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Fuzzy concept lattices and fuzzy relation equations in the retrieval processing of images and signals*

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This paper considers the introduced relations between fuzzy property-oriented concept lattices and fuzzy relation equations, on the one hand, and mathematical morphology, on the other hand, in the retrieval processing of images and signals. In the first part, it studies how the original images and signals can be retrieved using fuzzy property-oriented concept lattices and fuzzy relation equations. In the second one we analyze two of the most important tools in fuzzy mathematical morphology from the point of view of the fuzzy property-oriented concepts and the aforementioned study. Both parts are illustrated with practical examples.

Keywords: Rough sets; fuzzy sets; fuzzy mathematical morphology.

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1. Introduction

Fuzzy property-oriented concept lattices¹ provide a parallel framework, which arises as a fuzzy generalization of Rough Set Theory and in which a set of objects and a set of attributes are assumed, following the view point of Formal Concept Analysis. This theory has been considered to solve fuzzy relation equations^{2,3} and important results, in order to obtain the whole set of solutions, are given.

On the other hand, mathematical morphology (initiated by⁴ and⁵) is based on set theory, integral geometry and lattice algebra. This methodology is used in the recent years in general contexts related to activities as the information extraction in digital images, the noise elimination or the pattern recognition.

This theory was extended to a fuzzy setting, considering fuzzy subsets as objects, by^{6,7,8,9,10,11,12} using L -fuzzy sets as images and structuring elements, which was called *fuzzy morphological image processing*. These works have been generalized by¹³.

Both theories, fuzzy mathematical morphology and fuzzy property-oriented concept lattice, have been related by¹⁴, extending the initial relation given by^{15,13}. For example, it has been proved that the erosion and dilation operators are the necessity and possibility operators of the associated context, respectively. Moreover, closing and opening (images) are univocally related to the concepts of a fuzzy property-oriented concept lattice.

In mathematical morphology, the usual procedure is, given a structuring element, to obtain the dilation and the erosion from an initial image. But what happen if we lose the original image or, simply, we have not got it because we only know its corresponding dilation or erosion, how can the original image be obtained?

This paper studies the problem of objects retrieval in the framework of mathematical morphology. It is usual that there is noise in the transmission of information or, in several cases, it is easier to send a kind of image than another one. Hence, the received object is not equal to the original one. Note that this problem is also related to other settings, such as object recognition.

Hence, this paper is focused on solving the problem of obtaining the original object A from another one received B , assuming a structuring image and that B is the dilation or the erosion of the original image A . For that, this problem will be written as a fuzzy relation equation and the relationship introduced by¹⁴ and the results given by^{2,3} will be used to solve it.

Moreover, in the last section we will analyze two of the most useful tools in fuzzy mathematical morphology: the gradient and the top-hat transforms^{5,16,17,18,19,20,21}. In mathematical morphology, these operators are used to find relevant elements in an image, that is, to distinguish these relevant elements from the rest ones forming the background of the image. Specifically, we will define them in a fuzzy setting, we will present different properties and we will introduce these notions, interpreting their meaning, in a general fuzzy property-oriented concept lattice framework.

The work is organized as follows: in Section 2, we show some preliminary notions

about fuzzy property-oriented concept lattice, fuzzy mathematical morphology and the relation between both theories. Also fuzzy relation equations are presented. In Section 3, we carry out a study on the application of fuzzy relation equation to objects retrieval in the fuzzy property-oriented concept lattice. Then, in Section 4, we introduce the notions of morphological gradient and top-hat transforms in the fuzzy setting and we use the relationship between fuzzy property-oriented concept lattice and fuzzy mathematical morphology to obtain interesting results. In all the paper, some examples of images and signals are shown to illustrate the results. Finally we present some conclusions and future lines of work.

2. Preliminaries

This section recalls the fuzzy property-oriented concept lattice framework¹, fuzzy mathematical morphology^{6,7,8,9,10,11,12}, the relationship between them, introduced by¹⁴, and fuzzy relation equations²².

2.1. Fuzzy property-oriented concept lattice

In this framework a complete residuated lattice $(L, \vee, \wedge, *, I, 0, 1, \leq)$ is considered as algebraic structure.

Definition 1. A *complete residuated lattice* is a tuple $(L, \vee, \wedge, *, I, 0, 1, \leq)$, where $(L, \vee, \wedge, 0, 1, \leq)$ is a complete lattice and $*$: $L \times L \rightarrow L$, I : $L \times L \rightarrow L$ are mappings verifying the called *adjunction property*:

$$x * y \leq z \quad \text{if and only if} \quad x \leq I(y, z)$$

for all $x, y, z \in L$.

The pair $(*, I)$ is called *residuated pair*.

For example, every triangular norm defined on the unit interval, together with its residuated implication, satisfies the adjunction property.

Example 1. The pair $(*_L, I_L)$, where $*_L: [0, 1] \times [0, 1] \rightarrow [0, 1]$ and $I_L: [0, 1] \times [0, 1] \rightarrow [0, 1]$ are defined by

$$x *_L y = \max\{x + y - 1, 0\}, \quad I_L(x, y) = \min\{1 + y - x, 1\}$$

is a residuated pair called the Łukasiewicz residuated pair.

From now on, a residuated lattice $(L, \vee, \wedge, *, I, 0, 1, \leq)$ will be fixed.

A fuzzy context is assumed, (X, Y, R) , where $R: X \times Y \rightarrow L$ is an L -fuzzy relation between the sets X and Y , where X can be interpreted as a set of objects and Y as a set of properties (attributes).

