# Strategy-proof location of public facilities*

Jorge Alcalde–Unzu† and Marc Vorsatz‡

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**Abstract**

Consider the problem of locating a public facility taking into account the agents' preferences. To construct strategy-proof social choice rules, we propose a new preference domain that allows agents to have any single-peaked or any single-dipped preference on the location of the facility such that the peak/dip of the preference is in her own location. We characterize all strategy-proof rules in this general framework and study the conditions under which this family of strategy-proof rules includes non-dictatorial rules that have more than two alternatives in the range or that are Pareto efficient. Finally, we characterize for some focal cases all strategy-proof and Pareto efficient rules.

*Keywords:* social choice rule, strategy-proofness, Pareto efficiency, single-peaked preferences, single-dipped preferences.

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†Department of Economics, Universidad Pública de Navarra, Campus Arrosadia, 31006 Pamplona, Spain. `Email: jorge.alcalde@unavarra.es`. Financial support from Fundación Ramón Areces and the Spanish Ministry of Economy and Competitiveness, through the project ECO2015–64330-P, is gratefully acknowledged.

‡Corresponding author. Departamento de Análisis Económico, Universidad Nacional de Educación a Distancia (UNED), Paseo Senda del Rey 11, 28040 Madrid, Spain. `Email: mvorsatz@cee.uned.es`. Financial support from Fundación Ramón Areces and the Spanish Ministry of Economy and Competitiveness, through the project ECO2015–65701-P, is gratefully acknowledged.

# 1 Introduction

### Motivation and contribution

Governments frequently decide where to locate public facilities like schools, hospitals, prisons, nuclear plants, or industrial parks. In order to select a location for a particular facility, the public decision makers have to take into account not only technical or environmental constraints, but also the preferences of the agents that form the society: ideally, one would like to construct a facility near the locations of the agents that consider the facility a good but far away from those that consider it a bad. Since preferences are private information and agents have incentives to reveal them truthfully or not depending on how their reports are incorporated in the selection process, the objective is to construct a social choice rule that induces agents to reveal their true preferences, a property known as strategy-proofness.

It is well-known that if preferences are unrestricted, then there are no strategy-proof rules that have more than two alternatives in the range with the exception of the dictatorial rules (see, Gibbard [14] and Satterthwaite [19]). Therefore, to construct non-dictatorial social choice rules that induce truth telling, it is necessary to restrict the domain of admissible preferences. We propose in this paper a new preference domain that fits the problem of locating public facilities (on any subset of the real line) very naturally. In this domain, an agent's set of admissible preferences corresponds to all single-peaked and all single-dipped preferences so that the peak/dip is situated in her own location. Including both single-peaked and single-dipped preferences in the domain allows agents to express either a positive or a negative sentiment. The necessary richness of the domain is obtained by considering all preferences that satisfy the above-mentioned requirement.

Our main result is a characterization of all strategy-proof rules on this preference domain. Thereby it will become clear that it is possible to escape from the Gibbard-Satterthwaite impossibility in most instances. In particular, we will see that all strategy-proof rules share the following common structure: (*i*) first, agents are only asked about their types of preferences (single-peaked or single-dipped), and the answers determine one or two preselected locations; (*ii*) if only one location has been preselected, then it is finally chosen; and (*iii*) if two locations have been preselected, then all agents that are situated strictly between the preselected alternatives have to indicate their ordinal preferences between them (and, in case of indifference, their entire preferences) in order to choose the winning location. The particular ways the preselected locations that pass to the second phase are picked and the form in which the final location is chosen have to satisfy some monotonicity conditions. These conditions are generally not too restrictive and leave often space for the construction of non-dictatorial strategy-proof rules that are also Pareto efficient. For example, one particular class of rules that is strategy-proof and Pareto efficient if the set of alternatives is a closed interval and all agents are situated in this interval is what we call the *conditional two-step rules*. One prominent member of this family is the following rule: if there are agents that declare to have single-peaked preferences, the mean of the locations of these agents is selected; otherwise, vote by simple majority between the two extremes.

