Convolution lattices

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Abstract

We propose two convolution operations on the set of functions between two bounded lattices and investigate the algebraic structure they constitute, in particular the lattice laws they satisfy. Each of these laws requires the restriction to a specific subset of functions, such as normal, idempotent or convex functions. Combining all individual results, we identify the maximal subsets of functions resulting in a bounded lattice, and show this result to be equivalent to the distributivity of the lattice acting as domain of the functions. Furthermore, these lattices turn out to be distributive as well. Additionally, we show that for the larger subset of idempotent functions, although not satisfying the absorption laws, the convolution operations satisfy the Birkhoff equation.

Keywords: Algebra, Convolution operations, Lattice

1. Introduction

The mathematical operation of convolution and related operations play a pivotal role in science, engineering and mathematics [1, 2]. In the standard setting, convolution takes two real functions as input and outputs a third real function that represents the integral of the pointwise multiplication of the two functions as a function of the amount that one of the original functions is translated. More formally, given two real functions f and g, their convolution is the function $f * g$ defined by

$$(f * g)(t) = \int f(\lambda)g(t-\lambda)\mathrm{d}\lambda\,.$$

The convolution operation has applications in probability and statistics, differential equations, signal processing, natural language processing, image pro-

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cessing and computer vision, and engineering. It has attractive mathematical properties, including commutativity, associativity, distributivity w.r.t. pointwise addition of functions, and has an identity element (Dirac's delta function) and an absorbing element (the constant function equal to 0) [3]. Obviously, convolution is not defined in general, and needs a restriction to an appropriate subset of functions. This notion of convolution has been generalized to various other types of object, such as distribution functions, probability measures, complex functions, functions defined on a group endowed with a measure, and so on.

Similar operations are at the basis of mathematical morphology, a particular direction in image processing [4]. Recall that for subsets A and B of $\mathbb{R}^n$, the Minkowski addition $A \oplus B$ and subtraction $A \ominus B$ are defined as

$$\begin{aligned} A \oplus B &= \{y \mid (\exists x \in B)(y - x \in A)\} \\ A \ominus B &= \{y \mid (\forall x \in B)(y - x \in A)\} . \end{aligned}$$

Defining $-B = \{x \mid -x \in B\}$, the dilation $D(A, B)$ and erosion $E(A, B)$ of A by B are defined as $D(A, B) = A \oplus (-B)$ and $E(A, B) = A \ominus (-B)$. Identifying sets with their characteristic mapping, we can also write

$$\begin{aligned} (A \oplus B)(y) &= \sup_{x \in \mathbb{R}^n} \min(B(x), A(y - x)) \\ (A \ominus B)(y) &= \inf_{x \in \mathbb{R}^n} \max(1 - B(x), A(y - x)) . \end{aligned}$$

The subsets A and B represent the sets of black pixels in a black-and-white image. Treating gray-scale images as fuzzy subsets of $\mathbb{R}^n$ through rescaling, the above expressions are immediately applicable to gray-scale images, which is at the basis of first approaches to fuzzy mathematical morphology [5, 6] and further generalizations [7, 8]. Clearly, they bear a striking similarity with the traditional convolution operation. An alternative approach to gray-scale morphology is based on functions taking values in $\overline{\mathbb{R}} = [-\infty, +\infty]$, leading to expressions that are even more similar to the traditional convolution operation [9]. Interestingly, the theory of mathematical morphology has been further generalized to the lattice-theoretic setting [4, 10, 11].

In fuzzy set theory, a similar convolutional spirit can be recognized in Zadeh's seminal extension principle [12, 13]. This principle allows to extend any function $f : X \to Y$ between two universes X and Y to a function between $\mathcal{F}(X)$ (the fuzzy subsets of X) and $\mathcal{F}(Y)$ (the fuzzy subsets of Y) in the following natural way:

$$f(A)(y) = \sup_{f(x)=y} A(x) ;$$

and, similarly, for a composite universe $X = X_1 \times X_2$:

$$f(A_1, A_2)(z) = \sup_{f(x,y)=z} \min(A_1(x), A_2(y)) .$$

In particular, this principle is invoked to extend Moore's interval calculus to the computation with fuzzy intervals, leading to fuzzy interval arithmetic [14, 15].

For instance, the maximum, minimum and sum of two fuzzy intervals A and B are defined by

$$\max(A,B)(z) = \sup_{\max(x,y)=z} \min(A(x),B(y))$$
$$\min(A,B)(z) = \sup_{\min(x,y)=z} \min(A(x),B(y))$$
$$(A+B)(z) = \sup_{x} \min(A(x),B(z-x))\,.$$

The convolutional spirit is most easily recognized in the latter expression of the sum. Further generalizations of the extension principle, replacing $\min(A(x),B(y))$ by $T(A(x),B(y))$, with T a more general triangular norm, have been developed as well [16]. Additionally, settings in which the fuzzy intervals correspond to more general interactive fuzzy variables have been explored [17].

In particular, the extension principle can be used to extend Boolean operations [15, 18]. For instance, the Boolean operations OR and AND, two binary operations on the set of truth values $\{\mathrm{F},\mathrm{T}\}$, can be extended to binary operations on $\mathcal{F}(\{\mathrm{F},\mathrm{T}\})$ (the fuzzy subsets of $\{\mathrm{F},\mathrm{T}\}$) as follows:

$$\mathrm{OR}(f,g)(\mathrm{F}) = \min(f(\mathrm{F}),g(\mathrm{F}))$$
$$\mathrm{OR}(f,g)(\mathrm{T}) = \max(\min(f(\mathrm{F}),g(\mathrm{T})),\min(f(\mathrm{T}),g(\mathrm{F})),\min(f(\mathrm{T}),g(\mathrm{T})))$$

and

$$\mathrm{AND}(f,g)(\mathrm{F}) = \max(\min(f(\mathrm{F}),g(\mathrm{F})),\min(f(\mathrm{F}),g(\mathrm{T})),\min(f(\mathrm{T}),g(\mathrm{F})))$$
$$\mathrm{AND}(f,g)(\mathrm{T}) = \min(f(\mathrm{T}),g(\mathrm{T}))\,.$$

Note that if f and g are possibilistic truth values, i.e. $\max(f(\mathrm{F}),f(\mathrm{T}))=1$ and $\max(g(\mathrm{F}),g(\mathrm{T}))=1$, then $\mathrm{OR}(f,g)$ and $\mathrm{AND}(f,g)$ are possibilistic truth values as well. This approach goes back to the early years of fuzzy set theory [18, 19], and has been subject of further generalizations [20].

Zadeh's extension principle has been used in an extensive series of contributions on type-2 fuzzy sets [21, 22, 23, 24, 25], where the elements of $\mathcal{F}([0,1])$, i.e. functions from [0,1] to [0,1], play the role of fuzzy truth values. In particular, the Boolean operations OR and AND have been extended to binary operations on $\mathcal{F}([0,1])$ as follows, which can be considered convolution operations as well

$$(f \sqcup g)(x) = \sup_{\max(u,v)=x} \min(f(u),g(v))$$
$$(f \sqcap g)(x) = \sup_{\min(u,v)=x} \min(f(u),g(v))\,.$$

In particular, C. and E. Walker and coworkers have profoundly studied the algebraic/lattice-theoretic properties of these convolution operations [26, 27, 28, 29]. A similar approach replacing the unit interval $[0,1]$ by a finite chain has been developed in [30, 31]. Notwithstanding these remarkable achievements, the results obtained cannot easily be extended to a non-linear framework, as the proof methods are heavily based on the distributivity and linearity (of the unit interval or a finite chain). This renders these results of little use for other generalizations of the theory of fuzzy sets or for more general lattice-theoretic purposes. The core aim of this paper is therefore to identify the most general lattice-theoretic framework in which it is meaningful to study the above convolution operations and identify the conditions under which they constitute a lattice. We also study when the convolution operations satisfy the Birkhoff equation in order to identify when they constitute a Birkhoff system (a more general algebraic structure than a lattice).

This paper is organized as follows. In Section 2 we recall the necessary basic notions from lattice theory and introduce the subject of our study, the set of functions between lattices. In Section 3 we introduce two convolution operations on the set of functions between two bounded lattices, while in Section 4 we identify for each of the lattice laws the maximal subset(s) of functions on which these operations satisfy the law under consideration. After studying the closedness of these subsets of functions in Section 5, we combine all preceding results to conclude which algebraic structure is generated by the lattice functions together with the convolution operations. The key result of this paper is that the broad subset of normal, idempotent and convex functions yields a bounded lattice if and only if the domain of the functions is a distributive lattice. Consequently, the distributivity of this bounded lattice of lattice functions is further investigated in Section 6, while in Section 7 we show that for the broader class of idempotent functions, although the absorption laws do not hold, the convolution operations satisfy the Birkhoff equation. Finally, we present some concluding remarks and open problems in Section 8.

2. Lattice functions

2.1. Basic notions from lattice theory

In algebraic terms, a bounded lattice $\mathbb{L} = (L, \vee, \wedge, 0_L, 1_L)$ is a set L equipped with two binary operations $\vee$, called join, and $\wedge$, called meet, that satisfy the following conditions: for any $a, b, c \in L$, it holds that

(i) $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$ (*commutativity laws*);

(ii) $a \vee (b \vee c) = (a \vee b) \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ (*associativity laws*);

(iii) $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$ (*absorption laws*);

(iv) $a \vee 0_L = a$ and $a \wedge 1_L = a$ (*identity laws*).

The absorption laws imply the *idempotence* of the operations $\vee$ and $\wedge$, i.e., for any $a \in L$, it holds that

$$a \vee a = a \quad \text{and} \quad a \wedge a = a \,.$$

The element 0_L is the identity (neutral) element of the operation $\vee$, whereas the element 1_L is the identity element of the operation $\wedge$. Moreover, it holds that the element 1_L is the absorbing (annihilating) element of the operation $\vee$, whereas the element 0_L is the absorbing element of the operation $\wedge$, i.e., for any $a \in L$, it holds that $a \vee 1_L = 1_L$ and $a \wedge 0_L = 0_L$.

Let $(L, \leq)$ be a poset and $B \subseteq L$. An element $u \in L$ is said to be an *upper bound* of B if $b \leq u$ for any $b \in B$. A set may have many upper bounds, or none at all. An upper bound u^* of B is said to be a *least upper bound* of B if $u^* \leq u$ for any upper bound u of B. If a least upper bound of B exists, then it is unique, it is called the *supremum* of B and is denoted by $\sup B$. Analogously, $\ell \in L$ is said to be a *lower bound* of B if $\ell \leq b$ for any $b \in B$. A lower bound ℓ_* of B is said to be a *greatest lower bound* of B if $\ell \leq \ell_*$ for any lower bound ℓ of B. If a greatest lower bound of B exists, then it is unique, it is called the *infimum* of B and is denoted by $\inf B$. If any two-element subset $\{a, b\} \subseteq L$ has a supremum, denoted $a \vee b$, and an infimum, denoted $a \wedge b$, then the poset $(L, \leq)$ can be seen as a lattice $\mathbb{L} = (L, \vee, \wedge)$. If the poset $(L, \leq)$ is bounded, i.e., there exist two elements 0_L and 1_L such that $0_L \leq a \leq 1_L$ for any $a \in L$, then $\mathbb{L} = (L, \vee, \wedge, 0_L, 1_L)$ is a bounded lattice. Moreover, the order relation $\leq$ and the operations $\vee$ and $\wedge$ are connected as follows:

$$a \leq b \quad \Longleftrightarrow \quad a \vee b = b \quad \Longleftrightarrow \quad a \wedge b = a\,.$$

Conversely, for a given bounded lattice $\mathbb{L} = (L, \vee, \wedge, 0_L, 1_L)$, the relation $\leq$ defined by $a \leq b$ if $a \vee b = b$, and the relation $\leq'$ defined by $a \leq' b$ if $a \wedge b = a$, coincide and turn $\mathbb{L}$ into a bounded poset. A bounded lattice is called a chain if the corresponding order relation $\leq$ is linear, i.e., for any $a, b \in L$, it holds that $a \leq b$ or $b \leq a$. Note that in this paper, we will switch back and forth between the algebraic and order-theoretic interpretation when it is convenient.

Additionally, a bounded lattice $\mathbb{L} = (L, \vee, \wedge, 0_L, 1_L)$ is said to be *complete* if $\sup B$ and $\inf B$ exist for any $B \subseteq L$. Note that any finite bounded lattice is complete. For the sake of convenience, in this paper, instead of $\sup B$, we will also use the more explicit notation $\bigvee_{b \in B} b$. A complete lattice is called a *frame* [32] if it satisfies the *meet continuity* property: for any $a \in L$ and any $\emptyset \subset B \subseteq L$, it holds that

$$a \wedge \left(\bigvee_{b \in B} b \right) = \bigvee_{b \in B} (a \wedge b)\,. \tag{1}$$

Note that any frame is *distributive* [33, 34], i.e., for any $a, b, c \in L$, it holds that

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \quad \text{and} \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)\,.$$

Any finite distributive lattice satisfies the meet continuity property. Depending on the source, frames are also called *complete Heyting algebras* or *complete Brouwerian lattices* [33], due to the fact that in complete lattices, meet continuity and the *residuation property* [35] are equivalent (see, for example p. 128 in [33]). In this paper, we will only make use of the meet continuity property, and we will therefore stick to the term frame.

