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Orness measurements for lattice m-dimensional interval-valued OWA operators

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Abstract

Ordered weighted average (OWA) operators are commonly used to aggregate information in multiple situations, such as decision making problems or image processing tasks.

The great variety of weights that can be chosen to determinate an OWA operator provides a broad family of aggegating functions, which obviously give different results in the aggregation of the same set of data.

In this paper, some possible classifications of OWA operators are suggested when they are defined on m-dimensional intervals taking values on a complete lattice satisfying certain local conditions. A first classification is obtained by means of a quantitative orness measure that gives the proximity of each OWA to the OR operator. In the case in which the lattice is finite, another classification is obtained by means of a qualitative orness measure. In the present paper, several theoretical results are obtained in order to perform this qualitative value for each OWA operator.

Keywords: OWA operator, lattice-valued fuzzy sets, interval-valued fuzzy sets, orness, t-norm, t-conorm.

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1. Introduction

Interval-valued fuzzy sets have shown to be a good tool for modeling some situations in which uncertainty is present [3]. This class of fuzzy sets allows us to assign a whole interval to each element of the set, which is more flexible than a single value to represent the reality. However, the aggregation of intervals, necessary in most decision making problems or image processing techniques in order to obtain a global value from several data, is not always an easy task.

Ordered weighted average (OWA) operators are commonly used when the fuzzy sets that are involved in this case of problems take single real values instead of intervals ([7], [20], [21]). These aggregation functions, introduced by Yager [18], merge the data after modulating them by means of some weights, but in such a way that the weight affecting to each datum only depends on the place it takes in the descending chain of the arranged data. Hence Yager's OWA operators are symmetric, i.e., the global value that they obtain from a collection of data does not depend on either the expert or the resource that has provided each datum.

One of the advantages of OWA operators is their flexibility. The different weighting vectors provide a broad family of aggregation functions, varying from an OR aggregation (maximum) to an AND aggregation (minimum). One of the most difficult tasks for using OWA operators is the choice of its weighting vector. For this reason, Yager gives a classification of OWA operators by assigning an orness measure to each one of them. This value gives an idea of the proximity of each OWA operator to the OR one. Specifically, orness yields the maximum value (1) to the OR operator while it yields the minimum value (0) to the AND one.

Similar to other aggregation functions, see [12], OWA operators were generalized by Lizasoain and Moreno in [13] from the real unit interval to a general complete lattice L endowed with a t-norm T and a t-conorm S, whenever the weighting vector satisfies a distributivity condition with respect to T and S. Moreover, a qualitative parametrization of OWA operators, based on their proximity to the OR operator, but only in those cases in which the lattice L is finite, is studied in [16].

In [17], a quantitative parametrization of OWA operators is proposed for a wider family of lattices L: those containing a Maximal Finite Chain between any two elements. These lattices have been referred to as (MFC)-lattices and they comprise in particular all the finite lattices. The quantitative orness on these (MFC) lattices is defined in the following way:

First, for each weighting vector $\alpha = (\alpha_1, \dots, \alpha_n) \in L^n$, a qualitative quantifier $Q : \{0, 1, \dots, n\} \to L$ is defined by means of $Q_\alpha(0) = 0_L$ and $Q_\alpha(k) = S(\alpha_1, \dots, \alpha_k)$ for $1 \leq k \leq n$.

Then, instead of merging the weights as it had been done by Yager in the real case, the formula for the orness of OWA operators on lattices considers, for each $k \in \{1, \dots, m\}$, the length of the shortest maximal chain $\mu(k)$ between $Q_\alpha(k-1)$ and $Q_\alpha(k)$. Then it aggregates them according to Yager's formula:

$$\text{orness}(F_\alpha) = \frac{1}{n-1}\sum_{k=1}^{n}(n-k)\frac{\mu(k)}{\mu(1) + \cdots + \mu(m)}.$$

The present paper is devoted to the classification of m-dimensional interval-valued OWA operators. It deals with OWA operators defined on the lattice L^{I_m} comprising all m-dimensional intervals $[a_1, \dots, a_m]$ with $a_1 \leq_L \cdots \leq_L a_m$ belonging to a lattice L. The name of m-dimensional interval responds to the following reasons.

In the context of real-valued fuzzy sets, binary intervals are commonly used to express the membership degree of an element to a fuzzy set when some uncertainty or noise is present. As a generalization of them, m-dimensional real intervals are introduced in [2] to express membership degrees given by m different evaluation processes ordered by rigidity. For a general complete lattice L, m-dimensional intervals are studied in [14].

In the present paper, the case in which L is an (MFC)-lattice is considered. In particular, it is shown that, in that case, L^{I_m} is also an (MFC)-lattice. The quantitative orness defined in [17] for each lattice m-dimensional interval OWA operator is performed as a weighted average of the orness measures carried out componentwise. When $m = 1$, the results obtained here agree with those of [17].

In a complementary way, in those cases in which L is finite, it is obvious that L^{I_m} is also finite and the qualitative orness given in [16] is well-defined on this new lattice. However, the calculation of the elements belonging to L^{I_m} that occurs in the qualitative orness formula is not an easy task. We have achieved a formula for these elements in two common cases of L: when L is a distributive and complemented finite lattice and when L is a finite chain, which are shown in several examples of decision making problems. Also in this case, the results when $m = 1$ agree with those of [16].

The remainder of the paper is organized as follows. Section 2 provides some preliminary concepts and results regarding OWA operators defined on a complete lattice. We show that, if a lattice L satisfies some local finiteness condition, then so does L^{I_m}, in Section 3. In this section, we also study the relationship between the quantitative L^{I_m}-orness and the quantitative L-orness of OWA operators and we apply it to a decision making problem. In Section 4, we consider the particular case of finite lattices as well as we shown how to find the elements in L^{I_m} that are necessary to calculate the qualitative orness of OWA operators defined on it. In Section 5, we analyze those cases in which the finite lattice is distributive and complemented and show an application of this modelization to a decision making problem. We study those cases in which the lattice is a finite chain and apply the results to some decision making problems in Section 6. We finishes with some conclusions.

2. Preliminaries

Throughout this paper $(L, \leq_L)$ will denote a complete lattice, i.e., a partially ordered set, finite or infinite, for which all subsets have both a supremum (least upper bound) and an infimum (greatest lower bound). We denote 0_L and 1_L respectively as the least and the greatest elements in L.

Recall that an n-ary aggregation function is a function $M : L^n \to L$ such that:

(i) $M(a_1, \dots, a_n) \leq_L M(a'_1, \dots, a'_n)$ whenever $a_i \leq_L a'_i$ for $1 \leq i \leq n$.
(ii) $M(0_L, \dots, 0_L) = 0_L$ and $M(1_L, \dots, 1_L) = 1_L$.

It is said to be *idempotent* if $M(a, \dots, a) = a$ for every $a \in L$ and it is called *symmetric* if, for every permutation σ of the set $\{1, \dots, n\}$, $M(a_1, \dots, a_n) = M(a_{\sigma(1)}, \dots, a_{\sigma(n)})$.

Definition 2.1 (see [5]). A map $T : L \times L \to L$ is said to be a t-norm (resp. t-conorm) on $(L, \leq_L)$ if it is commutative, associative, increasing in each component and has a neutral element 1_L (resp. 0_L).

Remark 2.2. If $T : L \times L \to L$ is a t-norm on $(L, \leq_L)$, then for any $a, b \in L$, $T(a, b) \leq_L a \wedge b$ and, hence, $T(0_L, b) = 0_L$. Analogously, if $S : L \times L \to L$ is a t-conorm on $(L, \leq_L)$, then for any $a, b \in L$, $a \vee b \leq_L S(a, b)$ and hence $S(1_L, b) = 1_L$.
For any $n > 2$, $S(a_1, \dots, a_n)$ will denote $S(\dots(S(S(a_1, a_2), a_3), \dots a_{n-1}), a_n)$. Note that any t-conorm S is symmetric.

A wide family of both symmetric and idempotent aggregation functions was introduced by Yager [18] on the lattice $I = [0,1]$, the real unit interval with the usual order.

Definition 2.3 (Yager [18])**.** Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in [0,1]^n$ be a weighting vector with $\alpha_1 + \cdots + \alpha_n = 1$. An n-ary *ordered weighted average operator* or *OWA operator* is a map $F_\alpha : [0,1]^n \to [0,1]$ given by

$$F_\alpha(a_1, \cdots, a_n) = \alpha_1 b_1 + \cdots + \alpha_n b_n,$$

where $(b_1, \ldots, b_n)$ is a rearrangement of $(a_1, \cdots, a_n)$ satisfying that $b_1 \geq \cdots \geq b_n$.

It is easy to check that OWA operators form a family of aggregation functions bounded between the AND-operator (or minimum), given by the weighting vector $\alpha = (0, \ldots, 0, 1)$,

$$F_\alpha(a_1, \cdots, a_n) = a_1 \wedge \cdots \wedge a_n \text{ for any } (a_1, \ldots, a_n) \in [0,1]^n$$

and the OR-operator (or maximum), given by the weighting vector $\alpha = (1, 0, \ldots, 0)$,

$$F_\alpha(a_1, \cdots, a_n) = a_1 \vee \cdots \vee a_n \text{ for any } (a_1, \ldots, a_n) \in [0,1]^n.$$

With the purpose of classifying these operators, Yager [19] assigned an orness measure to each OWA operator F_α, which depends only on the weighting vector $\alpha = (\alpha_1, \ldots, \alpha_n)$, in the following way:

$$\text{orness}(F_\alpha) = \frac{1}{n-1} \sum_{i=1}^{n} (n-k)\alpha_k. \tag{1}$$

It is easy to check that the orness of each operator is a real value situated between 0, corresponding to the AND-operator, and 1, corresponding to the OR-operator. In general, the orness is a measure of the proximity of each OWA operator to the OR-operator. For instance, the orness of the arithmetic mean, provided by the weighting vector $(1/n, \ldots, 1/n)$, is equal to $1/2$.

In addition, Yagger defined, for each weighting vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in [0,1]^n$, a quantifier function $Q_\alpha : \{0, 1, \ldots, n\} \to [0,1]$ by means of:

$$Q_\alpha(k) = \begin{cases} 0 & \text{if } k = 0 \\ \alpha_1 + \cdots + \alpha_k & \text{otherwise} \end{cases} \tag{2}$$

Note that Q_α is a monotonically increasing function. Moreover, given a monotonically increasing function $Q : \{0, 1, \ldots, n\} \to [0, 1]$ with $Q(0) = 0$ and $Q(n) = 1$, then there exists a unique weighting vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in [0, 1]^n$ with $Q_\alpha = Q$. Indeed, for any $k = 1, \ldots, n$, put $\alpha_k = Q(k) - Q(k-1)$ and check that $Q_\alpha = Q$.

In [13] n-ary ordered weighted average (OWA) operators are extended from the unit interval to any complete lattice $(L, \leq_L)$ endowed with a t-norm T and a t-conorm S, which is the framework chosen for our study. We recall some concepts before writing the definition of lattice OWA operators.

Notation: Troughout this paper, a *quadruple* $(L, \leq_L, T, S)$ is a complete lattice $(L, \leq_L)$ endowed with a t-norm T and a t-conorm S.

The quadruple $(L, \leq_L, T, S)$ is said to satisfy the distributive property if

(D) $T(a, S(b, c)) = S(T(a, b), T(a, c))$ for any $a, b, c \in L$.