**Related literature**

Our paper belongs to the literature that searches for strategy-proof social choice rules on restricted preference domains that appear in socio-economic contexts —see, Barberà [2] for a survey—, and relates in particular to models in which the set of feasible alternatives can be linearly ordered; *i.e.*, from "left" to "right" in political applications or according to some kind of index (from "north" to "south" or from "east" to "west") if the objective is to find a location for some facility. In these settings, one can naturally define when an alternative lies between two other alternatives. Using this notion, a preference is then said to be *single-peaked* whenever the following two conditions are met: (a) there is a single most preferred alternative (called the peak) and (b) if alternative $x$ is between another alternative $y$ and the peak, then $x$ is preferred to $y$. Similarly, a preference is *single-dipped* whenever (a) there is a single worst alternative (called the dip) and (b) if alternative $x$ is between another alternative $y$ and the dip, then $y$ is preferred to $x$.

The domain of single-peaked preferences is discussed for the first time by Black [8, 9], who shows that the median voter rule, which selects the median of the declared peaks, is strategy-proof and selects the Condorcet winner. Moulin [17] and Barberà and Jackson [3] characterize the set of all strategy-proof rules on this domain (called *generalized median voter rules*). Like the median voter rule, the generalized median voter rules require agents only to indicate their peaks and not their entire preference structures.[1] In our domain, an extreme location of the preference of each agent (her location) is already known by the planner, but the planner is uncertain whether it is a peak or a dip. This is the reason why this information has to be elicited in the first step of the strategy-proof mechanisms in our domain. With respect to the domain of single-dipped preferences, Manjunath [15] and

[1]These original studies have triggered a lot of related research and extensions. For example, Border and Jordan [7], Barberà, Gul, and Stacchetti [5], and Chichilnisky and Heal [11] analyze the implications of strategy-proofness for different extensions of single-peaked preferences to multiple dimensions. A particularly interesting case is the one of separable preferences, introduced by Barberà, Sonnenschein, and Zhou [6]. Danilov [13] studies single-peaked preferences when alternatives are located on a tree, generalized by Schummer and Vohra [20] to networks. Finally, Miyagawa [16] and, more recently, Reffgen and Svensson [18] study the problem of locating multiple public goods.

Barberà, Berga, and Moreno [4] show that all strategy-proof and unanimous rules have to always select one of the two extreme feasible points. In the mechanisms characterized in our domain, the second stage is also reduced to a voting problem between at most two different locations, but there is a wide variety of alternatives that are potentially preselected in the first stage (not only the extremes).

The idea of considering a domain that includes both single-peaked and single-dipped preferences fits many problems of locating public facilities. Consider, for example, the case when the local authorities have to decide where to locate a new football stadium or a dog park. Then, one can easily imagine that many dog owners and football lovers would like to see the facility built close to their own home, yet the associated negative externalities are more important for other social groups. One might even argue that some facilities that are normally associated with the domain of only single-peaked preferences or with the domain of only single-dipped preferences may not be as clear-cut as they a priori look like: people without children living close to a school might suffer from the traffic noise, while some inhabitants of a town might put more emphasis on the job opportunities created by a near construction of a nuclear power plant than on the associated long-term health risks. Since it is not possible to include all single-peaked and all single-dipped preferences in a domain that escapes from the Gibbard-Satterthwaite impossibility (see, Berga and Serizawa [10])[2], some further domain restriction is necessary. Two previous approaches in this direction for the particular case in which the set of feasible locations is a closed interval are Feigenbaum and Sethuraman [12] and Thomson [21]. The former restricts the domain in such a way that the preferences should be cardinally determined by the distance between each location and the peak/dip. The latter only studies the case of two agents, situated at the same location, requiring that the first (second) agent can only have single-peaked (single-dipped) preferences with the peak (dip) in her location. The main novelty of our study is that we fully characterize all strategy-proof rules under the sole condition that the peak/dip of an agent is situated in her own location.