(a) $\mathbb{L}_1$ (b) Function f

Figure 1: (a) Hasse diagram of a lattice $\mathbb{L}_1$ and (b) graphical representation of a function $f : L_1 \to [0, 1]$.

2.2. Functions between lattices

In this paper, we consider two bounded lattices $\mathbb{L}_1 = (L_1, \vee_1, \wedge_1, 0_1, 1_1)$ and $\mathbb{L}_2 = (L_2, \vee_2, \wedge_2, 0_2, 1_2)$, with corresponding order relations $\leq_1$ and $\leq_2$, and the set of functions $\mathcal{F} = \{f \mid f : L_1 \to L_2\}$ between them. The elements of $\mathcal{F}$ are called lattice functions. Note that in this paper, we will refer to $\mathcal{F}$ without explicitly indicating the bounded lattices $\mathbb{L}_1$ and $\mathbb{L}_2$.

In case L_1 is finite, a function $f : L_1 \to L_2$ can be conveniently visualised by replacing the elements of L_1 in the Hasse diagram of $\mathbb{L}_1$ by the corresponding function values in L_2. For instance, the function f from the bounded lattice $\mathbb{L}_1$ in Fig. 1(a) to $\mathbb{L}_2 = ([0, 1], \max, \min, 0, 1)$ defined as

$$f(x) = \begin{cases} 0.6 & , \text{ if } x = 0\,, \\ 0 & , \text{ if } x = x_1\,, \\ 0.8 & , \text{ if } x = x_2\,, \\ 0.1 & , \text{ if } x = 1\,, \end{cases}$$

is depicted in Fig. 1(b). From here on, we will use the notation $\mathbb{M}_2$ to refer to the lattice in Fig. 1(a) and the notation $[\![0, 1]\!]$ to refer to the bounded chain $\mathbb{L}_2$.

2.3. Pointwise operations on lattice functions

An obvious way to turn the set $\mathcal{F}$ into a lattice is by extending the lattice operations of $\mathbb{L}_2$ to operations on $\mathcal{F}$ in a pointwise manner.

Definition 1. *For any $f, g \in \mathcal{F}$,*

(i) the pointwise join of f and g is the lattice function $f \vee g$ defined by:

$$(f \vee g)(x) = f(x) \vee_2 g(x)\,;$$

(ii) the pointwise meet of f and g is the lattice function $f \wedge g$ defined by:

$$(f \wedge g)(x) = f(x) \wedge_2 g(x)\,.$$

Theorem 1. *Let $\vee$ and $\wedge$ be the pointwise operations on $\mathcal{F}$ introduced in Definition 1. Then the algebraic structure $\mathbb{P} = (\mathcal{F}, \vee, \wedge, \underline{\mathbf{0}}, \overline{\mathbf{1}})$ is a bounded lattice where $\underline{\mathbf{0}}$ and $\overline{\mathbf{1}}$ are defined as $\underline{\mathbf{0}}(x) = 0_2$ and $\overline{\mathbf{1}}(x) = 1_2$ for any $x \in L_1$. Moreover, the corresponding order relation $\leq$ is given by $f \leq g$ if $f(x) \leq_2 g(x)$ for any $x \in L_1$.*

3. Convolution operations on lattice functions

3.1. Definition of the convolution operations

As explained in the introduction of this paper, we will propose two new operations on lattice functions: the join-convolution and the meet-convolution. From here on, we consider a bounded lattice $\mathbb{L}_1 = (L_1, \vee_1, \wedge_1, 0_1, 1_1)$ and a frame $\mathbb{L}_2 = (L_2, \vee_2, \wedge_2, 0_2, 1_2)$, with corresponding order relations $\leq_1$ and $\leq_2$, and the set of functions $\mathcal{F} = \{f \mid f : L_1 \to L_2\}$ between them. As mentioned before, we will refer to $\mathcal{F}$ without explicitly indicating $\mathbb{L}_1$ and $\mathbb{L}_2$.

Definition 2. *For any $f, g \in \mathcal{F}$,*

(i) the join-convolution of f and g is the lattice function $f \sqcup g$ defined by:

$$(f \sqcup g)(x) = \bigvee_{u \vee_1 v = x} f(u) \wedge_2 g(v) := \sup\{f(u) \wedge_2 g(v) \mid u \vee_1 v = x\}\,;$$

(ii) the meet-convolution of f and g is the lattice function $f \sqcap g$ defined by:

$$(f \sqcap g)(x) = \bigvee_{u \wedge_1 v = x} f(u) \wedge_2 g(v) := \sup\{f(u) \wedge_2 g(v) \mid u \wedge_1 v = x\}\,.$$

Obviously, the suprema in the above definition are taken in the lattice $\mathbb{L}_2$. Note that Definition 2 generalizes the convolution operations studied in [21, 26, 29] where $\mathbb{L}_1 = \mathbb{L}_2 = [\![0, 1]\!]$. In order not to overload the notations and since no confusion can occur, we will drop the subindices 1 and 2 from here on.

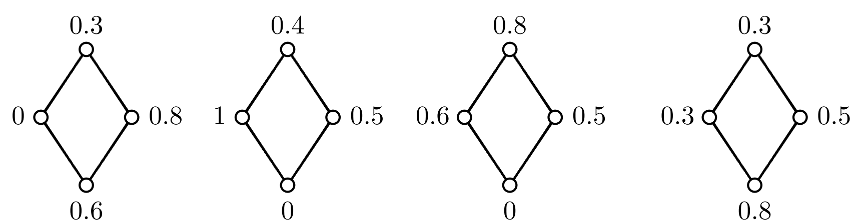
Remark 1. Note that these convolution operations can also be defined for functions between a bounded lattice $\mathbb{L}_1$ and a complete lattice $\mathbb{L}_2$ instead of restricting $\mathbb{L}_2$ to be a frame. However, in the study of the properties of the convolution operations, the meet-continuity of $\mathbb{L}_2$ will be instrumental. Since the latter will be used extensively, we will invoke it without explicitly mentioning. The same applies to the distributivity of $\mathbb{L}_2$.

In the following example, we illustrate the computation of the join- and meet-convolutions.

Example 1. *Let $\mathbb{L}_1 = \mathbb{M}_2$ and $\mathbb{L}_2 = [\![0, 1]\!]$. Consider the functions $f, g \in \mathcal{F}$ depicted in Figs. 2(a)–(b). Table 1 lists the calculations of the corresponding join-convolution $f \sqcup g$ depicted in Fig. 2(c). We have omitted the calculations of the corresponding meet-convolution $f \sqcap g$ depicted in Fig. 2(d).*

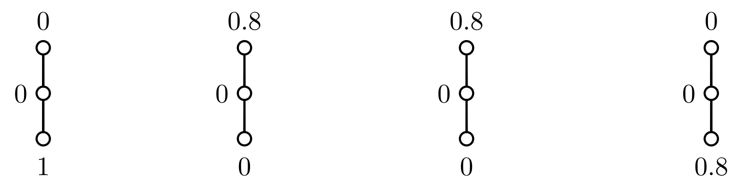
x	u	v	$f(u)$	$g(v)$	$f(u) \wedge g(v)$	$\bigvee_{u \vee v = x} f(u) \wedge g(v)$
0	0	0	0.6	0	**0**	0
x_1	x_1	x_1	0	1	0	0.6
	0	x_1	0.6	1	**0.6**	
	x_1	0	0	0	0	
x_2	x_2	x_2	0.8	0.5	**0.5**	0.5
	0	x_2	0.6	0.5	**0.5**	
	x_2	0	0.8	0	0	
1	1	1	0.3	0.4	0.3	0.8
	1	0	0.3	0	0	
	0	1	0.6	0.4	0.4	
	1	x_1	0.3	1	0.3	
	x_1	1	0	0.4	0	
	1	x_2	0.3	0.5	0.3	
	x_2	1	0.8	0.4	0.4	
	x_2	x_1	0.8	1	**0.8**	
	x_1	x_2	0	0.5	0	

Table 1: Computation of the join-convolution $f \sqcup g$ in Example 1.



(a) Function f (b) Function g (c) Function $f \sqcup g$ (d) Function $f \sqcap g$

Figure 2: Graphical representation of the functions in Example 1: (a) the function f, (b) the function g, (c) the join-convolution $f \sqcup g$, and (d) the meet-convolution $f \sqcap g$.



(a) Function f (b) Function g (c) Function $f \sqcup g$ (d) Function $f \sqcap g$

Figure 3: Graphical representation of the functions in Example 2: (a) the function f, (b) the function g, (c) the join-convolution $f \sqcup g$, and (d) the meet-convolution $f \sqcap g$.

3.2. The lack of a lattice structure in general

The main goal of this study is to unveil the conditions under which the algebraic structure $\mathbb{F} = (\mathcal{F}, \sqcup, \sqcap)$ is a bounded lattice. A first step consists of exploring the connection between the convolution operations and the corresponding relations in Definition 3.

Definition 3.

(i) With the join-convolution operation $\sqcup$ on $\mathcal{F}$, we associate the binary relation $\sqsubseteq_\sqcup$ on $\mathcal{F}$ defined by:

$$f \sqsubseteq_\sqcup g \text{ if } f \sqcup g = g\,.$$

(ii) With the meet-convolution operation $\sqcap$ on $\mathcal{F}$, we associate the binary relation $\sqsubseteq_\sqcap$ on $\mathcal{F}$ defined by:

$$f \sqsubseteq_\sqcap g \text{ if } f \sqcap g = f\,.$$

Example 2. *Let $\mathbb{L}_1 = (\{0, \frac{1}{2}, 1\}, \max, \min, 0, 1)$ and $\mathbb{L}_2 = [\![0, 1]\!]$. Consider the functions $f, g \in \mathcal{F}$ depicted in Figs. 3(a)–(b). The join- and meet-convolutions $f \sqcup g$ and $f \sqcap g$ are depicted in Figs. 3(c)–(d). One easily verifies that $f \sqcup g = g$, while $f \sqcap g \neq f$. Consequently, $f \sqsubseteq_\sqcup g$, while $f \not\sqsubseteq_\sqcap g$.*

Since the relations $\sqsubseteq_\sqcup$ and $\sqsubseteq_\sqcap$ do not coincide in general, as illustrated in Example 2, we can conclude that $\mathbb{F}$ is not a bounded lattice in general.

3.3. Pointwise operations on cumulative functions

Inspired by [26, 29], we analyse in this subsection whether the join- and meet-convolution of two functions can be equivalently formulated in terms of the cumulative functions introduced in the following definition.

Definition 4. *For any $f \in \mathcal{F}$,*

(i) the left-cumulative function f^L is the lattice function defined by:

$$f^L(x) = \bigvee_{y \le x} f(y)\,;$$

(ii) the right-cumulative function f^R is the lattice function defined by:

$$f^R(x) = \bigvee_{y \geq x} f(y)\,.$$

Proposition 1. *Let $f, g \in \mathcal{F}$. The following statements hold:*

(i) $(f \wedge g^L) \vee (f^L \wedge g) = (f \vee g) \wedge f^L \wedge g^L \leq f \sqcup g$;

(ii) $(f \wedge g^R) \vee (f^R \wedge g) = (f \vee g) \wedge f^R \wedge g^R \leq f \sqcap g$.

Proof. We only provide the proof of statement (i), the proof of statement (ii) being analogous. First, we show that the equality holds. For any $x \in L_1$, it holds that

$$\begin{aligned}
((f \wedge g^L) \vee (f^L \wedge g))(x) &= \big(f(x) \wedge g^L(x)\big) \vee \big(f^L(x) \wedge g(x)\big) \\
&= \Bigg(\big(f(x) \wedge g^L(x)\big) \vee f^L(x) \Bigg) \\
&\quad \wedge \Bigg(\big(f(x) \wedge g^L(x)\big) \vee g(x) \Bigg) \\
&= \Bigg(f(x) \vee f^L(x) \Bigg) \wedge \Bigg(g^L(x) \vee f^L(x) \Bigg) \\
&\quad \wedge \Bigg(f(x) \vee g(x) \Bigg) \wedge \Bigg(g^L(x) \vee g(x) \Bigg)\,.
\end{aligned}$$

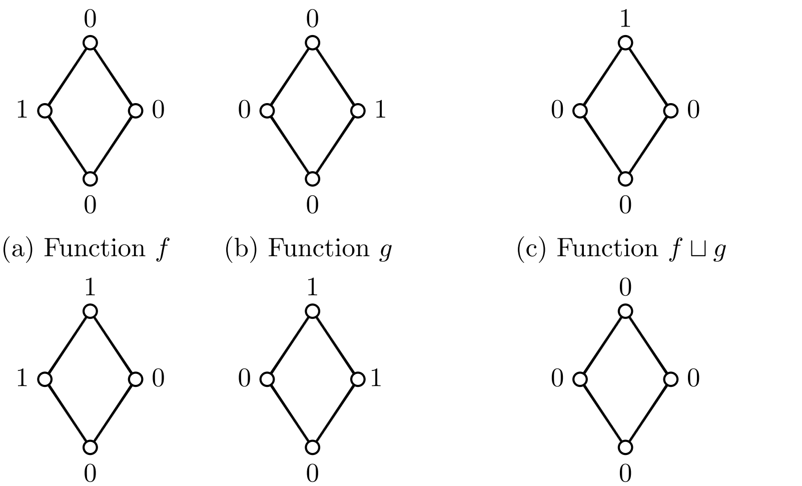
Taking into account that $f(x) \leq \bigvee_{y \leq x} f(y) = f^L(x)$ and $g(x) \leq \bigvee_{y \leq x} g(y) = g^L(x)$, it follows that $f(x) \vee g(x) \leq f^L(x) \vee g^L(x)$. Hence, it holds that

$$\begin{aligned}
((f \wedge g^L) \vee (f^L \wedge g))(x) &= f^L(x) \wedge \Bigg(f(x) \vee g(x) \Bigg) \wedge g^L(x) \\
&= \Bigg(\Bigg(f \vee g \Bigg) \wedge f^L \wedge g^L \Bigg)(x)\,.
\end{aligned}$$