Recall that, if $(L, \leq_L, T, S)$ satisfies the distributive property (D), then S is the t-conorm given by the join (see [6] Propositions 3.5, 3.6 and 3.7). In spite of this, the t-conorm given by the join is not always distributive with respect to an arbitrary t-norm T.

Definition 2.4 ([13])**.** Consider a quadruple $(L, \leq_L, T, S)$. A lattice vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in L^n$ is said to be a *weighting vector in* $(L, \leq_L, T, S)$ if $S(\alpha_1, \ldots, \alpha_n) = 1_L$ and it is referred to as a *distributive weighting vector in* $(L, \leq_L, T, S)$ if it also satisfies that

$$T(a, S(\alpha_1, \ldots, \alpha_n)) = S(T(a, \alpha_1), \ldots, T(a, \alpha_n)) \text{ for any } a \in L.$$

Note that, if $(L, \leq_L, T, S)$ satisfies property (D), then any weighting vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in L^n$, with $S(\alpha_1, \ldots, \alpha_n) = 1_L$, is distributive in $(L, \leq_L, T, S)$.

In order to define an OWA operator on a lattice L, which is not always totally ordered, we need to substitute the arrangement of the data in an increasing order with the construction of a chain starting from the data. We build it by means of the k-th statistics described below.

Definition 2.5. Let $(L, \leq_L, T, S)$ a quadruple. For each vector $(a_1, \ldots, a_n) \in L^n$, we consider

- $b_1 = a_1 \vee \cdots \vee a_n \in L$;

- $b_2 = [(a_1 \wedge a_2) \vee .. \vee (a_1 \wedge a_n)] \vee [(a_2 \wedge a_3) \vee .. \vee (a_2 \wedge a_n)] \vee .. \vee [a_{n-1} \wedge a_n] \in L$;

  $\vdots$

- $b_k = \bigvee \{a_{j_1} \wedge \cdots \wedge a_{j_k} \mid j_1 < \cdots < j_k \in \{1, \ldots, n\}\} \in L$;

  $\vdots$

- $b_n = a_1 \wedge \cdots \wedge a_n \in L$.

Remark 2.6. Let $(L, \leq_L, T, S)$ be a quadruple, $(a_1, \ldots, a_n) \in L^n$ and $\{b_1, \ldots, b_n\}$ as introduced in Definition 2.5.

(i) It is easy to check that $[b_n, \ldots, b_1]$ is indeed a chain:

$$a_1 \wedge \cdots \wedge a_n = b_n \leq_L b_{n-1} \leq \cdots \leq_L b_2 \leq_L b_1 = a_1 \vee \cdots \vee a_n.$$

(ii) If the set $\{a_1, \ldots, a_n\}$ is totally ordered, then $[b_n, \ldots, b_1]$ agrees with $[a_{\sigma(1)}, \ldots, a_{\sigma(n)}]$ for some permutation σ of $\{1, \ldots, n\}$.

(iii) For each $1 \leq k \leq n$, b_k is the k-th order statistic of the vector $(a_1, \ldots, a_n)$.

The generalization of OWA operators to an arbitrary complete lattice is based on the previous *rearrangement* of the data.

Definition 2.7 ([13])**.** Let $(L, \leq_L, T, S)$ be a quadruple. For each distributive weighting vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in L^n$, the function $F_\alpha : L^n \to L$ given by

$$F_\alpha(a_1, \ldots, a_n) = S\left(T(\alpha_1, b_1), \cdots, T(\alpha_n, b_n)\right) \quad (a_1, \ldots, a_n) \in L^n,$$

where the elements $b_n \leq_L \cdots \leq_L b_1$ are calculated according to Def. 2.5 for each $(a_1, \ldots, a_n) \in L^n$, is called an *$n$-ary OWA operator*.

If $I = [0, 1]$ with the usual order $\leq$, $T(a, b) = ab$ for every $a, b \in [0, 1]$ and $S(a, b) = \min\{a + b, 1\}$ for every $a, b \in [0, 1]$, then $\alpha = (\alpha_1, \ldots, \alpha_n) \in [0, 1]^n$ is a distributive weighting vector if and only if $\alpha_1 + \cdots + \alpha_n = 1$ (see [13]).

In this case, OWA operators $F_\alpha : I^n \to I$ coincide with those given by Yager (see [13]). In addition, the main properties of $[0, 1]$-valued OWA operators also hold for any quadruple $(L, \leq_L, T, S)$. Indeed, for any distributive weighting vector α, F_α is an idempotent symmetric n-ary aggregation function lying between the operators given by the meet (OR-operator) and the join (AND-operator) on L.

In those cases in which the lattice $(L, \leq_L)$ satisfies some certain local finiteness condition, denoted by (MFC), a quantitative orness was introduced in [17] in order to classify n-ary OWA operators:

(MFC) *For any $a, b \in L$ with $a \leq_L b$, there exists some maximal chain with a finite length l,*

$$a = a^0 <_L a^1 <_L \cdots <_L a^l = b,$$

where the maximality means that, for any $0 \leq i \leq l-1$, there is no $c \in L$ with $a^i <_L c <_L a^{i+1}$.

In such lattices, the *distance* $d_L(a, b)$ between two elements a and b, is considered to be the length of the shortest maximal chain between a and b.

Definition 2.8 ([17])**.** Let $(L, \leq_L, T, S)$ be a quadruple in which $(L, \leq_L)$ satisfies condition (MFC). For any distributive weighting vector in $(L, \leq_L, T, S)$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in L^n$, consider the qualitative quantifier $Q_\alpha : \{0, 1, \ldots, n\} \to L$ given by:

$$Q_\alpha(0) = 0_L$$
$$Q_\alpha(k) = S(\alpha_1, \cdots, \alpha_k) \text{ for } k = 1, \ldots, n.$$

For each $k = 1, \ldots, n$, denote $\mu(k) = d_L(Q_\alpha(k-1), Q_\alpha(k))$. If $\mu = \mu(1) + \cdots + \mu(n)$, then we define

$$\text{orness}(F_\alpha) = \frac{1}{n-1} \sum_{k=1}^{n} (n-k) \frac{\mu(k)}{\mu}. \tag{3}$$

Remark 2.9. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in L^n$ be a distributive weighting vector in $(L, \leq_L, T, S)$. Then Q_α is a monotonically increasing function. Indeed, for each $k \in \{1, \ldots, n\}$, $Q_\alpha(k) = S(Q_\alpha(k-1), \alpha_k) \geq_L Q_\alpha(k-1)$.

In addition, orness(F_α) is well-defined, i.e., if $F_\alpha = F_\beta$ for some distributive weighting vector $\beta \in L^n$, then orness$(F_\alpha) =$ orness(F_β).

Note that all the finite lattices satisfy condition (MFC). However, this is not the case for the real unit interval $[0, 1]$, in which, on the other hand, there is a natural definition of distance $d(a, b)$ between two elements a and b, named $d(a, b) = |a - b|$. Obviously, if this distance is placed instead of $d_L(Q_\alpha(k-1), Q_\alpha(k))$ in the previous formula, Yager's orness is obtained.

3. Quantitative orness of m-dimensional interval OWA operators defined on an (MFC) lattice

In this section, for each quadruple $(L, \leq_L, T, S)$, L^{I_m} will stand for the set of all the lattice m-dimensional intervals[2] $[a_1, \ldots, a_m]$ with $a_1 \leq_L \cdots \leq_L a_m$ contained in L. Note that $(L^{I_m}, \leq_{L^{I_m}})$ is also a complete lattice with the partial order relation $\leq_{L^{I_m}}$ given by

$$[a_1, \ldots, a_m] \leq_{L^{I_m}} [c_1, \ldots, c_m] \text{ if and only if } a_i \leq_L c_i \text{ for } i = 1, \ldots, m.$$

Furthermore, the map $\mathbb{T} : L^{I_m} \times L^{I_m} \to L^{I_m}$ given, for any $[a_1, \ldots, a_m]$, $[c_1, \ldots, c_m] \in L^{I_m}$ by

$$\mathbb{T}\left([a_1, \ldots, a_m], [c_1, \ldots, c_m]\right) = [T(a_1, c_1), \ldots, T(a_m, c_m)]$$

is a t-norm on L^{I_m} (see [14]). Similarly, the map $\mathbb{S} : L^{I_m} \times L^{I_m} \to L^{I_m}$ given, for any $[a_1, \ldots, a_m]$, $[c_1, \ldots, c_m] \in L^{I_m}$ by

$$\mathbb{S}\left([a_1, \ldots, a_m], [c_1, \ldots, c_m]\right) = [S(a_1, c_1), \ldots, S(a_m, c_m)]$$

is a t-conorm on L^{I_m} (see [14]).

Remark 3.1. It is easy to check that, if a quadruple $(L, \leq_L, T, S)$ satisfies distributive property (D), then the quadruple $(L^{I_m}, \leq_{L^{I_m}}, \mathbb{T}, \mathbb{S})$ also satisfies property (D).

Notation: Note that the symbol $\vee$ are used indistinctly for operators on both lattices L and L^{I_m}.

We show that, if $(L, \leq_L)$ satisfies the (MFC) property, then $(L^{I_m}, \leq_{L^{I_m}})$ also satisfies it, which allows us to calculate the quantitative orness of any lattice interval-valued OWA operator.

Theorem 3.2. *If $(L, \leq_L)$ satisfies condition (MFC), then so $(L^{I_m}, \leq_{L^{I_m}})$ does. Furthermore, if $[a_1, \ldots, a_m]$, $[c_1, \ldots, c_m] \in L^{I_m}$, then the distance $d_{L^{I_m}}$ defined as the length of any of the shortest maximal chain between them, satisfies*

$$d_{L^{I_m}}([a_1, \ldots, a_m], [c_1, \ldots, c_m]) = d_L(a_1, c_1) + \cdots + d_L(a_m, c_m).$$

[2] In the literature, m-dimensional intervals are also called chains, but we take the notation from the context of fuzzy sets, as has been explained in the Introduction

Proof. For each $1 \leq i \leq m$, let $a_i = a_i^0 < a_i^1 < \cdots < a_i^{l_i} = c_i$ be a shortest maximal chain between a_i and c_i in $(L, \leq_L)$. Consider the following chain $\mathcal{C}$ between $[a_1, \ldots, a_m]$ and $[c_1, \ldots, c_m]$ in $(L^{I_m}, \leq_{L^{I_m}})$:

$$\begin{aligned}
&[a_1, \ldots, a_{m-1}, a_m] = [a_1^0, \ldots, a_{m-1}^0, a_m^0] < [a_1^0, \ldots, a_{m-1}^0, a_m^1] < \cdots < \\
&[a_1^0, \ldots, a_{m-1}^0, a_m^{l_m}] = [a_1^0, \ldots, a_{m-1}^0, c_m] < [a_1^0, \ldots, a_{m-1}^1, c_m] < \cdots < \\
&[a_1^0, \ldots, a_{m-2}^0, a_{m-1}^{l_{m-1}}, c_m] = [a_1^0, \ldots, a_{m-2}^0, c_{m-1}, c_m] < \cdots < \\
&\vdots \\
&[a_1^0, c_2, \ldots, c_{m-1}, c_m] < [a_1^1, c_2, \ldots, c_{m-1}, c_m] < \cdots < \\
&[a_1^{l_1}, c_2, \ldots, c_{m-1}, c_m] = [c_1, c_2, \ldots, c_{m-1}, c_m].
\end{aligned}$$

Obviously, $\mathcal{C}$ is a maximal chain with length equal to $l_1 + \cdots + l_m = d(a_1, c_1) + \cdots + d(a_m, c_m)$. It only remains to prove that there is not a shorter chain between $[a_1, \ldots, a_m]$ and $[c_1, \ldots, c_m]$ in $(L^{I_m}, \leq_{L^{I_m}})$.

