LINKING MATHEMATICAL MORPHOLOGY AND *L*-FUZZY CONCEPTS

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In this paper we study the relation between L-fuzzy morphology and L-fuzzy concepts over complete lattices. In particular, we show how the erosion and dilation operators of the former can be understood in terms of the derivation operators of the latter, even when the set of objects is different from the set of attributes.

Keywords: L-fuzzy concept analysis; L-fuzzy mathematical morphology; formal concept analysis; implication operators

1. Introduction

In recent years, both formal concepts analysis and mathematical morphology have attracted the interest of many researchers [1;2;3;4;5;6;7;8;9]. These theories have their origin at problems which are very different to each other: deriving a concept hierarchy or formal ontology from a collection of objects and their properties, in the case of the former [6] and image processing in the case of the latter [1]. However, when both theories are considered in the general framework of bounded lattices, it comes out that it exists a mathematical relation between them.

In this sense, in [10], the authors consider a bounded lattice L and use a residuated pair to define L-fuzzy erosion and L-fuzzy dilation. Using a structuring element [11;12] $R \in L^{X\times X}$, in [13] the authors understand an image as a relation and use their theoretical developments for representing different effects over an initial fuzzy image. As a consequence, in that paper authors provide conditions which allow to identify erosion and dilation morphological operations defined in terms of a given structuring element and a residuated pair, on the one hand, and derivations over an appropriate L-fuzzy context, on the other hand. However, in this case the set of objects and the set of attributes were the same. Namely, we only consider relations defined over $X \times X$.

Our main goal in this paper is to go one step further in the study of the link between both theories. In particular, our main objective is to link fuzzy mathematical morphology and L-fuzzy context theory without imposing that the set of objects and the set of attributes of the latter are the same. In particular, we want to show that erosion and dilation morphological operators can be understood as derivations in an appropriate L-fuzzy context and conversely.

It is worth to notice that the study of the equivalence between morphological operators and other constructions which are, in principle, far from image processing, is not new. For instance, the equivalence between some morphological operators and rough sets defined from a relation has been considered by Bloch in [14]. Furthermore, the equivalence between mathematical morphology and formal concepts theory in a crisp setting have also been considered [15]. And, in [16], the relationship between both theories is also analyzed, but from a different perspective, since the authors relate algebraic mathematical morphology and fuzzy property-oriented concept lattices.

To achieve this objective, we start defining in an appropriate way the notion of residuated pair which is going to be the basis for building the different operators. After that, and for X and Y and an L-fuzzy relation $R \in L^{X\times Y}$, we introduce the basic morphological operators [17;18]: erosion, dilation, opening and closing. In particular, the opening and closing operators will allow us to characterize some relevant sets: the R-open sets and the R-closed sets. Finally, all this developments lead us to our main result: the equivalence between erosion and dilation operator, on the one hand, and derivation operators, on the other hand.

Note that, since the corresponding crisp theories are just particular cases of the fuzzy ones, we get in a straightforward way a link between (crisp) mathematical morphology and (crisp) formal concept analysis [6].

This work is organized as follows: in Section 2, we show some preliminary notions about L-fuzzy concept analysis and L-fuzzy mathematical morphology. In Section 3, we carry out a study on L-fuzzy implication functions in order to define appropriate residuated pairs. Then, in Section 4, we use such residuated pairs to analyze the relation between L-fuzzy mathematical morphology and L-fuzzy concept analysis. In Section 5 we propose a practical case to illustrate the results. Finally we present some conclusions and comments on future lines of work.

2. Preliminaries

2.1. *Basic concepts*

In the following, we provide some basic definitions which are necessary for understanding the present paper. $(L, \leq)$ is a complete lattice with top element given by 1_L and bottom element given by 0_L. Given two sets A, B we denote by A^B the set of mappings from B to A.

Definition 1. Let L be a complete lattice. A strong negation on L is a decreasing and involutive function $' : L \to L$.

For all strong negation, $0'_L = 1_L$ and $1'_L = 0_L$. Moreover if $'$ is a strong negation in the complete lattice L, X is a set and $A \in L^X$, we denote by $A' \in L^X$ the mapping defined by $A'(x) = (A(x))'$ for every $x \in X$.

We recall now the notion of implication operator over a complete lattice L or L-fuzzy implication function [19;20;21].

Definition 2. Let L be a complete lattice. An L-fuzzy implication function is a mapping $I : L \times L \to L$ such that

(i) $I(0_L, 0_L) = I(0_L, 1_L) = I(1_L, 1_L) = 1_L$;
(ii) $I(1_L, 0_L) = 0_L$;
(iii) I is decreasing (with respect to $\leq_L$) in its first component;
(iv) I is increasing (with respect to $\leq_L$) in its second component.

Given an L-fuzzy implication funcion I, it can be used to defined a conjunctive operator $\mathcal{C}$ (i.e., a commutative, associative and increasing operator) as follows.

Definition 3. Let L be a complete lattice and $I : L \times L \to L$ be an L-fuzzy implication function right-continuous in its second component (i.e. it preserves infima in the second component). The conjunction operator $\mathcal{C} : L \times L \to L$ associated with I by adjointness is given by:

$$\mathcal{C}(x, y) = \inf\{z \in L \mid y \leq_L I(x, z)\}$$

for every $x, y \in L$. We say that the pair $(I, \mathcal{C})$ is a residuated pair.

Remark 1. As a consecuence of the right continuity of I, Definition 3 is equivalent to:

$$I(\alpha, \beta) = \sup\{\omega \in L \mid \mathcal{C}(\alpha, \omega) \leq \beta\}, \ \forall \alpha, \beta \in L \tag{1}$$

Definition 4. Let X, Y be two sets and L a complete lattice. An L-fuzzy relation over $X \times Y$ is a mapping in $L^{X \times Y}$; that is, a mapping $R : X \times Y \to L$.

Note that the usual notion of fuzzy relation [22] is just a particular case of L-fuzzy relation with $L = [0, 1]$.

¿From this point on, in order to simplify the notation, we use the following convention. Let L be a complete lattice and I be an L-fuzzy implication function right-continuous in its second argument. Let $N \subseteq L$. The set $\{I(x, y) \mid y \in N\}$ will be denoted by $I(x, N)$. If $N = \varnothing$, then $I(x, \varnothing) = \varnothing$. In the same way, for $M \subseteq L$, we denote by $I(M, y)$ the set $\{I(x, y) \mid x \in M\}$, and $I(\varnothing, y) = \varnothing$.

2.2. *Formal concept analysis*

The theory of formal concept analysis of R. Wille [6;7] extracts information from a binary table that represents a formal context (X, Y, R), with X and Y finite sets of objects and attributes, respectively, and $R \subseteq X \times Y$. The hidden information is obtained by means of the formal concepts, which are pairs $(A, B) \subseteq X \times Y$ and such that A and B are related by means of the so-called derivation operator that associates an object set A with the attributes related to the elements of A (and, respectively, an attribute set B with the objects related to the elements of B), (see [6]). In this way, formal concepts can be interpreted as a group of objects A that shares the attributes of B.
In [8;23;24], the notion of an L-fuzzy context was introduced as follows.

Definition 5. An L-fuzzy context is an algebraic system (L, X, Y, R), where

(i) L a complete lattice;
(ii) X and Y are two (non-empty) sets, called set of objects and set of attributes, respectively;
(iii) R is an L-fuzzy relation between the set of objects and the set of attributes, which is called the incidence relation.

Remark 2. Definition 5 provides an extension of Wille's formal contexts when we want to study the relationship between the objects and the attributes with values in a complete lattice L, instead of binary ones. Other generalizations of formal concepts analysis using residuated implication operators are due to R. Belohlavek [25;26;27] and S. Pollandt [9].

In order to define L-fuzzy concepts, we also need to introduce the notion of a derivation operator which is going to connect objects and attributes. This can be done as follows.

Definition 6. Let X, Y be two sets, L a complete lattice, $R \in L^{X\times Y}$ an L-fuzzy relation and $I : L \times L \to L$ an L-fuzzy implication function.

(i) The first derivation operator $D_1^R : L^X \to L^Y$ is defined by

$$D_1^R(A)(y) = \inf_{x\in X}\{I(A(x), R(x, y))\}$$

for every $A \in L^X$.

(ii) The second derivation operator $D_2^R : L^Y \to L^X$ is defined by

$$D_2^R(B)(x) = \inf_{y\in Y}\{I(B(y), R(x, y))\}$$

for every $B \in L^Y$.

(iii) The operator D_{12}^R is defined by the composition $D_2^R \circ D_1^R$

In order to avoid notational complexity, if $A \in L^X$ and $B \in L^Y$, we denote $A_1 = D_1^R(A)$ and $B_2 = D_2^R(B)$.

Remark 3.

(i) In Definition 6, A_1 can be understood as a representation of the attributes related to the objects of A. Analogously, B_2 can be understood as a representation of the objects related to all the attributes of B.