Given a fuzzy context (X, Y, R) , two mappings $R_{\exists}: L^X \rightarrow L^Y$ and $R^{\forall}: L^Y \rightarrow L^X$ can be defined as:

$$R_{\exists}(A)(y) = \sup\{A(x) * R(x, y) \mid x \in X\} \quad (1)$$

$$R^{\forall}(B)(x) = \inf\{I(R(x, y), B(y)) \mid y \in Y\} \quad (2)$$

for all $A: X \rightarrow L$, $B: Y \rightarrow L$, $x \in X$ and $y \in Y$, where I is the residuated implication associated with the conjunctor $*$. Examples of these operators are given in ^{23,24}.

As a first result, the pair $(R_{\exists}, R^{\forall})$ forms an isotone Galois connection ²⁵. Therefore, a *fuzzy property-oriented concept* (or, a *fuzzy concept based on rough set theory*) of (X, Y, R) is a pair $(A, B) \in L^X \times L^Y$ such that $B = R_{\exists}(A)$ and $A = R^{\forall}(B)$.

The set of all fuzzy property-oriented concepts of (X, Y, R) is denoted by $\mathcal{P}(X, Y, R)$ and it is a complete lattice ¹, which is called the *fuzzy property-oriented concept lattice of (X, Y, R)* (or, the *fuzzy concept lattice of (X, Y, R) based on rough set theory*) ¹. For that isotone Galois connection $(R_{\exists}, R^{\forall})$ and lattice $\mathcal{P}(X, Y, R)$ interesting properties have been proven, e.g., in ^{23,26,25,1}.

2.2. Fuzzy mathematical morphology

Fuzzy morphological image processing has been developed using L -fuzzy sets $A \in L^X$ and S (with $X = \mathbb{R}^2$ or $X = \mathbb{Z}^2$) as images and structuring elements in ^{6,7,8,9,10,11,12}. The structuring image S represents the effect that we want to produce over the original image A .

Fuzzy morphological dilations $\delta_S : L^X \rightarrow L^X$ and *fuzzy morphological erosions* $\varepsilon_S : L^X \rightarrow L^X$ are defined using some operators of the fuzzy logic. In the literature (see ^{8,6,27,11}) erosion and dilation operators are introduced associated with the residuated pair $(*, I)$ as follows:

If $S: X \rightarrow L$ is an image that we take as *structuring element*, then we consider the following definitions associated with (L, X, S) , given by ⁶.

Definition 2. The fuzzy erosion of the image $A \in L^X$ by the structuring element S is the L -fuzzy set $\varepsilon_S(A) \in L^X$ defined as:

$$\varepsilon_S(A)(x) = \inf\{I(S(y-x), A(y)) \mid y \in X\} \quad \text{for all } x \in X$$

The fuzzy dilation of the image A by the structuring element S is the L -fuzzy set $\delta_S(A)$ defined as:

$$\delta_S(A)(x) = \sup\{S(x-y) * A(y) \mid y \in X\} \quad \text{for all } x \in X$$

From these definitions arise two mappings which will be called the fuzzy erosion and dilation operators $\varepsilon_S, \delta_S: L^X \rightarrow L^X$.

We can compose these operators dilation and erosion associated with the structuring element S and obtain the basic filters *morphological opening* $\gamma_S : L^X \rightarrow L^X$ and *morphological closing* $\phi_S : L^X \rightarrow L^X$ defined by:

$$\gamma_S = \delta_S \circ \varepsilon_S, \quad \phi_S = \varepsilon_S \circ \delta_S.$$

The opening and the closing operators verify the two conditions that characterize the morphological filters: they are isotone and idempotent operators. Moreover, for all $A, S \in L^X$ it is also verified $\gamma_S(A) \subseteq A \subseteq \phi_S(A)$.

These operators will characterize some special images: the S -open ($\gamma_S(A) = A$) and the S -closed ones ($\phi_S(A) = A$).

2.3. Relationship between both theories

In ¹⁴ these previous theories were related. For that, first of all, any fuzzy image $S \in L^X$ was associated with the fuzzy relation $R_S \in L^{X \times X}$, defined as:

$$R_S(x, y) = S(y - x)$$

for all $x, y \in X$. Hence, the fuzzy erosion and dilation of an L -fuzzy subset A of X are written as follows:

$$\varepsilon_S(A)(x) = \inf\{I(R_S(x, y), A(y)) \mid y \in X\} \quad (3)$$

$$\delta_S(A)(x) = \sup\{R_S(y, x) * A(y) \mid y \in X\} \quad (4)$$

and the following results were proved in ¹⁴.

Proposition 1. *Let (L, X, S) be the triple associated with the structuring element $S \in L^X$. Let (X, X, R_S) be the fuzzy property-oriented context whose incidence relation is the relation R_S associated with S . Then the erosion ε_S and dilation δ_S operators in (L, X, S) are related to the derivation operators $(R_S)^\forall$ and $(R_S)^\exists$ in the fuzzy property-oriented context (X, X, R_S) by:*

$$\varepsilon_S = (R_S)^\forall$$

$$\delta_S = (R_S)^\exists$$

This relation provides that the dilation and erosion are exactly the possibility and necessity operators associated with the context (X, X, R_S) . As a consequence, they have the properties of the isotone Galois connection $(R_S^\exists, R_S^\forall)$. The following result shows the connection between the outstanding morphological elements and the fuzzy property-oriented concepts.

Theorem 1. *Let $S \in L^X$ and its associated relation $R_S \in L^{X \times X}$, the following statements are equivalent:*

- (1) *The pair $(A, B) \in L^X \times L^X$ is a fuzzy property-oriented concept of the context (X, X, R_S) .*
- (2) *A is S -closed (i.e. $\varepsilon_S \circ \delta_S(A) = A$) and B is the S -dilation of A .*
- (3) *B is S -open (i.e. $\delta_S \circ \varepsilon_S(B) = B$) and A is the S -erosion of B .*

As a consequence, every S -closed (or S -open) set determines only one fuzzy property-oriented concept, and vice versa. This relation will be fundamental in the images and signals retrieval process we will present in this paper.