Second, we show that the inequality holds. For any $x \in L_1$, it holds that

$$\begin{aligned}
((f \wedge g^L) \vee (f^L \wedge g))(x) &= \big(f(x) \wedge g^L(x)\big) \vee \big(f^L(x) \wedge g(x)\big) \\
&= \Bigg(f(x) \wedge \bigvee_{y_1 \leq x} g(y_1) \Bigg) \vee \Bigg(\bigvee_{y_2 \leq x} f(y_2) \wedge g(x) \Bigg) \\
&= \Bigg(\bigvee_{x \vee y_1 = x} f(x) \wedge g(y_1) \Bigg) \vee \Bigg(\bigvee_{y_2 \vee x = x} f(y_2) \wedge g(x) \Bigg) \\
&\leq \bigvee_{x_1 \vee x_2 = x} f(x_1) \wedge g(x_2) = (f \sqcup g)(x)\,.
\end{aligned}$$

$\square$



(a) Function f (b) Function g (c) Function $f \sqcup g$

(d) Function f^L (e) Function g^L (f) Function $(f \wedge g^L) \vee (f^L \wedge g)$

Figure 4: Graphical representation of the functions in Example 3: (a) the function f, (b) the function g, (c) the join-convolution $f \sqcup g$, (d) the corresponding function f^L, (e) the corresponding function g^L and (f) the corresponding function $(f \wedge g^L) \vee (f^L \wedge g)$.

Note that in [26, 29], it is proven that when $\mathbb{L}_1 = \mathbb{L}_2 = [\![0, 1]\!]$, it holds that

$$\begin{aligned} (f \wedge g^L) \vee (f^L \wedge g) &= (f \vee g) \wedge f^L \wedge g^L = f \sqcup g \\ (f \wedge g^R) \vee (f^R \wedge g) &= (f \vee g) \wedge f^R \wedge g^R = f \sqcap g\,. \end{aligned} \tag{2}$$

However, these identities no longer hold when $\mathbb{L}_1$ is a bounded lattice and $\mathbb{L}_2$ is a frame as is shown in the following example.

Example 3. *Let $\mathbb{L}_1 = \mathbb{M}_2$ and $\mathbb{L}_2 = \mathbb{B} = (\{0, 1\}, \max, \min, 0, 1)$. Consider the functions $f, g \in \mathcal{F}$ depicted in Figs. 4(a)–(b). The corresponding functions $f \sqcup g$, f^L, g^L and $(f \wedge g^L) \vee (f^L \wedge g)$ are depicted in Figs. 4(c)–(f), respectively. One easily verifies that $(f \wedge g^L) \vee (f^L \wedge g) < f \sqcup g$. Similarly, it can be seen that $(f \wedge g^R) \vee (f^R \wedge g) < f \sqcap g$.*

Remark 2. In the study of the convolution operations for functions from $[\![0, 1]\!]$ to $[\![0, 1]\!]$, the identities in Eq. (2) are instrumental in proving the lattice properties of these operations [26]. For the general setting considered here, we will be compelled to base our proofs on the explicit expressions of the convolution operations.

3.4. Monotone functions

Obviously, in general there is no reason to expect the pointwise operations and the convolution operations to coincide. However, as we will show next, there exist subsets of $\mathcal{F}$ for which this effectively holds. In particular, we consider the set of *increasing* functions

$$\mathcal{M}_I = \{f \in \mathcal{F} \mid (\forall (x_1, x_2) \in L_1^2)(x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2))\}$$

and the set of *decreasing* functions

$$\mathcal{M}_D = \{f \in \mathcal{F} \mid (\forall (x_1, x_2) \in L_1^2)(x_1 \leq x_2 \Rightarrow f(x_2) \leq f(x_1))\}\,.$$

Proposition 2. *The following statements hold:*

(i) if $f, g \in \mathcal{M}_I$, then $f \sqcup g = f \wedge g$;

(ii) if $f, g \in \mathcal{M}_D$, then $f \sqcap g = f \wedge g$.

Proof. We only provide the proof of statement (i), the proof of statement (ii) being analogous. Let $f, g \in \mathcal{M}_I$, then for any $x \in L_1$, it holds that

$$\begin{aligned}
(f \sqcup g)(x) &= \bigvee_{u \vee v = x} f(u) \wedge g(v) \\
&= \bigvee_{\substack{u \vee v = x \\ u \leq x, v \leq x}} f(u) \wedge g(v) \\
&\leq \bigvee_{\substack{u \vee v = x \\ u \leq x, v \leq x}} f(x) \wedge g(x) \\
&= f(x) \wedge g(x)\,.
\end{aligned}$$

Moreover, since the couple $(u, v) = (x, x)$ satisfies $u \vee v = x$, it holds that

$$(f \sqcup g)(x) \geq f(x) \wedge g(x)\,,$$

and, hence, $(f \sqcup g)(x) = f(x) \wedge g(x)$. □

Corollary 1. The following statements hold:

(i) if $f, g \in \mathcal{M}_I$, then $f \sqsubseteq_\sqcup g$ if and only if $g \leq f$;

(ii) if $f, g \in \mathcal{M}_D$, then $f \sqsubseteq_\sqcap g$ if and only if $f \leq g$.

Proof. We only provide the proof of statement (i), the proof of statement (ii) being analogous. Due to Proposition 2, for any $f, g \in \mathcal{M}_I$ and for any $x \in L_1$, it holds that $(f \sqcup g)(x) = f(x) \wedge g(x)$. Hence, $(f \sqcup g)(x) = g(x)$ if and only if $g(x) \leq f(x)$, for any $x \in L_1$, i.e., $f \sqsubseteq_\sqcup g$ if and only if $g \leq f$. □

Consequently, the pointwise operations coincide with the convolution operations if we restrict to the subset $\mathcal{M}_I \cap \mathcal{M}_D$ of $\mathcal{F}$. However, this subset consists of constant functions only, and is of no further interest in this paper.

4. Lattice laws of the convolution operations

4.1. General properties

In this section, we study whether the convolution operations satisfy the algebraic laws of lattice operations. For some properties, we will have to restrict our attention to an appropriate subset of the set of lattice functions. An important question that then pops up is whether such subset is effectively closed under the convolution operations. As this question is not trivial at all, we will dedicate Section 5 to it.

For a lattice function $f \in \mathcal{F}$, the element $s_f \in L_2$ defined as $s_f := \bigvee_{x \in L_1} f(x)$ is called the supremum of f. We also consider the functions $\mathbf{0}_a$ and $\mathbf{1}_a$ (with $a \in L_2$) defined as:

$$\mathbf{0}_a(x) = \begin{cases} a & , \text{ if } x = 0\,, \\ 0 & , \text{ otherwise}\,; \end{cases} \qquad \mathbf{1}_a(x) = \begin{cases} a & , \text{ if } x = 1\,, \\ 0 & , \text{ otherwise}\,. \end{cases}$$

Theorem 2. *Let $f, g, h \in \mathcal{F}$. The following statements hold:*

(i) $f \sqcup g = g \sqcup f$ *and* $f \sqcap g = g \sqcap f$*;*

(ii) $f \sqcup (g \sqcup h) = (f \sqcup g) \sqcup h$ *and* $f \sqcap (g \sqcap h) = (f \sqcap g) \sqcap h$*;*

(iii) $f \sqcup \mathbf{0}_a = f$ *if and only if* $s_f \leq a$*;*

(iv) $f \sqcap \mathbf{1}_a = f$ *if and only if* $s_f \leq a$*;*

(v) $f \sqcup \underline{\mathbf{0}} = \underline{\mathbf{0}}$ *and* $f \sqcap \underline{\mathbf{0}} = \underline{\mathbf{0}}$*.*

Proof. Statement (i) is a direct consequence of the commutativity of the operations $\vee$ and $\wedge$ on L_1. Similarly, statement (v) is a direct consequence of the definition of the join- and meet-convolution operations. For statement (ii), we provide the proof of $f \sqcup (g \sqcup h) = (f \sqcup g) \sqcup h$, the proof of $f \sqcap (g \sqcap h) = (f \sqcap g) \sqcap h$ being analogous. For any $x \in L_1$, it holds that

$$\begin{aligned} ((f \sqcup g) \sqcup h)(x) &= \bigvee_{q_1 \vee w = x} ((f \sqcup g)(q_1) \wedge h(w)) \\ &= \bigvee_{q_1 \vee w = x} \left(\left(\bigvee_{u \vee v = q_1} f(u) \wedge g(v) \right) \wedge h(w) \right) \\ &= \bigvee_{u \vee v \vee w = x} f(u) \wedge g(v) \wedge h(w) \\ &= \bigvee_{u \vee q_2 = x} \left(f(u) \wedge \left(\bigvee_{v \vee w = q_2} g(v) \wedge h(w) \right) \right) \\ &= \bigvee_{u \vee q_2 = x} (f(u) \wedge (g \sqcup h)(q_2)) = (f \sqcup (g \sqcup h))(x)\,. \end{aligned}$$

Finally, we only provide the proof of statement (iii), the proof of statement (iv) being analogous. For any $x \in L_1$, it holds that

$$
\begin{aligned}
(f \sqcup \mathbf{0}_a)(x) &= \bigvee_{u \vee v = x} f(u) \wedge \mathbf{0}_a(v) \\
&= \left(\bigvee_{\substack{u \vee v = x \\ v = 0}} f(u) \wedge \mathbf{0}_a(v) \right) \vee \left(\bigvee_{\substack{u \vee v = x \\ v \neq 0}} f(u) \wedge \mathbf{0}_a(v) \right) \\
&= (f(x) \wedge a) \vee \left(\bigvee_{\substack{u \vee v = x \\ v \neq 0}} f(u) \wedge 0 \right) \\
&= f(x) \wedge a \,.
\end{aligned}
$$

Hence, $f \sqcup \mathbf{0}_a = f$ if and only if it holds that $f(x) \leq a$ for any $x \in L_1$, i.e., $s_f \leq a$. □

The following corollary is a direct consequence of Theorem 2.

Corollary 2. The following properties hold:

(i) The relations $\sqsubseteq_\sqcup$ and $\sqsubseteq_\sqcap$ are antisymmetric;

(ii) The relations $\sqsubseteq_\sqcup$ and $\sqsubseteq_\sqcap$ are transitive;

(iii) It holds that $\mathbf{0}_1 \sqsubseteq_\sqcup f$ and $f \sqsubseteq_\sqcap \mathbf{1}_1$ for any $f \in \mathcal{F}$.

Remark 3. Note that as a consequence of statements (iii) and (iv) of Theorem 2 we have the following equivalence: the lattice function $\mathbf{0}_a$ is the neutral element of the join-convolution if and only if the lattice function $\mathbf{1}_a$ is the neutral element of the meet-convolution.

As mentioned before, in a bounded lattice the identity element of the join operation is the absorbing element of the meet operation, while the identity element of the meet operation is the absorbing element of the join operation. However, the join- and meet-convolution on $\mathcal{F}$ have the same absorbing element $\underline{\mathbf{0}}$. Since an element cannot be identity and absorbing element at the same time (unless the lattice consists of a single element), we will need to study when the lattice function $\mathbf{1}_a$ is the absorbing element of the join-convolution as well as when the lattice function $\mathbf{0}_a$ is the absorbing element of the meet-convolution.

Proposition 3. *Let $f \in \mathcal{F}$. The following statements hold:*

(i) $f \sqcup \mathbf{1}_a = \mathbf{1}_a$ *if and only if* $a \leq s_f$*;*

(ii) $f \sqcap \mathbf{0}_a = \mathbf{0}_a$ *if and only if* $a \leq s_f$*.*

Proof. We only provide the proof of statement (i), the proof of statement (ii) being analogous. If $x \neq 1$, then

$$(f \sqcup \mathbf{1}_a)(x) = \bigvee_{u \vee v = x} f(u) \wedge \mathbf{1}_a(v) = \bigvee_{\substack{u \vee v = x \\ v \neq 1}} f(u) \wedge \mathbf{1}_a(v) = \bigvee_{\substack{u \vee v = x \\ v \neq 1}} f(u) \wedge 0 = 0\,.$$

If $x = 1$, then

$$\begin{aligned}
(f \sqcup \mathbf{1}_a)(1) &= \bigvee_{u \vee v = 1} f(u) \wedge \mathbf{1}_a(v) \\
&= \left(\bigvee_{\substack{u \vee v = 1 \\ v = 1}} f(u) \wedge \mathbf{1}_a(v) \right) \vee \left(\bigvee_{\substack{u \vee v = 1 \\ v \neq 1}} f(u) \wedge \mathbf{1}_a(v) \right) \\
&= \left(\bigvee_{u \in L_1} f(u) \wedge a \right) \vee \left(\bigvee_{\substack{u \vee v = 1 \\ v \neq 1}} f(u) \wedge 0 \right) = s_f \wedge a\,.
\end{aligned}$$

Consequently, $f \sqcup \mathbf{1}_a = \mathbf{1}_a$ if and only if $a \leq s_f$. □

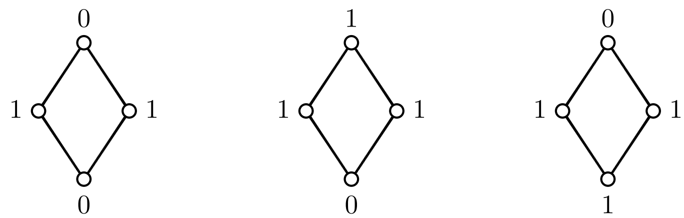
Remark 4. Note that as a consequence of Proposition 3, we have the following equivalence: $f \sqcup \mathbf{1}_a = \mathbf{1}_a$ if and only if $f \sqcap \mathbf{0}_a = \mathbf{0}_a$.

