzy Entropy-Set $E_{T2}(A_2)$ built as explained in Section IV.

Proposition 5.1: Let $M : [0,1]^n \to [0,1]$ be a function such that it satisfies $(M1) - (M3)$ of Theorem 2.1. Let $A_2 \in T2FS(U)$ and its corresponding $E_{T2}(A_2) \in FS(U)$ constructed with the method developed in Section IV. Under these conditions the function

$$Pm : T2FS(U) \to [0,1] \text{ given by}$$
$$Pm(A_2) = \underset{i=1}{\overset{n}{M}} E_{T2}(A_2)(u_i)$$

satisfies the following properties:

(Pm1) $Pm(A_2) = 0$ if and only if for every $u_i \in U$, $E^*(A_2(u_i)) = 0$; namely, for every $u_i \in U$, $H_{A_2(u_i)}$ has null Lebesgue measure;

(Pm2) $Pm(A_2) = 1$ if and only if for every $u_i \in U$, $E^*(A_2(u_i)) = 1$; namely, for every $u_i \in U$, $A_2(u_i)(x) = 0.5$ a.e. in $[0,1]$;

(Pm3) If $A_2, B_2 \in T2FS(U)$, satisfy that for every $u_i \in U$: for all $x \in [0,1]$

$$\left.\begin{array}{l} A_2(u_i)(x) \leq B_2(u_i)(x) \leq \frac{1}{2} \\ \text{or} \\ A_2(u_i)(x) \geq B_2(u_i)(x) \geq \frac{1}{2} \end{array}\right\} \text{ then}$$
$$Pm(A_2) \leq Pm(B_2);$$

(Pm4) $Pm(A_2) = Pm(N(A_2))$ for all $A_2 \in T2FS$, where $N(A_2) = \{(u_i, N(A(u_i)))\}$.

Proof. It is just a straight calculation.

Remark 2: In this way, Pm does not measure the classical concept of entropy, in the sense that it does not measure how far a T2FS is from a crisp one. However, it gives a global value of the uncertainty associated with which values should represent the membership degrees of u_i for all $u_i \in U$. In particular, if there is no doubt about the membership degrees of any element $u_i \in U$ independently if they are crisp, Fuzzy Set, IVFS, etc, then the punctual measure Pm returns 0.

VI. An illustrative example in image thresholding

In this section we develop an example of application of Type-2 Fuzzy Entropy-Set. We present an adaptation of Huang and Wang's method [7] to segment images in grayscale. To do so, we build a T2FS associated with the image and we calculate its Type-2 Fuzzy Entropy-Set.

Image segmentation consists of dividing an image into regions (objects) that compound it [19]. More specifically, it consists of assigning a label to each pixel of the image, so that all the pixels which share certain properties have the same label. One of the most used techniques in image segmentation is thresholding or segmentation by gray levels [20], [21], [22]. It is based on the assumption that the objects of the image are only characterized by the intensity of their pixels. When the image has only two objects (called object and background), this thresholding technique consists of finding an intensity value (t) to be considered the threshold. Using that value, we label all the pixels whose intensities are lower or equal than t as background and all the pixels whose intensities are greater than t as object (or vice versa). When there are more than two objects in the image, we need more thresholds, in such a way that all the pixels whose intensities are between two consecutive thresholds belong to the same object.

The results of thresholding are limited when comparing with other segmentation techniques, because the single characteristic they take into account is the intensity of every pixel. However, its advantages are the simplicity and low computational cost. This is why this procedure is commonly used as a first step of more complex segmentation algorithms.

We consider an image as a set of elements arranged in N rows and M columns. Each element of a grayscale image has a value of intensity q between 0 and $L-1$ (usually $L = 256$). However, we work with normalized images $\frac{q}{L-1}$ in such a way that $q \in [0, 1]$.

As we have said in the introduction, we rewrite Huang and Wang's algorithm [7] using T2FSs and Type-2 Fuzzy Entropy-Sets (see Algorithm 1).