2.4. Fuzzy relation equations and concept lattices

Fuzzy relation equations have been widely studied, for instance in ^{28,29,30}. This section recalls these kind of equations in the particular case in which the unknown and independent fuzzy relations have only one argument (only one column), which will be the case needed in this paper.

Given two sets U, V , two fuzzy relations $R \in L^{U \times V}$ and $T \in L^U$, and an unknown fuzzy relation $Z \in L^V$, a *fuzzy relation equation with sup- $*$ -composition* (FRE*), is the equation

$$R \circ Z = T \quad (5)$$

where the composition \circ is defined as $R \circ Z(u) = \bigvee_{v \in V} (R(u, v) * Z(v))$, for all $u \in U$.

Assuming the same sets and fuzzy relations, its counterpart is a *fuzzy relation equation with inf- I -composition* (FREI), that is,

$$R \triangleleft Z = T \quad (6)$$

where the composition \triangleleft is defined as $R \triangleleft Z(u) = \bigwedge_{v \in V} I(R(u, v), Z(v))$, for all $u \in U$.

In ², the authors related the solvability of the previous fuzzy relation equations to the fuzzy property-oriented concept lattice theory, considering the context (V, R, R^{-1}) associated with Equation 5, where R^{-1} represents the inverse relation of R , that is, $R^{-1}(v, u) = R(u, v)$, for all $(u, v) \in U \times V$, and the context (U, V, R) associated with Equation 6. Several results introduced in the aforementioned paper will be needed in the following section and so, they will be recalled below.

Theorem 2. *Considering the above environment and consideration, Equation (5) can be solved if and only if $(R^{-1})_{\exists}(R^{-1})^{\forall}(T) = T$.*

Analogously, Equation (6) can be solved if and only if $R^{\forall}(R_{\exists}(T)) = T$.

When Equation (5) (resp. Equation (6)) is solvable, a greatest (resp. least) solution exists, as the following result shows.

Proposition 2. *If Equation (5) can be solved, then $(R^{-1})^{\forall}(T)$ is the greatest solution. Analogously, if Equation (6) can be solved, then $R_{\exists}(T)$ is the least solution.*

The following result provides a characterization of the solutions of Equation (5). A similar result can be given for Equation (6).

Theorem 3. *Let $(\mu_1, \lambda_1), (\mu_2, \lambda_2), \dots, (\mu_r, \lambda_r) \in \mathcal{P}(V, U, R^{-1})$ the lower neighbors of the concept $((R^{-1})^{\forall}(T), T)$, if an element $\mu \in L^V$ is a solution of Equation (5) then either $\mu_i < \mu \leq (R^{-1})^{\forall}(T)$, for some $i \in \{1, \dots, r\}$ or μ is incomparable with μ_i , for all $i \in \{1, \dots, r\}$.*

3. Images and signals retrieval

This section introduces an application of fuzzy relation equation to objects retrieval in the fuzzy mathematical morphology setting. From the relationship between fuzzy mathematical morphology and fuzzy property-oriented concept lattice, recalled in Section 2.3, and the relationship between fuzzy property-oriented concept lattice and fuzzy relation equations ², we will solve the problem of obtaining the original

object $A: X \rightarrow L$ from another one received $B: X \rightarrow L$ and a fixed structuring image $S: X \rightarrow L$.

Specifically, given an image $B: X \rightarrow L$, we can consider a structuring image $S: X \rightarrow L$ and ask if there exists $A: X \rightarrow L$ such that $\delta_S(A) = B$ and, if there exists, how to obtain it. Analogously, for each image $A: X \rightarrow L$, we can ask if there exists $B: X \rightarrow L$ such that $\varepsilon_S(B) = A$, for a structuring image S , and, if there exists, how to obtain it.

First of all, we will write this mathematical morphology problem in terms of fuzzy relation equations using Equations (4) and (3), and the definition of the corresponding compositions, Equations (5) and (6), respectively.

Given an image $B: X \rightarrow L$ and a structuring image $S: X \rightarrow L$, if we want to obtain an image $A: X \rightarrow L$ such that $\delta_S(A) = B$, then we need to solve the following equation:

$$R_S^{-1} \circ A = B \quad (7)$$

Analogously, given an image $A: X \rightarrow L$ and a structuring image $S: X \rightarrow L$, obtaining an image $B: X \rightarrow L$, such that $\varepsilon_S(B) = A$, is equivalent to solve the equation:

$$R_S \triangleleft B = A \quad (8)$$

Next, several results will be presented in the fuzzy mathematical morphology framework, based on the properties introduced in ² and recalled previously. The first one is about the solvability of Equations (7) and (8).

Theorem 4. *Equation (7) can be solved if and only if B is S -open in X . In that case, $\varepsilon_S(B) \in L^X$ is the greatest solution.*

Analogously, Equation (8) can be solved if and only if A is S -closed in X . In that case, $\delta_S(A) \in L^X$ is the least solution.

Proof. From Theorem 2, we have Equation (7) can be solved if and only if $(R_S)_{\exists}((R_S)^{\forall}(B) = B$, and, by Proposition 1 and Theorem 1, this is equivalent to B is an S -open image. Now, applying Proposition 2 and Theorem 2 we obtain that $\varepsilon_S(B) \in L^X$ is the greatest solution of Equation (7).

The other equivalence is similarly proved. \square

The second result relates the independent term and greatest solution to a fuzzy property-oriented concept.