Suppose, contrary to the hypothesis, that there exists some maximal chain $\mathcal{D}$,

$$[a_1, \ldots, a_m] = [d_1^0, \ldots, d_m^0] < [d_1^1, \ldots, d_m^1] < \cdots < [d_1^l, \ldots, d_m^l] = [c_1, \ldots, c_m],$$

between $[a_1, \ldots, a_m]$ and $[c_1, \ldots, c_m]$ with $l < l_1 + \cdots + l_m$. For each $j \in \{0, \ldots, l-1\}$, $[d_1^j, \ldots, d_m^j] < [d_1^{j+1}, \ldots, d_m^{j+1}]$. Then there is some $i \in \{1, \ldots, m\}$ with $d_i^j < d_i^{j+1}$. Moreover:

(i) The previous index i is unique (for each j). Otherwise, consider, for each $j \in \{0, \ldots, l-1\}$, the minimum index i with $d_i^j < d_i^{j+1}$ and the minimum i' in $\{i+1, \ldots, m\}$ with $d_{i'}^j < d_{i'}^{j+1}$, then the interval $\delta = [d_1^j, \ldots, d_i^j, \ldots, d_{i'}^{j+1}, \ldots, d_m^{j+1}] \in L^{I_m}$ would satisfy

$$[d_1^j, \ldots, d_i^j, \ldots, d_{i'}^j, \ldots d_m^j] < \delta < [d_1^{j+1}, \ldots, d_i^{j+1}, \ldots, d_{i'}^{j+1}, \ldots, d_m^{j+1}],$$

contradicting the maximality of $\mathcal{D}$.

(ii) There is no $e \in L$ with $d_i^j < e < d_i^{j+1}$ because it would imply that

$$[d_1^j, \ldots, d_i^j, \ldots, d_m^j] < [d_1^j, \ldots, e, \ldots, d_m^j] < [d_1^{j+1}, \ldots, d_i^{j+1}, \ldots, d_m^{j+1}],$$

contrary to the maximality of $\mathcal{D}$ again.

Now, consider a fixed $i \in \{1, \ldots, m\}$ with $a_i < c_i$ and call $\{j_1, \ldots, j_{r_i}\}$ all the indexes $j \in \{1, \ldots, l\}$ with $d_i^{j-1} < d_i^j$. By previous remarks (i) and (ii), the length of the chain $\mathcal{D}$ will be the sum of all the cardinals r_i. It is obvious that $a_i = d_i^0 < d_i^{j_1} < \cdots < d_i^{j_{r_i}} = c_i$ is a maximal chain between a_i and c_i in L. Hence, $r_i \geq l_i$, the length of the shortest maximal chain between a_i and c_i in L. Then the sum of all these r_i is greater or equal than $l_1 + \cdots + l_m = l$, which is an absurdity by the choice of $\mathcal{D}$.

Therefore, chain $\mathcal{C}$ is a shortest maximal chain between $[a_1, \ldots, a_m]$ and $[c_1, \ldots, c_m]$ in $(L^{I_m}, \leq_{L^{I_m}})$. □

Proposition 3.3. *Let $\alpha = (\alpha^1 = [\alpha_1^1, \ldots, \alpha_m^1], \ldots, \alpha^n = [\alpha_1^n, \ldots, \alpha_m^n])$ be a weighting vector in $(L^{I_m}, \leq_{L^{I_m}})$. Then α is a distributive weighting vector in $(L^{I_m}, \leq_{L^{I_m}}, \mathbb{T}, \mathbb{S})$ if and only if, for each $1 \leq i \leq m$, $\alpha_i = (\alpha_i^1, \ldots, \alpha_i^n)$ is a distributive weighting vector in $(L, \leq_L, T, S)$.*

Proof. This is easy to check. □

The following result shows that the quantitative orness of a lattice interval-valued OWA operator can also be performed componentwise.

Theorem 3.4. *Let $(L, \leq_L, T, S)$ be a quadruple in which $(L, \leq_L)$ satisfies condition (MFC) and let $\alpha = (\alpha^1 = [\alpha_1^1, \ldots, \alpha_m^1], \ldots, \alpha^n = [\alpha_1^n, \ldots, \alpha_m^n])$ be a distributive weighting vector in $(L^{I_m}, \leq_{L^{I_m}})$. Then*

$$orness_{L^{I_m}}(F_\alpha) = \frac{1}{\mu_1 + \cdots + \mu_m} \left(\mu_1 orness_L(F_{\alpha_1}) + \cdots + \mu_m orness_L(F_{\alpha_m})\right),$$

where for each $i \in \{1, \ldots, m\}$, the vector $\alpha_i = (\alpha_i^1, \ldots, \alpha_i^n)$ and $\mu_i = \sum_{k=1}^n d_L\left(Q_{\alpha_i}(k-1), Q_{\alpha_i}(k)\right)$.

Proof. For each $i = 1, \ldots, m$, consider the distributive weighting vector in $(L, \leq_L, T, S)$, $\alpha_i = (\alpha_i^1, \ldots, \alpha_i^n)$, and build the qualitative quantifier $Q_{\alpha_i} : \{0, 1, \ldots, n\} \to L$ given by:

$$\begin{aligned} &Q_{\alpha_i}(0) = 0_L \\ &Q_{\alpha_i}(k) = S(\alpha_i^1, \cdots, \alpha_i^k) \text{ for } k = 1, \ldots, n. \end{aligned}$$

For each $k = 1, \ldots, n$, call $\mu_i(k) = d_L\left(Q_{\alpha_i}(k-1), Q_{\alpha_i}(k)\right)$ and $\mu_i = \mu_i(1) + \cdots + \mu_i(n)$.

If we consider now the qualitative quantifier $Q_\alpha : \{0, 1, \ldots, n\} \to L^{I_m}$ given by:

$$\begin{aligned} &Q_\alpha(0) = [0_L, \ldots, 0_L] \\ &Q_\alpha(k) = \mathbb{S}([\alpha_1^1, \ldots, \alpha_m^1], \ldots, [\alpha_1^k, \ldots, \alpha_m^k]) \text{ for } k = 1, \ldots, n, \end{aligned}$$

then it is clear that for each $k = 0, 1, \dots, n$,

$$Q_\alpha(k) = [Q_{\alpha_1}(k), \dots, Q_{\alpha_m}(k)] \text{ with } Q_{\alpha_1}(k) \leq_L \cdots \leq_L Q_{\alpha_m}(k)$$

because for each $1 \leq j \leq n$, we have $\alpha_1^j \leq_L \cdots \leq_L \alpha_m^j$.

By Theorem 3.2,

$$\begin{aligned}
&\mu(k) = d_{L^{I_m}}\left(Q_\alpha(k-1), Q_\alpha(k)\right) = \\
&d_L\left(Q_{\alpha_1}(k-1), Q_{\alpha_1}(k)\right) + \cdots + d_L\left(Q_{\alpha_m}(k-1), Q_{\alpha_m}(k)\right) \\
&= \mu_1(k) + \cdots + \mu_m(k).
\end{aligned}$$

Therefore $\mu(1) + \cdots + \mu(n) = \mu_1 + \cdots + \mu_m$ and then

$$\begin{aligned}
&\mathrm{orness}_{L^{I_m}}(F_\alpha) = \frac{1}{n-1}\sum_{k=1}^{n}(n-k)\frac{\mu(k)}{\mu(1)+\cdots+\mu(n)} \\
&= \frac{1}{n-1}\sum_{k=1}^{n}(n-k)\frac{\mu_1(k)+\cdots+\mu_m(k)}{\mu(1)+\cdots+\mu(n)} = \\
&\frac{1}{n-1}\sum_{k=1}^{n}(n-k)\frac{\mu_1(k)}{\mu_1+\cdots+\mu_m} + \cdots + \frac{1}{n-1}\sum_{k=1}^{n}(n-k)\frac{\mu_m(k)}{\mu_1+\cdots+\mu_m} \\
&= \frac{1}{\mu_1+\cdots+\mu_m}\frac{\mu_1}{n-1}\sum_{k=1}^{n}(n-k)\frac{\mu_1(k)}{\mu_1} + \cdots + \frac{\mu_m}{n-1}\sum_{k=1}^{n}(n-k)\frac{\mu_m(k)}{\mu_m} \\
&= \frac{1}{\mu_1+\cdots+\mu_m}\left(\mu_1 \mathrm{orness}_L(F_{\alpha_1}) + \cdots + \mu_m \mathrm{orness}_L(F_{\alpha_m})\right).
\end{aligned}$$

□

Remark 3.5. Recall that a distributive and complemented finite lattice $(L, \leq_L)$ with p atoms (minimal elements in $L \setminus \{0_L\}$) is a lattice in which each element can be written in an only way as the join of r atoms of L for some $0 \leq r \leq p$. Such number r is referred to as the heigth of the element. The unique element with heigth equal to 0 is 0_L. We denote the elements with heigth equal to 1, i.e., the atoms, by t_i ($i = 1, \dots, p$). Each element with heigth equal to 2, $t_i \vee t_j$ with $1 \leq i, j \leq p$, $i \neq j$, is denoted by $t_i t_j$ and so on. In this manner, 1_L will be denoted by $t_1 \dots t_p$.

The order relation $\leq_L$ is given by the obvious $x \leq_L y$ if and only if the set of atoms of x is contained in the set of atoms of y.

Since $(L, \leq_L)$ is a distributive lattice, then any weighting vector in $(L, \leq_L, \wedge, \vee)$ is distributive, and so is any weighting vector in $(L^{I_m}, \leq_{L^{I_m}}, \wedge, \vee)$ by Proposition 3.3.

Proposition 3.6. *Let $(L, \leq_L)$ be a distributive and complemented finite lattice with p atoms. Then, for any weighting vector in $(L^{I_m}, \leq_{L^{I_m}}, \wedge, \vee)$,*

$$\alpha = (\alpha^1 = [\alpha_1^1, \dots, \alpha_m^1], \dots, \alpha^n = [\alpha_1^n, \dots, \alpha_m^n]),$$

(i) *the value μ_i considered in Definition 3.4 is equal to p.*
(ii) $orness_{L^{I_m}}(F_\alpha) = \frac{1}{m}\left(orness_L(F_{\alpha_1}) + \cdots + orness_L(F_{\alpha_m})\right)$.