Algorithm 1 Thresholding algorithm

INPUT: Image to segment
OUTPUT: t the best threshold
1: {Construction of the T2FS}
2: **for** each intensity level $t \in \{0, 1/255, \ldots, 254/255\}$ (For every possible threshold) **do**
3: Construct a FS on the universe $[0, 1]$ associated with the intensity level t
4: **end for**
5: Calculate the Type-2 Fuzzy Entropy-Set of the resulting T2FS
6: Select as best threshold t the one associated with the lowest element in the Type-2 Fuzzy Entropy-Set

The main idea of this procedure consists in creating a T2FS associated with the image and calculating its entropy set. One of the most difficult tasks is the construction of the T2FS. It should represent the information of how would be the image if we segment it with every possible threshold. For this purpose, we start by fixing the referential set of the T2FS as the set of all possible thresholds in the image: $U = \{0/255, 1/255, \ldots, 254/255\}$ (remember the image is normalized). For every element in U, its membership degree is given by a Fuzzy Set. This set has a continuous referential set from 0 to 1. In Figure 12 we show a T2FS that fulfills our conditions.

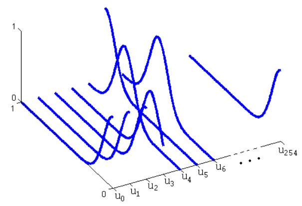

Fig. 12. Example of a Type-2 Fuzzy Set.

Each of these functions represents, for a fixed threshold, the membership degree of every possible intensity either to the object or to the background. To construct each of these sets, following [8], we start by calculating the average intensity of the pixels lower or equal than the studied threshold (denoted as $m_B(t)$) and the average intensity of the pixels greater than the studied threshold (denoted as $m_O(t)$).

The membership function quantifies how close is every posible value (q) to the average of the background or to the average of the object, by means of restricted equivalence functions:

$$\mathcal{A}(u_i)(q) = \begin{cases} R(q, m_B(u_i)) & \text{if } q \leq u_i \\ R(q, m_O(u_i)) & \text{if } q > u_i. \end{cases} \quad (3)$$

We linearly interpolate between every pair of consecutive points $(q_i, (\mathcal{A}(u_i)(q_i)))$ and $(q_{i+1}, (\mathcal{A}(u_i)(q_{i+1})))$ with $i \in \{0, \cdots, 254\}$. That is, we take the points $(0, \mathcal{A}(u_i)(0))$ and $(1/255, \mathcal{A}(u_i)(1/255))$ and, for each $s \in [0, 1/255]$, we define its membership as:

$$A_2(s) = 255(\mathcal{A}(u_i)(1/255) - \mathcal{A}(u_i)(0))s + \mathcal{A}(u_i)(0)\ .$$

Next, we repeat this procedure for each interval$[j/255, (j+1)/255]$, $(j = 0, \ldots, 254)$, calculating in each case the equation of the line which passes through the points $(j/255, \mathcal{A}(u_i)(j/255))$ and $((j+1)/255, \mathcal{A}(u_i)((j+1)/255))$.

In this way, we get a continuous membership function defined over the whole universe $[0, 1]$. This membership function is piecewise linear and it has only two points where its value is 1: the average of the background ($m_B(t)$) and the average of the object ($m_O(t)$).

To select the best threshold from the T2FS we use its Type-2 Fuzzy Entropy-Set. We are looking for the threshold whose membership function is as higher as possible for all the pixels in the image. The entropy is minimum when the membership is 0 or 1, and maximum in the middle point. To adapt this concept to our problem, we scale our membership function to $[0.5, 1]$, in such a way that the minimum entropy is only achieved when the membership degree is 1.