(ii) Although Definition 6 is provided in terms of a general L-fuzzy implication function, in the following we assume that, unless otherwise stated, I is a residuated L-fuzzy implication function.

Finally, in order to define the L-fuzzy concept lattice, we recall that for a given mapping $f : X \to X$, the set of fixed points of f is

$$fix(f) = \{x \in X \mid f(x) = x\}.$$

Then the following result holds.

Proposition 1.[8;23] *Let L be a complete lattice, $R \in L^{X\times Y}$ be an L-fuzzy relation and $I : L \times L \to L$ be an L-fuzzy implication function. Then, the set $\mathcal{L} = \{(A, A_1) \mid A \in \mathrm{fix}(D_{12}^R)\}$ with the order relation $\leq$ defined as:*

$$(A, A_1) \leq (C, C_1) \text{ if } A \leq C \quad (\text{ or, equivalently, } A_1 \geq C_1) \tag{2}$$

is a complete lattice, where $\leq$ denotes the usual (pointwise) ordering between mappings.

Definition 7.[8;23] Let L be a complete lattice, $R \in L^{X\times Y}$ be an L-fuzzy relation and $I : L \times L \to L$ be an L-fuzzy implication function. Then $\mathcal{L}$ constructed as in Proposition 1 is called the L-fuzzy concept lattice.

The information stored in the context is shown by means of the L-fuzzy concepts, which are the pairs $(A, A_1) \in L^X \times L^Y$ with $A \in \mathrm{fix}(\varphi)$, where $\varphi(A) = (A_1)_2$. These pairs, whose first and second components are said to be the fuzzy extension and intension respectively, represent a set of objects that shares a set of attributes.

Other extensions of formal concept analysis to the interval-valued case are in[28;29;30] and to the fuzzy property-oriented concept lattice framework in[31;32;33]. Also we can see a study of L-fuzzy contexts using two relations in[34].

Example 1. A very interesting particular case of L-fuzzy contexts appears trying to analyze situations where the object and the attribute sets are the same[35], that is, L-fuzzy contexts (L, X, X, R) with $R \in L^{X \times X}$ (this relation can be reflexive, symmetrical ...). In these situations, the L-fuzzy concepts are pairs (A, B) such that $A, B \in L^X$.

These are the L-fuzzy contexts that were used in[10] to obtain the main results of the work. Moreover, in the case of the use of the L-fuzzy contexts $(L, \mathbb{R}^n, \mathbb{R}^n, R)$ or $(L, \mathbb{Z}^n, \mathbb{Z}^n, R)$, the L-fuzzy concepts (A, B) are interpreted as signal or image pairs or digital versions of these signals or images, respectively.

2.3. *Mathematical morphology*

Mathematical morphology is a theory concerned with the processing and analysis of images or signals using filters and operators that modify them. The fundamentals of this theory (initiated by G. Matheron[36;37] and J. Serra[1]), are in affine space theory, integral geometry and lattice algebra. Nowadays, this methodology is used in general contexts related to activities as information extraction in digital images, noise elimination or pattern recognition[38;39].

Mathematical morphology was originally conceived for the processing of binary images and later extended to gray-scale images[40;1]. This theory defines some tools (morphological filters) for image processing and computer vision. These morphological filters are obtained by means of two basic operators called erosion and dilation, that are defined in the case of binary images with the sum and difference of Minkowski[1], respectively.

The previous approach for binary images was extended into a more general framework, the Fuzzy mathematical morphology[41;17;18;42;2;43;44;5;45] with some links with the gray-scale mathematical morphology. There are also extensions of fuzzy mathematical morphology based on discrete t-norms in[46], and to bipolar or interval-valued fuzzy sets in[47;48].

One of the main papers about this extension of mathematical morphology is due to P. Sussner et al.[49]. These authors investigate a number of theoretical aspects of L-fuzzy mathematical morphology. The paper studies interval-valued and intuitionistic fuzzy mathematical morphology as special cases of these L-fuzzy mathematical morphology.

Furthermore, we can take $(L_1, \leq_1)$ and $(L_2, \leq_2)$ complete lattices and define the filters as operators $F : L_1 \to L_2$ with properties related to the order in these lattices[1;50;44]. Then, an erosion is defined as an operator $\varepsilon : L_1 \longrightarrow L_2$ that commutes with the infimum operator and a dilation $\delta : L_1 \longrightarrow L_2$ as an operator that commutes with the supremum:

$$\varepsilon(\bigwedge Y) = \bigwedge_{y\in Y} \varepsilon(y), \delta(\bigvee Y) = \bigvee_{y\in Y} \delta(y), \forall Y \subseteq L_1$$

In the literature (see[42;17;51]), fuzzy erosion and fuzzy dilation operators are introduced associated with the residuated pair $(I, \mathcal{C})$. Although this paper uses the extensions to L-fuzzy sets of the definitions given by Bloch and Maître[17], there is another extension due to DeBaets[42]. The general framework proposed by De Baets (see[52] for a nice compilation) considers a general fuzzy conjunction and a general fuzzy implication function in the fuzzy dilation and erosion, respectively. Indeed, in some applications, a residuated pair is not the best choice (see[53]).

In the lattice morphological setting, an image is just a mapping $A : X \to L$ from a set X to a complete lattice L. The set X is always taken to be either $\mathbb{R}^2$ or $\mathbb{Z}^2$. Let us start with some notions of this theory.

Definition 8. Let L be a complete lattice and X be a set. A structuring element (over X) is an element $S \in L^X$.

The structuring elements are the building blocks for constructing L- fuzzy erosion and L-fuzzy dilation operators, as follows.

Definition 9. Let S be a structuring element (over a set X), let L be a complete lattice and let $(I, \mathcal{C})$ be a residuated pair in L.

(i) The L-fuzzy erosion operator associated with S is the mapping $\varepsilon_S : L^X \to L^X$, where, for each $A \in L^X$, $\varepsilon_S(A) \in L^X$ is defined as:

$$\varepsilon_S(A)(x) = \inf\{I(S(y-x), A(y)) \mid y \in X\} \quad \forall x \in X$$

(ii) The L-fuzzy dilation operator associated with S is the mapping $\delta_S : L^X \to L^X$, where, for each $A \in L^X$, $\delta_S(A) \in L^X$ is defined as:

$$\delta_S(A)(x) = \sup\{\mathcal{C}(S(x-y), A(y)) \mid y \in X\} \quad \forall x \in X$$

Note that we have the following relation between erosion and dilation.

Proposition 2. *Let X be a set. Let L be a totally ordered lattice. Let's denote by $\leq$ the order obtained in L^X by extending the order $\leq_L$. Then the pair $(\varepsilon_S, \delta_S)$ satisfies that*

$$\delta_S(A) \leq B \Longleftrightarrow A \leq \varepsilon_S(B), \forall A, B \in L^X \tag{3}$$

Remark 4. Pairs of operators which verify Eq. 3 in a given complete lattice are said to be an adjunction in that lattice.

2.4. *The relation between L-fuzzy mathematical morphology and L-fuzzy contexts*

In this subsection we make a short review of some already existing results which link L-fuzzy mathematical morphology and L-fuzzy contexts theory.

In [10], the authors define L-fuzzy erosion and L-fuzzy dilation of images associated with a residuated pair $(I, \mathcal{C})$ in $([0,1], X, X, R_S)$, with the set X equal to $\mathbb{R}^n$ or to the digitalized space $\mathbb{Z}^n$. In particular, in that paper $R_S \in [0,1]^{X\times X}$ is a fuzzy relation associated with a structuring image $S \in [0,1]^X$ and it is defined as

$$R_S(x,y) = S(x-y), \forall (x,y) \in X \times X \qquad (4)$$

In [13], the authors make use of structuring elements $R \in L^{X\times X}$ for representing different effects over an initial fuzzy image $A \in L^X$ with appropriate residuated pairs. In this framework, fuzzy erosion and dilation operators associated with the pair $(I, \mathcal{C})$ were defined as follows:

$$\begin{aligned}\varepsilon_R(A)(x) &= \inf\{I(R(y,x), A(y)) \mid y \in X\}\\ &= \inf\{I(R^{op}(x,y), A(y)) \mid y \in X\},\ \forall x \in X\\ \delta_R(A)(x) &= \sup\{\mathcal{C}(R(x,y), A(y)) \mid y \in X\},\ \forall x \in X\end{aligned}$$

One of the main results presented there, which sets up the relation between L-fuzzy mathematical morphology and L-fuzzy concept analysis, is the following.