Proof. (i) For any $1 \leq i \leq m$, consider the qualitative quantifier $Q_{\alpha_i} : \{0, 1, \dots, n\} \to L$ given by:

$$\begin{aligned} &Q_{\alpha_i}(0) = 0_L \\ &Q_{\alpha_i}(k) = \alpha_i^1 \vee \cdots \vee \alpha_i^k \text{ for } k = 1, \dots, n. \end{aligned}$$

Observe that $0_L = Q_{\alpha_i}(0) \leq_L Q_{\alpha_i}(1) \leq_L \cdots Q_{\alpha_i}(n) = 1_L$ is a chain connecting 0_L and 1_L and that all the maximal chains connecting 0_L and 1_L inside L have the same length. Therefore,

$$\begin{aligned} &\mu_i = \mu_i(1) + \cdots + \mu_i(n) = d_L\left(Q_{\alpha_i}(0), Q_{\alpha_i}(1)\right) + d_L\left(Q_{\alpha_i}(1), Q_{\alpha_i}(2)\right) \\ &+ \cdots + d_L\left(Q_{\alpha_i}(n-1), Q_{\alpha_i}(n)\right) \\ &= d_L(0_L, 1_L) = p \end{aligned}$$

(ii) Now, by Theorem 3.4, $\mathrm{orness}_{L^{I_m}}(F_\alpha)$

$$\begin{aligned} &= \frac{1}{\mu_1 + \cdots + \mu_m}\left(\mu_1 \mathrm{orness}_L(F_{\alpha_1}) + \cdots + \mu_m \mathrm{orness}_L(F_{\alpha_m})\right) \\ &= \frac{p}{mp}\left(\mathrm{orness}_L(F_{\alpha_1}) + \cdots + \mathrm{orness}_L(F_{\alpha_m})\right) \\ &= \frac{1}{m}\left(\mathrm{orness}_L(F_{\alpha_1}) + \cdots + \mathrm{orness}_L(F_{\alpha_m})\right) \end{aligned}$$

□

Remark 3.7. Proposition 3.6 also holds when L is a finite chain with length equal to p.

Example 3.8. Let $(L, \leq_L)$ be a distributive and complemented finite lattice with 4 atoms. Consider the lattice $(L^{I_3}, \leq_{L^{I_3}})$ comprising all the 3-dimensional lattice intervals,

$$L^{I_3} = \{[a_1, a_2, a_3] \mid a_1,\ a_2,\ a_3 \in L \text{ with } \ a_1 \leq_L a_2 \leq_L a_3\},$$

and the following weighting vector α in $(L^{I_3}, \leq_{L^{I_3}}, \wedge, \vee)$:

$$(\alpha^1, \alpha^2, \alpha^3, \alpha^4) = ([t_1, t_1t_2, t_1t_2],\ [t_2, t_2t_4, t_2t_3t_4],\ [t_3t_4, t_3t_4, t_3t_4],\ [0_L, 0_L, 1_L])$$

with $\alpha^1 \vee \alpha^2 \vee \alpha^3 \vee \alpha^4 = [1_L, 1_L, 1_L]$.

Note that both symbols $\wedge$ and $\vee$ are used indistinctly on L or on L^{I_3}.

Finding out the quantitative orness of the OWA operator $F_\alpha : (L^{I_3})^4 \to L^{I_3}$ is easier if we apply Theorem 3.4. First, we must obtain the quantitative orness of the following OWA operators defined on L:

(i) F_{α_1} with $\alpha_1 = (t_1, t_2, t_3t_4, 0_L)$.

The quantifier $Q_{\alpha_1} : \{0, 1, 2, 3, 4\} \to L$ is given by $Q_{\alpha_1}(0) = 0_L$, $Q_{\alpha_1}(1) = t_1$, $Q_{\alpha_1}(2) = t_1t_2$, $Q_{\alpha_1}(3) = Q_{\alpha_1}(4) = 1_L$, whence $\mu_1(1) = \mu_1(2) = 1$; $\mu_1(3) = 2$; $\mu_1(4) = 0$ and $\mu_1 = 1 + 1 + 2 + 0 = 4.$ Therefore $\text{orness}_L(F_{\alpha_1}) =$

$$\frac{1}{n-1}\sum_{k=1}^{4}(n-k)\frac{\mu_1(k)}{\mu_1} = \frac{1}{3}\left(3 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + 1 \cdot \frac{2}{4}\right) = \frac{7}{12}.$$

(ii) F_{α_2} with $\alpha_2 = (t_1t_2, t_2t_4, t_3t_4, 0_L)$.

The quantifier $Q_{\alpha_2} : \{0, 1, 2, 3, 4\} \to L$ is given by $Q_{\alpha_2}(0) = 0_L$, $Q_{\alpha_2}(1) = t_1t_2$, $Q_{\alpha_2}(2) = t_1t_2t_4$, $Q_{\alpha_2}(3) = Q_{\alpha_2}(4) = 1_L$, whence $\mu_2(1) = 2$, $\mu_2(2) = 1$, $\mu_2(3) = 1$, $\mu_2(4) = 0$ and $\mu_2 = 2 + 1 + 1 + 0 = 4.$ Therefore $\text{orness}_L(F_{\alpha_2}) =$

$$\frac{1}{n-1}\sum_{k=1}^{4}(n-k)\frac{\mu_2(k)}{\mu_2} = \frac{1}{3}\left(3 \cdot \frac{2}{4} + 2 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4}\right) = \frac{3}{4}.$$

(iii) F_{α_3} with $\alpha_3 = (t_1t_2, t_2t_3t_4, t_3t_4, 1_L)$.

The quantifier $Q_{\alpha_3} : \{0, 1, 2, 3, 4\} \to L$ is given by
$Q_{\alpha_3}(0) = 0_L$, $Q_{\alpha_3}(1) = t_1 t_2$, $Q_{\alpha_3}(2) = 1_L$, $Q_{\alpha_3}(3) = Q_{\alpha_3}(4) = 1_L$,
whence $\mu_3(1) = 2$, $\mu_3(2) = 2$, $\mu_3(3) = 0$, $\mu_3(4) = 0$ and
$\mu_3 = 2 + 2 + 0 + 0 = 4$. Therefore $\mathrm{orness}_L(F_{\alpha_3}) =$

$$\frac{1}{n-1}\sum_{k=1}^{4}(n-k)\frac{\mu_3(k)}{\mu_3} = \frac{1}{3}\left(3\cdot\frac{2}{4} + 2\cdot\frac{2}{4} + 1\cdot\frac{0}{4}\right) = \frac{5}{6}.$$

Now, Proposition 3.6 gives:

$$\mathrm{orness}_{L^{I_3}}(F_\alpha) = \frac{1}{3}\left(\mathrm{orness}_L(F_{\alpha_1}) + \mathrm{orness}_L(F_{\alpha_2}) + \mathrm{orness}_L(F_{\alpha_3})\right)$$
$$= \frac{1}{3}\left(\frac{7}{12} + \frac{3}{4} + \frac{5}{6}\right) = \frac{13}{18}.$$

This result means that the OWA operator F_α is more similar to the OR-operator than to the AND-one. In Section 5, we show an application of these results in a decision making problem.

In some cases, it is possible to recover the weighting vector $(\alpha_1, \ldots, \alpha_n) \in L^n$ from the quantifier Q.

Theorem 3.9 ([17]). *Let $(L, \leq_L, T, \vee)$ be a quadruple satisfying distributive property (D).*

For each monotonically increasing function $Q : \{0, 1, \ldots, n\} \to L^{I_m}$ with $Q(0) = [0_L, \ldots, 0_L]$ and $Q(n) = [1_L, \ldots, 1_L]$, the following statements hold.

(i) *There exists some weighting vector α in $(L^{I_m}, \leq_{L^{I_m}}, \mathbb{T}, \vee)$ with $Q_\alpha = Q$.*
(ii) *Such a weighting vector α is not necessarily unique. However, if both α and β are weighting vectors in $(L^{I_m}, \leq_{L^{I_m}}, \mathbb{T}, \vee)$ with $Q_\alpha = Q_\beta$, then the OWA operators F_α and F_β agree on L^{I_m}.*

Proof. The assertion has been proved in [17] Theorem 4.5 for each function Q, satisfying the conditions given above, that takes values on a distributive quadruple $(L, \leq_L, T, \vee)$. Since the quadruple $(L^{I_m}, \leq_{L^{I_m}}, \mathbb{T}, \vee)$ is also distributive by Remark 3.1, the result follows. □

Example 3.8 above is a particular case of a distributive lattice.

4. Qualitative orness of m-dimensional interval OWA operators defined on a finite lattice

In this section, we only consider quadruples $(L, \leq_L, T, S)$ in which $(L, \leq_L)$ is a finite bounded lattice endowed with a t-norm T and a t-conorm S.

In [16] a qualitative orness is introduced for OWA operators defined on finite lattices in order to have some extra information regarding to the influence of the choice of the weighting vector in the aggregation result. The qualitative orness of an OWA operator is defined as an element of the lattice L and its position on the lattice gives an idea of the *tendency* that the aggregation of the results will have.

The aim of this section is extending that qualitative value to the case of the quadruple $(L^{I_m}, \leq_{L^{I_m}}, \mathbb{T}, \mathbb{S})$, in which L^{I_m} is also finite, which comprises all the m-dimensional intervals taking values on L.

It is clear that L^{I_m} is finite. However, calculating its cardinal is not an easy task. In this section we will find this cardinality for two special cases: the case in which $(L, \leq_L)$ is a distributive and complemented finite lattice and that case in which $(L, \leq_L)$ is a finite chain. In addition, we will calculate the elements of L^{I_m} necessary to obtain the qualitative orness of each m-dimensional interval OWA operator on each case.

Definition 4.1 ([16]). Let $L = \{a_1, \ldots, a_l\}$ be a finite lattice and call $b_1 \geq_L \cdots \geq_L b_l$ the descending chain introduced in Definition 2.5, involving all the elements of the lattice. For any distributive weighting vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in L^n$, a qualitative *orness* measure of the corresponding OWA operator F_α is calculated by means of a descending chain $d_1 \geq_L \cdots \geq_L d_n$ that consists of some *equidistant* elements in the lattice, which are performed following some steps:

(i) Call $s = l(n-1)$.

(ii) Consider the descending chain $c_1 \geq_L \cdots \geq_L c_s$ defined by $c_1 = \cdots = c_{n-1} = b_1$; $c_n = \cdots = c_{2(n-1)} = b_2$; ...; $c_{(l-1)(n-1)+1} = \cdots = c_{l(n-1)} = b_l$.
Note that $c_1 = b_1 = 1_L$ and $c_{l(n-1)} = b_l = 0_L$.

(iii) Build a descending subchain of $\{c_1, \ldots, c_s\}$, $d_1 \geq_L \cdots \geq_L d_n$, by means of

$$d_1 = 1_L,\ d_2 = c_l,\ d_3 = c_{2l}, \ldots,\ d_n = c_{(n-1)l} = 0_L,$$

i.e. , $d_1 = 1_L$ and, for each $j \in \{1, \dots, n-1\}$, $d_{j+1} = c_{jl} = b_k$ with $k = 1 + \left\lfloor \dfrac{jl-1}{n-1} \right\rfloor$, in which the symbol $\left\lfloor \dfrac{a}{b} \right\rfloor$ denotes the integer part of $\dfrac{a}{b}$.

(iv) Call orness$(F_\alpha) = S\left(T(\alpha_1, d_1), \dots, T(\alpha_n, d_n)\right)$.