With our membership construction, the calculation of our entropies is simple, since we can divide the area in 255 trapezoids and we just need to sum the entropy measure of each of these parts multiplied by the proportion of pixels with that intensity. That is, we calculate the entropy of each FS as

$E(A_2(u_i)) = \int g(A_2(u_i)(x))dx$ where $A_2(u_i)$ is the FS associated with u_i on the universe $[0,1]$ and $g(x) = R(x, 1-x)$.

In this way we obtain a set of entropies, each one associated with an element of the universe (possible thresholds based on our construction) and we can build the Type-2 Fuzzy Entropy-Set. Finally, we select as the best threshold, the one associated with the lowest entropy measure.

With an illustrative aim, we use this algorithm for thresholding the image in Figure 13.

Fig. 13. Original image to segment.

After constructing the T2FS for this image, we use $g(x) = R(x, 1-x) = 1 - |2x-1|$ to get its associated Type-2 Fuzzy Entropy-Set. The resulting set is as follows:

$$E_{T2} = \{(u_0, 0.6014), (u_1, 0.6010), (u_2, 0.6004), \ldots, (u_{254}, 0.5935)\}$$

For a better visualization of this set, in Figure 14 we show it graphically.

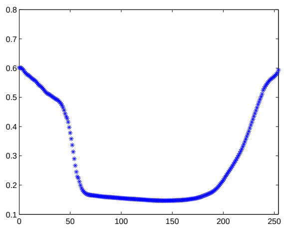

Fig. 14. Fuzzy Entropy-Set for thresholding the image of Figure 13.

The minimum of this Type-2 Fuzzy Entropy-Set corresponds to the element $(u_{143}, 0.1467)$. So the threshold used to segment the image is 143/255 and we get the image shown in Figure 15.

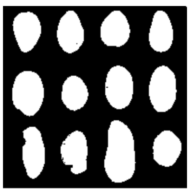

Fig. 15. Image of Figure 13 segmented with threshold 143.

To further extend this illustrative example, we consider now a set of 8 standard images for thresholding and their ideal segmentations; that is, the segmentation provided by an expert. For each of them (see Figure 16) we show the original image, the ideal segmentation and the segmented image obtained with our method using the function $E(A_2(u_i)) = \int g(A_2(u_i)(x))dx$ with $g(x) = R(x, 1-x) = 1 - |2x-1|$.

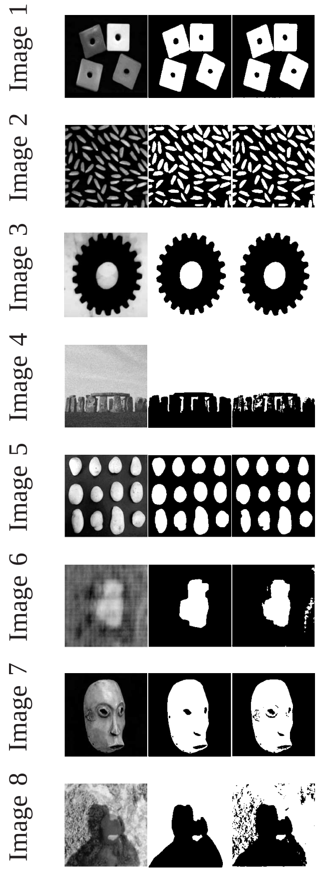

Fig. 16. Original images (first column), ideal segmentations (second column) and segmentations obtained by our proposal using the Type-2 Fuzzy Entropy-Set (third column)

Our proposed Algorithm 1 uses the Type-2 Fuzzy Entropy-Set to calculate the threshold for segmenting an image. In the fuzzy literature, there exist several fuzzy algorithms which use extensions of FSs (for instance [8], [23], [14]) for thresholding images. All of them, including our proposal, are based on Huang and Wang's algorithm [7], which is an adaptation of the classical method by Otsu [20]. It is important to notice that none of these algorithms is better than the others for every image. For this reason, we propose to use a combination of the results obtained with different algorithms, including our Algorithm 1. To show the goodness of this proposal, we use the following 5 thresholding algorithms.