Theorem 1. *Let (L, X, R) be the triple associated with the structuring element $R \in L^{X\times X}$. Let $'$ be a strong negation in L and I a fuzzy implication operator verifying the contrapositive symmetry. Let (L, X, X, R') be the L-fuzzy context whose incidence relation $R' \in L^{X\times X}$ is such that for all $(x,y) \in X \times X$, $R'(x,y) = (R(x,y))'$. Then, L-fuzzy erosion ε_R and L-fuzzy dilation δ_R operators in (L, X, R) are related to derivation operators $D_1^{R'}$ and $D_2^{R'}$ in the L-fuzzy context (L, X, X, R') by:*

$$\begin{aligned}\varepsilon_R(A) &= D_1^{R'}(A') \qquad \forall A \in L^X\\ \delta_R(A) &= (D_2^{R'^{op}}(A))' \quad \forall A \in L^X\end{aligned}$$

As we have already stated in the introduction, our goal in this paper is to extend this result to cover the case of structuring elements defined in $L^{X\times Y}$ with $X \neq Y$.

3. Construction of a Residuated Pair in a Complete Lattice

In order to relate the operators used in mathematical morphology and those used in formal concept analysis, a crucial step is the definition of an appropriate residuated pair. In this section, and starting from an operator which does not need to be a fuzzy implication function but which satisfies suitable properties, we show how we can arrive at the desired residuated pair. Our starting point is the following results where, for the moment, we do not require to deal with an implication.

Proposition 3. *Let $(L, \leq)$ be a complete lattice and let $I : L \times L \longrightarrow L$ be an internal operation in L such that:*

$$I(\alpha, I(\beta, \nu)) \leq I(\beta, I(\alpha, \nu)), \quad \forall \alpha, \beta, \nu \in L \tag{5}$$

$$\exists \boldsymbol{e} \in L : \boldsymbol{e} \leq I(\alpha, \beta) \text{ iff } \alpha \leq \beta, \quad \forall \alpha, \beta \in L \tag{6}$$

Then operation I verifies the following properties.

(I1) The exchange principle:

$$I(\alpha, I(\beta, \nu)) = I(\beta, I(\alpha, \nu)), \quad \forall \alpha, \beta, \nu \in L \tag{7}$$

(I2) $\alpha \leq I(\beta, \nu) \Longleftrightarrow \beta \leq I(\alpha, \nu)$.

(I3) $I(0_L, \beta) = 1_L, \forall \beta \in L$. *Specifically,* $I(0_L, 0_L) = I(0_L, 1_L) = 1_L$.

(I4) $\alpha \leq I(I(\alpha, \beta), \beta), \quad \forall \alpha, \beta \in L$.

(I5) The element $\boldsymbol{e}$ of (6) is a left neutral element for I and it is unique.

(I6) The element $\boldsymbol{e}$ is equal to 0_L, if and only if, $|L| = 1$.

(I7) The operation I is decreasing in the first argument:

$$\alpha_1 \leq \alpha_2 \Longrightarrow I(\alpha_1, \beta) \geq I(\alpha_2, \beta), \forall \beta \in L \tag{8}$$

and $I(\sup M, \beta) = \inf I(M, \beta), \forall M \subseteq L, \forall \beta \in L$ *holds.*

(I8) $\boldsymbol{e} = \min\{I(\alpha, \alpha) \mid \alpha \in L\} = \min\{I(\alpha, \beta) \mid \alpha, \beta \in L \text{ and } \alpha \leq \beta\}$.

(I9) $I(1_L, \alpha) \leq \alpha, \forall \alpha \in L$. *Specifically,* $I(1_L, 0_L) = 0_L$.

(I10) $I(\alpha, 0_L) = 0_L, \forall \alpha \geq \boldsymbol{e}$.

(I11) $I(\alpha, \beta) = I(I(I(\alpha, \beta), \beta), \beta), \ \forall \alpha, \beta \in L$.

Proof:

(I1) Interchanging α and β in (5) we obtain the opposite inequality, so the following equality holds:

$$I(\alpha, I(\beta, \nu)) = I(\beta, I(\alpha, \nu)), \ \forall \alpha, \beta, \nu \in L.$$

(I2) Suppose that $\alpha \leq I(\beta, \nu)$ is verified, then applying property (6), we know that $\boldsymbol{e} \leq I(\alpha, I(\beta, \nu))$ and, as a result, $\boldsymbol{e} \leq I(\beta, I(\alpha, \nu))$. So $\beta \leq I(\alpha, \nu)$.

(I3) From the inequality $0_L \leq I(1_L, \beta), \forall \beta \in L,$ and from (I2) we can deduce $1_L \leq I(0_L, \beta), \forall \beta \in L$ and therefore, $1_L = I(0_L, \beta), \forall \beta \in L$. As a result, as particular cases, if we take $\beta = 0_L$ and $\beta = 1_L$ we obtain that $I(0_L, 0_L) = I(0_L, 1_L) = 1_L$.

(I4) From $I(\alpha, \beta) \leq I(\alpha, \beta)$ we deduce $\boldsymbol{e} \leq I(I(\alpha, \beta), I(\alpha, \beta))$, that is, $\boldsymbol{e} \leq I(\alpha, I(I(\alpha, \beta), \beta))$ and therefore $\alpha \leq I(I(\alpha, \beta), \beta)$.

(I5) From $\alpha \leq \alpha$ we prove $\boldsymbol{e} \leq I(\alpha, \alpha)$, then $\alpha \leq I(\boldsymbol{e}, \alpha), \forall \alpha \in L$. On the other hand, by the previous paragraph, $\boldsymbol{e} \leq I(I(\boldsymbol{e}, \alpha), \alpha), \forall \alpha \in L$, which proves that $I(\boldsymbol{e}, \alpha) \leq \alpha$. We can conclude that $I(\boldsymbol{e}, \alpha) = \alpha, \forall \alpha \in L$ and so, $\boldsymbol{e}$ is a left neutral element for I.

For the uniqueness of an identity element, suppose that there is another $\boldsymbol{e}_1$ with the same property, then it also would be a left neutral element. Then, we have $\boldsymbol{e}_1 = I(\boldsymbol{e}, \boldsymbol{e}_1), \boldsymbol{e} = I(\boldsymbol{e}_1, \boldsymbol{e})$; so, $\boldsymbol{e} \leq \boldsymbol{e}_1$ and $\boldsymbol{e}_1 \leq \boldsymbol{e}$ which proves the equality $\boldsymbol{e} = \boldsymbol{e}_1$.

(I6) Suppose that $\boldsymbol{e} = 0_L$. Let be $\alpha, \beta \in L$, then, from $0_L \leq I(\alpha, \beta)$ and $0_L \leq I(\beta, \alpha)$, we deduce that $\alpha \leq \beta$ and $\beta \leq \alpha$, that is, $\alpha = \beta$. So, L has an only element, that is, $|L| = 1$.
On the other hand, if $|L| = 1$, then the property (6) is trivially verified.

(I7) In the first place, we see that I is decreasing in the first argument. Suppose that $\alpha_1 \leq \alpha_2$. From $\alpha_2 \leq I(I(\alpha_2, \beta), \beta), \forall \beta \in L$, it is proved that $\alpha_1 \leq I(I(\alpha_2, \beta), \beta), \forall \beta \in L$, and, as a result, $\boldsymbol{e} \leq I(\alpha_1, I(I(\alpha_2, \beta), \beta)), \forall \beta \in L$, that is, $\boldsymbol{e} \leq I(I(\alpha_2, \beta), I(\alpha_1, \beta))$. Therefore $I(\alpha_2, \beta) \leq I(\alpha_1, \beta)$.
Consider $M \subseteq L$, we can see that $I(\sup M, \beta) = \inf I(M, \beta), \forall M \subseteq L$. If $M = \varnothing$, then the property is trivial since the set $I(\varnothing, \beta)$ is just $\varnothing$ and $\sup \varnothing = 0_L, \inf \varnothing = 1_L$, then:

$$I(\sup \varnothing, \beta) = I(0_L, \beta) = 1_L = \inf \varnothing = \inf I(\varnothing, \beta).$$

If $M \neq \varnothing$: From $m \leq \sup M, \forall m \in M$, we can deduce that

$$I(\sup M, \beta) \leq I(m, \beta), \forall m \in M, \forall \beta \in L$$

then $I(\sup M, \beta)$ is a lower bound of $I(M, \beta)$. Let $\lambda \in L$ be another lower bound of $I(m, \beta)$, i.e., $\lambda \leq I(m, \beta), \forall m \in M$. It is verified that $m \leq I(\lambda, \beta), \forall m \in M$ and, as a result, $\sup M \leq I(\lambda, \beta)$ that is equivalent to $\lambda \leq I(\sup M, \beta)$. This expression proves that $I(\sup M, \beta)$ is the greatest lower bound, that is,

$$I(\sup M, \beta) = \inf I(M, \beta).$$

(I8) We have that $\boldsymbol{e} \leq \inf\{I(\alpha, \alpha) \mid \alpha \in L\}$ and $\boldsymbol{e} \leq \inf\{I(\alpha, \beta) \mid \alpha, \beta \in L \text{ and } \alpha \leq \beta\}$. Moreover, as $\boldsymbol{e} = I(\boldsymbol{e}, \boldsymbol{e})$, $\boldsymbol{e}$ is minimum

$$\boldsymbol{e} = \min\{I(\alpha, \alpha) \mid \alpha \in L\} = \min\{I(\alpha, \beta) \mid \alpha, \beta \in L \text{ and } \alpha \leq \beta\}$$

(I9) From $\boldsymbol{e} \leq 1_L$ we deduce that $I(1_L, \alpha) \leq I(\boldsymbol{e}, \alpha) = \alpha, \forall \alpha \in L$. In particular, $I(1_L, 0_L) \leq 0_L$, and so $I(1_L, 0_L) = 0_L$.