Remark 4.2. Note that the definition of orness(F_α) depends only on the vector $\alpha = (\alpha_1, \dots, \alpha_n)$. So, it is necessary to check that, if $\beta = (\beta_1, \dots, \beta_n)$ is another distributive weighting vector with $F_\alpha = F_\beta$, then orness$(F_\alpha) =$ orness(F_β). Indeed, note that

$$\text{orness}(F_\alpha) = S\left(T(\alpha_1, d_1), \dots, T(\alpha_n, d_n)\right) = F_\alpha(d_1, \dots, d_n).$$

Therefore, if $F_\alpha = F_\beta$, then orness$(F_\alpha) =$ orness(F_β) and this concept is well-defined.

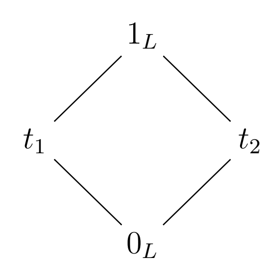
In a complementary way, note that the same chain $d_1 \geq_L \cdots \geq_L d_n$ is obtained if we let s be any common multiple of $\{l, n-1\}$. In that case, if $s = le$ with $e \in \mathbb{N}$, the chain $c_1 \geq_L \cdots \geq_L c_s$ must be

$c_1 = \cdots = c_e = b_1$; $c_{e+1} = \cdots = c_{2e} = b_2$; $\dots$; $c_{(l-1)e+1} = \cdots = c_{le} = b_l$ and, if $s = (n-1)h$ with $h \in \mathbb{N}$, then the chain $d_1 \geq_L \cdots \geq_L d_n$ must be

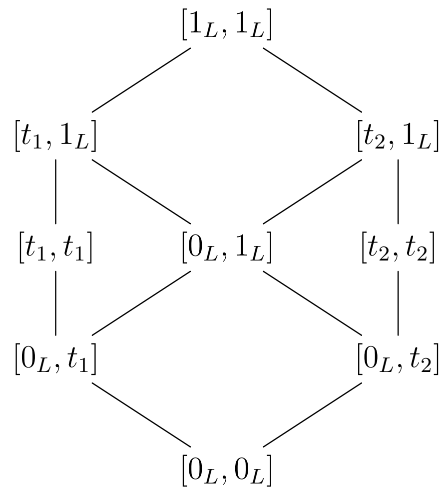
$$d_1 = 1_L,\ d_2 = c_h,\ d_3 = c_{2h}, \dots,\ d_n = c_{(n-1)h} = 0_L.$$

In this manner, the chain $d_1 \geq_L \cdots \geq_L d_n$ obtained in this way is the same as that obtained in Definition 4.1. Indeed, for each $j \in \{1, \dots, n-1\}$, $d_{j+1} = c_{jh} = b_k$ with $k = 1 + \left\lfloor \dfrac{jh-1}{e} \right\rfloor$.

Example 4.3. Let $(L, \leq_L)$ be the distributive and complemented lattice with exactly 2 atoms.



The lattice $(L^{I_2}, \leq_{L^{I_2}})$ is given by



Consider the weighting vector $\alpha = ([t_2, t_2], [t_1, 1_L], [t_1, t_1])$ with $[t_2, t_2] \vee [t_1, 1_L] \vee [t_1, t_1] = [1_L, 1_L]$. In order to calculate its qualitative orness, the chains defined in Definition 4.1 are obtained:

(i) Call $s = l(n-1) = 9 \cdot 2 = 18$.
(ii) First, consider the descending chain $b_1 \geq_{L^{I_2}} \cdots \geq_{L^{I_2}} b_9$ defined by $b_1 = b_2 = b_3 = [1_L, 1_L]$; $b_4 = b_5 = b_6 = [0_L, 1_L]$; $b_7 = b_8 = b_9 = [0_L, 0_L]$.
(iii) Then, obtain the descending chain $c_1 \geq_{L^{I_2}} \cdots \geq_{L^{I_2}} c_{18}$ defined by $c_1 = c_2 = b_1$; $c_3 = c_4 = b_2$; $c_5 = c_6 = b_3$; $c_7 = c_8 = b_4$; $c_9 = c_{10} = b_5$; $c_{11} = c_{12} = b_6$; $c_{13} = c_{14} = b_7$; $c_{15} = c_{16} = b_8$; $c_{17} = c_{18} = b_9$.
(iv) Build a descending subchain of $\{c_1, \ldots, c_{18}\}$ by means of

$$d_1 = [1_L, 1_L],\ d_2 = c_9 = b_5 = [0_L, 1_L],\ d_3 = c_{18} = [0_L, 0_L].$$

Therefore,

$$\begin{aligned}
&\text{orness}(F_\alpha) = (\alpha_1 \wedge d_1) \vee (\alpha_2 \wedge d_2) \vee (\alpha_3, d_3)\\
&= ([t_2, t_2] \wedge [1_L, 1_L]) \vee ([t_1, 1_L] \wedge [0_L, 1_L]) \vee ([t_1, t_1] \wedge [0_L, 0_L])\\
&= [t_2, t_2] \vee [0_L, 1_L] \vee [0_L, 0_L] = [t_2, 1_L].
\end{aligned}$$

Remark 4.4. The previous example shows that

$$\text{orness}_{L^{I_2}}(F_\alpha) \neq [\text{orness}_L(F_{\alpha_1}), \text{orness}_L(F_{\alpha_2})].$$

Indeed, if $\alpha_1 = (t_2, t_1, t_1)$ and $\alpha_2 = (t_2, 1_L, t_1)$, then

$$\begin{aligned}
\text{orness}_L(F_{\alpha_1}) &= (t_2 \wedge d_1) \vee (t_1 \wedge d_2) \vee (t_1 \wedge d_3) = 1_L \text{ and} \\
\text{orness}_L(F_{\alpha_2}) &= (t_2 \wedge d_1) \vee (1_L \wedge d_2) \vee (t_1 \wedge d_3) = 1_L
\end{aligned}$$

where the chains considered in Definition 4.1 for the lattice L are:

$$\begin{aligned}
&b_1 = 1_L,\ b_2 = 1_L,\ b_3 = 0_L,\ b_4 = 0_L; \\
&c_1 = b_1 = 1_L,\ c_2 = b_2 = 1_L,\ c_3 = b_3 = 0_L,\ c_4 = b_4 = 0_L; \\
&d_1 = 1_L,\ d_2 = c_2 = 1_L,\ d_3 = b_4 = 0_L.
\end{aligned}$$

5. The case in which L is a distributive and complemented finite lattice. An application in a decision making problem

In this section, L is a distributive and complemented finite lattice with p atoms, as described in Remark 3.5 .

Note that, for each $0 \leq r \leq p$, there are exactly $\binom{p}{r}$ elements in L with heigth equal to r. Recall that $\sum_{r=0}^{p} \binom{p}{r}$ is equal to 2^p, the number of elements of L.

The lattice consisting of all the m-dimensional intervals with elements in L, $(L^{I_m}, \leq_{L^{I_m}}, \wedge, \vee)$, is also distributive by Remark 3.1.

Note that the symbols $\wedge$ and $\vee$ are used indistinctly on both lattices L and L^{I_m}.

Proposition 5.1. *Let L be a distributive and complemented finite lattice with p atoms. For each $m \geq 1$, the lattice L^{I_m} has exactly $(m+1)^p$ elements.*

Proof. By using induction on m. If $m = 1$, then $L^{I_m} = L$, which has 2^p elements.

Let $m > 1$. The set L^{I_m} consists of all the m-dimensional intervals $[a_1, \ldots, a_m]$ with $a_1 \leq_L \cdots \leq_L a_m$ in L. Now, for each fixed a_m of height $r \in \{0, \ldots, p\}$, write $a_m = t_1 \vee \cdots \vee t_r$, where $t_1, \ldots, t_r$ are atoms of L. It is clear that all the possible $[a_1, \ldots, a_m] \in L^{I_m}$ must consist only of joins of atoms occurring in $\{t_1, \ldots, t_r\}$, i.e., $[a_1, \ldots, a_{m-1}]$ must be an $(m-1)$-dimensional interval on the sublattice of L consisting of all the joins of atoms in $\{t_1, \ldots, t_r\}$ together with 0_L. The inductive hypothesis says that there is exactly m^r such $(m-1)$-dimensional intervals for each a_m.

But now a_m must run all the elements of L: For each $0 \leq r \leq p$, there is $\binom{p}{r}$ possible a_m of height r and, fixed a_m, there is m^r possible intervals $[a_1, \ldots, a_{m-1}, a_m]$. Therefore, the whole number of elements in L^{I_m} is

$$\sum_{r=0}^{p} \binom{p}{r} m^r = (m+1)^p$$

because this is an special case of Newton's binomial theorem. $\square$

The next results are addressed to calculate the chain $b_1 \geq_{L^{I_m}} b_2 \geq_{L^{I_m}} \cdots \geq_{L^{I_m}} b_{(m+1)^p}$ considered in Definition 4.1, which involves all the elements of the lattice L^{I_m}. This chain must be performed in order to find the qualitative orness of any lattice-interval OWA operator.

Lemma 5.2. *Let t be an atom of L. For each $i \in \{0, 1, \ldots, m\}$, call $e_i(t) = [0_L, \ldots, 0_L, t, \ldots, t]$, the element in L^{I_m} with i bounds equal to 0_L and $(m-i)$ bounds equal to t. Call $M_i(t)$ the set*

$$M_i(t) = \{[a_1, \ldots, a_m] \in L^{I_m} \mid e_i(t) \leq_{L^{I_m}} [a_1, \ldots, a_m]\}.$$

Then, $M_i(t)$ has exactly $(i+1)(m+1)^{p-1}$ elements.

Proof. Call L_t the sublattice of L generated by all the atoms in L, except t, and consider the map $f_0 : L_t^{I_m} \to M_0(t)$ given by

$$f_0([c_1, \ldots, c_m]) = [c_1 \vee t, \ldots, c_m \vee t].$$

Note that, if $c, c' \in L_t$ with $c \leq_L c'$, then $c \vee t \leq_L c' \vee t$, i.e., f_0 is well-defined. Moreover, if $a \in L \setminus L_t$, then there exists an only $c \in L_t$ with $c \vee t = a$, i.e., the map f_0 is bijective. As a consequence, $M_0(t)$ has as many elements as $L_t^{I_m}$, which has exactly $(m+1)^{p-1}$ elements by Proposition 5.1.

Now, for each $i \in \{0, 1, \ldots, m-1\}$, define $f_{i+1} : L_t^{I_m} \to M_{i+1}(t) \setminus M_i(t)$ by

$$f_{i+1}([c_1, \ldots, c_m]) = [c_1, \ldots, c_{i+1}, c_{i+2} \vee t, \ldots, c_m \vee t].$$

Note that f_{i+1} is well-defined because, for every $[c_1, \ldots, c_m] \in L_t^{I_m}$, $e_{i+1}(t) \leq_{L^{I_m}} f_{i+1}([c_1, \ldots, c_m])$ but $e_i(t) \not\leq_{L^{I_m}} f_{i+1}([c_1, \ldots, c_m])$, i.e., $f_{i+1}([c_1, \ldots, c_m]) \in M_{i+1}(t) \setminus M_i(t)$.

It is easy to check that f_{i+1} is one-to-one. In addition, for each interval $[a_1,\dots,a_i,a_{i+1},\dots,a_m] \in M_{i+1}(t)\setminus M_i(t)$, it is clear that $a_1,\dots,a_i \in L_t$ whereas $a_{i+1},\dots,a_m \notin L_t$. Write $a_{i+1} = c_{i+1} \vee t$,..., $a_m = c_m \vee t$. Then $[a_1,\dots,a_i,a_{i+1},\dots,a_m] = f_{i+1}([a_1,\dots,a_i,c_{i+1},\dots,c_m]$, which means that f_{i+1} is surjective and consequently it is bijective.