**Remainder**

The remainder of the paper is organized as follows. Section 2 introduces the necessary notation, definitions and some focal examples that illustrate our main insights. Section 3 develops the main structure of the strategy-proof social choice rules. Section 4 provides a complete characterization of all strategy-proof rules. In Section 5, we first explain when this characterized family includes non-dictatorial rules that have more than two alternatives in

[2]See Arribillaga and Massò [1] for an approach to assess the manipulability of different generalized median voter rules.

the range or that are Pareto efficient. Afterwards, we characterize for some focal cases the rules that are Pareto efficient and strategy-proof. Finally, Section 6 concludes. All proofs are relegated to the Appendix.

# 2 Notation, definitions, and focal examples

Consider a social planner that wants to locate a public facility in a point on a set $T \subseteq \mathbb{R}$ of feasible locations. There is a finite group of agents $N$ of size $|N| \geq 2$. Each agent is situated at a point on the real line. We denote the agent situated at $i \in \mathbb{R}$ by $i \in N$.[3] For the moment, we do not impose any restriction on $N$ or $T$, or on the relation between these sets.

Let $R_i$ be the weak preference relation of agent $i \in N$ on $T$. Formally, $R_i$ is a complete and transitive binary relation. The strict preference relation induced by $R_i$ is denoted by $P_i$. We then say that $R_i$ is a single-peaked preference with peak $i$ if for all $x, y \in T$ such that $i \geq x > y$ or $i \leq x < y$, we have that $x \, P_i \, y$. Similarly, $R_i$ is a single-dipped preference with dip $i$ if for all $x, y \in T$ such that $i \geq x > y$ or $i \leq x < y$, we have that $y \, P_i \, x$.[4] The preference domain of agent $i$ is $\mathcal{R}_i = \mathcal{R}_i^+ \cup \mathcal{R}_i^-$, where $\mathcal{R}_i^+$ ($\mathcal{R}_i^-$) is the set of all single-peaked (single-dipped) preferences with peak (dip) $i$. Observe that the preference domains are personalized.

A preference profile is a set of preferences $R = (R_i)_{i \in N}$. The domain of all admissible preference profiles is denoted by $\mathcal{R} = \times_{i \in N} \mathcal{R}_i$. Let $\mathcal{R}^A = \times_{i \in A} \mathcal{R}_i^+ \times \times_{j \in (N \setminus A)} \mathcal{R}_j^-$ be the set of preference profiles such that the agents in $A \subseteq N$ have single-peaked and the agents in $N \setminus A$ have single-dipped preferences. Sometimes, we will write $\mathcal{R}_i^A$ to indicate $\mathcal{R}_i^+$ whenever $i \in A$ and $\mathcal{R}_i^-$ whenever $i \notin A$. Similarly, $R_S$ and $R_{-S}$ are the restrictions of $R$ to the agents in $S \subseteq N$ and $N \setminus S$, respectively. We will write $R_{-i}$ instead of $R_{-\{i\}}$.

The solution concept is a *social choice rule*, a function $f : \mathcal{R} \to T$ that selects for each preference profile $R \in \mathcal{R}$ a feasible location $f(R) \in T$. We denote the range of $f$ by $\Omega$ and its range in the subdomain $\mathcal{R}^A$ by $\Omega_A$.[5] We say that $f$ is *manipulable by agent* $i \in N$

[3] This notation assumes implicitly that there is at most one agent at any point of the real line. Our results also hold when multiple agents are situated at the same point.

[4] Technically speaking, these preferences only have a maximal/minimal alternative at $i$ if $i \in T$. If $i \notin T$ and $T$ is a compact set, the most/least preferred feasible alternative is either the closest one at $i$'s left or the closest one at $i$'s right. If $i \notin T$ and $T$ is not compact, the most/least preferred feasible alternative may not be well-defined.