Theorem 5. *Equation (7) can be solved if and only if $(\varepsilon_S(B), B)$ is a fuzzy property-oriented concept of the context (X, X, R_S) .*

Similarly, Equation (8) can be solved if and only if $(A, \delta_S(A))$ is a fuzzy property-oriented concept of the context (X, X, R_S) .

Proof. The first equivalence is obtained from Theorem 4, Proposition 1 and the property of Galois connections, which ensure that $\varepsilon_S(\delta_S(\varepsilon_S(B))) = \varepsilon_S(B)$. The second one is similarly proved. \square

Now, we present an application to digital signals.

Example 2. We consider in this example the particular residuated lattice $(L, \vee, \wedge, *_L, I_L, 0, 1, \leq)$, where $L = \{0, 0.1, 0.2, \dots, 0.9, 1\}$ and $(*_L, I_L)$ the restriction of the Lukasiewicz residuated pair $(*_L, I_L)$ on L .

Let us assume the set $X = \{0, 1, 2, \dots, 21, 22\} \subseteq \mathbb{Z}$, the mapping $B: X \rightarrow L$, which is represented in Figure 2, and the structuring set $S = \{-1, 0, 1\}$. Note that B can be interpreted as a 1-D discrete signal.

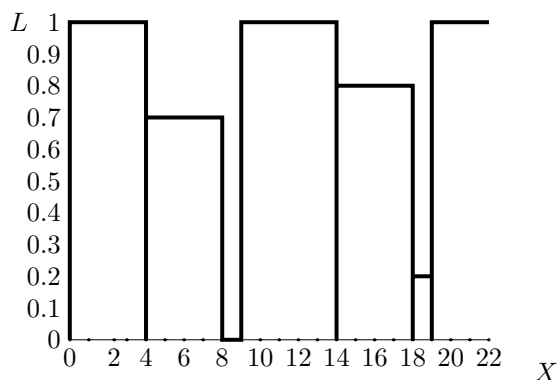


Fig. 1. Discrete signal received

From this environment a fuzzy relation equations similar to Equation (7) is considered, in order to obtain a signal A with dilation B .

First of all, we need to check if this equation has a solution. Hence, we consider the context (X, X, R_S) , where the fuzzy relation $R_S \subseteq X \times X$ is defined, for each $(x, y) \in X \times X$, as

$$R_S(x, y) = S(y - x) = \begin{cases} 1 & \text{if } |y - x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Since the signal B is S -open in X , that is $(\delta_S \circ \varepsilon_S)(B) = B$, by Theorems 4 and 5, we have that the considered equation has, at least, one solution and the greatest solution A_g is $\varepsilon_S(B)$, which is given in Figure 2 and defined as

$$\varepsilon_S(B)(x) = \inf\{I(R_S(x, y), B(y)) \mid y \in X\} = \inf\{B(y) \mid |y - x| \leq 1\}$$

for all $x \in X$. Moreover, (A_g, B) is a fuzzy property-oriented concept.

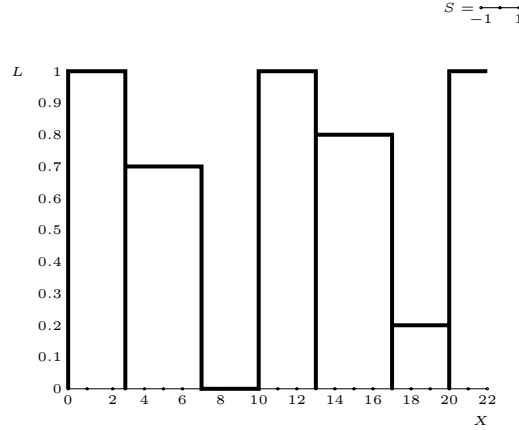


Fig. 2. Greatest solution

It is clear, in Example 2, that the original signals A may not be the greatest solution, but another one of the proposed Equation. In order to obtain the whole set of solutions the following result is introduced.

Theorem 6. *Given an S -open object $B \in L^X$, if $A \in L^X$ is the original object, such that $\delta_S(A) = B$, then either $A^l < A \leq \varepsilon_S(B)$ for some (A^l, B^l) lower neighbour of $(\varepsilon_S(B), B)$ in $\mathcal{P}(X, X, R_S)$; or $A < \varepsilon_S(B)$ and A is incomparable with A^l , for all (A^l, B^l) lower neighbour of $(\varepsilon_S(B), B)$ in $\mathcal{P}(X, X, R_S)$.*

Analogously, given an S -closed object $A \in L^X$, if $B \in L^X$ is the original object, such that $\varepsilon_S(B) = A$, then either $\delta_S(A) \leq B < B^u$ for some upper neighbour (A^u, B^u) of $(A, \delta_S(A))$ in $\mathcal{P}(X, X, R_S)$; or $\delta_S(A) < B$ and B is incomparable with B^u , for all (A^u, B^u) upper neighbour of $(A, \delta_S(A))$ in $\mathcal{P}(X, X, R_S)$.

Proof. Given an S -open object $B \in L^X$, by Theorem 5, we have that $(\varepsilon_S(B), B)$ is a fuzzy property-oriented concept of the context (X, X, R_S) , that is $(\varepsilon_S(B), B) \in \mathcal{P}(X, X, R_S)$, and by Proposition 1 we have $\varepsilon_S = (R_S)^\forall$. Hence, since the original object $A \in L^X$ is a solution of Equation 7, applying Theorem 3, we obtain the result.

The second part is similarly obtained. \square

Therefore, Theorem 6 can be applied in order to obtain the whole set of solutions of the system given in Example 2.

Notice that the original image can be a minimal solution instead of the greatest solution. Hence, an suitable methodology should consider the whole set of solutions in order to detect the original image. More results related to the computation of the whole set of solutions and minimal solutions are given in ^{31,32,33}.