Therefore, the number of elements in $M_{i+1}(t)\setminus M_i(t)$ agrees with that in $L_t^{I_m}$, which is $(m+1)^{p-1}$ by Proposition 5.1. Finally, for $i \in \{1,\dots,m\}$,

$$|M_i(t)| = |M_i(t)\setminus M_{i-1}(t)|+|M_{i-1}(t)\setminus M_{i-2}(t)|+\dots+|M_0(t)| = (i+1)(m+1)^{p-1}.$$

□

Theorem 5.3. *Let L be a distributive and complemented finite lattice with a set A of p atoms and consider the lattice L^{I_m} for some $m \geq 1$. For each $k \in \{1,\dots,(m+1)^p\}$, consider the join of all the possible meets of k different elements in L^{I_m}, which is b_k according to Definition 4.1. If we write $k = i(m+1)^{p-1}+j$ with $i \in \{0,1,\dots,m\}$ and $j \in \{1,\dots,(m+1)^{p-1}\}$, then*

$$b_k = e_i(1) = [0_L,\dots,\overset{(i)}{0_L},1_L,\dots,1_L].$$

Proof. Consider first any index k associated with the greatest possible index j, i.e., $k = i(m+1)^{p-1}+(m+1)^{p-1}$ with $i \in \{0,\dots,m-1\}$. For each atom t in L, Lemma 5.2 asserts that $e_i(t)$ is the least element of the set $M_i(t)$, which has exactly $(i+1)(m+1)^{p-1} = k$ elements. Hence, $e_i(t)$ is the meet of the k different elements in $M_i(t)$, whence $e_i(t) \leq_{L^{I_m}} b_k$ and consequently

$$\bigvee_{t\in A} e_i(t) = \bigvee_{t\in A}[0_L,\dots,\overset{(i)}{0_L},t,\dots,t] = [0_L,\dots,\overset{(i)}{0_L},1_L,\dots,1_L] \leq_{L^{I_m}} b_k.$$

If $i=0$, the inequality $[1_L,\dots,1_L] \leq_{L^{I_m}} b_k$ gives $b_k = [1_L,\dots,1_L]$.

For $i \in \{1,\dots,m-1\}$, consider $b_k = a_k^1 \vee\dots\vee a_k^{k_0}$, where $\{a_k^1,\dots,a_k^{k_0}\}$ is the set comprising the meets of all the possible sets in L^{I_m} that have exactly k different elements.

We show that, for each $1 \leq s \leq k_0$, the first i components in a_k^s must be equal to 0_L: otherwise, there would be some s and some element $[0_L,\dots,\overset{(r)}{0_L}, c_{r+1},\dots,c_m] \leq_{L^{I_m}} a_k^s$ with $c_{r+1} \neq 0_L$ and $r < i$. Consequently, there would be some atom t in L with $e_r(t) = [0_L,\dots,\overset{(r)}{0_L},t,\dots,t] \leq_{L^{I_m}} a_k^s$, which is an absurdity because there are only $(r+1)(m+1)^{p-1} < k$ different elements greater than $e_r(t)$ by Lemma 5.2.

Since $b_k = a_k^1 \vee \cdots \vee a_k^{k_0}$, then the first i components in b_k must be equal to 0_L and, since $[0_L, \ldots, \overset{(i)}{0_L}, 1_L, \ldots, 1_L] \leq_{L^{I_m}} b_k$, the equality holds, as desired.

We show now that $b_{k+1} = [0_L, \ldots, \overset{(i+1)}{0_L}, 1_L, \ldots, 1_L]$: Clearly,

$$b_{k+1} \geq_{L^{I_m}} b_{k+(m+1)^{p-1}} = b_{(i+1)(m+1)^{p-1}+(m+1)^{p-1}} = [0_L, \ldots, \overset{(i+1)}{0_L}, 1_L, \ldots, 1_L]$$

by the previous reasoning. But $b_k = a_{k+1}^1 \vee \cdots \vee a_{k+1}^{(k+1)_0}$, where $\{a_{k+1}^1, .., a_{k+1}^{(k+1)_0}\}$ is the set comprising the meets of all the possible sets in L^{I_m} with $(k+1)$ different elements. Reasoning as before, it can be seen that each a_{k+1}^s has its first $(i+1)$ components equal to 0_L. Hence, $b_{k+1} = [0_L, \ldots, \overset{(i+1)}{0_L}, f_{i+2}, \ldots, f_m]$ for some $f_{i+2}, \ldots, f_m \in L$. Consequently, we have $b_{k+1} \leq_{L^{I_m}} [0_L, \ldots, \overset{(i+1)}{0_L}, 1_L, \ldots, 1_L]$ and the equality holds. $\square$

Example 5.4. Let L be the distributive and complemented finite lattice with 4 atoms described in Example 3.8 and consider again the weighting vector α in $(L^{I_3}, \leq_{L^{I_3}}, \wedge, \vee)$ given in Example 3.8,

$$(\alpha^1, \alpha^2, \alpha^3, \alpha^4) = ([t_1, t_1t_2, t_1t_2], [t_2, t_2t_4, t_2t_3t_4], [t_3t_4, t_3t_4, t_3t_4], [0_L, 0_L, 1_L]).$$

In order to calculate the qualitative orness of the OWA operator F_α, we must find out all the elements occurring in Definition 4.1:

Number of options: $n = 4$.
Length of the lattice L^{I_3} (by Theorem 5.1): $l = (m+1)^p = (3+1)^4 = 256$.
Length of the chain $\{c_i\}$: $l(n-1) = 256(4-1) = 768$.
First chain: $c_1 = c_2 = c_3 = b_1$; $c_4 = c_5 = c_6 = b_2$; $\ldots$; $c_{766} = c_{767} = c_{768} = b_{256}$.
Second chain (by using Theorem 5.3): $d_1 = [1_L, 1_L, 1_L]$; $d_2 = c_l = c_{256} = b_{86}$ $= [0_L, 1_L, 1_L]$; $d_3 = c_{512} = b_{171} = [0_L, 0_L, 1_L]$; $d_4 = b_{768} = [0_L, 0_L, 0_L]$.

Now, the qualitative orness is defined in 4.1 as:

$$
\begin{aligned}
&\text{orness}(F_\alpha) = \\
&= (d_1 \wedge \alpha^1) \vee (d_2 \wedge \alpha^2) \vee (d_3 \wedge \alpha^3) \vee (d_4 \wedge \alpha^4) \\
&= [t_1, t_1t_2, t_1t_2] \vee [0_L, t_2t_4, t_2t_3t_4] \vee [0_L, 0_L, t_3t_4] \vee [0_L, 0_L, 0_L] \\
&= [t_1, t_1t_2t_4, 1_L].
\end{aligned}
$$

We will show in the application given in Section 5 that this qualitative orness tells us which is the tendency in the aggregation result of four 3-dimensional intervals by means of the OWA operator F_α.

5.1. An application in a decision making problem

The following example shows an application of this theory in a decision making problem. It is extendible to all the cases in which several experts are asked to express an opinion about different aspects involved in the decision problem: different complementary tasks to be assigned (or not assigned) to a worker, different accessories to be incorporated (or not incorporated) to a prototype... The only condition is that each expert is required to answer *yes*, *no* or *optional* to the incorporation of each aspect in an independent way.

The mathematical model considers the distributive and complemented finite lattice L with 4 atoms considered in Example 3.8.

A car factory has to decide which accessories will be included in each range of a specific car model. Before reaching a decision, they ask for the services of four consultancy firms. Each firm has to make a proposal about which accessories would be standard in the low-end car, in the middle-range car and in the high-end car. In addition, the standard accessories in each range that are not standard in the immediately lower range are offered as optional there. The accessories non included as standard in the high-end car are offered as optional in this range of cars.

The possible accessories are:

$$
\begin{aligned}
&t_1 = \text{GPS navigation system}, \; t_2 = \text{multifunction steering wheel}, \\
&t_3 = \text{glazed panoramic roof}, \; t_4 = \text{stability control systems}.
\end{aligned}
$$

The answers are collected in the following table:

	Low-end car standard access.	Middle-range car standard access.	High-end car standard accessories
Firm A	t_1	t_1, t_2	t_1, t_2, t_3, t_4
Firm B	t_1, t_2	t_1, t_2, t_3	t_1, t_2, t_3, t_4
Firm C	t_2	t_2, t_3	t_2, t_3, t_4
Firm D	t_1	t_1, t_3, t_4	t_1, t_2, t_3, t_4

The mathematical model represents each possible car as an element in the complemented and distributive lattice L with four atoms, $\{t_1, t_2, t_3, t_4\}$, described in Example 3.8. For instance, a car with accessories t_1 and t_3 (without t_2 and t_4) would be represented by $t_1t_3 \in L$ while a car with the four accessories is represented as $t_1t_2t_3t_4$ or simply by 1_L.

In this manner, the answer of each consultancy can be seen as a three-dimensional interval $[a_1, a_2, a_3] \in L^{I_3}$, with $a_1 \leq_L a_2 \leq_L a_3$, where the interval $[a_1, a_2]$ models all the possible cars included in the low-end range, the interval $[a_2, a_3]$ models all the possible cars included in the middle range and $[a_3, 1_L]$ models the high-end car.

The aggregation of the four options is carried out by means of an OWA operator defined on the lattice L^{I_3}, which comprises the 3-dimensional intervals. The aim is obtaining a single interval in L^{I_3} that determines which accessories are suitable for the cars of each range. The first step in the aggregation is obtaining the chain considered in Definition 2.5:

$$
\begin{aligned}
& b_1 = [t_1, t_1t_2, 1_L] \vee [t_1t_2, t_1t_2t_3, 1_L] \vee [t_2, t_2t_3, t_2t_3t_4] \vee [t_1, t_1t_3t_4, 1_L] \\
& = [t_1t_2, 1_L, 1_L], \\
& b_2 = [t_1, t_1t_2, 1_L] \vee [0_L, t_2, t_2t_3t_4] \vee [t_1, t_1, 1_L] \vee [t_2, t_2t_3, t_2t_3t_4] \\
& \vee [t_1, t_1t_3, 1_L] \vee [0_L, t_3, t_2t_3t_4] = [t_1t_2, t_1t_2t_3, 1_L], \\
& b_3 = [0_L, t_2, t_2t_3t_4] \vee [t_1, t_1, 1_L] \vee [0_L, 0_L, t_2t_3t_4] \vee [0_L, t_3, t_2t_3t_4] \\
& = [t_1, t_1t_2t_3, 1_L], \\
& b_4 = [t_1, t_1t_2, 1_L] \wedge [t_1t_2, t_1t_2t_3, 1_L] \wedge [t_2, t_2t_3, t_2t_3t_4] \wedge [t_1, t_1t_3t_4, 1_L] \\
& = [0_L, 0_L, t_2t_3t_4].
\end{aligned}
$$

Obviously, the aggregation result of the options given by the consultancy firms will depend on the weighting vector chosen by the car factory. We will show four possible aggregations of the possible options, by means of four different weighting vectors in L^{I_3}.