If we want to ensure that $\mathbf{0}_a$ is both the neutral element of the join-convolution and the absorbing element of the meet-convolution, it follows from Theorem 2 and Proposition 3 that it should hold that $s_f = a$. Analogously, if we want to ensure that $\mathbf{1}_a$ is both the neutral element of the meet-convolution and the absorbing element of the join-convolution, it should hold that $s_f = a$ as well. Consequently, we are forced to consider a subset of functions that share the same supremum, i.e., we are forced to consider a set $\mathcal{N}_a = \{f \in \mathcal{F} \mid s_f = a\}$ for some $a \in L_2$. Note that the functions in the set $\mathcal{N}_1$ are commonly called normal functions [26].

4.2. Idempotency laws

The only lattice laws not studied so far are the absorption laws. However, as mentioned before, in a lattice the absorption laws imply the idempotence of the join and the meet operations. Since the idempotency laws are easier to study than the absorption laws, before checking whether the absorption laws hold, we will therefore study whether the convolution operations are idempotent. Although in the preceding subsection, we have shown that for the constitution of a bounded lattice, we will be forced to restrict our attention to a subset $\mathcal{N}_a$ for some $a \in L_2$, we make an independent study of the idempotency laws of the convolution operations. In general, the idempotency laws do not hold. However, the following inequalities hold.

Proposition 4. *Let $f \in \mathcal{F}$. The following statements hold:*



(a) Function f (b) Function $f \sqcup f$ (c) Function $f \sqcap f$

Figure 5: Graphical representations of the functions in Example 4: (a) the function f, (b) the join-convolution $f \sqcup f$, and (c) the meet-convolution $f \sqcap f$.

(i) $f \leq f \sqcup f$;

(ii) $f \leq f \sqcap f$.

Proof. We only provide the proof of statement (i), the proof of statement (ii) being analogous. For any $x \in L_1$, since the couple $(u, v) = (x, x)$ satisfies $u \vee v = x$, it holds that

$$(f \sqcup f)(x) = \bigvee_{u \vee v = x} f(u) \wedge f(v) \geq f(x) \wedge f(x) = f(x)\,.$$

Hence, $f \leq f \sqcup f$. $\square$

In the following example, we show that the inequality in Proposition 4 can be strict. Note that this implies that the convolution operations are indeed not idempotent in general.

Example 4. *Let $\mathbb{L}_1 = \mathbb{M}_2$ and $\mathbb{L}_2 = \mathbb{B}$. Consider the function $f \in \mathcal{F}$ depicted in Fig. 5(a). The join- and meet-convolution $f \sqcup f$ and $f \sqcap f$ are depicted in Figs. 5(b)–(c). One easily verifies that $f < f \sqcup f$ and $f < f \sqcap f$.*

In order to have the idempotency laws satisfied, we consider the following subsets of $\mathcal{F}$

$$\mathcal{I}_{\sqcup} = \left\{ f \in \mathcal{F} \mid (\forall (x, y) \in L_1^2)(f(x) \wedge f(y) \leq f(x \vee y)) \right\}$$

and

$$\mathcal{I}_{\sqcap} = \left\{ f \in \mathcal{F} \mid (\forall (x, y) \in L_1^2)(f(x) \wedge f(y) \leq f(x \wedge y)) \right\}\,.$$

We also use the notation $\mathcal{I} := \mathcal{I}_{\sqcup} \cap \mathcal{I}_{\sqcap}$ and refer to its members as *idempotent* functions, as is justified by the following theorem.

Theorem 3. *Let $f \in \mathcal{F}$. The following statements hold:*

(i) $f \sqcup f = f$ *if and only if* $f \in \mathcal{I}_{\sqcup}$;

(ii) $f \sqcap f = f$ *if and only if* $f \in \mathcal{I}_{\sqcap}$;

(iii) $f \sqcup f = f$ *and* $f \sqcap f = f$ *if and only if* $f \in \mathcal{I}$.

Proof. We only provide the proof of statement (i), the proof of statement (ii) being analogous. Statement (iii) is a direct consequence of (i) and (ii).

$\Rightarrow$ Suppose that $f \sqcup f = f$, while $f \notin \mathcal{I}_\sqcup$. Then there exist $x, y \in L_1$ such that $f(x) \wedge f(y) \not\leq f(x \vee y)$. Note that this means that $(f(x) \wedge f(y)) \vee f(x \vee y) > f(x \vee y)$.

Further, since both couples $(u, v) = (x, y)$ and $(u, v) = (x \vee y, x \vee y)$ satisfy $u \vee v = x \vee y$, it holds that

$$\begin{aligned}(f \sqcup f)(x \vee y) &= \bigvee_{u \vee v = x \vee y} f(u) \wedge f(v) \\ &\geq (f(x) \wedge f(y)) \vee (f(x \vee y) \wedge f(x \vee y)) \\ &= (f(x) \wedge f(y)) \vee f(x \vee y) > f(x \vee y)\,,\end{aligned}$$

which contradicts $f \sqcup f = f$.

$\Leftarrow$ Due to Proposition 4, it holds that $f \leq f \sqcup f$ and, hence, it only remains to prove that $f \sqcup f \leq f$. If $f \in \mathcal{I}_\sqcup$, then it holds that

$$(f \sqcup f)(x) = \bigvee_{u \vee v = x} f(u) \wedge f(v) \leq \bigvee_{u \vee v = x} f(u \vee v) = \bigvee_{u \vee v = x} f(x) = f(x).$$

Hence, $f \sqcup f = f$. $\square$

Consequently, if we want to ensure the idempotence of the convolution operations, we are forced to consider the set of idempotent functions $\mathcal{I}$ (or a subset of it). Note that in case $\mathbb{L}_1$ is a bounded chain, it holds that $\mathcal{I}_\sqcup = \mathcal{I}_\sqcap = \mathcal{I} = \mathcal{F}$, and the convolution operations are idempotent (as in [21, 26, 29]).

Corollary 3. The following properties hold:

(i) The relation $\sqsubseteq_\sqcup$ is reflexive on $\mathcal{I}_\sqcup$;

(ii) The relation $\sqsubseteq_\sqcap$ is reflexive on $\mathcal{I}_\sqcap$.

Hence, due to Corollaries 2 and 3, the relation $\sqsubseteq_\sqcup$ constitutes a partial order on $\mathcal{I}_\sqcup$, while the relation $\sqsubseteq_\sqcap$ constitutes a partial order on $\mathcal{I}_\sqcap$.

4.3. Absorption laws

As the convolution operations are not idempotent in general, the absorption laws surely do not hold in general either. However, the following result holds.

Proposition 5. *Let* $f, g \in \mathcal{F}$. *The following statements are equivalent:*

(i) $f \leq f \sqcup (f \sqcap g)$;

(ii) $f \leq f \sqcap (f \sqcup g)$;

(iii) $s_f \leq s_g$.

Proof. We only provide the proof of the equivalence of statements (i) and (iii), the proof of the equivalence of statements (ii) and (iii) being analogous.

$\Rightarrow$ Suppose that $f \leq f \sqcup (f \sqcap g)$. For any $x \in L_1$, it holds that

$$(f \sqcup (f \sqcap g))\,(x) = \bigvee_{u_1 \vee (u_2 \wedge v) = x} f(u_1) \wedge f(u_2) \wedge g(v) \leq \bigvee_{u_1 \vee (u_2 \wedge v) = x} g(v) \leq \bigvee_{v \in L_1} g(v)\,.$$

Consequently, $f(x) \leq (f \sqcup (f \sqcap g))\,(x) \leq \bigvee_{v \in L_1} g(v)$ for any $x \in L_1$, and, hence, $s_f \leq s_g$.

$\Leftarrow$ Suppose that $s_f \leq s_g$, then it holds that $f(x) \leq \bigvee_{y \in L_1} g(y)$, for any $x \in L_1$. Due to the absorption laws in $\mathbb{L}_1$, for any $x, y \in L_1$, it holds that $x \vee (x \wedge y) = x$. Hence, for any $x \in L_1$, it holds that

$$\begin{aligned}
(f \sqcup (f \sqcap g))\,(x) &= \bigvee_{u_1 \vee (u_2 \wedge v) = x} f(u_1) \wedge f(u_2) \wedge g(v) \\
&\geq \bigvee_{\substack{u_1 \vee (u_2 \wedge v) = x \\ u_1 = x,\, u_2 = x}} f(x) \wedge f(x) \wedge g(v) \\
&= f(x) \wedge \left(\bigvee_{v \in L_1} g(v) \right) = f(x)\,.
\end{aligned}$$

□

Corollary 4. Let $f, g \in \mathcal{F}$. The following statements hold:

(i) If $f \sqcup (f \sqcap g) = f$ and $g \sqcup (g \sqcap f) = g$, then $s_f = s_g$;

(ii) If $f \sqcap (f \sqcup g) = f$ and $g \sqcap (g \sqcup f) = g$, then $s_f = s_g$.

Proof. We only provide the proof of statement (i), the proof of statement (ii) being analogous. Suppose that $f = f \sqcup (f \sqcap g)$ and $g = g \sqcup (g \sqcap f)$. Due to Proposition 5, it then holds that $s_f \leq s_g$ and $s_g \leq s_f$, i.e., $s_f = s_g$. □

Consequently, if we want to ensure that the absorption laws hold for the convolution operations, we are again forced to consider a subset $\mathcal{N}_a = \{f \in \mathcal{F} \mid s_f = a\}$ for some $a \in L_2$. Moreover, since the idempotency laws only hold when restricting to the set of idempotent functions $\mathcal{I}$, the same restriction is additionally required for the absorption laws. As the following theorem will show, a further restriction will even be necessary. Indeed, we will be forced to consider the set of *(order-)convex* functions

$$\mathcal{C} = \{f \in \mathcal{F} \mid (\forall (x_1, x_2, x_3) \in L_1^3)(x_1 \leq x_2 \leq x_3 \Rightarrow f(x_1) \wedge f(x_3) \leq f(x_2))\}\,.$$

Theorem 4. *Let $f \in \mathcal{F}$. The following statements hold:*

(i) $f \sqcup (f \sqcap g) = f$, for any $g \in \mathcal{N}_{s_f}$, if and only if $f \in \mathcal{I}_\sqcup \cap \mathcal{C}$;

(ii) $f \sqcap (f \sqcup g) = f$, for any $g \in \mathcal{N}_{s_f}$, if and only $f \in \mathcal{I}_\sqcap \cap \mathcal{C}$;

(iii) $f \sqcup (f \sqcap g) = f$ *and* $f \sqcap (f \sqcup g) = f$*, for any* $g \in \mathcal{N}_{s_f}$*, if and only* $f \in \mathcal{I} \cap \mathcal{C}$*.*

Proof. We only provide the proof of statement (i), the proof of statement (ii) being analogous. Statement (iii) is a direct consequence of statements (i) and (ii).

$\Rightarrow$ Suppose that $f \sqcup (f \sqcap g) = f$, for any $g \in \mathcal{N}_{s_f}$, while $f \notin \mathcal{I}_\sqcup \cap \mathcal{C}$. We distinguish two different cases.

(a) The case $f \notin \mathcal{I}_\sqcup$. Let $g = \mathbf{1}_{s_f} \in \mathcal{N}_{s_f}$. Due to Theorem 2 (iv), it holds that $f \sqcap g = f$. Hence, it holds that

$$f \sqcup (f \sqcap g) = f \sqcup f\,.$$

Since $f \sqcup f \neq f$, it follows that $f \sqcup (f \sqcap g) \neq f$, a contradiction.