[5] Obviously, $\Omega$ and $\Omega_A$ depend on the particular rule $f$ that is being employed. Since the rule in use is always clear throughout the paper, we do not include this dependence in the notation.

if there is a preference profile $R \in \mathcal{R}$ and an alternative preference $R_i' \in \mathcal{R}_i$ such that $f(R_i', R_{-i})\, P_i\, f(R)$. Then, $f$ is *strategy-proof* if it is not manipulable by any agent $i \in N$. The social choice rule $f$ is *Pareto efficient* if for all $R \in \mathcal{R}$, there is no $x \in T$ such that $x\, R_i\, f(R)$ for all $i \in N$ and $x\, P_j\, f(R)$ for some $j \in N$. Finally, $f$ is *dictatorial* if there exists an agent $i \in N$ (called the dictator) such that for all $R \in \mathcal{R}$ and $x \in \Omega$, $f(R)\, R_i\, x$.

We now present three examples of different structures of $N$ and $T$ and their effects on the possibilities of constructing non-dictatorial strategy-proof social choice rules. These examples provide some first insights about the results that will be developed in the following sections. In the first example, each of the three agents is situated between two of the three feasible locations and no agent is situated at a feasible location.

**Example 1** *Suppose that $N = \{3, 4, 5\}$ and $T = \{1, 2, 6\}$. Then, each agent $i \in N$ has six possible strict preferences over $T$ under $\mathcal{R}$. If she has single-peaked preferences, her possible strict preferences are $2\, P_i\, 1\, P_i\, 6$, $2\, P_i\, 6\, P_i\, 1$, or $6\, P_i\, 2\, P_i\, 1$. If she has single-dipped preferences, her possible strict preferences are $1\, P_i\, 2 P_i\, 6$, $1\, P_i\, 6\, P_i\, 2$, or $6\, P_i\, 1\, P_i\, 2$. Thus, the strict preferences of $\mathcal{R}$ coincide with the universal strict preference domain and the Gibbard-Satterthwaite impossibility applies.*

Fortunately, Example 1 is an extreme case and mostly we are going to be able to construct non-dictatorial strategy-proof rules. We show that for a particular case in which each agent is either situated to the left or to the right of all feasible locations.

**Example 2** *Suppose that $N = \{1, 2, 8, 9\}$ and $T = \{3, 4, 5, 6, 7\}$. Let the social choice rule $f$ be such that for all $R \in \mathcal{R}$, $f(R) = 7 - |\{i \in N : 3\, P_i\, 7\}|$. To select a location, this rule starts at point 7 and moves one unit to the left for each agent of $\{1, 2\}$ with single-peaked and for each agent of $\{8, 9\}$ with single-dipped preferences. It can be checked that $f$ is strategy-proof, Pareto efficient, non-dictatorial, and has more than two alternatives in its range.*[6]

In the last example, the set of feasible locations is infinite and each agent is situated at a feasible location. Then, it is again possible to find rules that satisfy the desired properties.

[6] Observe that in Example 2, agents only have two admissible preferences and both are single-peaked *on the set of feasible locations* with the peak in one of the extremes of the range, 3 or 7. The rule $f$ in this example is a generalized median voter defined by Moulin [17]: it selects the median between the tops of the agents' preferences and the fixed ballots 4, 5, and 6. We will see in Section 5 that if each agent is situated weakly to the left or weakly to the right of all feasible locations, then all strategy-proof rules are generalized median voter rules.

**Example 3** *Suppose that* $N = \{1, 2, 3\}$ *and* $T = [0, 4]$*. Consider the social choice rule* $f$ *defined through the following procedure: if the set of agents with single-peaked preferences is non-empty, choose the mean location of these agents; otherwise, choose by simple majority (with* 4 *as tie-breaker) between the extremes* 0 *and* 4*. It can be checked that* $f$ *is strategy-proof, Pareto efficient, non-dictatorial, and has more than two alternatives in its range.*

Examples 1 to 3 show the importance of the structure of $N$ and $T$. In Sections 3 and 4, we analyze the general problem without making any assumption on $N$ or $T$. Since in this general analysis there is no closed-form representation of the set of strategy-proof rules, we put in Section 5 some more structure on $N$ and $T$ —covering, among others, the most natural and common cases of the literature (*i.e.*, when $T$ is a closed interval and $N \subset T$)— in order to obtain particular and simpler characterizations.