First, consider the weighting vector α given in Example 3.8:

$$(\alpha^1, \alpha^2, \alpha^3, \alpha^4) = ([t_1, t_1t_2, t_1t_2],\, [t_2, t_2t_4, t_2t_3t_4],\, [t_3t_4, t_3t_4, t_3t_4],\, [0_L, 0_L, 1_L])$$

with $\alpha^1 \vee \alpha^2 \vee \alpha^3 \vee \alpha^3 = [1_L, 1_L, 1_L]$.

The aggregation of the three options given in the table is

$$\begin{aligned}
&F_\alpha([t_1, t_1t_2, 1_L],\, [t_1t_2, t_1t_2t_3, 1_L],\, [t_2, t_2t_3, t_2t_3t_4],\, [t_1, t_1t_3t_4, 1_L]) = \\
&(b_1 \wedge \alpha^1) \vee (b_2 \wedge \alpha^2) \vee (b_3 \wedge \alpha^3) \vee (b_4 \wedge \alpha^4) = \\
&[t_1, t_1t_2, t_1t_2] \vee [t_2, t_2, t_2t_3t_4] \vee [0_L, t_3, t_3t_4] \vee [0_L, 0_L, t_2t_3t_4] \\
&= [t_1t_2, t_1t_2t_3, 1_L].
\end{aligned}$$

This result means that, if the factory uses the weighting vector α, it would be inclined to make low-end cars with both GPS navigation system and multifunction steering wheel as standard accessories, middle-range cars with a glazed panoramic roof, added to the previous ones, as standard accessories, and high end cars with stability control systems as standard accessories.

The following table compares this aggregation result with that obtained by means of the OWA operators determined by other weighting vectors β, γ, δ and ϵ below. Both quantitative and qualitative orness of each OWA operator have been calculated in order to understand their influence on the aggregation result. The last two columns in the table show the aggregation of some different elements in L^{I_3} by means of each OWA operator.

$$\begin{aligned}
&\alpha = ([t_1, t_1t_2, t_1t_2],\, [t_2, t_2t_4, t_2t_3t_4],\, [t_3t_4, t_3t_4, t_3t_4],\, [0_L, 0_L, 1_L]), \\
&\beta = ([t_1t_2t_3, t_1t_2t_3, 1_L],\, [t_3t_4, 1_L, 1_L],\, [t_1t_2, t_1t_2t_3, 1_L],\, [1_L, 1_L, 1_L]), \\
&\gamma = ([t_3, t_3, t_3t_4],\, [t_4, t_4, t_3t_4],\, [t_1, t_1, t_1],\, [t_2, t_2, t_2t_4]), \\
&\delta = ([t_3, t_3, t_3],\, [t_4, t_4, t_4],\, [t_2, t_2, t_2],\, [t_1, t_1, t_1]), \\
&\epsilon = ([t_3, t_3, t_3],\, [t_3, t_3, t_3],\, [t_1t_2, t_1t_2, t_1t_2],\, [t_4, t_4, t_4]).
\end{aligned}$$

	Quantitative orness of each OWA	Qualitative orness of each OWA	Aggregation result of (*) by means of each OWA	Aggregation result of (**) by means of each OWA
α	$\frac{13}{18}$	$[t_1, t_1t_2t_4, 1_L]$	$[t_1t_2, t_1t_2t_3, 1_L]$	$[t_1t_2, 1_L, 1_L]$
β	$\frac{17}{18}$	$[t_1t_2t_3, 1_L, 1_L]$	$[t_1t_2, t_1t_2t_3, 1_L]$	$[t_1t_2t_3, 1_L, 1_L]$
γ	$\frac{19}{36}$	$[t_3, t_3t_4, t_1t_3t_4]$	$[t_1, t_1t_3, 1_L]$	$[t_3, t_3t_4, 1_L]$
δ	$\frac{1}{2}$	$[t_3, t_3t_4, t_2t_3t_4]$	$[0_L, t_2t_3, t_2t_3t_4]$	$[t_3, t_2t_3t_4, t_2t_3t_4]$
ϵ	$\frac{5}{12}$	$[t_3, t_3, t_1t_2t_3]$	$[t_1, t_1t_2t_3, 1_L]$	$[t_3, t_2t_3, 1_L]$

(*) $[t_1, t_1t_2, 1_L]$, $[t_1t_2, t_1t_2t_3, 1_L]$, $[t_2, t_2t_3, t_2t_3t_4]$, $[t_1, t_1t_3t_4, 1_L]$

(**) $[t_3, t_3t_4, 1_L]$, $[t_1t_3, t_1t_2t_3, 1_L]$, $[t_2, t_2t_4, 1_L]$, $[t_2, t_2t_3, t_2t_3t_4]$

6. The case in which L is a finite chain. Applications to several decising making problems

In this subsection, we consider $(L, \leq)$ to be a finite chain. The symbols $0 \leq 1 \leq \cdots \leq p$ denote the elements of L. We only consider the t-norm given by the minimum, $\wedge$, and the t-conorm given by the maximum, $\vee$.

For each integer $m \geq 1$, the quadruple $(L^{I_m}, \leq, \wedge, \vee)$ will refer to the lattice $(L^{I_m}, \leq)$ comprising all the m-dimensional intervals as defined in Section 3, endowed with the t-norm and the t-conorm given respectively by the join and the meet. The cardinal of L^{I_m} is equal to the number of ways to sample m elements from a set of $p+1$ elements allowing for duplicates, i.e., the m-combinations with repetition of $(p+1)$ elements.

$$CR(p+1, m) = \binom{p+m}{m}.$$

The following results are addressed to calculate the chain $b_1 \geq_{L^{I_m}} b_2 \geq_{L^{I_m}}$

$\cdots \geq_{L^{I_m}} b_{CR(p+1,m)}$ considered in Definition 4.1, which is necessary in order to find the qualitative orness of any lattice-interval OWA operator.

Lemma 6.1. *Fix $r \in L$ with $r \neq 0$. For each $1 \leq i \leq m$, let $c_i(r) = [\underbrace{0, \ldots, 0}_{i-1}, \underbrace{r, \ldots, r}_{m-i+1}]$. Then*

$$|\{d \in L^{I_m} \mid c_i(r) \leq d\}| = \sum_{j=0}^{i-1} CR(r, j) \cdot CR(p - r + 1, m - j),$$

which will be denoted by $f(i, r)$.

Proof. If $i = 1$, $c_1 = [r, \ldots, r]$ and $f(1, r)$ is the number of m-dimensional intervals with all its coordinates greater than or equal to r. This number is equal to the number of ways of sampling m elements from the set $\{r, r + 1, \ldots, p\}$ allowing for duplicates, i.e., $CR(p - r + 1, m)$. Since $CR(r, 0) = 1$, the formula works for $i = 1$.

Suppose, by using induction on i, that the formula works for i. We show that it works for $i + 1$:

$$\begin{aligned}&|\{d \in L^{I_m} \mid c_{i+1}(r) \leq d\}| = \\ &|\{d \in L^{I_m} \mid c_i(r) \leq d\}| + |\{d \in L^{I_m} \mid c_i(r) \not\leq d,\, c_{i+1}(r) \leq d\}|.\end{aligned}$$

The induction hypothesis asserts that the first summand on the right hand is equal to $f(i, r)$. The second is the cardinal of the set of m-intervals in which both the i-th component (and then the first i components) are less than r and the other $(m - i)$ components are greater than or equal to r. This cardinal is equal to $CR(r, i) \cdot CR(p - r + 1, m - i)$, i.e., the number of ways of sampling the first i coordinates from $\{0, 1, \ldots, r - 1\}$ multiplied by the number of ways of sampling the last $(m - i)$ coordinates from $\{r, r + 1, \ldots, p\}$. □

Remark 6.2. For each $1 \leq i \leq m$, call $f(i, 0) = |\{d \in L^{I_m} \mid [0, \ldots, 0] \leq d\}|$. It is obvious that

$$f(i, 0) = |L^{I_m}| = CR(p + 1, m) \text{ for any } 1 \leq i \leq m.$$

Note that the definition of $f(i, r)$ forces this function to be strictly increasing in the first component and strictly decreasing in the second one.

Theorem 6.3. *For each $k \in \{1, 2, \dots, CR(p+1, m)\}$, call $b_k \in L^{I_m}$ the m-dimensional interval which is the join of all the meets of k different m-dimensional intervals. Then $b_k = [r_1, r_2, \dots, r_m]$, where for each $i \in \{1, \dots, m\}$,*

$$r_i = max\{r \in \{0, 1, \dots, p\} \mid k \leq f(i, r)\}.$$

Remark 6.4. Note that $b_k = [r_1, r_2, \dots, r_m]$ is an m-dimensional interval for any $k \in \{1, 2, \dots, CR(p+1, m)\}$. Indeed, $r_i \leq r_{i+1}$ because $f(i, r) \leq f(i+1, r)$ and then $\{r \in \{0, 1, \dots, p\} \mid k \leq f(i, r)\} \subseteq \{r \in \{0, 1, \dots, p\} \mid k \leq f(i+1, r)\}$ and consequently,

$$\max\{r \in \{0, 1, \dots, p\} \mid k \leq f(i, r)\} \leq \max\{r \in \{0, 1, \dots, p\} \mid k \leq f(i+1, r)\}.$$

Proof. Fix $k \in \{1, 2, \dots, CR(p+1, m)\}$. For each $1 \leq i \leq m$, $[\underbrace{0, \dots, 0}_{i-1}, r_i, \dots, r_i]$ has $f(i, r_i)$, and, hence, k, elements greater than or equal to itself. This means that $[\underbrace{0, \dots, 0}_{i-1}, r_i, \dots, r_i] \leq b_k$ and hence

$$[r_1, r_2, \dots, r_m] = \bigvee_{i=1}^{m} [\underbrace{0, \dots, 0}_{i-1}, r_i, \dots, r_i] \leq b_k.$$

In order to prove the equality, suppose that, for some $i \in \{1, \dots, m\}$, the i-th coordinate of b_k was equal to a certain $s > r_i$. In this case, $c_i(s) = [\underbrace{0, \dots, 0}_{i-1}, s, \dots, s] \leq b_k$. But $f(i, s) < f(i, r_i)$ by Remark 6.2 and, since $f(i, r_i) = \min\{f(i, r) \mid k \leq f(i, r)\}$, it would be $k > f(i, s)$, i.e., any sampling of k elements in L^{I_m} would contain some element neither greater than nor equal to $c_i(s)$. This would mean that its i-th coordinate would be less than s. In other words, the i-th coordinate of the meet of any sampling of k elements in L^{I_m} would be less than s and hence, the i-th coordinate of the join of all these meets would be less than s too, contradicting the hypothesis of s being the i-th coordinate of b_k. □

We show two examples of how this modelization can be applied in decision making problems.

Example 6.5. Consider the finite chain L of linguistic terms,

$$\textit{very bad} < \textit{bad} < \textit{fair} < \textit{good} < \textit{very good},$$

which is frequently used in problems of decision making in order to express the opinion of an expert about the quality of some product or the satisfaction degree of a client, among others. According to the notation of Theorem 6.3, $p = 4$ and the elements of L are represented as $0 < 1 < 2 < 3 < 4$.