- Otsu's algorithm [20];
- area algorithm [8] with $\varphi_1(x) = x^2$ and $\varphi_2(x) = x$;
- ignorance functions based algorithm [23] with $G_u(x,y) = 2\sqrt{(1-x)(1-y)}$ if $(1-x)(1-y) \leq 0.25$ and $G_u(x,y) = 1/(2\sqrt{(1-x)(1-y)})$ otherwise
- Algorithm 1 with $E(A_2(u_i)) = \int g(A_2(u_i)(x))dx$ and $g(x) = R(x, 1-x) = 1 - |2x-1|$
- Algorithm 1 with $E(A_2(u_i)) = \int g(A_2(u_i)(x))dx$ and $g(x) = R(x, 1-x) = 1 - (2x-1)^2$

In Table I we study the obtained thresholds as well as the percentage of pixels correctly segmented with respect to the ideal segmentation for each of the algorithms and each of the 8 images shown in Figure 16. For the sake of simplicity, thresholds have been multiplied by 255. Moreover, we consider the combination of all the obtained thresholds using the arithmetic mean and we also calculate for the latter the percentage of well segmented pixels.

As we can see in Table I, it does not exist one single method which is the best for every possible image. However, when we take the mean of several methods we get good results, which are even the best ones for 4 of the 8 images. So, after combining the results of several algorithms (including Algorithm 1), we see that the obtained segmentations are very good. These segmentations can be taken as a first step in the calculation of segmentations which take into account more properties of the images, apart from the intensity of the pixels.

VII. Conclusions and Future Work

The construction of entropy measures for Fuzzy Sets with infinite universes results intricate. In this direction one of the main novelties of this study is the introduction of the concept of quasi-entropy. Defined slightly different than the fuzzy entropy given by De Luca and Termini it is proven that both concepts are equivalent if we restrict to continuous membership functions. The quasi-entropy measure has been applied to a T2FS (whose membership degree for an element x in the universe X is a $FS([0,1])$), generating the novel concepts of Type-2 Fuzzy Entropy-Set and pointwise measure. Finally, we have shown the usefulness of the Type-2 Fuzzy Entropy-Set in an illustrative example in Huang and Wang's algorithm for image thresholding.

Due to the relevance of a theoretical method to calculate the entropy of T2FSs we leave for a future work the deeper study of the application, i.e. we leave for future work the deep analysis of the conditions under which the algorithms considered in the illustrative example (Algorithm 1) can improve the thresholds usually calculated.

Appendix
Proofs of the Theorems

Theorem 3.2 Let $\Gamma : FS([0,1]) \mapsto [0,1]$ be a function given by Eq. (1)*. Then*

$$\Gamma \textit{ satisfies } (E1^*) \textit{ if and only if } g(z) \neq 0 \textit{ for all } z \in]0,1[.$$

Proof.

$\Rightarrow$) Let Γ satisfy $(E1^*)$.

Suppose that $g(z_0) = 0$ for some $z_0 \in]0,1[$. Let $A \in FS([0,1])$ be given by $A(z) = z_0$ for all $z \in [0,1]$. Then, $\Gamma(A) = \int_{H_A} g(A(y))dy = \int_0^1 g(z_0) = 0$ and Γ does not satisfy $(E1^*)$.

$\Leftarrow$) Take $g(z) \neq 0$ for all $z \in]0,1[$.

- *If H_A has Lebesgue measure 0 then $\Gamma(A) = \int_{H_A} g(A(y))dy = 0$.*
- *If $\Gamma(A) = \int_{H_A} g(A(y))dy = 0$, since $g(z) \neq 0$ for all $z \in]0,1[$, then $g(A(y)) \neq 0$ for all $y \in H_A$. Consequently, $\Gamma(A) = \int_{H_A} g(A(y))dy = 0$ can only hold if $m(H_A) = 0$.*

Thus, Γ satisfies $(E1^*)$.