The following example focus on images retrieval.

Example**3.**

In this example, we consider the residuated lattice $(L, \vee, \wedge, *_L, I_L, 0, 1, \leq)$, where $L = \{0, \frac{1}{256}, \frac{2}{256}, \frac{3}{256}, \dots, 1\}$ (1 represents the white color and 0 the black color), $(*_L, I_L)$ is the restriction of the Łukasiewicz residuated pair on L .

A two dimensional pixelated image in a 8-bits grayscale will be represented by a mapping $A: X \rightarrow L$, where $X = \mathbb{Z}^2$. The elements in X are denoted as $\bar{x} = (x_1, x_2) \in \mathbb{Z}^2$.

In this case, the image $B: \mathbb{Z}^2 \rightarrow L$, given in Figure 3, is obtained and we want to retrieve the original image or a good approximation.



Fig. 3. Original image

The structuring element is a fuzzy disk of radius $r = 5$ with center in the origin $(0,0)$, where the belonging value of center is 1 and this is progressively decreasing to 0 outside the circle, as follows:

$$S(x, y) = \begin{cases} 1 - \frac{x^2 + y^2}{r^2 + 1} & \text{if } x^2 + y^2 \leq r^2 \\ 0 & \text{otherwise} \end{cases}$$

Now, we consider a Equation (7) and a context $(\mathbb{Z}^2, \mathbb{Z}^2, R_S)$, where the associated incidence relation $R_S \subseteq \mathbb{Z}^2 \times \mathbb{Z}^2$ is defined, for each $\bar{x} = (x_1, x_2)$, $\bar{y} = (y_1, y_2)$, as

$$R_S(\bar{x}, \bar{y}) = S(\bar{y} - \bar{x}) = \begin{cases} 1 - \frac{(y_1 - x_1)^2 + (y_2 - x_2)^2}{5^2 + 1} & \text{if } (y_1 - x_1)^2 + (y_2 - x_2)^2 \leq 5^2 \\ 0 & \text{otherwise} \end{cases}$$

Given the image $B: \mathbb{Z}^2 \rightarrow L$, which has been obtained as the dilation, $B = \delta_S(A)$, of an initial image, A , by the fuzzy structuring element S , the considered

problem is to find the image A , or the best possible approximation. The considered operators are the Lukasiewicz t-norm and its residuated implication.

In this example the initial image, B , is given in Figure 4.



Fig. 4. Initial image, B .

Therefore, by the previous results, the greatest solution $A_g: \mathbb{Z}^2 \rightarrow L$, associated with B and the structuring image S , can be a good approximation of B . This image is given in Figure 5 and it is defined as:

$$\begin{aligned} \varepsilon_S(B)(\bar{x}) &= (R_S)^\vee(B)(\bar{x}) \\ &= \inf\{I(R_S(\bar{x}, \bar{y}), B(\bar{y}) \mid \bar{y} \in X\} \\ &= \inf\left\{\frac{(y_1 - x_1)^2 + (y_2 - x_2)^2}{5^2 + 1} + B((y_1, y_2)) \mid (y_1 - x_1)^2 + (y_2 - x_2)^2 \leq 5^2\right\} \end{aligned}$$

By Theorem 5, the pair (A, B) is a fuzzy property-oriented concept of the context $(\mathbb{Z}^2, \mathbb{Z}^2, R_S)$.

The best approximation of the original image we can obtain is given by the erosion $\varepsilon_S(B)$, represented in the Fig.5. This is the greatest solution of the problem, that is, the greatest image (the most white) which dilation is the initial image. However, in this case, the considered original image is not the greatest solution A_g but a solution less than it.

The use of fuzzy structuring elements allows the computation of good results because several gray levels can be considered. For instance, if we use the crisp disk



Fig. 5. Greatest solution, A_g .

of radius 5 as structuring element in the previous example:

$$S(x, y) = \begin{cases} 1 & \text{if } x^2 + y^2 \leq 5^2 \\ 0 & \text{otherwise} \end{cases}$$

the obtained image by the dilation of the original image will be the one given in Figure 6, and the computed (greatest) solution (applying the erosion with the crisp structuring element) is the image represented in Figure 7, which is clearly worse than the one obtained from the fuzzy case.

4. Some elements of Fuzzy Mathematical Morphology in fuzzy property-oriented concept lattices

The morphological gradient and the top-hat transforms^{16,20,21,5} are tools defined in mathematical morphology. In this section, we will introduce their fuzzy definitions and we will study their meaning from the viewpoint of the fuzzy property-oriented concept lattice framework. Hereon, a residuated lattice $(L, \vee, \wedge, *, I, 0, 1, \leq)$, a set $X = \mathbb{R}^2$ or $X = \mathbb{Z}^2$ and a structuring element $S \in L^X$ are fixed.

4.1. Morphological Gradient in (L, X, S) .

Erosions and dilations are the basic elements in mathematical morphology and they can be combined defining morphological gradient operators.

In image analysis, the objects are considered as areas of rather homogeneous grey levels. Then, object boundaries or edges are located where there are high grey



Fig. 6. Initial image, B , with the crisp structuring element.



Fig. 7. Gratest solution, A_g , with the crisp structuring element

level variations. Gradient operators^{16,18,20} are used to locate these variations. The morphological gradient outputs the maximum variation of the grey level intensities within the neighborhood defined by the structuring element.