(I10) If $\boldsymbol{e} \leq \alpha$, then $I(\alpha, 0_L) \leq I(\boldsymbol{e}, 0_L) = 0_L$. So, $I(\alpha, 0_L) = 0_L$.

(I11) As we prove in I4, it is verified that $I(\alpha, \beta) \leq I(I(I(\alpha, \beta), \beta), \beta)$.
We now prove the other inequality. From $\alpha \leq I(I(\alpha, \beta), \beta)$ and (I7) we deduce that $I(\alpha, \beta) \geq I(I(I(\alpha, \beta), \beta), \beta)$, then the equality holds.

□

Remark 5. The advantage of the previous proposition is that it allows us to consider, in principle, more general applications than L-fuzzy implication functions. Note that, in fuzzy logic, for the complete lattice L it is always assumed that $|L| > 1$ so, in particular, $\mathbf{2} = \{0_L, 1_L\}$ is included in L.

In this work, we assume that the elements 0_L and $\boldsymbol{e}$ can be different. We use the chain $F = \{0_L, \boldsymbol{e}\} \subseteq L$ to represent the so-called "flat-sets" of a referential set X.

These flat sets can be considered as maps $\psi : X \longrightarrow F$ and they are very useful in mathematical morphology. In the case of $\boldsymbol{e} = 1_L$, that is, if $F = \mathbf{2}$, the "flat-sets" are the "crisp sets".

Now, the key point is the assumption that the so-called contrapositive symmetry with respect to a strong negation $'$ holds, i.e.,

$$I(\alpha, \beta) = I(\beta', \alpha'), \ \forall \alpha, \beta \in L \tag{9}$$

We prove next that if we use a strong negation and Equation 9 is verified, then I is an L-fuzzy implication function.

Proposition 4. *Let L be a complete lattice and $' : L \longrightarrow L$ be a strong negation in L. Consider a mapping $I : L \times L \to L$ such that (5), (6) and (9) properties are verified. Then:*

(i) I is an L-fuzzy implication function which is right continuous in the second argument.
(ii) $I(\alpha, \boldsymbol{e}^{'}) = \alpha^{'}$, $I(\alpha, 0_L) \leq \alpha^{'}$, $\forall \alpha \in L$.

Proof:

(i) By the results of Proposition 3, to prove that I is an implication operator, we have to see that it is increasing in the second argument. Suppose that $\beta_1 \leq \beta_2$, then $\beta_2' \leq \beta_1'$, and so

$$I(\alpha, \beta_1) = I(\beta_1', \alpha') \leq I(\beta_2', \alpha') = I(\alpha, \beta_2)$$

which proves that it is increasing in the second argument and, as a result, an L-fuzzy implication function.
Let be now $\alpha \in L$, $I(\alpha, 1_L) = I(0_L, \alpha')$ by *(9)* and then $I(\alpha, 1_L) = I(0_L, \alpha') = 1_L$ by Proposition 3 (I3). As a particular case, $I(1_L, 1_L) = 1_L$.
Take now $N \subseteq L$. If $N = \varnothing$, then $\inf \varnothing = 1_L$ and $I(\alpha, \varnothing) = \varnothing$, so

$$I(\alpha, \inf \varnothing) = I(\alpha, 1_L) = 1_L = \inf \varnothing = \inf I(\alpha, \varnothing)$$

If $N \neq \varnothing$, as $(\inf N)' = \sup N'$ since $'$ is a strong negation, then

$$\begin{aligned} I(\alpha, \inf N) = & I((\inf N)', \alpha') = I(\sup N', \alpha') = \inf I(N', \alpha') = \\ = & \inf I(\alpha, N) \end{aligned}$$

(ii) $\forall \alpha \in L$, it is verified that $I(\alpha, \boldsymbol{e}^{'}) = I(\boldsymbol{e}, \alpha^{'}) = \alpha^{'}$. Moreover, $I(\alpha, 0_L) \leq I(\alpha, \boldsymbol{e}^{'}) = \alpha^{'}$.

□

Now we can define the operation $\mathcal{C}$ given in Definition 3 and we have the following result:

Proposition 5. *In the setting of Proposition 4, the pair $(I, \mathcal{C})$ verifies:*

$$\mathcal{C}(\alpha, \beta) \leq \sigma \Longleftrightarrow \beta \leq I(\alpha, \sigma) \tag{10}$$

Proof:

$\Longrightarrow$)Suppose that $\mathcal{C}(\alpha,\beta)\leq\sigma$, then $\omega\leq\sigma$ for some ω such that $\beta\leq I(\alpha,\omega)$. From the monotonicity in the second argument, $I(\alpha,\omega)\leq I(\alpha,\sigma)$. Therefore, $\beta\leq I(\alpha,\omega)\leq I(\alpha,\sigma)$, as we wanted to see.
$\Longleftarrow$) Consider $\beta\leq I(\alpha,\sigma)$, then $\sigma\in\{\omega\in L\mid\beta\leq I(\alpha,\omega)\}$ and so,

$$\mathcal{C}(\alpha,\beta)=\inf\{\omega\in L\mid\beta\leq I(\alpha,\omega)\}\leq\sigma$$

□

We are going to see now that, in fact, $\mathcal{C}$ provides us with a residuated pair.

Proposition 6. *Let $\mathcal{C}$ be defined in Definition 3. Then $\mathcal{C}:L\times L\to L$ verifies the following properties:*

(i) It is commutative.
(ii) It is associative.
(iii) $\boldsymbol{e}$ is the neutral element: $\mathcal{C}(\alpha,\boldsymbol{e})=\mathcal{C}(\boldsymbol{e},\alpha)=\alpha$, $\forall\alpha\in L$.
(iv) $\mathcal{C}(\alpha,0_L)=0_L,\ \forall\alpha\in L$.
(v) It is increasing in both arguments.
(vi) It is left-continuous in both arguments.

Proof:

(i) $\mathcal{C}(\alpha,\beta)=\inf\{\omega\mid\beta\leq I(\alpha,\omega)\}=\inf\{\omega\mid\alpha\leq I(\beta,\omega)\}=\mathcal{C}(\beta,\alpha),\ \forall\alpha,\beta\in L$.
(ii) It is verified that $\forall\alpha,\beta,\gamma\in L$:

$$\begin{aligned}&\mathcal{C}(\alpha,\mathcal{C}(\beta,\gamma))=\mathcal{C}(\alpha,\mathcal{C}(\gamma,\beta))=\inf\{\omega\mid\mathcal{C}(\gamma,\beta)\leq I(\alpha,\omega)\}=\\&\inf\{\omega\mid\beta\leq I(\gamma,I(\alpha,\omega))\}=\inf\{\omega\mid\beta\leq I(\alpha,I(\gamma,\omega))\}=\\&\inf\{\omega\mid\mathcal{C}(\alpha,\beta)\leq I(\gamma,\omega)\}=\mathcal{C}(\gamma,\mathcal{C}(\alpha,\beta))=\mathcal{C}(\mathcal{C}(\alpha,\beta),\gamma)\end{aligned}$$

(iii) $\mathcal{C}(\alpha,\boldsymbol{e})=\mathcal{C}(\boldsymbol{e},\alpha)=\inf\{\omega\mid\alpha\leq I(\boldsymbol{e},\omega)\}=\inf\{\omega\mid\alpha\leq\omega\}=\alpha,\ \forall\alpha\in L$.
(iv) $\mathcal{C}(\alpha,0_L)=\inf\{\omega\mid 0_L\leq I(\alpha,\omega)\}=\inf L=0_L,\ \forall\alpha\in L$.
(v) Immediate by proposition 5.
(vi) Immediate by proposition 5.