Theorem 3.5 Let $\Gamma : FS([0,1]) \mapsto [0,1]$ be a function given by Eq. (1)*. Then,*

$$\Gamma \textit{ satisfies } (E2^*) \textit{ if and only if } g^{-1}(1) = \left\{\frac{1}{2}\right\}$$

Proof.

$\Rightarrow$) Let Γ satisfy $(E2^*)$.

- *Suppose that $g(\frac{1}{2}) \neq 1$. Let the FS A be given by $A(x) = \frac{1}{2}$ for all $x \in [0,1]$. Then $\Gamma(A) = \int_{H_A} g(A(y))dy = \int_0^1 g(\frac{1}{2}) = g(\frac{1}{2}) \neq 1$, which is in contradiction with $(E2^*)$.*
- *Suppose $g(z_0) = 1$ for some $z_0 \neq \frac{1}{2}$. Given $A(x) = z_0$ for all $x \in [0,1]$ we have $\Gamma(A) = \int_{H_A} g(A(y))dy = \int_0^1 g(z_0) = g(z_0) = 1$, which is again in contradiction with $(E2^*)$.*

$\Leftarrow$) Let g satisfy $g^{-1}(1) = \{\frac{1}{2}\}$.

- *If $A(x) = \dfrac{1}{2}$ a.e. in $[0,1]$, then $m(\{x \in H_A \mid A(x) \neq \dfrac{1}{2}\}) \leq m(\{x \mid A(x) \neq \dfrac{1}{2}\}) = 0$ and $m(\{x \mid A(x) = \dfrac{1}{2})\}) = 1$. Thus, $\Gamma(A) = \int_{H_A} g(A(y))dy = \int_{\{x \in H_A \mid A(x) \neq \frac{1}{2}\}} g(A(y))dy + \int_{\{x \mid A(x) = \frac{1}{2}\}} g(A(y))dy = 0 + \int_{\{x \mid A(x) = \frac{1}{2}\}} g(\frac{1}{2})dy = g(\frac{1}{2}) = 1$.*
- *Now take $\Gamma(A) = \int_{H_A} g(A(y))dy = 1$. Since $m(H_A) \leq 1$ and $g(z) \leq 1$ then $\Gamma(A) = 1$ can only hold if $m(H_A) = 1$ and $g(A(y)) = 1$ for all $y \in H_A$. But given $y \in H_A$, $g(A(y)) = 1$ only if $A(y) = \frac{1}{2}$. Since the measure of H_A is 1, this means that $A = \frac{1}{2}$ a.e. in $[0,1]$.*

Consequently, Γ satisfies $(E2^*)$.

Theorem 3.7 Let $\Gamma : FS([0,1]) \mapsto [0,1]$ be a function given by Eq. (1)*. Then, Γ satisfies $(E3^*)$ if and only if g is increasing on $\left]0, \frac{1}{2}\right]$ and decreasing on $\left[\frac{1}{2}, 1\right[$.*

Proof.

$\Rightarrow$) Let Γ satisfy $(E3^*)$.

1) *Suppose g is not increasing in $]0, \frac{1}{2}]$. Then, there exist z_1, z_2 such that $0 < z_1 < z_2 \leq \frac{1}{2}$ and $g(z_1) > g(z_2)$.* *Let $A(x) = z_1$ for all $x \in [0,1]$ and $B(x) = z_2$ for all $x \in [0,1]$. As $A(x) \leq B(x) \leq \frac{1}{2}$ for all $x \in [0,1]$, by $(E3^*)$ it must be satisfied that $\Gamma(A) \leq \Gamma(B)$.* *But $\Gamma(A) = \int_{H_A} g(A(y))dy = \int_0^1 g(z_1)dy = g(z_1)$ and $\Gamma(B) = \int_{H_B} g(B(y))dy = \int_0^1 g(z_2)dy = g(z_2)$, which is in contradiction with $g(z_1) > g(z_2)$.*
2) *Suppose that g is not decreasing in $[\frac{1}{2}, 1[$. Then, there exist z_1, z_2 such that $\frac{1}{2} \leq z_1 < z_2 < 1$ and $g(z_1) < g(z_2)$.* *Let $A(x) = z_2$ for all $x \in [0,1]$ and $B(x) = z_1$ for all $x \in [0,1]$. Since $\frac{1}{2} \leq B(x) \leq A(x)$ for all $x \in [0,1]$, by $(E3^*)$ $\Gamma(A) \leq \Gamma(B)$ must be satisfied.* *But $\Gamma(A) = \int_{H_A} g(A(y))dy = \int_0^1 g(z_2)dy = g(z_2)$ and $\Gamma(B) = \int_{H_B} g(B(y))dy = \int_0^1 g(z_1)dy = g(z_1)$, which is in contradiction with $g(z_1) < g(z_2)$.*