Many gradient operators have been proposed in image analysis because there

is no unique discrete equivalent of the gradient operator defined for differentiable continuous functions. Three combinations are currently used:

- The arithmetic difference between the original image and the eroded image. This operator enhances the internal boundaries of the objects of the image and it is said to be the *internal gradient*.
- The *external gradient* is defined as the arithmetic difference between the dilated image and the original one. This gradient extracts the external boundaries of the objects in the image. Internal and external gradients are also called *half gradients* and they are used when thin contours are needed.
- The basic *morphological gradient*, also called *Beucher gradient*^{16,20}, is defined as the arithmetic difference between the dilation and the erosion of the image. This operator gives the maximum variation of grey level in a region defined by the structuring element.

Only structuring elements containing the origin are considered to make sure that the arithmetic difference is always non negative.

Extending this idea to the L -fuzzy case, we can define the following operators, which are not the simple consideration of the fuzzy definitions of the dilation and erosion as was considered in³⁴:

Definition 3. Let (L, X, S) be a tuple associated with the structuring element $S \in L^X$. We define the *internal gradient* of $A \in L^X$ as

$$\text{GRAD}_{\bar{S}}(A) = A * (\varepsilon_S(A))'$$

being $*$ a t-norm and $'$ an involutive negation defined on L^X .

Definition 4. The *external gradient* of the set $A \in L^X$ with the structuring element $S \in L^X$ is defined as

$$\text{GRAD}_S^+(A) = \delta_S(A) * A'$$

Finally, the Beucher gradient can be extended as follows:

Definition 5. Consider the structuring element $S \in L^X$ and (L, X, S) a tuple associated with S . The *morphological gradient* of an L -fuzzy set $A \in L^X$ is defined as

$$\text{GRAD}_S(A) = \delta_S(A) * (\varepsilon_S(A))'$$

These three definitions for the gradient operator are related in the following proposition.

Proposition 3. *If the structuring element S is such that it contains the origin, i.e. $S(0) = 1$, then for any L -fuzzy set $A \in L^X$ it is fulfilled that:*

$$\sup\{\text{GRAD}_{\bar{S}}(A), \text{GRAD}_S^+(A)\} \leq \text{GRAD}_S(A)$$

Proof. Let us denote by \mathbb{O} the L -fuzzy set that represents the origin, that is, $\mathbb{O}: X \rightarrow L$, defined, for all $x \in X$, as:

$$\mathbb{O}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

Since the structuring element S contains the origin, the inequality $\mathbb{O}(x) \leq S(x)$ holds, for all $x \in X$, and, by the monotonicity of I and the definition of \mathbb{O} , we have

$$\begin{aligned} \varepsilon_S(A)(x) &= \inf\{I(S(y-x), A(y)) \mid y \in X\} \leq \inf\{I(\mathbb{O}(y-x), A(y)) \mid y \in X\} = \\ &= I(\mathbb{O}(x-x), A(x)) = A(x) \end{aligned}$$

On the other hand,

$$\begin{aligned} \delta_S(A)(x) &= \sup\{S(x-y) * A(y) \mid y \in X\} \geq \sup\{\mathbb{O}(x-y) * A(y) \mid y \in X\} = \\ &= \mathbb{O}(x-x) * A(x) = A(x) \end{aligned}$$

Therefore,

$$\text{GRAD}_S(A) = \delta_S(A) * (\varepsilon_S(A))' \geq A * (\varepsilon_S(A))' = \text{GRAD}_S^-(A)$$

and

$$\text{GRAD}_S(A) = \delta_S(A) * (\varepsilon_S(A))' \geq \delta_S(A) * A' = \text{GRAD}_S^+(A) \quad \square$$

From the proof of the previous result, the following interesting property arises.

Corollary 1. *If the structuring element S contains the origin, then*

$$\varepsilon_S(A) \leq A \leq \delta_S(A)$$

for all $A \in L^X$.

The gradient is an useful mapping in order to know the possible error given by the approximation in Section 3. From this value we have a bounded of the maximum difference between the approximation given by the greatest solution of Equation (7) or the minimal solutions and the original image.

As we explained in Section 3, the greatest solution is $\varepsilon_S(B)$ and the equality $\delta_S(A) = \delta_S(\varepsilon_S(B)) = B$ holds. On the other hand, since a minimal solution M exists, such as $M \leq A$, then $\varepsilon_S(M) \leq \varepsilon_S(A)$. Therefore,

$$\text{GRAD}_S(A) \leq B * (\varepsilon_S(M))'$$

Other useful operators in mathematical morphology are the top-hat transforms, which will be studied in the fuzzy case in the following section.

4.2. Top-Hat Transforms in (L, X, S)

In most of the cases, the choice of a morphological filter is due to the available knowledge about the shape, size and orientation of the elements we would like to filter. Morphological Top-Hat transforms^{17,19,21,35} proceed in a different way since the approach undertaken with these transforms consists in using knowledge about the shape characteristics that are not shared by the relevant image elements.^a In this sense, we use opening or closing with a structuring element that does not fit the relevant image structures in order to remove them from the image.

These operators are useful when variations in the background mean that extraction of relevant structures in an image cannot be achieved by a simple threshold.

In mathematical morphology two types of top-hat transform are defined:

- The *top-hat by opening* is defined as the difference between the original image and its opening by a structuring element. This transform is appropriate for finding bright features in an image, this is why it is also called *white top-hat*.
- The *top-hat by closing* is obtained when the original image is subtracted from the closing by a structuring element. Since the top-hat by closing returns an image containing those elements that are darker than their surroundings, it is called *black top-hat*.

The extension of these definitions to the L -fuzzy framework can be done as follows:

Definition 6. Let (L, X, S) be a tuple associated with the structuring element $S \in L^X$. We define the *top-hat by opening* $\text{TH}_S^\gamma: L^X \rightarrow L^X$ and *top-hat by closing* $\text{TH}_S^\phi: L^X \rightarrow L^X$ as:

$$\begin{aligned}\text{TH}_S^\gamma(A) &= A * (\gamma_S(A))' \\ \text{TH}_S^\phi(A) &= \phi_S(A) * A'\end{aligned}$$

for all $A \in L^X$, where $'$ is an involutive negation defined on L^X and $*$ is a t-norm that can be the same or different from the one used to obtain the fuzzy erosion and dilation.