□

L-fuzzy implication functions and conjunctions have been also studied by other authors in[49]. Moreover, the operation $\mathcal{C}$ can also be also expressed in terms of I and a strong negation $'$:

Proposition 7. *If $(I,')$ verifies* (5), (6) *and* (9), *then:*

$$\mathcal{C}(\alpha,\beta)=(I(\alpha,\beta'))',\ \forall\alpha,\beta\in L \tag{11}$$

Proof: Let be $\alpha, \beta \in L$,

$$\begin{aligned}&\mathcal{C}(\alpha,\beta) = \inf\{\omega \mid \beta \leq I(\alpha,\omega)\} = \inf\{\omega \mid \beta \leq I(\omega',\alpha')\} = \\ &\inf\{\omega \mid \omega' \leq I(\beta,\alpha')\} = \inf\{\omega \mid \omega' \leq I(\alpha,\beta')\} = \\ &\inf\{s' \mid s \leq I(\alpha,\beta')\} = (\sup\{s \mid s \leq I(\alpha,\beta')\})' = (I(\alpha,\beta'))'\end{aligned}$$

□

Remark 6. In the literature, if $L = [0,1]$, then an operation $\mathcal{C} : [0,1] \times [0,1] \longrightarrow [0,1]$ that is associative, commutative, has a neutral element, and is increasing in both arguments, is said to be a *uninorm*[54]. If also $\mathcal{C}(1,0) = 0$, then it is said to be a *conjunctive uninorm.* We use this definition in the more general case of complete lattices. As a result, the operator $\mathcal{C} : L \times L \longrightarrow L$ associated with the pair $(I,')$ verifying (5), (6) and (9) properties is a conjunctive uninorm in L with neutral element $\boldsymbol{e}$. Moreover, it is left-continuous. It plays the role of a conjunction in L. Note that if $\boldsymbol{e} = 1$, the uninorm is a t-norm. One can find some papers that characterize the uninorms in $L = [0,1]$[54;55], in finite chains[56] and in complete lattices[57;58].

Next, we see some examples of these operators $(I, \mathcal{C})$.

Example 2. Let be $L = [0,1]$ or a finite chain $L = \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\}$. Consider the Lukasiewicz implication $I(\alpha,\beta) = \min(1, 1-\alpha+\beta)$, $\forall \alpha,\beta \in L$, with the Zadeh negation $\alpha' = 1-\alpha, \forall \alpha \in L$, if $L = [0,1]$, or the only strong negation if L is a finite chain. We obtain the operator $\mathcal{C}$ such that $\mathcal{C}(\alpha,\beta) = \max(0, \alpha+\beta-1)$ that is a t-norm. The neutral element is $\boldsymbol{e} = 1$. These implication and t-norm are well known in the literature[22]. In this case, flat sets are the same as crisp sets since $F = \{0,1\} = \mathbf{2}$.

Example 3. Let $L = [0,1]$, with the negation of Zadeh. Let I be:

$$I(\alpha,\beta) = \begin{cases} 1 & if\ (\alpha,\beta) \in \{(0,0),(1,1)\} \\ \frac{(1-\alpha)\beta}{(1-\alpha)\beta+\alpha(1-\beta)} & \text{otherwise} \end{cases}$$

The operation $\mathcal{C}$ is *the triple* Π *operator*[59]

$$\mathcal{C}(\alpha,\beta) = \begin{cases} 0 & if\ (\alpha,\beta) \in \{(0,1),(1,0)\} \\ \frac{\alpha\beta}{\alpha\beta+(1-\alpha)(1-\beta)} & \text{otherwise,} \end{cases}$$

which is a conjunctive uninorm with neutral element $\boldsymbol{e} = 1/2$. In this case, $F = \{0, 1/2\} \neq \{0,1\}$. So, a nonvoid flat set $\psi : X \longrightarrow F$ is not a crisp set. This example appears in[60].

Example 4. If $(L, \leq)$ is a complete Boolean Algebra with the negation $\alpha' = \alpha^c$, then:

$$I(\alpha,\beta) = \alpha^c \vee \beta,\ \forall \alpha,\beta \in L$$

The operation $\mathcal{C}$ is $\mathcal{C}(\alpha,\beta) = \alpha \wedge \beta$. The conjunction is a t-norm in L with neutral element $\boldsymbol{e} = 1$ ([22]). So the flat sets $\psi : X \longrightarrow F$ are crisp sets.

Example 5. Consider a pair $(I, ')$ defined in $[0,1]$ verifying (5), (6) and (9). Consider also $\mathcal{J}([0,1])$, the set of intervals on $[0,1]$, with the strong negation $'$ defined for intervals $[\alpha] = [\underline{\alpha}, \overline{\alpha}]$ as $[\alpha]' = [\overline{\alpha}', \underline{\alpha}']$, and the operation $\widehat{I} : \mathcal{J}([0,1]) \times \mathcal{J}([0,1]) \rightarrow \mathcal{J}([0,1])$, which extends I to $\mathcal{J}([0,1])$:

$$\widehat{I}([\alpha],[\beta]) = [I(\underline{\alpha},\underline{\beta}) \wedge I(\overline{\alpha},\overline{\beta}), I(\underline{\alpha},\overline{\beta})]$$

Here we are considering the order $[\underline{\alpha},\overline{\alpha}] \leq_L [\underline{\beta},\overline{\beta}]$ iff $\underline{\alpha} \leq \underline{\beta}$ and $\overline{\alpha} \leq \overline{\beta}$. Then, the pair $(\widehat{I}, ')$ also verifies (5), (6) and (9). The associated operation $\widehat{\mathcal{C}}$ is an extension of $\mathcal{C}$ given by:

$$\widehat{\mathcal{C}}([\alpha],[\beta]) = [\mathcal{C}(\underline{\alpha},\underline{\beta}), \mathcal{C}(\underline{\alpha},\overline{\beta}) \vee \mathcal{C}(\overline{\alpha},\underline{\beta})]$$

These operators are known in the literature as the *optimistic implication* and the *pessimistic conjunction* (See[61]).

4. *L*-fuzzy Mathematical Morphology and *L*-fuzzy Concept Analysis

Let L be a complete lattice. Once we have the residuated pair $(I, \mathcal{C})$, consisting of an L-fuzzy implication function I and a strong negation satisfying the conditions of Proposition 4, and the operation $\mathcal{C}$ defined by Definition 3, we can define L-fuzzy erosion and L-fuzzy dilation as follows:

Definition 10. Let X, Y be two sets and $R \in L^{X\times Y}$ a structuring element.

(i) The L-fuzzy erosion operator associated with R is a mapping $\varepsilon_{1R} : L^X \rightarrow L^Y$, such that, for each $A \in L^X$, $\varepsilon_{1R}(A)$ is given by:

$$\varepsilon_{1R}(A)(y) = \inf\{I(R(x,y), A(x)) \mid x \in X\} \quad \forall y \in Y \tag{12}$$

(ii) The L-fuzzy dilation operator associated with R is a mapping $\delta_{1R} : L^X \rightarrow L^Y$ such that, for each $A \in L^X$, $\delta_{1R}(A) \in L^Y$ is given by:

$$\delta_{1R}(A)(y) = \sup\{\mathcal{C}(R(x,y), A(x)) \mid x \in X\} \quad \forall y \in Y \tag{13}$$

In the same way, we can define L-fuzzy erosion and L-fuzzy dilation operators with domain L^Y and range L^X by

$$\varepsilon_{2R}(B)(x) = \inf\{I(R(x,y), B(y)) \mid y \in Y\} \quad \forall x \in X \tag{14}$$

$$\delta_{2R}(B)(x) = \sup\{\mathcal{C}(R(x,y), B(y)) \mid y \in Y\} \quad \forall x \in X \tag{15}$$

for every $B \in L^Y$.

These definitions are a reinterpretation of the operators used in[31].

Remark 7. If the structuring element R is a flat set, that is, $R(x,y) \in \{0_L, \boldsymbol{e}\}$ $\forall (x,y) \in X \times Y$, then the expressions for L-fuzzy erosion and L-fuzzy dilation are simplified:

$$\varepsilon_{1R}(A)(y) = \inf\{A(x) \mid x \in X \text{ and } R(x,y) = \boldsymbol{e}\} \quad \forall y \in Y$$
$$\delta_{1R}(A)(y) = \sup\{A(x) \mid x \in X \text{ and } R(x,y) = \boldsymbol{e}\} \quad \forall y \in Y$$

And, analogously,

$$\varepsilon_{2R}(B)(x) = \inf\{B(y) \mid y \in Y \text{ and } R(x,y) = \boldsymbol{e}\} \quad \forall x \in X$$
$$\delta_{2R}(B)(x) = \sup\{B(y) \mid y \in Y \text{ and } R(x,y) = \boldsymbol{e}\} \quad \forall x \in X$$

The previous operators are related by adjunction, as can be seen in the following proposition:

Proposition 8. *The pair* $(\varepsilon_{1R}, \delta_{2R})$ *(and, analogously, the pair* $(\varepsilon_{2R}, \delta_{1R})$*) is an adjunction between complete lattices, that is:*

$$\delta_{2R}(B) \leq A \iff B \leq \varepsilon_{1R}(A), \forall A \in L^X, \forall B \in L^Y \tag{16}$$

(In the same way, $\delta_{1R}(A) \leq B \iff A \leq \varepsilon_{2R}(B), \forall A \in L^X, \forall B \in L^Y$*.)*

Proof: Suppose that $\delta_{2R}(B) \leq A$. Then $\delta_{2R}(B)(x) \leq A(x)$ $\forall x \in X$. That is,

$$\mathcal{C}(R(x,y), B(y)) \leq A(x) \ \forall (x,y) \in X \times Y$$

From these inequalities and from the equivalence $\mathcal{C}(\alpha,\beta) \leq \gamma \Leftrightarrow \beta \leq I(\alpha,\gamma)$ we have

$$B(y) \leq I(R(x,y), A(x)), \forall (x,y) \in X \times Y$$

and we have:

$$B(y) \leq \inf\{I(R(x,y), A(x)) \mid x \in X\}, \forall y \in Y$$

That is, $B(y) \leq \varepsilon_{1R}(A)(y), \forall y \in Y$, which shows that $B \leq \varepsilon_{1R}(A)$.