# 3 The main structure of the strategy-proof rules

Our first result is a necessary condition on the range of $f$ that facilitates the further analysis. The condition states that if a social choice rule is strategy-proof, then, for a given partition of the set of agents into types (single-peaked and single-dipped), the range of the rule can include at most two alternatives. That is, the cardinality of the range of the rule in the subdomain $\mathcal{R}^A$, $|\Omega_A|$, is at most 2 for any $A \subseteq N$.

**Proposition 1** *Suppose that* $f$ *is strategy-proof. Then, for all* $A \subseteq N$, $|\Omega_A| \leq 2$*.*

Proposition 1 implies that any strategy-proof rule can be divided into two steps. In the first step, agents have to declare only their types of preferences (single-peaked or single-dipped) and, depending on the answers, a set of at most two locations is preselected. If one alternative is preselected, this alternative is finally implemented. If two alternatives are preselected, the alternative that is finally implemented has to be determined in the second step of the procedure.[7] Since $\Omega_A$ contains at most two preselected locations, we indicate by $l_A$ and $h_A$ the elements of $\Omega_A$ such that $l_A \leq h_A$. Then, $N_A = N \cap (l_A, h_A)$ corresponds to the set of agents that are situated strictly between the preselected alternatives of $\Omega_A$.

Next, we derive several conditions the second step of a strategy-proof social choice rule has to satisfy. To do so, let $f_A : \mathcal{R}^A \to \Omega_A$ be the binary decision function associated with

[7] Example 3 illustrates that there are strategy-proof social choice rules such that $|\Omega_A| = 2$ for some $A \subseteq N$.

$f$ that chooses between $l_A$ and $h_A$ when the set of agents that declared to have single-peaked preferences is equal to $A$. The next proposition shows that only the preferences of the agents belonging to $N_A$ can affect the outcome of $f_A$. The underlying intuition is as follows. Since the two preselected locations are both weakly to the left or both weakly to the right of an agent $i \notin N_A$, $i$ has the same ordinal preferences over $l_A$ and $h_A$ in all preference profiles of $\mathcal{R}^A$. Hence, the binary decision function $f_A$ must be independent of these preferences in order to guarantee the strategy-proofness of $f$.

**Proposition 2** *Suppose that $f$ is strategy-proof. Then, for all $A \subseteq N$ and all preference profiles $R, R' \in \mathcal{R}^A$ such that $R_{N_A} = R'_{N_A}$, $f_A(R) = f_A(R')$.*

Proposition 2 implies that if $l_A \neq h_A$, then at least one agent has to be situated strictly between the two preselected alternatives. Also, we partition $N_A$ for a given profile $R \in \mathcal{R}^A$ into three groups, depending on the ordinal preferences over the two preselected alternatives: $L_A(R) = \{i \in N_A : l_A \, P_i \, h_A\}$, $H_A(R) = \{i \in N_A : h_A \, P_i \, l_A\}$, and $I_A(R) = \{i \in N_A : h_A \, R_i \, l_A \text{ and } l_A \, R_i \, h_A\}$.

Any binary decision function $f_A$ can be defined by specifying for each profile $R \in \mathcal{R}^A$, a set of coalitions $\mathcal{G}^A(R) \subseteq 2^N$ (called $l_A$–decisive sets) such that $f_A$ chooses $l_A$ if the set of agents of $N_A$ that prefer $l_A$ to $h_A$, $L_A(R)$, belongs to $\mathcal{G}^A(R)$, and $h_A$ otherwise. This definition incorporates irrelevant information because it only matters whether or not $L_A(R)$ belongs to $\mathcal{G}^A(R)$, and not which other coalitions form part of $\mathcal{G}^A(R)$. However, it facilitates the description of the conditions that strategy-proofness imply. To describe these conditions, let $R_{I_A(R')}$ be the preferences at profile $R \in \mathcal{R}^A$ of the agents belonging to $N_A$ that are indifferent between $l_A$ and $h_A$ at profile $R'$. If $R' = R$, then we drop the parenthesis and write $R_{I_A}$.