(b) The case $f \notin \mathcal{C}$. This means that there exist $x_1, x_2, x_3 \in L_1$ such that $x_1 \leq x_2 \leq x_3$ and $f(x_1) \wedge f(x_3) \nleq f(x_2)$. Consequently, $f(x_2) < f(x_2) \vee (f(x_1) \wedge f(x_3))$. Let $g \in \mathcal{N}_{s_f}$ be the function

$$g(x) = \begin{cases} s_f & \text{, if } x = x_2\,, \\ 0 & \text{, otherwise}\,, \end{cases}$$

Since the triplets $(u_1, u_2, v) = (x_2, x_2, x_2)$ and $(u_1, u_2, v) = (x_1, x_3, x_2)$ satisfy $u_1 \vee (u_2 \wedge v) = x_2$, it holds that

$$\begin{aligned}
(f \sqcup (f \sqcap g))(x_2) &= \bigvee_{u_1 \vee (u_2 \wedge v) = x_2} f(u_1) \wedge f(u_2) \wedge g(v) \\
&\geq (f(x_2) \wedge f(x_2) \wedge g(x_2)) \vee (f(x_1) \wedge f(x_3) \wedge g(x_2)) \\
&= (f(x_2) \wedge f(x_2) \wedge s_f) \vee (f(x_1) \wedge f(x_3) \wedge s_f) \\
&= s_f \wedge (f(x_2) \vee (f(x_1) \wedge f(x_3))) \\
&= f(x_2) \vee (f(x_1) \wedge f(x_3)) > f(x_2)\,,
\end{aligned}$$

a contradiction.

$\Leftarrow$ Suppose that $f \in \mathcal{I}_\sqcup \cap \mathcal{C}$. Due to Proposition 5, for any $g \in \mathcal{N}_{s_f}$, it holds that $f \leq f \sqcup (f \sqcap g)$ and, hence, it only remains to prove that $f \sqcup (f \sqcap g) \leq f$, i.e., we need to verify that for any $x \in L_1$, it holds that

$$(f \sqcup (f \sqcap g))(x) = \bigvee_{u_1 \vee (u_2 \wedge v) = x} f(u_1) \wedge f(u_2) \wedge g(v) \leq f(x)\,.$$

If $u_1 \vee (u_2 \wedge v) = x$, then $u_1 \leq u_1 \vee (u_2 \wedge v) = x$ and $u_1 \leq x$. Similarly, $x = u_1 \vee (u_2 \wedge v) \leq u_1 \vee u_2$ and we find that $u_1 \leq x \leq u_1 \vee u_2$. For any $x \in L_1$, it holds that

$$\begin{aligned}
(f \sqcup (f \sqcap g))(x) &= \bigvee_{u_1 \vee (u_2 \wedge v) = x} f(u_1) \wedge f(u_2) \wedge g(v) \\
&= \bigvee_{u_1 \vee (u_2 \wedge v) = x} f(u_1) \wedge f(u_1) \wedge f(u_2) \wedge g(v) \\
&\leq \bigvee_{u_1 \vee (u_2 \wedge v) = x} f(u_1) \wedge f(u_1 \vee u_2) \wedge g(v) \\
&\leq \bigvee_{u_1 \vee (u_2 \wedge v) = x} f(x) \wedge g(v) \\
&\leq \bigvee_{u_1 \vee (u_2 \wedge v) = x} f(x) \\
&= f(x)\,,
\end{aligned}$$

where the first inequality is because $f \in \mathcal{I}_\sqcup$ and the second one because $f \in \mathcal{C}$. □

Consequently, if we want to ensure the absorption laws for the convolution operations, we are forced to consider the set $\mathcal{N}_a \cap \mathcal{I} \cap \mathcal{C}$ for some $a \in L_2$ (or a subset of it).

5. Algebraic structure

5.1. Closedness of the sets of functions considered

In the study of the lattice laws of the convolution operations, some restrictions on the set of lattice functions have appeared naturally. More specifically, the subsets $\mathcal{N}_a$ (with $a \in L_2$), $\mathcal{I}_\sqcup$, $\mathcal{I}_\sqcap$, $\mathcal{I}$ and $\mathcal{C}$ have been considered, as well as the subsets $\mathcal{M}_I$ and $\mathcal{M}_D$. However, we have not yet verified whether these subsets of $\mathcal{F}$ are closed under the convolution operations, or, in other words, we have not yet verified whether the convolution operations are internal on these subsets. This issue is addressed in this subsection.

Proposition 6. *The sets $\mathcal{N}_a$ (with $a \in L_2$) are closed under join- and meet-convolution.*

Proof. Let $f, g \in \mathcal{N}_a$ for some $a \in L_2$. We only provide the proof that $\mathcal{N}_a$ is closed under join-convolution, the proof that $\mathcal{N}_a$ is closed under meet-convolution being analogous. For any $x \in L_1$, it holds that

$$\begin{aligned}
\bigvee_{x\in L_1} (f \sqcup g)(x) &= \bigvee_{x\in L_1} \bigvee_{u\vee v=x} f(u) \wedge g(v) \\
&= \bigvee_{(u\vee v)\in L_1} f(u) \wedge g(v) \\
&= \bigvee_{\substack{u\in L_1\\ v\in L_1}} f(u) \wedge g(v) \\
&= \left(\bigvee_{u\in L_1} f(u)\right) \wedge \left(\bigvee_{v\in L_1} g(v)\right) \\
&= a \wedge a = a\,.
\end{aligned}$$

□

Proposition 7. *The set $\mathcal{I}_\sqcup$ is closed under join-convolution and the set $\mathcal{I}_\sqcap$ is closed under meet-convolution.*

Proof. We only provide the proof that $\mathcal{I}_\sqcup$ is closed under join-convolution, the proof that $\mathcal{I}_\sqcap$ is closed under meet-convolution being analogous.

Let $f, g \in \mathcal{I}_\sqcup$. For any $x_1, x_2 \in L_1$, it holds that

$$\begin{aligned}
(f \sqcup g)(x_1) \wedge (f \sqcup g)(x_2) &= \left(\bigvee_{u_1\vee v_1=x_1} f(u_1) \wedge g(v_1)\right) \wedge \left(\bigvee_{u_2\vee v_2=x_2} f(u_2) \wedge g(v_2)\right) \\
&= \bigvee_{\substack{u_1\vee v_1=x_1\\ u_2\vee v_2=x_2}} f(u_1) \wedge g(v_1) \wedge f(u_2) \wedge g(v_2) \\
&\leq \bigvee_{\substack{u_1\vee v_1=x_1\\ u_2\vee v_2=x_2}} f(u_1 \vee u_2) \wedge g(v_1 \vee v_2)\,,
\end{aligned}$$

where the inequality holds due to $f, g \in \mathcal{I}_\sqcup$. Further, since $(u_1 \vee u_2) \vee (v_1 \vee v_2) = (u_1 \vee v_1) \vee (u_2 \vee v_2) = x_1 \vee x_2$, we find that

$$(f \sqcup g)(x_1) \wedge (f \sqcup g)(x_2) \leq (f \sqcup g)(x_1 \vee x_2)\,.$$

□

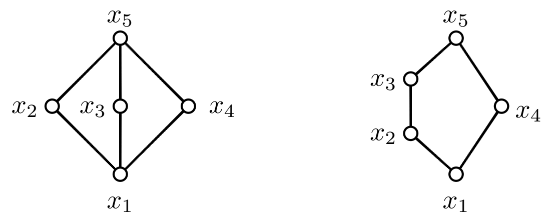
Proposition 8. *The set $\mathcal{M}_I$ is closed under join-convolution and the set $\mathcal{M}_D$ is closed under meet-convolution.*

Proof. We only provide the proof that $\mathcal{M}_I$ is closed under join-convolution, the proof that $\mathcal{M}_D$ is closed under meet-convolution being analogous.

Let $f, g \in \mathcal{M}_I$. Due to Proposition 2, for any $x_1, x_2 \in L_1$ such that $x_1 \leq x_2$, it holds that

$$(f \sqcup g)(x_1) = f(x_1) \wedge g(x_1) \leq f(x_2) \wedge g(x_2) = (f \sqcup g)(x_2)\,.$$

Hence, the set $\mathcal{M}_I$ is closed under join-convolution. □



(a) $\mathbb{M}_3$ (b) $\mathbb{N}_5$

Figure 6: Hasse diagram of: (a) the sublattice $\mathbb{M}_3$, and (b) the sublattice $\mathbb{N}_5$.

We will prove the closedness of the set $\mathcal{M}_D$ under join-convolution and of the set $\mathcal{M}_I$ under meet-convolution under the additional assumption that $\mathbb{L}_1$ is a distributive lattice; the latter will turn out to be a necessary and sufficient condition. Our proofs make extensive use of the famous $\mathbb{M}_3$–$\mathbb{N}_5$ theorem [33].

Theorem 5. *A lattice $\mathbb{L}$ is not distributive if and only if it has a sublattice that is isomorphic to the lattice $\mathbb{M}_3$ or the lattice $\mathbb{N}_5$ (see Figs. 6(a)–(b)).*

Proposition 9.

(i) The set $\mathcal{M}_D$ is closed under join-convolution if and only if $\mathbb{L}_1$ is a distributive lattice.

(ii) The set $\mathcal{M}_I$ is closed under meet-convolution if and only if $\mathbb{L}_1$ is a distributive lattice.

Proof. We only provide the proof that $\mathcal{M}_D$ is closed under join-convolution if and only if $\mathbb{L}_1$ is a distributive lattice, the proof that $\mathcal{M}_I$ is closed under meet-convolution if and only if $\mathbb{L}_1$ is a distributive lattice being analogous.

$\Rightarrow$ Suppose that $\mathcal{M}_D$ is closed under join-convolution, while $\mathbb{L}_1$ is not distributive. Due to Theorem 5, $\mathbb{L}_1$ has a sublattice that is isomorphic to $\mathbb{M}_3$ or to $\mathbb{N}_5$. We distinguish two cases.

(a) The case that $\mathbb{L}_1$ has a sublattice isomorphic to $\mathbb{M}_3$. We refer to the elements of this sublattice as in Fig. 6(a). We consider the functions $f, g \in \mathcal{M}_D$ defined as:

$$f(x) = \begin{cases} 1 & , \text{ if } x \leq x_2 \,, \\ 0 & , \text{ otherwise} \,; \end{cases} \qquad g(x) = \begin{cases} 1 & , \text{ if } x \leq x_3 \,, \\ 0 & , \text{ otherwise} \,. \end{cases}$$

It holds that $(f \sqcup g)(x_5) \geq f(x_2) \wedge g(x_3) = 1$. Moreover,

$$
\begin{aligned}
(f \sqcup g)(x_4) &= \bigvee_{u \vee v = x_4} f(u) \wedge g(v) \\
&= \left(\bigvee_{\substack{u \vee v = x_4 \\ u \nleq x_2 \text{ or } v \nleq x_3}} f(u) \wedge g(v) \right) \vee \left(\bigvee_{\substack{u \vee v = x_4 \\ u \leq x_2 \text{ and } v \leq x_3}} f(u) \wedge g(v) \right) \\
&= 0 \vee \left(\bigvee_{\substack{u \vee v = x_4 \\ u \leq x_2 \text{ and } v \leq x_3}} 1 \right),
\end{aligned}
$$

which equals 1 unless the set

$$U = \{(u, v) \in L_1^2 \mid u \vee v = x_4\,, u \leq x_2 \text{ and } v \leq x_3\}$$

is empty.

For any $u \in L_1$ such that $u \vee v = x_4$ and $u \leq x_2$, it follows that $u \leq x_2 \wedge x_4 = x_1$. Analogously, for any $v \in L_1$ such that $u \vee v = x_4$ and $v \leq x_3$, it follows that $v \leq x_3 \wedge x_4 = x_1$. It then follows that $x_4 = u \vee v \leq x_1 \vee x_1 = x_1$, a contradiction. Consequently, $U = \emptyset$ and $(f \sqcup g)(x_4) = 0$. We conclude that $(f \sqcup g)(x_4) = 0 < 1 = (f \sqcup g)(x_5)$, and, hence, $f \sqcup g \notin \mathcal{M}_D$.

(b) The case that $\mathbb{L}_1$ has a sublattice isomorphic to $\mathbb{N}_5$. We refer to the elements of this sublattice as in Fig. 6(b). We consider the functions $f, g \in \mathcal{M}_D$ defined as:

$$f(x) = \begin{cases} 1 & , \text{ if } x \leq x_2\,, \\ 0 & , \text{ otherwise}\,; \end{cases} \qquad g(x) = \begin{cases} 1 & , \text{ if } x \leq x_4\,, \\ 0 & , \text{ otherwise}\,. \end{cases}$$

It holds that $(f \sqcup g)(x_5) \geq f(x_2) \wedge g(x_4) = 1$. Moreover,

$$
\begin{aligned}
(f \sqcup g)(x_3) &= \bigvee_{u \vee v = x_3} f(u) \wedge g(v) \\
&= \left(\bigvee_{\substack{u \vee v = x_3 \\ u \nleq x_2 \text{ or } v \nleq x_4}} f(u) \wedge g(v) \right) \vee \left(\bigvee_{\substack{u \vee v = x_3 \\ u \leq x_2 \text{ and } v \leq x_4}} f(u) \wedge g(v) \right) \\
&= 0 \vee \left(\bigvee_{\substack{u \vee v = x_3 \\ u \leq x_2 \text{ and } v \leq x_4}} 1 \right),
\end{aligned}
$$

which equals 1 unless the set

$$U = \{(u, v) \in L_1^2 \mid u \vee v = x_3\,, u \leq x_2 \text{ and } v \leq x_4\}$$

is empty.