Suppose now that 4 experts are asked about the quality of a product. Each opinion is represented by a 2-dimensional interval, such as [*fair*, *good*], denoted by $[2, 3]$, or [*very bad*, *fair*], denoted by $[0, 2]$. The aggregation of their opinions can be carried out by means of an OWA operator $F_\alpha : (L^{I_2})^4 \to L^{I_2}$, where α is a weighting vector in $\big(L^{I_2}, \leq_{L^{I_2}}, \wedge, \vee\big)$.

In order to find out the qualitative orness of F_α, the chain $d_1 \geq d_2 \geq d_3 \geq d_4$ must be calculated according to Definition 4.1. The first step is performing the chain $\{b_k \mid 1 \leq k \leq 15\}$, where $15 = |L^{I_2}|$.

This task is easy due to Theorem 6.3 and the following table, which shows the values of $\{f(i, r) \mid 1 \leq i \leq 2,\, 0 \leq r \leq 4\}$:

	r=0	r=1	r=2	r=3	r=4
i=1	15	10	6	3	1
i=2	15	14	12	9	5

Now, $b_k = [r_1, r_2]$ with

$$r_i = \max\{r \in \{0, 1, 2, 3, 4\} \mid f(i, r) \geq k\} \text{ for each } i = 1,\, 2.$$

Therefore

$$\begin{aligned} &b_1 = [4, 4],\, b_2 = [3, 4],\, b_3 = [3, 4],\, b_4 = [2, 4],\, b_5 = [2, 4],\\ &b_6 = [2, 3],\, b_7 = [1, 3],\, b_8 = [1, 3],\, b_9 = [1, 3],\, b_{10} = [1, 2],\\ &b_{11} = [0, 2],\, b_{12} = [0, 2],\, b_{13} = [0, 1],\, b_{14} = [0, 1],\, b_{15} = [0, 0]. \end{aligned}$$

Consider, for instance, the weighting vector:
$(\alpha^1, \alpha^2, \alpha^3, \alpha^4) = ([0, 1],\, [1, 2],\, [3, 3],\, [4, 4])$ with $\alpha^1 \vee \alpha^2 \vee \alpha^3 \vee \alpha^3 = 1_{L^{I_2}}$.
In order to calculate the qualitative orness of the OWA operator F_α, we

must find out all the elements occurring in Definition 4.1:

Dimension of the lattice intervals: $m = 2$.
Number of options: $n = 4$.
Length of the lattice L^{I_2}: $l = CR(4+1, 2) = 15$.
Length of the chain $\{c_i\}$: $\text{lcm}(l, n-1) = 15$.
First chain: $c_1 = b_1$; $c_2 = b_2$; $\ldots$; $c_{15} = b_{15}$.
Second chain: $d_1 = [4, 4]$; $d_2 = c_5 = b_5 = [2, 4]$;
$d_3 = c_{10} = b_{10} = [1, 2]$; $d_4 = b_{15} = [0, 0]$.

Now, the qualitative orness of α is equal to:

$$(\alpha^1 \wedge d_1) \vee (\alpha^2 \wedge d_2) \vee (\alpha^3 \wedge d_3) \vee (\alpha^4 \wedge d_4) =$$
$$([0, 1] \wedge [4, 4]) \vee ([1, 2] \wedge [2, 4]) \vee ([3, 3] \wedge [1, 2]) \vee ([4, 4] \wedge [0, 0])$$
$$= [0, 1] \vee [1, 2] \vee [1, 2] \vee [0, 0] = [1, 2].$$

The following table shows the influence of both the quantitative and the qualitative orness of F_α on the aggregation result of two possible collections of expert opinions. It can be compared with the result provided by other OWA operators, defined by the weighting vectors β, γ and δ below.

$$\alpha = ([0, 1],\ [1, 2],\ [3, 3],\ [4, 4]),$$
$$\beta = ([1, 2],\ [3, 3],\ [4, 4],\ [0, 1]),$$
$$\gamma = ([3, 3],\ [4, 4],\ [0, 1],\ [1, 2]),$$
$$\delta = ([4, 4],\ [0, 1],\ [1, 2],\ [3, 3]).$$

	Quantitative orness of each OWA	Qualitative orness of each OWA	Aggregation result of (*) by means of each OWA	Aggregation result of (**) by means of each OWA
α	$\frac{5}{12}$	$[1,2]$	$[1,2]$	$[2,3]$
β	$\frac{17}{24}$	$[2,3]$	$[1,3]$	$[3,3]$
γ	$\frac{11}{12}$	$[3,4]$	$[1,3]$	$[3,4]$
δ	1	$[4,4]$	$[1,3]$	$[3,4]$

(*) $[0,1]$, $[1,2]$, $[1,3]$, $[0,3]$, with $b_1 = b_2 = [1,3]$, $b_3 = [0,2]$ and $b_4 = [0,1]$.

(**) $[3,4]$, $[2,3]$, $[2,4]$, $[3,3]$, with $b_1 = b_2 = [3,4]$, $b_3 = b_4 = [2,3]$.

Note that weighting vectors with higher orness give a more optimistic aggregation, while weighting vectors with lower orness provide a more pessimistic result.

Example 6.6. The public health department of a certain country decides to launch a vaccination campaign targeting to children against a determinate desease. Its application requires a primary shot of the vaccine and two booster vaccinations, all of them before the children are 10 years old. The health department commisiones four independent studies in order to determine which age is most suitable to apply both the primary and the two booster vaccinations.

The first ten years of a child life must be represented as $0 < 1 < \cdots < 9$, i.e., as the elements of a chain L with $p = 9$ according to the notation of Theorem 6.3. The results of the studies are collected in the following table.

	Primary vaccination date	First booster vaccination date	Second booster vaccination date
Study A	0	4	7
Study B	1	4	6
Study C	1	3	9

The result of each study can be represented by a 3-dimensional interval, i.e., by an element of the lattice L^{I_3}. The aggregation of the three results is made by means of an OWA operator, which allows the health department to prioritise either the earlier ages, by choosing for instance the vector α below, or the older ones, by choosing for instance the vector β:

$$\alpha = ([1,2,3],\ [2,3,4],\ [9,9,9]) \text{ with } \alpha_1 \vee \alpha_2 \vee \alpha_3 = 1_{L^{I_3}},$$
$$\beta = ([9,9,9],\ [3,4,4],\ [1,1,3]) \text{ with } \beta_1 \vee \beta_2 \vee \beta_3 = 1_{L^{I_3}}.$$

The first step in the aggregation by means of any OWA operator is obtaining the chain considered in Definition 2.5:

$$\begin{aligned} b_1 &= [0,4,7] \vee [1,4,6] \vee [1,3,9] = [1,4,9], \\ b_2 &= [0,4,6] \vee [0,3,7] \vee [1,3,6] = [1,4,7], \\ b_3 &= [0,4,7] \wedge [1,4,6] \wedge [1,3,9] = [0,3,6]. \end{aligned}$$

Now, the aggregation result by means of the OWA F_α is

$$\begin{aligned} &F_\alpha([0,4,7],[1,4,6],[1,3,9]) = \\ &(b_1 \wedge \alpha^1) \vee (b_2 \wedge \alpha^2) \vee (b_3 \wedge \alpha^3) = \\ &([1,4,9] \wedge [1,2,3]) \vee ([1,4,7] \wedge [2,3,4]) \vee ([0,3,6] \wedge [9,9,9]) \\ &= [1,2,3] \vee [1,3,4] \vee [0,3,6] = [1,3,6] \end{aligned}$$

while the aggregation result by means of F_β is

$$\begin{aligned} &F_\beta([0,4,7],[1,4,6],[1,3,9]) = \\ &(b_1 \wedge \beta^1) \vee (b_2 \wedge \beta^2) \vee (b_3 \wedge \beta^3) = \\ &([1,4,9] \wedge [9,9,9]) \vee ([1,4,7] \wedge [3,4,4]) \vee ([0,3,6] \wedge [1,1,3]) \\ &= [1,4,9] \vee [1,4,4] \vee [0,1,3] = [1,4,9] \end{aligned}$$

In order to calculate the qualitative orness of the OWA operators F_α, $F_\beta : (L^{I_3})^3 \to L^{I_3}$, we must find out all the elements occurring in Definition 4.1:

Dimension of the intervals: $m = 3$.

Number of options: $n = 3$.

Length of the lattice L^{I_3}: $l = CR(p+1, m) = 220$.

Length of the chain $\{c_i\}$: $lcm(l, n-1) = 220$.

First chain: $c_1 = b_1$; $c_2 = b_2$; $\ldots$; $c_{220} = b_{220}$.

Second chain: $d_1 = [9,9,9]$; $d_2 = c_{110} = b_{110}$; $d_3 = c_{220} = b_{220} = [0,0,0]$,

where the term $b_{110} = [r_1, r_2, r_3]$ is given by Theorem 6.3, which says that $r_i = \max\{r \in \{0,1,\ldots,9\} \mid f(i,r) \geq k\}$ for $i = 1$, 2, 3, and the following table, which shows the values of $\{f(i,r)\}$ for $m = 3$.

	r=0	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9
i=1	220	165	120	84	56	35	20	10	4	1
i=2	220	210	192	168	140	110	80	52	28	10
i=3	220	219	216	210	200	185	164	136	100	55

Observe that, according to the table above, $b_{110} = [2,5,7]$. Hence, the qualitative orness of F_α is equal to:

$$\begin{aligned}
&(\alpha^1 \wedge d_1) \vee (\alpha^2 \wedge d_2) \vee (\alpha^3 \wedge d_3) \\
&= ([1,2,3] \wedge [9,9,9]) \vee ([2,3,4] \wedge [2,5,7]) \vee ([9,9,9] \wedge [0,0,0]) \\
&= [1,2,3] \vee [2,3,4] \vee [0,0,0] = [2,3,4]
\end{aligned}$$

In the same manner, the qualitative orness of F_β is

$$\begin{aligned}
&(\beta^1 \wedge d_1) \vee (\beta^2 \wedge d_2) \vee (\beta^3 \wedge d_3) \\
&= ([9,9,9] \wedge [9,9,9]) \vee ([3,4,4] \wedge [2,5,7]) \vee ([1,1,3] \wedge [0,0,0]) \\
&= [9,9,9] \vee [3,4,4] \vee [0,0,0] = [9,9,9].
\end{aligned}$$

Note the influence of the qualitative orness of each OWA operator on the aggregation result. If the weighting vector chosen was α, with a low qualitative orness, then the primary vaccunation would be applied at the age of 1 and the two booster vaccunations at the ages of 3 and 6 years old respectively. However, if the weighting vector chosen was β, with a high qualitative orness, then the two booster vaccunations would be applied at the ages of 4 and 9 years old respectively.

7. Conclusions

The quantitative orness measure defined for OWA operators with values on the lattice of all the m-dimensional intervals with bounds in a complete lattice endowed with a t-norm T and a t-conorm S, whenever it satisfies some local finiteness condition, can be calculated in terms of the orness measures of their components in L. This quantitative orness measure gives some idea of the proximity of each OWA operator to the OR-operator and allows us to classify all of these lattice interval-valued OWA operators.

In a complementary way, the qualitative orness measure defined for OWA operators on a finite lattice has also sense for OWA operators defined on the lattice comprising all the lattice-valued m-dimensional intervals. The elements necessary to carry out the qualitative orness can easily be performed thanks to the main results obtained in this paper for both the cases of a distributive and complemented finite lattice and a finite chain.

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