**Definition 1** *Given $A \subseteq N$, the binary decision function $f_A : \mathcal{R}^A \to \Omega_A$ is called a voting by collections of $l_A$–decisive sets if there is a correspondence $\mathcal{G}^A : \mathcal{R}^A \rightrightarrows 2^N$ such that for each $R \in \mathcal{R}^A$,*

$$f_A(R) = \begin{cases} l_A & \textit{if } L_A(R) \in \mathcal{G}^A(R) \\ h_A & \textit{otherwise} \end{cases}$$

*and the following conditions are satisfied:*

- $\mathcal{G}^A(R) \in 2^{N_A \setminus I_A(R)}$.
- *For all $R' \in \mathcal{R}^A$ such that $R'_{I_A} = R_{I_A}$, $\mathcal{G}^A(R') = \mathcal{G}^A(R)$.*
- *If $B \in \mathcal{G}^A(R)$ and $B \subseteq C \in 2^{N_A \setminus I_A(R)}$, then $C \in \mathcal{G}^A(R)$.*

- *For all* $R' \in \mathcal{R}^A$ *and* $C \subseteq N_A \backslash I_A(R)$ *such that* $R'_{I_A(R)} = R_{I_A}$ *and* $I_A(R') = I_A(R) \cup C$*:*
  - *If* $B \cap C = \emptyset$ *and* $B \cup C \notin \mathcal{G}^A(R)$*, then* $B \notin \mathcal{G}^A(R')$*.*
  - *If* $B \in \mathcal{G}^A(R)$ *and* $B \cap C = \emptyset$*, then* $B \in \mathcal{G}^A(R')$*.*
- *If* $I_A(R) = \emptyset$*, then* $\emptyset \notin \mathcal{G}^A(R) \neq \emptyset$*.*

The first condition on the correspondence $\mathcal{G}^A$ requires that the $l_A$–decisive sets at a profile $R$ have to be subsets of $N_A$ formed by agents with strict preferences over $\Omega_A$ at $R$. The second condition indicates that the $l_A$–decisive sets at $R$ only depend on the complete preferences of the agents belonging to $I_A(R)$. Furthermore, the $l_A$–decisive sets at any profile have to satisfy three intuitive monotonicity properties and a non-emptiness condition. First, all supersets of a $l_A$–decisive set at $R$ are also decisive at $R$. Second, if a set of agents $B \cup C$ cannot impose $l_A$ when only the agents of $B \cup C$ prefer $l_A$ to $h_A$, then the set $B$ is also not able to impose $l_A$ when the agents of $C$ switch their preferences and become indifferent between the two preselected alternatives. Third, if a set of agents $B$ can impose $l_A$ when only the agents of $B$ prefer $l_A$ to $h_A$ and the agents of the set $C$ prefer $h_A$ to $l_A$, then $B$ is also able to impose $l_A$ when the agents of $C$ become indifferent between the two preselected alternatives. Finally, the non-emptiness condition guarantees that each of the preselected alternatives is implemented at least in one profile in which all agents have strict preferences over $\Omega_A$.

Observe that the outcome of a voting by collections of $l_A$–decisive sets at a profile depends on $(i)$ the complete preferences of the agents belonging to $N_A$ that are indifferent between $l_A$ and $h_A$, and $(ii)$ the ordinal preferences only between $l_A$ and $h_A$ of the remaining agents of $N_A$. We also note that the family of voting by collections of $l_A$–decisive sets is almost identical to the family introduced under the same name in Manjunath [15], where it is shown that these are the unique type of rules that are strategy-proof and unanimous in the domain of single-dipped preferences when the two alternatives to choose from are the extreme locations $\min T$ and $\max T$. The only difference is that the non-emptiness condition of $\mathcal{G}^A(R)$ is needed in Manjunath [15] unless $I_A(R) = N_A$ and not only when $I_A(R) = \emptyset$. Our next result shows that for each $A \subseteq N$, the binary decision function $f_A$ associated with any strategy-proof social choice rule $f$ has to be a voting by collections of $l_A$–decisive sets.