The following result relates both top-hat operators from a *symmetrical structuring element* S , that is, a structuring element S verifying that $S(-x) = S(x)$, for all $x \in X$.

Proposition 4. *If $A \in L^X$ is an S -open fuzzy set and A' is S -closed, then*

$$\text{TH}_S^\gamma(A) = \text{TH}_S^\phi(A')$$

^aNote that, in mathematical morphology, the relevant images are the images that are not part of the background of the image.

Proof. Since A is S -open and A' is S -closed, we have that $A = \gamma_S(A)$ and $A' = \phi_S(A')$.

Therefore,

$$\text{TH}_S^\gamma(A) = A * (\gamma_S(A))' = A * A' = A' * A = \phi_S(A') * A = \text{TH}_S^\phi(A') \quad \square$$

Note that, if the negation $'$ associated with the implication of the residuated lattice (that is, $x' = I(x, 0)$, for all $x \in [0, 1]$) is involutive and S is symmetric, then we have that $(\varepsilon_S(A))' = \delta_S(A')$ and $(\delta_S(A))' = \varepsilon_S(A')$. See ^{13,36} for more details. As a consequence, we obtain the following result.

Corollary 2. *Given a symmetrical structuring element S . If the negation $'$ associated with the residuated implication I is involutive and $A \in L^X$ is an S -open fuzzy set, then*

$$\text{TH}_S^\gamma(A) = \text{TH}_S^\phi(A')$$

Proof. If S is symmetrical, $'$ is the negation associated with I and it is involutive, we have that A is S -open if and only if A' is S -closed. Therefore, the result is obtained from Theorem 4. \square

When the original image is approximated, for instance, by the greatest solution of Equation (7) ($A \approx \varepsilon_S(B)$), the top hat mappings can also be computed using this approximation as:

$$\begin{aligned} \text{TH}_S^\gamma(A) &= A * (\gamma_S(A))' \approx \varepsilon_S(B) * (\gamma_S(A))' \\ \text{TH}_S^\phi(A) &= \phi_S(A) * A' \approx \varepsilon_S(B) * (\varepsilon_S(B))' \end{aligned}$$

From the last equation we can conclude that the approximation of the top hat by closing is not optimal by the greatest solutions and a better approximation would be given by a minimal solution of Equation (7).

The next section introduces the gradient and top-hat transforms in the fuzzy property-oriented concept lattice framework.

4.3. Application in the fuzzy property-oriented concept lattice

Let (L, X, S) be the tuple associated with the structuring element $S \in L^X$ and let the fuzzy property-oriented context (X, X, R_S) be.

Using Proposition 1 we can analyze the effect of the transformations defined in the previous paragraph when we are working with the particular fuzzy property-oriented context (X, X, R_S) and how can be extended to a general framework. First of all, we need to note that in Rough Set Theory the notion of gradient is similar to the difference between the upper and lower approximations of a given set of objects and, specifically, in the fuzzy property-oriented context (X, X, R_S) , the gradient can be considered as the intersection between the 'upper approximations' of the sets of objects A and A' .

Proposition 5. *If S is a symmetrical structuring element and the negation $'$ associated with the residuated implication I is involutive, then for all $A \in L^X$*

$$GRAD_S(A) = (R_S)_{\exists}(A) * (R_S)_{\exists}(A')$$

Proof. If S is a symmetrical structuring element, then $(\varepsilon_S(A))' = \delta_S(A')$.

Therefore, the gradient can be obtained as

$$GRAD_S(A) = \delta_S(A) * \delta_S(A') = (R_S)_{\exists}(A) * (R_S)_{\exists}(A') \quad \square$$

The following example shows the applicability of the morphological gradient.

Example 4. In the context $(\mathbb{Z}, \mathbb{Z}, R_S)$ of the two dimensional 8-bits grayscale images, let us consider the initial image represented in Figure 8.



Fig. 8. Initial image

Taking the following fuzzy ball as the structuring element:

$$S(x, y) = \begin{cases} 1 - \frac{x^2 + y^2}{5^2 + 1} & \text{if } x^2 + y^2 \leq 5^2 \\ 0 & \text{otherwise} \end{cases}$$

and the negation $' : L \rightarrow L$ associated with the residuated implication I_L , which is defined by $x' = 1 - x$, for all $x \in L$, and it is involutive, the gradient of Figure 8 is shown in Figure 9. It represents the points belonging to the derived of the image A and to the derived of the negative image of A .

As in Example 3, if we use the crisp disk of radius 5 as structuring element, the result is clearly worse as it can be seen in Figure 10,



Fig. 9. Gradient



Fig. 10. Initial image with the clear-cut structuring element.

This notion can straightforwardly be translated to a general fuzzy property-oriented context (X, Y, R) .

Definition 7. Given a complete residuated lattice $(L, \vee, \wedge, *, I, 0, 1, \leq)$, a fuzzy

property-oriented context (X, Y, R) , an involutive negation $'$ on L and $A \in L^X$, the *gradient of the fuzzy subset of objects* A is defined as

$$\text{GRAD}_X(A) = R_{\exists}(A) * R_{\exists}(A')$$

Notice that an analogous notion related to a subset of attributes $B \in L^Y$ cannot be considered since the operator R^{\vee} must be used and, in this case, we have $R_{\vee}(B) * (R_{\vee}(B))'$ that coincides with the empty set in the classical case.