For any $u \in L_1$ such that $u \vee v = x_3$ and $u \leq x_2$, it follows that $u \leq x_2 \wedge x_3 = x_2$. Analogously, for any $v \in L_1$ such that $u \vee v = x_3$ and $v \leq x_4$, it follows that $v \leq x_3 \wedge x_4 = x_1$. It then follows that $x_3 = u \vee v \leq x_2 \vee x_1 = x_2$, a contradiction. Consequently, $U = \emptyset$ and $(f \sqcup g)(x_3) = 0$. We conclude that $(f \sqcup g)(x_3) = 0 < 1 = (f \sqcup g)(x_5)$, and, hence, $f \sqcup g \notin \mathcal{M}_D$.

$\Leftarrow$ Let $\mathbb{L}_1$ be a distributive lattice and $f, g \in \mathcal{M}_D$. For any $x_1, x_2 \in L_1$ such that $x_1 \leq x_2$ and for any couple (u_2, v_2) such that $u_2 \vee v_2 = x_2$, it holds that $u_2 \wedge x_1 \leq u_2$ and $v_2 \wedge x_1 \leq v_2$. Hence, since $f, g \in \mathcal{M}_D$, it holds that

$$\begin{aligned}(f \sqcup g)(x_2) &= \bigvee_{u_2 \vee v_2 = x_2} f(u_2) \wedge g(v_2) \\ &\leq \bigvee_{u_2 \vee v_2 = x_2} f(u_2 \wedge x_1) \wedge g(v_2 \wedge x_1)\,.\end{aligned}$$

Further, since $\mathbb{L}_1$ is distributive, it follows that $(u_2 \wedge x_1) \vee (v_2 \wedge x_1) = (u_2 \vee v_2) \wedge x_1 = x_2 \wedge x_1 = x_1$ and, hence, we find that

$$(f \sqcup g)(x_2) \leq \bigvee_{u \vee v = x_1} f(u) \wedge g(v) = (f \sqcup g)(x_1)\,.$$

□

Note that in Proposition 7 we have neither studied the closedness of the set $\mathcal{I}_\sqcap$ under join-convolution nor the closedness of the set $\mathcal{I}_\sqcup$ under meet-convolution. Moreover, the only set of which we have not yet studied the closedness is $\mathcal{C}$. As we show in the following example, neither $\mathcal{I}_\sqcap$ nor $\mathcal{C}$ is closed under join-convolution. Similarly, it can be shown that neither $\mathcal{I}_\sqcup$ nor $\mathcal{C}$ is closed under meet-convolution.

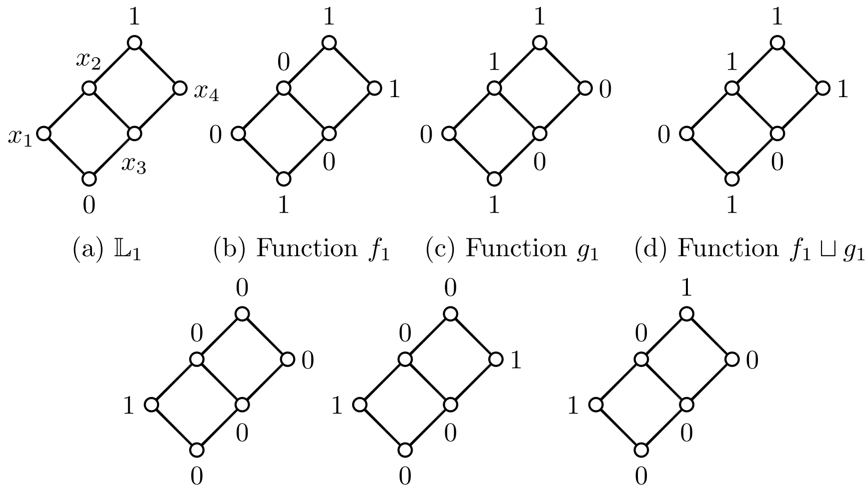
Example 5. *Let $\mathbb{L}_1$ be the distributive lattice with Hasse diagram depicted in Fig. 7(a) and $\mathbb{L}_2 = \mathbb{B}$.*

(i) Consider the functions $f_1, g_1 \in \mathcal{I}_\sqcap$ depicted in Figs. 7(b)–(c). The join-convolution $f_1 \sqcup g_1$ is depicted in Fig. 7(d). One easily verifies that $x_2 \wedge x_4 = x_3$, while $(f_1 \sqcup g_1)(x_2) \wedge (f_1 \sqcup g_1)(x_4) = 1 > (f_1 \sqcup g_1)(x_3) = 0$. Hence, $f_1 \sqcup g_1 \notin \mathcal{I}_\sqcap$.

(ii) Consider the functions $f_2, g_2 \in \mathcal{C}$ depicted in Figs. 7(e)–(f). The join-convolution $f_2 \sqcup g_2$ is depicted in Fig. 7(g). One easily verifies $x_1 \leq x_2 \leq 1$, while $(f_2 \sqcup g_2)(x_1) \wedge (f_2 \sqcup g_2)(1) = 1 > 0 = (f_2 \sqcup g_2)(x_2)$. Hence, $f_2 \sqcup g_2 \notin \mathcal{C}$.

Note that since the absorption laws surely do not hold outside the subset of lattice functions $\mathcal{I} \cap \mathcal{C}$, the closedness of the set $\mathcal{C}$ (under the convolution operations) and the closedness of the sets $\mathcal{I}_\sqcap$ (under join-convolution) and $\mathcal{I}_\sqcup$ (under meet-convolution) are crucial.

However, the functions f_1, g_1 in Example 5(i) are not convex, while the function g_2 in Example 5(ii) is not idempotent. We could therefore investigate



(a) $\mathbb{L}_1$ (b) Function f_1 (c) Function g_1 (d) Function $f_1 \sqcup g_1$

(e) Function f_2 (f) Function g_2 (g) Function $f_2 \sqcup g_2$

Figure 7: Graphical representation of the functions in Example 5: (a) the Hasse diagram of the lattice $\mathbb{L}_1$, (b) the function f_1, (c) the function g_1, (d) the join-convolution $f_1 \sqcup g_1$, (e) the function f_2, (f) the function g_2, and (g) the join-convolution $f_2 \sqcup g_2$.

the closedness of $\mathcal{I} \cap \mathcal{C}$. We will prove the closedness of this subset under the additional assumption that $\mathbb{L}_1$ is a distributive lattice; the latter will turn out to be a necessary and sufficient condition once again.

Theorem 6.

(i) The set $\mathcal{I} \cap \mathcal{C}$ is closed under join-convolution if and only if $\mathbb{L}_1$ is a distributive lattice.

(ii) The set $\mathcal{I} \cap \mathcal{C}$ is closed under meet-convolution if and only if $\mathbb{L}_1$ is a distributive lattice.

Proof. We only provide the proof that $\mathcal{I} \cap \mathcal{C}$ is closed under join-convolution if and only if $\mathbb{L}_1$ is a distributive lattice, the proof that $\mathcal{I} \cap \mathcal{C}$ is closed under meet-convolution if and only if $\mathbb{L}_1$ is a distributive lattice being analogous.

$\Rightarrow$ Suppose that $\mathcal{I} \cap \mathcal{C}$ is closed under join-convolution, while $\mathbb{L}_1$ is not distributive. Due to Theorem 5, $\mathbb{L}_1$ has a sublattice that is isomorphic to $\mathbb{M}_3$ or to $\mathbb{N}_5$. We distinguish two cases.

(a) The case that $\mathbb{L}_1$ has a sublattice isomorphic to $\mathbb{M}_3$. We refer to the elements of this sublattice as in Fig. 6(a). We consider the functions $f, g \in \mathcal{I} \cap \mathcal{C}$ defined as:

$$f(x) = \begin{cases} 1 & \text{, if } x \in \{x_1, x_2\}\,, \\ 0 & \text{, otherwise}\,; \end{cases} \qquad g(x) = \begin{cases} 1 & \text{, if } x \in \{x_1, x_3\}\,, \\ 0 & \text{, otherwise}\,. \end{cases}$$

It holds that $(f \sqcup g)(x) = 0$ for any $x \in L_1$ unless $x \in \{x_1, x_2, x_3, x_5\}$, where $f \sqcup g$ takes the value 1. Since $x_1 \leq x_4 \leq x_5$ and $(f \sqcup g)(x_4) = 0 < 1 = (f \sqcup g)(x_1) \wedge (f \sqcup g)(x_5)$, we conclude that $f \sqcup g \notin \mathcal{C}$, a contradiction.

(b) The case that $\mathbb{L}_1$ has a sublattice isomorphic to $\mathbb{N}_5$. We refer to the elements of this sublattice as in Fig. 6(b). We consider the functions $f, g \in \mathcal{I} \cap \mathcal{C}$ defined as:

$$f(x) = \begin{cases} 1 & \text{, if } x = x_2\,, \\ 0 & \text{, otherwise}\,; \end{cases} \qquad g(x) = \begin{cases} 1 & \text{, if } x \in \{x_1, x_4\}\,, \\ 0 & \text{, otherwise}\,. \end{cases}$$

It holds that $(f \sqcup g)(x) = 0$ for any $x \in L_1$ unless $x \in \{x_2, x_5\}$, where $f \sqcup g$ takes the value 1. Since $x_2 \leq x_3 \leq x_5$ and $(f \sqcup g)(x_3) = 0 < 1 = (f \sqcup g)(x_2) \wedge (f \sqcup g)(x_5)$, we conclude that $f \sqcup g \notin \mathcal{C}$, a contradiction.

$\Leftarrow$ Let $\mathbb{L}_1$ be a distributive lattice and $f, g \in \mathcal{I} \cap \mathcal{C}$. Since $\mathcal{I}_{\sqcup}$ is closed under join-convolution, it holds that $f \sqcup g \in \mathcal{I}_{\sqcup}$ and we only need to show that $f \sqcup g \in \mathcal{I}_{\sqcap} \cap \mathcal{C}$.

Firstly, we prove that $f \sqcup g \in \mathcal{I}_{\sqcap}$. For any $x_1, x_2 \in L_1$, it holds that

$$\begin{aligned}(f \sqcup g)(x_1) \wedge (f \sqcup g)(x_2) &= \left(\bigvee_{u_1 \vee v_1 = x_1} f(u_1) \wedge g(v_1)\right) \wedge \left(\bigvee_{u_2 \vee v_2 = x_2} f(u_2) \wedge g(v_2)\right) \\ &= \bigvee_{\substack{u_1 \vee v_1 = x_1 \\ u_2 \vee v_2 = x_2}} f(u_1) \wedge f(u_2) \wedge g(v_1) \wedge g(v_2)\,.\end{aligned}$$

Since $f \in \mathcal{I}$, it holds that $f(u_1) \wedge f(u_2) \leq f(u_1 \vee u_2)$ and $f(u_1) \wedge f(u_2) \leq f(u_1 \wedge u_2)$, and, hence, $f(u_1) \wedge f(u_2) \leq f(u_1 \wedge u_2) \wedge f(u_1 \vee u_2)$. Similarly, since $g \in \mathcal{I}$, it holds that $g(v_1) \wedge g(v_2) \leq g(v_1 \wedge v_2) \wedge g(v_1 \vee v_2)$. This leads to

$$\begin{aligned}&(f \sqcup g)(x_1) \wedge (f \sqcup g)(x_2) \\ &\qquad \leq \bigvee_{\substack{u_1 \vee v_1 = x_1 \\ u_2 \vee v_2 = x_2}} f(u_1 \wedge u_2) \wedge f(u_1 \vee u_2) \wedge g(v_1 \wedge v_2) \wedge g(v_1 \vee v_2)\,.\end{aligned}$$

Taking into account that $u_1 \leq u_1 \vee v_1 = x_1$ and $u_2 \leq u_2 \vee v_2 = x_2$, it holds that $u_1 \wedge u_2 \leq x_1 \wedge x_2$. Moreover, since $u_1 \wedge u_2 \leq u_1 \vee u_2$, we find that

$$u_1 \wedge u_2 \leq (x_1 \wedge x_2) \wedge (u_1 \vee u_2) \leq (u_1 \vee u_2)\,.$$

Analogously, it follows that

$$v_1 \wedge v_2 \leq (x_1 \wedge x_2) \wedge (v_1 \vee v_2) \leq (v_1 \vee v_2)\,.$$

Since $f, g \in \mathcal{C}$, it holds that

$$\begin{aligned}&(f \sqcup g)(x_1) \wedge (f \sqcup g)(x_2) \\ &\qquad \leq \bigvee_{\substack{u_1 \vee v_1 = x_1 \\ u_2 \vee v_2 = x_2}} f((x_1 \wedge x_2) \wedge (u_1 \vee u_2)) \wedge g((x_1 \wedge x_2) \wedge (v_1 \vee v_2))\,.\end{aligned}$$

Finally, since $\mathbb{L}_1$ is a distributive lattice, it holds that

$$\begin{aligned}((x_1 \wedge x_2) \wedge (u_1 \vee u_2)) \vee ((x_1 \wedge x_2) \wedge (v_1 \vee v_2))&\\ &= (x_1 \wedge x_2) \wedge ((u_1 \vee u_2) \vee (v_1 \vee v_2))\\ &= (x_1 \wedge x_2) \wedge ((u_1 \vee v_1) \vee (u_2 \vee v_2))\\ &= (x_1 \wedge x_2) \wedge (x_1 \vee x_2) = x_1 \wedge x_2\,.\end{aligned}$$

Denoting $u = (x_1 \wedge x_2) \wedge (u_1 \vee u_2)$ and $v = (x_1 \wedge x_2) \wedge (v_1 \vee v_2)$, it holds that

$$(f \sqcup g)(x_1) \wedge (f \sqcup g)(x_2) \leq \bigvee_{u \vee v = x_1 \wedge x_2} f(u) \wedge g(v) = (f \sqcup g)(x_1 \wedge x_2)\,.$$

Consequently, $f \sqcup g \in \mathcal{I}_{\sqcap}$.