**Proposition 3** *If* $f$ *is strategy-proof, the family of binary decision functions* $\{f_A : \mathcal{R}^A \to \Omega_A\}_{A \subseteq N}$ *is such that for each* $A \subseteq N$*,* $f_A$ *is a voting by collections of* $l_A$*–decisive sets.*

Propositions 1 and 3 describe the basic structure any strategy-proof social choice rule has to satisfy. It is summarized in the next corollary.

**Corollary 1** *Suppose that $f$ is strategy-proof. Then, there is a decomposition of $f$ into a function $\omega : 2^N \to T^2$ and a family $\{f_A : \mathcal{R}^A \to \Omega_A\}_{A \subseteq N}$ of votings by collections of $l_A$–decisive sets such that for all $A \subseteq N$ and all preference profiles $R \in \mathcal{R}^A$, $\omega(A) = \Omega_A$ and $f(R) = f_A(R)$.*

# 4 A complete characterization of the strategy-proof rules

To obtain a complete characterization of all strategy-proof rules, we have to derive additional necessary conditions on top of Corollary 1 in each of the two steps.

## Conditions on the first step

We analyze the function $\omega$ that is applied in the first step of the two-step procedure. In particular, we are going to explain how the preselected alternatives may change as more agents declare to have single-peaked preferences. To do that, the next result shows, for any agent $i \in N$ and any set $A \subseteq N \setminus \{i\}$, how $\omega(A) = \Omega_A = \{l_A, h_A\}$ and $\omega(A \cup \{i\}) = \Omega_{A\cup\{i\}} = \{l_{A\cup\{i\}}, h_{A\cup\{i\}}\}$ could be related.

**Proposition 4** *Suppose that $f$ is strategy-proof. Then, for all $A \subset N$ and all $i \in N \setminus A$:*

- *If $h_A \leq i$, then $l_{A\cup\{i\}} \in [l_A, i]$ and $h_{A\cup\{i\}} \in [h_A, i]$.*
- *If $l_A \geq i$, then $l_{A\cup\{i\}} \in [i, l_A]$ and $h_{A\cup\{i\}} \in [i, h_A]$.*
- *If $i \in N_A$, then $i \in [l_{A\cup\{i\}}, h_{A\cup\{i\}}] \subseteq [l_A, h_A]$.*
- *If $i \in N_A$ and $i \in \Omega_{A\cup\{i\}}$, then $\Omega_{A\cup\{i\}} = \{i\}$ or $\Omega_{A\cup\{i\}} \cap \Omega_A \neq \emptyset$.*

Proposition 4 establishes that for any set of agents $A \subseteq N \setminus \{i\}$ with single-peaked preferences, if agent $i$ passes from having single-dipped to single-peaked preferences, the preselected alternatives of $\Omega_{A\cup\{i\}}$ have to be *closer* to $i$ than those in $\Omega_A$. On the one hand, if agent $i$ is situated weakly to the left or weakly to the right of the preselected alternatives of $\Omega_A$, then each of the preselected alternatives of $\Omega_{A\cup\{i\}}$ has to be weakly between the location of agent $i$ and the corresponding preselected location of $\Omega_A$. If, on the other hand, agent $i$ is situated strictly between the two preselected alternatives of $\Omega_A$, then $l_A \leq l_{A\cup\{i\}} \leq i$ and $h_A \geq h_{A\cup\{i\}} \geq i$. That is, agent $i$ is situated weakly between the preselected alternatives of $\Omega_{A\cup\{i\}}$, and the preselected alternatives move weakly into the direction of the location of agent $i$. Moreover, if this location $i$ belongs to $\Omega_{A\cup\{i\}}$, then either $i$ is the unique preselected location or the second preselected alternative already belonged to $\Omega_A$.