$\Leftarrow$) Let g be increasing in $]0, \frac{1}{2}]$ and decreasing in $[\frac{1}{2}, 1[$.

	Otsu		Area		Ignorance		Alg1v1		Alg1v2		Average	
	u	%	u	%	u	%	u	%	u	%	u	%
Im. 1	79	93.6614	50	97.3064	13	96.6738	50	97.3064	29	97.2375	44	**97.4059**
Im. 2	74	92.2227	56	92.7227	11	90.8861	58	92.7074	47	92.7099	49	**92.7762**
Im. 3	104	98.0148	87	98.2887	13	97.6454	96	98.1731	88	98.2887	77	**98.3741**
Im. 4	136	**95.8283**	135	95.7278	135	95.7278	135	95.7278	134	95.5912	135	95.7278
Im. 5	127	95.8474	140	95.9545	177	93.3757	143	**95.9621**	157	95.2479	148	95.7224
Im. 6	134	95.6408	138	96.4085	97	64.4245	138	96.4085	141	**96.7835**	129	94.9316
Im. 7	71	95.9748	50	96.6721	3	92.2469	52	96.6337	49	96.7208	45	**96.8029**
Im. 8	123	89.0935	121	**89.5726**	121	**89.5726**	121	**89.5726**	121	**89.5726**	121	**89.5726**

TABLE I

THRESHOLDS (MULTIPLIED BY 255) AND PERCENTAGE OF WELL CLASSIFIED PIXELS. (OTSU) RESULTS OBTAINED WITH OTSU'S METHOD. (AREA) RESULTS OBTAINED AREA ALGORITHM AND $\varphi_1(x) = x^2$ AND $\varphi_2(x) = x$. (IGNORANCE) RESULTS OBTAINED WITH WITH THE ALGORITHM BASED ON THE IGNORANCE AND $G_u(x, y) = 2\sqrt{(1-x)(1-y)}$ IF $(1-x)(1-y) \leq 0.25$ AND $G_u(x, y) = 1/(2\sqrt{(1-x)(1-y)})$ OTHERWISE. (ALG2V1) RESULTS OBTAINED WITH OUR PROPOSAL, USING $E = \int g(A(x))dx$ WITH $g(x) = R(x, 1-x) = 1 - |2x-1|$. (ALG2V2) RESULTS OBTAINED WITH OUR PROPOSAL, USING $E = \int g(A(x))dx$ WITH $g(x) = R(x, 1-x) = 1 - (2x-1)^2$.

First of all, notice that g has a maximum on $\frac{1}{2}$.

Suppose that $A, B \in FS([0,1])$ satisfy that for all $x \in [0,1]$

$$\left.\begin{array}{l} A(x) \leq B(x) \leq \frac{1}{2} \\ \text{or} \\ A(x) \geq B(x) \geq \frac{1}{2} \end{array}\right\} \tag{4}$$

and let us see that $E^*(A) \leq E^*(B)$.