The other implication can be proved in a similar way.

□

As a corollary of this proposition, we can see that Definition 10 is coherent with the usual definition of erosion and dilation in mathematical morphology for complete lattices [1;50;44]:

Corollary 1. *In the setting of Definition 10:*

(i) *The operators ε_{1R} and ε_{2R} preserve infima; i.e.,*

$$\varepsilon_{1R}(\bigwedge_{j\in J} A_j) = \bigwedge_{j\in J} \varepsilon_{1R}(A_j) \quad \forall \{A_j\}_{j\in J} \subseteq L^X \tag{17}$$

$$\varepsilon_{2R}(\bigwedge_{k\in K} B_k) = \bigwedge_{k\in K} \varepsilon_{2R}(B_k) \quad \forall \{B_k\}_{k\in K} \subseteq L^Y . \tag{18}$$

and $\varepsilon_{1R}(X) = Y$ and $\varepsilon_{2R}(Y) = X$.

(ii) *The operators δ_{1R} and δ_{2R} preserve suprema, i.e.,*

$$\delta_{1R}(\bigvee_{j\in J} A_j) = \bigvee_{j\in J} \delta_{1R}(A_j) \quad \forall \{A_j\}_{j\in J} \subseteq L^X ; \tag{19}$$

$$\delta_{2R}(\bigvee_{k\in K} B_k) = \bigvee_{k\in K} \delta_{2R}(B_k) \quad \forall \{B_k\}_{k\in K} \subseteq L^Y . \tag{20}$$

and $\delta_{1R}(\mathbb{O}) = \mathbb{O}$, $\delta_{2R}(\mathbb{O}) = \mathbb{O}$, where $\mathbb{O}$ represents the mapping $\mathbb{O} : X \to L$ (or, respectively, $\mathbb{O} : Y \to L$) which is identically constant and equal to 0_L.

Remark 8. It is trivial to prove that in the particular case where $X = Y \in \{\mathbb{R}^n, \mathbb{Z}^n\}$ and $R(x, y) = S(x - y)$, the operators ε_{1R} and δ_{2R} coincide with the erosion and dilation operators ε_R and δ_R defined in section 2.4.

Also the following corollary is obtained:

Corollary 2. *The L-fuzzy erosions ε_{1R} and ε_{2R}, and L-fuzzy dilations δ_{1R} and δ_{2R} are increasing operators, that is, $\forall A, C \in L^X$, $\forall B, D \in L^Y$:*

$$A \le C \Longrightarrow (\varepsilon_{1R}(A) \le \varepsilon_{1R}(C)) \text{ and } (\delta_{1R}(A) \le \delta_{1R}(C)) \tag{21}$$

$$B \le D \Longrightarrow (\varepsilon_{2R}(B) \le \varepsilon_{2R}(D)) \text{ and } (\delta_{2R}(B) \le \delta_{2R}(D)) \tag{22}$$

L-fuzzy erosion and L-fuzzy dilation operators are dual with respect to the negation $'$:

Proposition 9. *If $A^{'} \in L^X$ and $B^{'} \in L^Y$ are the strong negation of the subsets A and B respectively, and $R \in L^{X\times Y}$, then,*

$$\varepsilon_{1R}(A^{'}) = (\delta_{1R}(A))^{'} \tag{23}$$

$$\varepsilon_{2R}(B^{'}) = (\delta_{2R}(B))^{'} \tag{24}$$

Proof: For every $y \in Y$ it is verified that

$$\begin{aligned}\varepsilon_{1R}(A^{'})(y) &= \inf\{I(R(x, y), A^{'}(x)) \mid x \in X\} = \\ &= \inf\{(\mathcal{C}(R(x, y), A(x)))^{'} \mid x \in X\} = \\ &= (\sup\{\mathcal{C}(R(x, y), A(x)) \mid x \in X\})^{'} = \\ &= (\delta_{1R}(A)(y))^{'} = (\delta_{1R}(A))'(y)\end{aligned}$$

The second equality is proved analogously.

□

Using erosion and dilation operators we can construct the initial morphological filters: opening and closing operators.

Definition 11. Let L be a complete lattice and $R \in L^{X\times Y}$ a structuring element. Assume that erosion and dilation have been defined as in Definition 10.

An L-fuzzy opening (in X) is an operator $\gamma_{1R} : L^X \to L^X$ defined by the composition $\delta_{2R} \circ \varepsilon_{1R}$; that is, for every $A \in L^X$:

$$\gamma_{1R}(A) = \delta_{2R}(\varepsilon_{1R}(A)) \tag{25}$$

Analogously, an L-fuzzy opening (in Y) is an operator $\gamma_{2R} : L^Y \to L^Y$ defined by the composition $\delta_{1R} \circ \varepsilon_{2R}$; that is, for every $B \in L^Y$:

$$\gamma_{2R}(B) = \delta_{1R}(\varepsilon_{2R}(B)) \tag{26}$$

Definition 12. Let L be a complete lattice and $R \in L^{X\times Y}$ a structuring element. Assume that erosion and dilation have been defined as in Definition 10. An L-fuzzy closing (in X) is an operator $\phi_{1R} : L^X \to L^X$ defined by the composition $\varepsilon_{2R} \circ \delta_{1R}$; that is, for every $A \in L^X$:

$$\phi_{1R}(A) = \varepsilon_{2R}(\delta_{1R}(A)) \tag{27}$$

Analogously, an L-fuzzy closing (in Y) is an operator $\phi_{2R} : L^Y \to L^Y$ defined by the composition $\varepsilon_{1R} \circ \delta_{2R}$; that is, for every $B \in L^Y$:

$$\phi_{2R}(B) = \varepsilon_{1R}(\delta_{2R}(B)) \tag{28}$$

Proposition 10. *For any $R \in L^{X\times Y}$, the opening and closing operators γ_{1R} and ϕ_{1R} (and, analogously, the operators γ_{2R} and ϕ_{2R}) are morphological filters, that is, they are increasing and idempotent. That is, for every $A, C \in L^X$,*

(i) $A \le C \Longrightarrow (\gamma_{1R}(A) \le \gamma_{1R}(C))$ *and* $(\phi_{1R}(A) \le \phi_{1R}(C))$;
(ii) $\gamma_{1R}(\gamma_{1R}(A)) = \gamma_{1R}(A)$;
(iii) $\phi_{1R}(\phi_{1R}(A)) = \phi_{1R}(A)$.

Moreover, these filters verify that:

(i) $\gamma_{1R}(A) \le A \le \phi_{1R}(A) \quad \forall A \in L^X$;
(ii) $\gamma_{2R}(B) \le A \le \phi_{2R}(B) \quad \forall B \in L^Y$.

Proof: It is a consequence of Proposition 8.

□

An analogous result holds for erosion and dilation operators, as we show in the next proposition.

Proposition 11. *Let $'$ be a strong negation. For any $R \in L^{X\times Y}$,*

$$\gamma_{1R}(A') = (\phi_{1R}(A))' \tag{29}$$
$$\gamma_{2R}(B') = (\phi_{2R}(B))' \tag{30}$$

Proof:

$$\gamma_{1R}(A') = \delta_{2R}(\varepsilon_{1R}(A')) = \delta_{2R}((\delta_{1R}(A))') =$$
$$= (\varepsilon_{2R}(\delta_{1R}(A))' = (\phi_{1R}(A))'.$$

The other equality is proved in the same way.

□

By Tarski's fixed point theorem[62], as the opening and closing operators are increasing in L, their fixed points sets have a complete lattice structure. Moreover, the fixed points of these operators exist. This leads us to introduce the following definition.

Definition 13. An L-fuzzy set $A \in L^X$ (or, analogously, $B \in L^Y$) is said to be R-*open* if it coincides with its L-fuzzy opening by the relation $R \in L^{X\times Y}$, and it is said to be R-*closed* if it coincides with its L-fuzzy closing by the relation R.

R-open and R-closed sets provide the link between L-fuzzy mathematical morphology and L-fuzzy concept analysis. Thus, given the complete lattice L and the sets X and Y, an L-fuzzy context (L, X, Y, R') is defined such that the extension and intension of the L-fuzzy concepts are related to the R-open and the R-closed sets, as we show in the next theorem.