Secondly, we prove that $f \sqcup g \in \mathcal{C}$. For any $x_1, x_2, x_3 \in L_1$ such that $x_1 \leq x_2 \leq x_3$, it holds that

$$\begin{aligned}(f \sqcup g)(x_1) \wedge (f \sqcup g)(x_3) &= \left(\bigvee_{u_1 \vee v_1 = x_1} f(u_1) \wedge g(v_1)\right) \wedge \left(\bigvee_{u_3 \vee v_3 = x_3} f(u_3) \wedge g(v_3)\right)\\ &= \bigvee_{\substack{u_1 \vee v_1 = x_1\\ u_3 \vee v_3 = x_3}} f(u_1) \wedge f(u_3) \wedge g(v_1) \wedge g(v_3)\,.\end{aligned}$$

Analogously to the case $\mathcal{I}_{\sqcap}$, since $f, g \in \mathcal{I}$, it holds that

$$f(u_1) \wedge f(u_3) \leq f(u_1 \wedge u_3) \wedge f(u_1 \vee u_3)$$

and

$$g(v_1) \wedge g(v_3) \leq g(v_1 \wedge v_3) \wedge g(v_1 \vee v_3)\,.$$

This leads to

$$\begin{aligned}&(f \sqcup g)(x_1) \wedge (f \sqcup g)(x_3)\\ &\qquad\leq \bigvee_{\substack{u_1 \vee v_1 = x_1\\ u_3 \vee v_3 = x_3}} (f(u_1 \wedge u_3) \wedge f(u_1 \vee u_3)) \wedge (g(v_1 \wedge v_3) \wedge g(v_1 \vee v_3))\,.\end{aligned}$$

Taking into account that $u_1 \wedge u_3 \leq u_1 \leq u_1 \vee v_1 = x_1 \leq x_2$ and $u_1 \wedge u_3 \leq u_1 \vee u_3$, it holds that

$$u_1 \wedge u_3 \leq x_2 \wedge (u_1 \vee u_3) \leq u_1 \vee u_3\,.$$

Analogously, it follows that

$$v_1 \wedge v_3 \leq x_2 \wedge (v_1 \vee v_3) \leq v_1 \vee v_3\,.$$

Since $f, g \in \mathcal{C}$, it holds that

$$\begin{aligned}&(f \sqcup g)(x_1) \wedge (f \sqcup g)(x_3)\\ &\qquad\leq \bigvee_{\substack{u_1 \vee v_1 = x_1\\ u_3 \vee v_3 = x_3}} (f(x_2 \wedge (u_1 \vee u_3))) \wedge (g(x_2 \wedge (v_1 \vee v_3)))\,.\end{aligned}$$

Finally, since $\mathbb{L}_1$ is a distributive lattice, it holds that

$$\begin{aligned}(x_2 \wedge (u_1 \vee u_3)) \vee (x_2 \wedge (v_1 \vee v_3)) &= x_2 \wedge ((u_1 \vee u_3) \vee (v_1 \vee v_3)) \\ &= x_2 \wedge ((u_1 \vee v_1) \vee (u_3 \vee v_3)) \\ &= x_2 \wedge (x_1 \vee x_3) = x_2 \wedge x_3 = x_2 \,.\end{aligned}$$

Denoting $u_2 = x_2 \wedge (u_1 \vee u_3)$ and $v_2 = x_2 \wedge (v_1 \vee v_3)$, it holds that

$$(f \sqcup g)(x_1) \wedge (f \sqcup g)(x_3) \leq \bigvee_{u_2 \vee v_2 = x_2} f(u_2) \wedge g(v_2) = (f \sqcup g)(x_2)\,.$$

Consequently, $f \sqcup g \in \mathcal{C}$. $\square$

5.2. Algebraic structures

Finally, in this subsection we conclude which types of algebra the convolution operations constitute on the different subsets of lattice functions considered. The following results are direct consequences of Section 4 and Subsection 5.1.

Recall that a monoid is a set equipped with an associative binary operation that has an identity element [32]. Moreover, if this operation is commutative, then the monoid is called commutative as well. In general, due to Theorem 2, the following proposition holds.

Proposition 10. *The algebraic structures* $\mathbb{F} = (\mathcal{F}, \sqcup, \mathbf{0}_1)$ *and* $\mathbb{F} = (\mathcal{F}, \sqcap, \mathbf{1}_1)$ *are commutative monoids.*

Propositions 3, 6 and Theorem 2 lead to the following observation.

Proposition 11. *Let* $a \in L_2$*.*

(i) The algebraic structure $(\mathcal{N}_a, \sqcup, \mathbf{0}_a)$ *is a commutative monoid with absorbing element* $\mathbf{1}_a$*.*

(ii) The algebraic structure $(\mathcal{N}_a, \sqcap, \mathbf{1}_a)$ *is a commutative monoid with absorbing element* $\mathbf{0}_a$*.*

Recall that a semilattice is a set equipped with an idempotent, commutative and associative binary operation [32]. Theorems 2 and 3 and Proposition 7 lead to the following observation.

Proposition 12. *The algebraic structures* $(\mathcal{I}_\sqcup, \sqcup)$ *and* $(\mathcal{I}_\sqcap, \sqcap)$ *are semilattices.*

Recall that in case $\mathbb{L}_1$ is a bounded chain, it holds that $\mathcal{I}_\sqcup = \mathcal{I}_\sqcap = \mathcal{F}$. Hence, if $\mathbb{L}_1$ is a chain, then the algebraic structures $(\mathcal{F}, \sqcup)$ and $(\mathcal{F}, \sqcap)$ are semilattices.

Finally, Theorems 4 and 6 and Propositions 11 and 12 lead to the central result of this paper.

Theorem 7. *The algebraic structure* $\mathbb{F} = (\mathcal{N}_a \cap \mathcal{I} \cap \mathcal{C}, \sqcup, \sqcap, \mathbf{0}_a, \mathbf{1}_a)$ *(with* $a \in L_2$*) is a bounded lattice if and only if* $\mathbb{L}_1$ *is a distributive lattice.*

The preceding result justifies the name *convolution lattice* for the algebraic structure $\mathbb{F} = (\mathcal{N}_a \cap \mathcal{I} \cap \mathcal{C}, \sqcup, \sqcap, \mathbf{0}_a, \mathbf{1}_a)$ (with $a \in L_2$).

Remark 5. Note that the preceding theorem expresses that the convolution operations constitute a bounded lattice on the maximal set $\mathcal{N}_a \cap \mathcal{I} \cap \mathcal{C}$ (with $a \in L_2$) if and only if $\mathbb{L}_1$ is a distributive lattice. However, even if $\mathbb{L}_1$ is not distributive, we can still find a smaller set $\mathcal{G} \subset \mathcal{N}_a \cap \mathcal{I} \cap \mathcal{C}$, closed under the convolution operations, such that these operations constitute a bounded lattice on $\mathcal{G}$. For instance, one easily verifies that the sets $\mathcal{S}_a \subset \mathcal{N}_a \cap \mathcal{I} \cap \mathcal{C}$ (with $a \in L_2$) given by

$$\mathcal{S}_a = \{f \in \mathcal{F} \mid (\exists x^* \in L_1)(f(x^*) = a \,\text{and}\, (\forall x \in L_1)(x \neq x^* \Rightarrow f(x) = 0))\}\,,$$

are closed under the convolution operations (whether or not $\mathbb{L}_1$ is distributive). Moreover, since $\mathbf{0}_a, \mathbf{1}_a \in \mathcal{S}_a$, we find that the algebraic structure $\mathbb{F} = (\mathcal{S}_a, \sqcup, \sqcap, \mathbf{0}_a, \mathbf{1}_a)$ (with $a \in L_2$) constitutes a bounded lattice independently of the distributivity of $\mathbb{L}_1$. Unfortunately, from an algebraic point of view, the sets $\mathcal{S}_a$ are of no real interest.

6. Distributivity laws

The distributivity of $\mathbb{L}_1$ plays a decisive role in the constitution of the bounded lattice $\mathbb{F} = (\mathcal{N}_a \cap \mathcal{I} \cap \mathcal{C}, \sqcup, \sqcap, \mathbf{0}_a, \mathbf{1}_a)$ (with $a \in L_2$). A natural question that arises is whether or not the convolution operations satisfy the distributivity laws. In general, the following inequalities hold.

Proposition 13. *Let $\mathbb{L}_1$ be a distributive lattice and $f, g, h \in \mathcal{F}$. The following statements hold:*

(i) $f \sqcup (g \sqcap h) \leq (f \sqcup g) \sqcap (f \sqcup h)$;

(ii) $f \sqcap (g \sqcup h) \leq (f \sqcap g) \sqcup (f \sqcap h)\,.$

Proof. We only provide the proof of statement (i), the proof of statement (ii) being analogous. For any $x \in L_1$, due to the distributivity of $\mathbb{L}_1$, it holds that

$$\begin{aligned}
(f \sqcup (g \sqcap h))(x) &= \bigvee_{u \vee (v \wedge w) = x} f(u) \wedge g(v) \wedge h(w) \\
&= \bigvee_{(u \vee v) \wedge (u \vee w) = x} (f(u) \wedge g(v)) \wedge (f(u) \wedge h(w)) \\
&\leq ((f \sqcup g) \sqcap (f \sqcup h))\,(x)\,.
\end{aligned}$$

□

In the following example, we show that Proposition 13 no longer holds in general when $\mathbb{L}_1$ is not distributive.

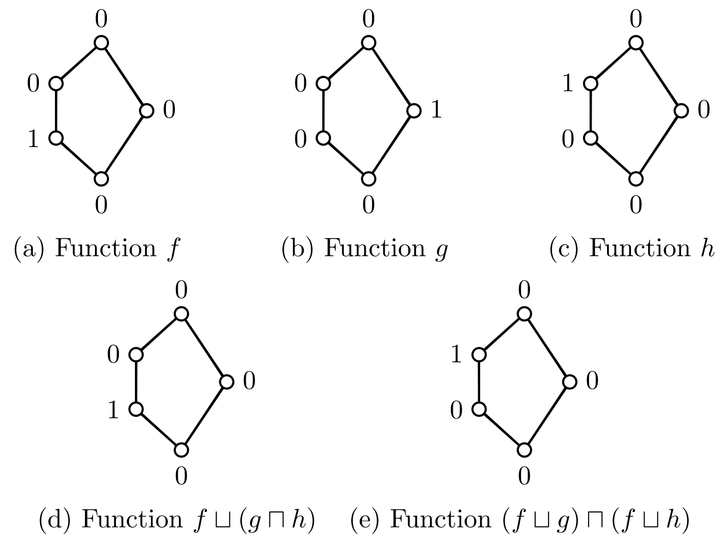


Figure 8: Graphical representation of the functions in Example 6: (a) the function f, (b) the function g, (c) the function h, (d) the corresponding function $f \sqcup (g \sqcap h)$, and (e) the corresponding function $(f \sqcup g) \sqcap (f \sqcup h)$.

Example 6. *Let $\mathbb{L}_1$ be the non-distributive lattice $\mathbb{N}_5$ and $\mathbb{L}_2 = \mathbb{B}$. Consider the functions $f, g, h \in \mathcal{F}$ depicted in Figs. 8(a)–(c). The corresponding functions $f \sqcup (g \sqcap h)$ and $(f \sqcup g) \sqcap (f \sqcup h)$ are depicted in Figs. 8(d)–(e). One easily verifies that neither $f \sqcup (g \sqcap h) \leq (f \sqcup g) \sqcap (f \sqcup h)$ nor $f \sqcup (g \sqcap h) \geq (f \sqcup g) \sqcap (f \sqcup h)$ holds. A similar example can be given where the roles of the convolution operations are exchanged.*

In the following example, we show that the inequality in Proposition 13 can be strict.

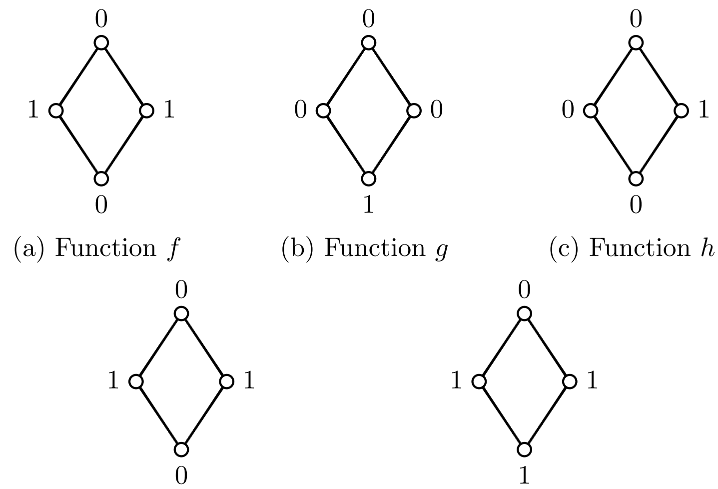
Example 7. *Let $\mathbb{L}_1 = \mathbb{M}_2$ and $\mathbb{L}_2 = \mathbb{B}$. Consider the function $f \notin \mathcal{I}_\sqcup$ depicted in Fig. 9(a) and the functions $g, h \in \mathcal{F}$ depicted in Figs. 9(b)–(c). The corresponding functions $f \sqcup (g \sqcap h)$ and $(f \sqcup g) \sqcap (f \sqcup h)$ are depicted in Figs. 9(d)–(e). One easily verifies that $f \sqcup (g \sqcap h) < (f \sqcup g) \sqcap (f \sqcup h)$.*

In the following theorem, we show that the inequalities in Proposition 13 turn into equalities when restricting to the set of functions that are idempotent and convex.