First, we prove $H_A \subseteq H_B$. Take $x \in H_A$, by the Definition of H_A then $A(x) \neq 0$ and $A(x) \neq 1$. There are three different cases:

- *If $A(x) < \frac{1}{2}$ then $0 < A(x) \leq B(x) \leq \frac{1}{2}$, so $0 < B(x) < 1$ and $x \in H_B$.*
- *If $A(x) > \frac{1}{2}$ then $1 > A(x) \geq B(x) \geq \frac{1}{2}$, so $0 < B(x) < 1$ and $x \in H_B$.*
- *If $A(x) = \frac{1}{2}$ then $\frac{1}{2} \leq B(x) \leq \frac{1}{2}$, so $0 < B(x) = \frac{1}{2} < 1$ and $x \in H_B$.*

Thus, $H_A \subseteq H_B$. Thereby,

$$\begin{aligned} \Gamma(A) &= \int_{H_A} g(A(y))dy \leq \int_{H_B} g(A(y))dy \\ &= \int_{\{x|0<B(x)<\frac{1}{2}\}} g(A(y))dy + \int_{\{x|B(x)=\frac{1}{2}\}} g(A(y))dy \\ &+ \int_{\{x|\frac{1}{2}<B(x)<1\}} g(A(y))dy \leq \int_{\{x|0<B(x)<\frac{1}{2}\}} g(B(y))dy \\ &+ \int_{\{x|B(x)=\frac{1}{2}\}} g(B(y))dy + \int_{\{x|\frac{1}{2}<B(x)<1\}} g(B(y))dy \\ &= \int_{H_B} g(B(y))dy = \Gamma(B) \end{aligned}$$

where the first inequality holds due to $H_A \subseteq H_B$ and the second one because g is an increasing function on $]0, \frac{1}{2}]$, because g has a maximum on $\frac{1}{2}$ and because g is decreasing on $[\frac{1}{2}, 1[$, respectively.

Theorem 3.9 Let $\Gamma : FS([0,1]) \mapsto [0,1]$ be a function given by Eq. (1). *Then,*

Γ satisfies $(E4^*)$ if and only if g is a symmetric function with respect to $z = \frac{1}{2}$, i.e., $g(z) = g(1-z)$ for all $z \in]0,1[$.

Proof. *First of all, notice that $H_{N(A)} = \{x \mid N(A(x)) \in]0,1[\} = \{x \mid 1 - A(x) \in]0,1[\} = \{x \mid A(x) \in]0,1[\} = H_A$.*

$\Rightarrow$) Let Γ satisfy $(E4^*)$.

Suppose that g is not symmetric, then there exists $z_0 \in]0,1[$ such that $g(z_0) \neq g(1-z_0)$. Let $A(x) = z_0$ for all $x \in [0,1]$, then $N(A(x)) = 1 - z_0$ for all $x \in [0,1]$. However, function Γ yields

$\Gamma(A) = \int_{H_A} g(A(y))dy = \int_0^1 g(z_0)dy = g(z_0)$ and

$$\begin{aligned} \Gamma(N(A)) &= \int_{H_{N(A)}} g(N(A(y)))dy \\ &= \int_{H_{N(A)}} g(1-z_0)dy = g(1-z_0), \end{aligned}$$

which is in contradiction with $(E4^*)$.

$\Leftarrow$) Let g be a symmetric function with respect to $z = \frac{1}{2}$. Then

$$\begin{aligned} \Gamma(A) &= \int_{H_A} g(A(y))dy \\ &= \int_{H_{N(A)}} g(A(y))dy = \int_{H_{N(A)}} g(1-A(y))dy \\ &= \int_{H_{N(A)}} g(N(A(y)))dy = \Gamma(N(A)) \end{aligned}$$

where the second equality holds because $H_A = H_{N(A)}$, the third one holds because g is symmetric and the fourth one by the expression of negation.

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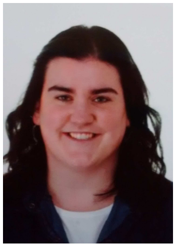

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