Proposition 12. *Let $'$ be a strong negation. Consider the L-fuzzy context (L, X, Y, R') where $R' \in L^{X\times Y}$ is such that for all $(x, y) \in X \times Y$ $R'(x, y) = (R(x, y))'$. Then, the L-fuzzy erosion and L-fuzzy dilation operators are related to the derivation operators D_1^R and D_2^R in the L-fuzzy context (L, X, Y, R') by the following expressions:*

$$\varepsilon_{1R}(A) = (A')_1 \quad \forall A \in L^X \tag{31}$$
$$\delta_{2R}(B) = (B_2)' \quad \forall B \in L^Y \tag{32}$$
$$\varepsilon_{2R}(B) = (B')_2 \quad \forall B \in L^Y \tag{33}$$
$$\delta_{1R}(A) = (A_1)' \quad \forall A \in L^X \tag{34}$$

Proof: Consider $A \in L^X$. For any $y \in Y$ it is verified that,

$$\varepsilon_{1R}(A)(y) = \inf\{I(R(x, y), A(x)) \mid x \in X\} =$$
$$= \inf\{I(A'(x), R'(x, y)) \mid x \in X\} = (A')_1(y).$$

Analogously, $\forall x \in X$,

$$\delta_{2R}(B)(x) = \sup\{\mathcal{C}(R(x, y), B(y)) \mid y \in Y\} =$$
$$= \sup\{(I(R(x, y), B'(y)))' \mid y \in Y\} =$$
$$= (\inf\{I(R(x, y), B'(y)) \mid y \in Y\})' =$$
$$= (\inf\{I(B(y), R'(x, y)) \mid y \in Y\})' =$$
$$= (B_2(x))' = (B_2)'(x).$$

The other two equalities can be proved in the same way.

□

As a consequence, a relation between some morphological elements and L-fuzzy concepts is found:

Theorem 2. *Let $R \in L^{X\times Y}$. The following propositions are equivalent:*

(t1) The pair $(A,B) \in L^X \times L^Y$ is an L-fuzzy concept of the L-fuzzy context (L,X,Y,R').
(t2) The pair $(A,B) \in L^X \times L^Y$ is such that the strong negation A' of A is R-open ($\gamma_{1R}(A') = A'$) and B is the L-fuzzy erosion of A' (that is, $B = \varepsilon_{1R}(A')$).
(t3) The pair $(A,B) \in L^X \times L^Y$ is such that B is R-closed ($\phi_{2R}(B) = B$) and A is the strong negation of the L-fuzzy dilation of B, that is, $A = (\delta_{2R}(B))'$.

Proof:

(t1)$\Longrightarrow$ (t2) Let $R \in L^{X\times Y}$ be the structuring element. Let us consider an L-fuzzy concept (A,B) of the L-fuzzy context (L,X,Y,R'). By the definition of L-fuzzy concept, it is verified that $B = A_1$ and $A = B_2$, and, applying the previous proposition, $\varepsilon_{1R}(A') = A_1 = B$.
Moreover, it is fulfilled that

$$\gamma_{1R}(A') = \delta_{2R}(\varepsilon_{1R}(A')) = \delta_{2R}(B) = (B_2)' = A'$$

which proves that A' is R-open.

(t2) $\Longrightarrow$ (t3) Let us suppose that the pair $(A,B) \in L^X \times L^Y$ is such that $\gamma_{1R}(A') = A'$ and $B = \varepsilon_{1R}(A')$. Then,

$$\begin{aligned}\phi_{2R}(B) =&\varepsilon_{1R}(\delta_{2R}(B)) = \varepsilon_{1R}(\delta_{2R}(\varepsilon_{1R}(A'))) = \\ =&\varepsilon_{1R}(\gamma_{1R}(A')) = \varepsilon_{1R}(A') = B\end{aligned}$$

which proves that B is R-closed.
On the other hand, from the hypothesis $B = \varepsilon_{1R}(A')$. It can be deduced that

$$\delta_{2R}(B) = \delta_{2R}(\varepsilon_{1R}(A')) = \gamma_{1R}(A')$$

and, taking into account that A' is R-open, that $\delta_{2R}(B) = A'$. Finally, $A = (\delta_{2R}(B))'$.

(t3) $\Longrightarrow$ (t1) Let (A,B) be a pair fulfilling that $\phi_{2R}(B) = B$ and $A = (\delta_{2R}(B))'$. Let us consider the L-fuzzy context (L,X,Y,R'). Then, by the previous theorem we can deduce that $B_2 = (\delta_{2R}(B))' = A$.
On the other hand, applying Proposition 12 and the hypothesis,

$$A_1 = \varepsilon_{1R}(A') = \varepsilon_{1R}(\delta_{2R}(B)) = \phi_{2R}(B) = B$$

therefore, as B is the derived set of A, the pair (A, B) is an L-fuzzy concept of the L-fuzzy context (L, X, Y, R').

□

This existing relation between L-fuzzy mathematical morphology and L-fuzzy concept analysis allows working with examples that are not very common in L-fuzzy mathematical morphology, such as data tables.

5. Practical Case

A big company is evaluating the work of the teams in the different departments trying to set up comparisons among them. To do this, the person in charge evaluates its team by means of a test. Our theory can be useful in order to study the strengths and weaknesses of the different departments.

We are going to take the set X formed by the different people in charge, the set Y that is the set of the questions of the test and a relation R with the answers of the managers to the questions. In this case, these answers will be intervals formed by the worst and the best opinion of the leaders about people in their teams, varying from $[0, 0]$ (totally disagree) to $[1, 1]$ (totally agree). Therefore, the relation $R \in L^{X\times Y}$, where $L = \mathcal{J}([0, 1])$ is the set of closed intervals in $[0, 1]$ endowed with the usual order relation:

$$[a, b] \leq [c, d] \iff a \leq c \text{ and } b \leq d.$$

The questions of the test are:

y_1 : How skilled at their jobs are the members of your team
y_2 : How professional are the members of your team
y_3 : How honest with each other are the members of your team
y_4 : How well do members of your team share responsibility for tasks
y_5 : How well do members of your team communicate with each other
y_6 : How often does your team meet its deadlines
y_7 : How professionally do members of your team deal with each other's mistakes
y_8 : How quickly does your team adjust to changing priorities
y_9 : How quickly does your team act on its decisions

The relation R that appears in Table 1 is obtained by asking the opinion of the different leaders of the work teams using the previous questions.

This example shows the relationship between interval-valued fuzzy mathematical morphology[49] and interval-valued L-fuzzy concept analysis[28].

In this practical case, the structuring element is R and that can be interpreted as the strengths of the different work teams. This relation represents the effect that we want to produce on a starting set formed by a group of people in charge ($A \in L^X$)

Table 1. Opinion of the leaders about their teams

R	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9
x_1	[0.7, 1]	[0.4, 0.8]	[0, 0.3]	[0, 0.2]	[0. 8, 1]	[0. 9, 0.9]	[0. 7, 1]	[0, 0.2]	[1, 1]
x_2	[0.5, 1]	[0.8, 0.8]	[0.6, 0.8]	[0.8, 1]	[0.3,0.3]	[0.7, 0.8]	[0.6, 0.6]	[1,1]	[0.0.2]
x_3	[0.5, 0.7]	[0.9, 1]	[1, 1]	[0.2, 0.2]	[0, 0]	[0.8, 1]	[0.9, 1]	[0.6, 0.8]	[0.2, 0.2]
x_4	[0, 0]	[0.1, 0.3]	[0, 0.2]	[0, 0]	[0, 0.2]	[0, 0]	[0, 0.3]	[0.2, 0.4]	[0, 0.3]
x_5	[0.7, 0.7]	[0, 0]	[0.8, 1]	[0.3, 0.5]	[0, 0.1]	[0, 0]	[0.8, 0.9]	[1, 1]	[0, 0.1]
x_6	[0, 0]	[0, 0]	[0, 0]	[0, 0.2]	[0, 0.2]	[0, 0.2]	[0, 0]	[0, 0.2]	[0, 0]
x_7	[0.6, 0.8]	[1, 1]	[1, 1]	[0.8, 1]	[0, 0.1]	[1, 1]	[1, 1]	[0.7, 1]	[0, 0]
x_8	[0, 0.3]	[0, 0]	[0, 0]	[0, 0]	[0, 0.1]	[0, 0.4]	[0.2, 0.2]	[0, 0]	[0.2, 0.5]
x_9	[0.5, 0.7]	[0, 0]	[0.3, 0.3]	[0.3, 0.5]	[0, 0]	[1, 1]	[0.5, 0.8]	[0, 0.2]	[0.8, 1]
x_{10}	[0.3, 0.5]	[0, 0]	[0.8, 1]	[0, 0]	[0.3, 0.5]	[1, 1]	[1, 1]	[0.7, 1]	[0, 0.1]

or by a set of questions of the test ($B \in L^Y$). Then, we can extend the basic notions of mathematical morphology (erosion, dilation, opening, closing) using a non-usual structuring element (relation R). The operators $(I, \mathcal{C})$ that we have used to obtain these morphological elements are those of Example 5 which extend the implication and t-norm of Lukasiewicz to the interval-valued case. The used strong negation is the usual one defined in $L = \mathcal{J}([0,1])$, that is $[a,b]' = [1-b, 1-a]$ for all $[a,b] \in \mathcal{J}([0,1])$.