Theorem 8. *Let $\mathbb{L}_1$ be a distributive lattice. If $f \in \mathcal{I} \cap \mathcal{C}$, then the following statements hold:*

(i) $f \sqcup (g \sqcap h) = (f \sqcup g) \sqcap (f \sqcup h)$*, for any* $g, h \in \mathcal{F}$*;*

(ii) $f \sqcap (g \sqcup h) = (f \sqcap g) \sqcup (f \sqcap h)$*, for any* $g, h \in \mathcal{F}$*.*



(a) Function f (b) Function g (c) Function h

(d) Function $f \sqcup (g \sqcap h)$ (e) Function $(f \sqcup g) \sqcap (f \sqcup h)$

Figure 9: Graphical representation of the functions in Example 7: (a) the function f, (b) the functions g, (c) the functions h. (d) the corresponding function $f \sqcup (g \sqcap h)$, and (e) the corresponding function $(f \sqcup g) \sqcap (f \sqcup h)$.

Proof. We only provide the proof of statement (i), the proof of statement (ii) being analogous. Suppose $f \in \mathcal{I} \cap \mathcal{C}$. Due to Proposition 13, it holds that $f \sqcup (g \sqcap h) \leq (f \sqcup g) \sqcap (f \sqcup h)$, so it only remains to prove that $(f \sqcup g) \sqcap (f \sqcup h) \leq f \sqcup (g \sqcap h)$, i.e., we need to verify that, for any $x \in L_1$, it holds that

$$\begin{aligned}
((f \sqcup g) \sqcap (f \sqcup h))(x) &= \bigvee_{(u_1 \vee v) \wedge (u_2 \vee w) = x} f(u_1) \wedge g(v) \wedge f(u_2) \wedge h(w) \\
&= \bigvee_{(u_1 \vee v) \wedge (u_2 \vee w) = x} f(u_1) \wedge f(u_2) \wedge g(v) \wedge h(w) \\
&\leq \bigvee_{u \vee (v \wedge w) = x} f(u) \wedge g(v) \wedge h(w) = (f \sqcup (g \sqcap h))(x)\,.
\end{aligned}$$

Since $f \in \mathcal{I}$, it holds that $f(u_1) \wedge f(u_2) \leq f(u_1 \vee u_2)$ and $f(u_1) \wedge f(u_2) \leq f(u_1 \wedge u_2)$, and hence $f(u_1) \wedge f(u_2) \leq f(u_1 \wedge u_2) \wedge f(u_1 \vee u_2)$. This leads to

$$\begin{aligned}
((f \sqcup g) \sqcap (f \sqcup h))(x) &= \bigvee_{(u_1 \vee v) \wedge (u_2 \vee w) = x} f(u_1) \wedge f(u_2) \wedge g(v) \wedge h(w) \\
&\leq \bigvee_{(u_1 \vee v) \wedge (u_2 \vee w) = x} f(u_1 \wedge u_2) \wedge f(u_1 \vee u_2) \wedge g(v) \wedge h(w)\,.
\end{aligned}$$

Taking into account that $u_1 \wedge u_2 \leq (u_1 \vee v) \wedge (u_2 \vee w) = x$, as well as $u_1 \wedge u_2 \leq u_1 \vee u_2$, it follows that

$$u_1 \wedge u_2 \leq x \wedge (u_1 \vee u_2) \leq u_1 \vee u_2\,.$$

Since $f \in \mathcal{C}$, it holds that

$$
\begin{aligned}
((f \sqcup g) \sqcap (f \sqcup h))(x) &\leq \bigvee_{(u_1 \vee v) \wedge (u_2 \vee w) = x} f(u_1 \wedge u_2) \wedge f(u_1 \vee u_2) \wedge g(v) \wedge h(w) \\
&\leq \bigvee_{(u_1 \vee v) \wedge (u_2 \vee w) = x} f(x \wedge (u_1 \vee u_2)) \wedge g(v) \wedge h(w) \,.
\end{aligned}
$$

Taking into account that $(u_1 \vee v) \wedge (u_2 \vee w) = x$, it holds that

$$
\begin{aligned}
& v \wedge w \leq (v \vee u_1) \wedge (w \vee u_2) = x \\
& x = (v \vee u_1) \wedge (w \vee u_2) \leq v \vee u_1 \\
& x = (v \vee u_1) \wedge (w \vee u_2) \leq w \vee u_2 \,.
\end{aligned} \tag{3}
$$

On the one hand, since $x \wedge (u_1 \vee u_2) \leq x$ and $v \wedge w \leq x$ (Eq. (3)), we find that

$$
(x \wedge (u_1 \vee u_2)) \vee (v \wedge w) \leq x \,. \tag{4}
$$

On the other hand, since L_1 is a distributive lattice, it follows that

$$
\begin{aligned}
(x \wedge (u_1 \vee u_2)) \vee (v \wedge w) &= (x \vee (v \wedge w)) \wedge ((u_1 \vee u_2) \vee (v \wedge w)) \\
&= (x \vee (v \wedge w)) \wedge ((u_1 \vee u_2) \vee v) \wedge ((u_1 \vee u_2) \vee w) \\
&= (x \vee (v \wedge w)) \wedge \underbrace{((u_1 \vee v) \vee u_2)}_{(*)} \wedge \underbrace{(u_1 \vee (u_2 \vee w))}_{(**)} \\
&\geq x \wedge x \wedge x = x \,,
\end{aligned} \tag{5}
$$

where $(*)$ and $(**)$ are greater than or equal to x due to Eq. (3).

Due to Eqs. (4) and (5), it holds that $(x \wedge (u_1 \vee u_2)) \vee (v \wedge w) = x$. Denoting $u = x \wedge (u_1 \vee u_2)$, it follows that

$$
\begin{aligned}
((f \sqcup g) \sqcap (f \sqcup h))(x) &\leq \bigvee_{(u_1 \vee v) \wedge (u_2 \vee w) = x} f(x \wedge (u_1 \vee u_2)) \wedge g(v) \wedge h(w) \\
&\leq \bigvee_{u \vee (v \wedge w) = x} f(u) \wedge g(v) \wedge h(w) = (f \sqcup (g \sqcap h))(x) \,.
\end{aligned}
$$

□

In view of Theorem 6, the preceding theorem implies that if $\mathbb{L}_1$ is a distributive lattice, then the distributivity laws are satisfied in the set $\mathcal{I} \cap \mathcal{C}$. This allows us to further refine Theorem 7 into the following theorem.

Theorem 9. *The algebraic structure* $\mathbb{F} = (\mathcal{N}_a \cap \mathcal{I} \cap \mathcal{C}, \sqcup, \sqcap, \mathbf{0}_a, \mathbf{1}_a)$ *(with* $a \in L_2$*) is a bounded distributive lattice if and only if* $\mathbb{L}_1$ *is a distributive lattice.*

7. Birkhoff systems

A Birkhoff system is a more general algebraic structure than a lattice. It is defined as a set L equipped with two binary operations $\vee$ and $\wedge$ such that both $(L, \vee)$ and $(L, \wedge)$ are semilattices and they satisfy the Birkhoff equation, i.e., for any $a, b, c \in L$, it holds that $a \vee (a \wedge b) = a \wedge (a \vee b)$.

In this section we study for which subset of functions the convolution operations satisfy the Birkhoff equation.

Proposition 14. *Let $\mathbb{L}_1$ be a distributive lattice and $f \in \mathcal{F}$. The equality $f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g)$ holds for any $g \in \mathcal{F}$ if and only if $f \in \mathcal{I}$.*

Proof. $\Rightarrow$ Suppose that, for any $g \in \mathcal{F}$, it holds that

$$f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g)\,, \tag{6}$$

while $f \notin \mathcal{I}$. We distinguish two different cases.

(a) The case $f \notin \mathcal{I}_{\sqcup}$. Let $g = \mathbf{1}_{s_f}$. It holds that $f \sqcap g = f$ and $f \sqcup g = g$. Hence, it holds that

$$f \sqcup (f \sqcap g) = f \sqcup f$$

and

$$f \sqcap (f \sqcup g) = f \sqcup g = f\,.$$

Consequently, it holds that $f \sqcup f = f$, which contradicts $f \notin \mathcal{I}_{\sqcup}$.

(b) The case $f \notin \mathcal{I}_{\sqcap}$. Let $g = \mathbf{0}_{s_f}$. It holds that $f \sqcup g = f$ and $f \sqcap g = g$. Hence, it holds that

$$f \sqcup (f \sqcap g) = f \sqcup g = f\,,$$

and

$$f \sqcap (f \sqcup g) = f \sqcap f\,.$$

Consequently, it holds that $f = f \sqcap f$, which contradicts $f \notin \mathcal{I}_{\sqcap}$.

$\Leftarrow$ Let $f \in \mathcal{I}$. Due to the distributivity of $\mathbb{L}_1$ and the fact that $f \in \mathcal{I}_{\sqcup}$, for any $x \in L_1$, it holds that

$$\begin{aligned}
(f \sqcup (f \sqcap g))(x) &= \bigvee_{u_1 \vee (u_2 \wedge v) = x} f(u_1) \wedge f(u_2) \wedge g(v) \\
&= \bigvee_{(u_1 \vee u_2) \wedge (u_1 \vee v) = x} f(u_1) \wedge f(u_2) \wedge g(v) \\
&= \bigvee_{(u_1 \vee u_2) \wedge (u_1 \vee v) = x} f(u_1) \wedge f(u_2) \wedge f(u_1) \wedge g(v) \\
&\leq \bigvee_{(u_1 \vee u_2) \wedge (u_1 \vee v) = x} f(u_1 \vee u_2) \wedge f(u_1) \wedge g(v) \\
&\leq (f \sqcap (f \sqcup g))(x)\,.
\end{aligned}$$

Similarly, due to the distributivity of $\mathbb{L}_1$ and the fact that $f \in \mathcal{I}_\sqcap$, for any $x \in L_1$, it holds that

$$
\begin{aligned}
(f \sqcap (f \sqcup g))(x) &= \bigvee_{u_1 \wedge (u_2 \vee v) = x} f(u_1) \wedge f(u_2) \wedge g(v) \\
&= \bigvee_{(u_1 \wedge u_2) \vee (u_1 \wedge v) = x} f(u_1) \wedge f(u_2) \wedge g(v) \\
&= \bigvee_{(u_1 \wedge u_2) \vee (u_1 \wedge v) = x} f(u_1) \wedge f(u_2) \wedge f(u_1) \wedge g(v) \\
&\leq \bigvee_{(u_1 \wedge u_2) \vee (u_1 \wedge v) = x} f(u_1 \wedge u_2) \wedge f(u_1) \wedge g(v) \\
&\leq (f \sqcup (f \sqcap g))(x)\,.
\end{aligned}
$$

Hence, it holds that $f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g)$. □

Note that the Birkhoff equation is satisfied on the set $\mathcal{I}$ when $\mathbb{L}_1$ is distributive, while we have shown in Example 5(i) that the set of lattice functions $\mathcal{I}_\sqcap$ is not closed under join-convolution. Consequently, even if we restrict to a distributive lattice $\mathbb{L}_1$, the set $\mathcal{I}$ is not closed under the convolution operations. This means that the convolution operations do not generate a Birkhoff system on $\mathcal{I}$. However, as we have mentioned before, if $\mathbb{L}_1$ is a bounded chain, then $\mathcal{F} = \mathcal{I}$ and we can state the following interesting result.

Corollary 5. Let $\mathbb{L}_1$ be a chain. Then the algebra $\mathcal{F} = (\mathcal{F}, \sqcup, \sqcap)$ is a Birkhoff system.

8. Conclusions

In this paper, we have introduced two convolution operations on the set of lattice functions and have studied their algebraic properties. In particular, we have studied for which subsets of functions they generate a bounded lattice, which we have coined a convolution lattice. It has become clear that the distributivity of the lattice acting as domain of the functions is of primordial importance. We have been able to demonstrate that convolution lattices are distributive.

Several open problems and points of further interest are:

(i) A deeper study of the convolution operations when $\mathbb{L}_1$ is not distributive. Note that this study will include:

- (a) the search for subsets $\mathcal{G} \subset \mathcal{N}_a \cap \mathcal{I} \cap \mathcal{C}$ that are closed under the convolution operations and such that the operations constitute a bounded lattice on $\mathcal{G}$;
- (b) the study of the Birkhoff equation.

(ii) The search for specific classes of lattices such that the set of idempotent functions is closed under the convolution operations.

(iii) The study of the completeness of convolution lattices as well as the meet-continuity of the meet-convolution with the goal of characterizing when convolution lattices are frames.

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