The next result establishes that the top-hat by closing of the L -fuzzy set A can be interpreted as the existing difference between the ‘upper closure’ of the image A and the complement of A .

Proposition 6. *In the fuzzy property-oriented context (X, X, R_S) , the top-hat by closing of the L -fuzzy set $A \in L^X$ can be obtained as*

$$\text{TH}_S^{\phi}(A) = (R_S)^{\vee}((R_S)_{\exists}(A)) * A'$$

Proof. Since $\text{TH}_S^{\phi}(A) = \phi_S(A) * A'$ and $\phi_S = \varepsilon_S \circ \delta_S$, applying Proposition 1, we obtain the proposed equality. \square

Analogously, the top-hat by opening of the L -fuzzy set A represents the existing difference between an initial set A and the intension of the property-oriented concept obtained from A , as the following result explains.

Proposition 7. *The top-hat by opening of the L -fuzzy set $A \in L^X$ can be obtained as*

$$\text{TH}_S^{\gamma}(A) = A * ((R_S)_{\exists}((R_S)^{\vee}(A)))'$$

Proof. The proof is straightforwardly obtained from the definition of the top-hat by opening $\text{TH}_S^{\gamma}(A) = A * (\gamma_S(A))'$, the equality $\gamma_S = \delta_S \circ \varepsilon_S$ and Proposition 1. \square

Example 5. Returning to the context in the previous example, the top-hat transforms of the initial image are in Figure 11.

In order to interpret an image by a property-oriented concept the top-hat transform provides an interesting procedure. Given an image $A \in L^X$, if A is considered as a subset of attributes in the fuzzy property-oriented context (X, X, R_S) , then the associated concept is $\mathcal{C}_1 = ((R_S)^{\vee}(A), (R_S)_{\exists}((R_S)^{\vee}(A)))$. Otherwise, if A is assumed as a subset of objects, then the associated concept is $\mathcal{C}_2 = ((R_S)^{\vee}((R_S)_{\exists}(A)), (R_S)_{\exists}(A))$. Therefore, top-hat by opening compares the original image with the first concept and top-hat by closing compares the original image with the second one.

Hence, in both previous cases, the top-hat transform provide us a tool to find the *most robust* sets in the context.

For example, in the previous example we can see that the initial image (Figure 4) is very similar to the extension (morphological opening) of the first concept and to

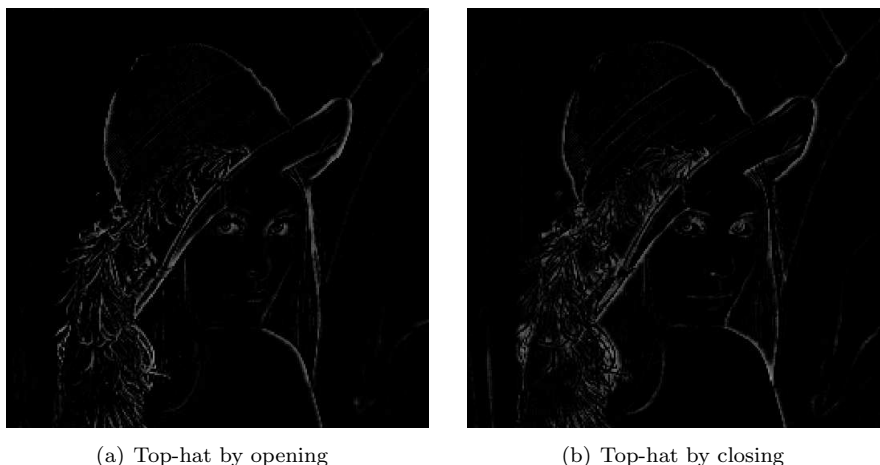


Fig. 11. Top-hat transforms

the intension (morphological closing) of the second one, since in both cases the obtained image only consists of a few points. Moreover, in order to chose the more representative concept, the second one will be chosen due to the top-hat by closing has less white points.

From these comments the top hat can be introduced in a general fuzzy property-oriented framework as follows.

Definition 8. Given a complete residuated lattice $(L, \vee, \wedge, *, I, 0, 1, \leq)$, a fuzzy property-oriented context (X, Y, R) , an involutive negation $'$ on L , $A \in L^X$ and $B \in L^Y$:

- the *object top hat of the fuzzy subset of objects* A is defined as

$$\text{TH}_X(A) = R^\vee (R_\exists(A)) * A'$$

- the *attribute top hat of the fuzzy subset of attributes* B is defined as

$$\text{TH}_Y(B) = B * (R_\exists(R^\vee(B)))'$$

5. Conclusions and future work

The usual procedure in mathematical morphology is, given a structuring element, obtaining the dilation and the erosion of an original image. This paper have studied the opposite problem, that is, given a fuzzy object $B: X \rightarrow L$ and a structuring element $S: X \rightarrow L$, find out the original object $A: X \rightarrow L$ such that B is the dilation of A , $\delta_S(A) = B$, or the erosion of A , $\varepsilon_S(A) = B$, and, if there exists, how to obtain it.

We have shown that this problem is associated with solving fuzzy relation equations. Therefore, the results given in ² and the relationship introduced in ¹⁴ have

been used to obtain the original image or a good approximation. Moreover, we have introduced some results focus on searching the whole set of possible original images.

Furthermore, the generalization of two important tools in mathematical morphology: gradient and top-hat transformations, have been presented and several properties of them have been introduced from the viewpoint of the analysis of fuzzy property-oriented concept lattices.

In the future, more tools, properties and applications will be studied in order to improve the existing mechanisms in image and signal processing, such as in object recognition. Moreover, we will study equations in which the input and output images are known and the structuring relation is unknown.

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