At the same time, we work with the L-fuzzy context (L, X, Y, R'), where R' could be interpreted as the answer of the managers about the weaknesses of the work teams. The information of this L-fuzzy context is obtained by means of the L-fuzzy concepts (A, B) that are groups of managers A which teams have as weak points B.

Before explaining the results obtained from Theorem 2 in this practical case, we show some preliminary results:

In Proposition 12 the existing relationship between L-fuzzy erosion and L-fuzzy dilation and the derivation operators in (L, X, Y, R') has been established. This relation allows to give an interpretation of the erosion and dilation as follows:

- Starting from a set of managers $A \in L^X$, the calculation of its L-fuzzy erosion $\varepsilon_{1R}(A)$ using (12) lies in obtaining the weaknesses of those managers that are in the complement of A.

  So, let us take the set of managers $\{x_1, x_4, x_9\}$ which can be represented by the L-fuzzy subset

$$A = \{(x_1, [1,1]), (x_2, [0,0]), (x_3, [0,0]), (x_4, [1,1]), (x_5, [0,0]), (x_6, [0,0]), (x_7, [0,0]), (x_8, [0,0]), (x_9, [1,1]), (x_{10}, [0,0])\}$$

  the obtained L-fuzzy erosion is:

$$\varepsilon_{1R}(A) = \{(y_1, [0,0.3]), (y_2, [0,0]), (y_3, [0,0]), (y_4, [0,0.2]), (y_5, [0.5,0.7]), (y_6, [0,0]), (y_7, [0,0]), (y_8, [0,0]), (y_9, [0.5,0.8])\}$$

and it can be deduced that y_5 and y_9 are the weaknesses of the teams led by the managers $x_2, x_3, x_5, x_6, x_7, x_8$ and x_{10}.

- The L-fuzzy dilation of the set $A \in L^X$, $\delta_{1R}(A)$, consists in selecting the set of skills in at least one manager in A, using (13).
For example, starting from the previous L-fuzzy set A, its dilation is:

$$\delta_{1R}(A) = \{(y_1, [0.7, 1]), (y_2, [0.4, 0.8]), (y_3, [0.3, 0.3]), (y_4, [0.3, 0.5]), (y_5, [0.8, 1]), (y_6, [1, 1]), (y_7, [0.7, 1]), (y_8, [0.2, 0.4]), (y_9, [1, 1])\}$$

what can be interpreted by saying that the features y_1, y_5, y_6, y_7 and y_9, and to a lesser extend y_3 are the skills in at least one manager in A.

- The L-fuzzy erosion $\varepsilon_{2R}(B)$ of an L-fuzzy subset $B \in L^Y$ using (14) represents the set of managers which skills are among the elements in B.
If we consider the questions represented by the set:

$$B = \{(y_1, [1, 1]), (y_2, [1, 1]), (y_3, [0, 0]), (y_4, [0, 0]), (y_5, [0, 0]), (y_6, [0, 0]), (y_7, [1, 1]), (y_8, [0, 0]), (y_9, [1, 1])\}$$

its L-fuzzy erosion is the set:

$$\varepsilon_{2R}(B) = \{(x_1, [0, 0.1]), (x_2, [0, 0]), (x_3, [0, 0]), (x_4, [0.6, 0.8]), (x_5, [0, 0]), (x_6, [0.8, 1]), (x_7, [0, 0]), (x_8, [0.6, 1]), (x_9, [0, 0]), (x_{10}, [0, 0])\}$$

and it can be deduced that x_4, x_6 and x_8 are the managers who have answered with a low score to the questions y_3, y_4, y_5, y_6, y_8 and y_9.

- Starting from the set $B \in L^Y$, its dilation $\delta_{2R}(B)$, using (15), lies in selecting those managers with high skills in at least one feature in B.
For the previous set B the obtained L-fuzzy dilation is:

$$\delta_{2R}(B) = \{(x_1, [1, 1]), (x_2, [0.8, 1]), (x_3, [0.9, 1]), (x_4, [0.1, 0.3]), (x_5, [0.8, 0.9]), (x_6, [0, 0]), (x_7, [1, 1]), (x_8, [0.2, 0.5]), (x_9, [0.8, 1]), (x_{10}, [1, 1])\}$$

Hence, we can deduce that the teams lead by the managers $x_1, x_2, x_3, x_5, x_7, x_9$ and x_{10} have some skills among the features of set B.

We could also obtain the L-fuzzy opening and L-fuzzy closing of a starting set ($A \in L^X$) or ($B \in L^Y$)

- The effect produced by the L-fuzzy opening lies in lowering the membership values of outstanding elements or even eliminating some of them. The opening keeps only the managers that are strong at least in one of the weak properties of the complementary of the managers.
Taking the previous set A,

$$A = \{(x_1, [1, 1]), (x_2, [0, 0]), (x_3, [0, 0]), (x_4, [1, 1]), (x_5, [0, 0]), (x_6, [0, 0]), (x_7, [0, 0]), (x_8, [0, 0]), (x_9, [1, 1]), (x_{10}, [0, 0])\}$$

the obtained L-fuzzy opening using (25) is:

$$\gamma_{1R}(A) = \{(x_1, [0.8, 0.8]), (x_2, [0, 0]), (x_3, [0, 0]), (x_4, [0, 0]), (x_5, [0, 0]), (x_6, [0, 0]), (x_7, [0, 0]), (x_8, [0, 0]), (x_9, [0.6, 0.6]), (x_{10}, [0, 0])\}$$

- Regarding the L-fuzzy closing, the obtained result is higher than the original set.
  Closing adds B the weakness of the managers that only have properties of the complementary of B, i.e. the weakness of the managers that are weak in every element of B.
  Starting from the previous set B and using (28)

$$B = \{(y_1, [1, 1]), (y_2, [1, 1]), (y_3, [0, 0]), (y_4, [0, 0]), (y_5, [0, 0]), (y_6, [0, 0]), (y_7, [1, 1]), (y_8, [0, 0]), (y_9, [1, 1])\}$$

  we obtain:

$$\phi_{2R}(B) = \{(y_1, [1, 1]), (y_2, [1, 1]), (y_3, [0.9, 1]), (y_4, [0.8, 1]), (y_5, [0.8, 1]), (y_6, [0.8, 1]), (y_7, [1, 1]), (y_8, [0.8, 0.9]), (y_9, [1, 1])\}$$

At this point, we are in a position to interpret Theorem 2:

We take the L-fuzzy context (L, X, Y, R') where R' is the strong negation of the structuring element R and the L-fuzzy sets $A \in L^X$ and $B \in L^Y$:

$$A = \{(x_1, [0.2, 0.2]), (x_2, [0.2, 0.4]), (x_3, [1, 1]), (x_4, [1, 1]), (x_5, [0.6, 0.6]), (x_6, [1, 1]), (x_7, [0.2, 0.4]), (x_8, [0.8, 1]), (x_9, [0.3, 0.4]), (x_{10}, [0.7, 0.7])\}$$
$$B = \{(y_1, [0.3, 0.5]), (y_2, [0, 0.1]), (y_3, [0, 0]), (y_4, [0.8, 0.8]), (y_5, [0.8, 1]), (y_6, [0, 0.2]), (y_7, [0, 0.1]), (y_8, [0.2, 0.4]), (y_9, [0.7, 0.8])\}$$

Then the three statements of Theorem 2 hold:

(t1) The pair $(A, B) \in L^X \times L^Y$ is an L-fuzzy concept of the L-fuzzy context (L, X, Y, R'). We can say that managers x_3, x_4, x_6 and x_8 make a negative evaluation of their teams in questions y_4 and y_5.

(t2) A' is an R-open set since $\gamma_{1R}(A') = A'$. Moreover, $B = \varepsilon_{1R}(A')$.

(t3) B is R-closed ($\phi_{2R}(B) = B$) and $A = (\delta_{2R}(B))'$.

In some way, we can say that the sets A' and B are robust with respect to the structuring element R (they are not modified by the opening or closing using relation R).

6. Conclusions and Future Work

In this work we have shown how to link the theories of L-fuzzy concept analysis and L-fuzzy mathematical morphology. This fact allows us to apply the algorithms of L-fuzzy concept analysis in L-fuzzy mathematical morphology and vice versa.

Of course, some deeper analysis on the possible implications of this link would be needed. We leave this for future works. Nevertheless, it is worth to remark that it would be interesting to study the morphological gradient and top-hat and hit-or-miss transformations when we work with structuring elements $R \in L^{X \times Y}$, and their interpretation in the field of the L-fuzzy concept analysis.

Finally, in future works the definitions of L-fuzzy erosion and L-fuzzy dilation operators will also be extended to the case of working with L-fuzzy relations. In particular, we expect that this extension will allow for applications in databases.

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