## Conditions on the second step

We now study additional necessary conditions that arise in the second step of the two-step procedure. In particular, we analyze how the collections of decisive sets change as an agent $i$ passes from having single-dipped to single-peaked preferences so that this agent cannot benefit from misrepresenting her preferences in some instance. That is, given some $A \subseteq N \setminus \{i\}$, some profile $R \in \mathcal{R}^A$, and some $R'_i \in \mathcal{R}_i^+$, we compare the $l_A$–decisive sets at $R$, $\mathcal{G}^A(R)$, with the $l_{A\cup\{i\}}$–decisive sets at $R' = (R'_i, R_{-i}) \in \mathcal{R}^{A\cup\{i\}}$, $\mathcal{G}^{A\cup\{i\}}(R')$. We only have to impose additional conditions if $|\Omega_A| = |\Omega_{A\cup\{i\}}| = 2$. However, given that we are studying the general case without any assumption on $N$ or $T$, there are many possibilities of how $\Omega_A$ and $\Omega_{A\cup\{i\}}$ could be related, all explained in Proposition 4. We introduce four propositions that cover all possible cases.

We start with the case in which agent $i$ is situated strictly between the two preselected alternatives when she declares to have single-dipped preferences (*i.e.*, $i \in N_A$) and the set of preselected alternatives changes when $i$ announces to have single-peaked preferences (*i.e.*, $\Omega_A \neq \Omega_{A\cup\{i\}}$) in such a way that agent $i$ is also situated strictly between $l_{A\cup\{i\}}$ and $h_{A\cup\{i\}}$ (*i.e.*, $i \in N_{A\cup\{i\}}$). It turns then out that agent $i$ is a dictator at $\Omega_A$ and at $\Omega_{A\cup\{i\}}$, that is, the alternative selected by $f_A$ and by $f_{A\cup\{i\}}$ is always weakly preferred by $i$ to the other alternative of $\Omega_A$ or $\Omega_{A\cup\{i\}}$, respectively. It is important to note that the fact of being a dictator at $\Omega_A$ or at $\Omega_{A\cup\{i\}}$ implies that this agent is selecting her best alternative of this set in *all* profiles of $\mathcal{R}^A$ or $\mathcal{R}^{A\cup\{i\}}$, and not only in $R$ or $R'$. Formally, given $S \subseteq N$, agent $i \in N_S$ is said to be a *dictator at* $\Omega_S$ if for all profiles $R \in \mathcal{R}^S$ such that $i \notin I_S(R)$, $B \in \mathcal{G}^S(R)$ if and only if $i \in B$.[8]

**Proposition 5** *Suppose that $f$ is strategy-proof. Then, for all $A \subset N$ and all $i \in N \setminus A$ such that $i \in N_A \cap N_{A\cup\{i\}}$ and $\Omega_A \neq \Omega_{A\cup\{i\}}$, $i$ is a dictator at $\Omega_A$ and at $\Omega_{A\cup\{i\}}$.*

[8] Observe that the notion of being a dictator at $\Omega_S$ is different from the notion of being a dictator defined in Section 2. The former selects the best alternative of $\Omega_S$ in all profiles of $\mathcal{R}^S$, while the latter selects the best alternative of the entire range $\Omega$ in all profiles of the entire domain $\mathcal{R}$. That is, if an agent is a dictator, she is also a dictator at $\Omega_S$ (whenever $|\Omega_S| = 2$), but not necessarily the other way around.

The next proposition focuses on the situation in which $\Omega_{A\cup\{i\}} = \Omega_A$. First, if agent $i$ is situated weakly to the left (respectively, right) of the two preselected locations, it has to be "easier" (respectively, "more difficult") to select the left alternative when agent $i$ changes her type of preferences from single-dipped to single-peaked. Here, "easier" (respectively, "more difficult") means that the set of coalitions that can impose $l_{A\cup\{i\}} = l_A$ when agent $i$ declares to have single-peaked preferences has to be a weak superset (respectively, subset) of the set when she declares to have single-dipped preferences: $\mathcal{G}^{A\cup\{i\}}(R') \supseteq \mathcal{G}^A(R)$ (respectively, $\mathcal{G}^{A\cup\{i\}}(R') \subseteq \mathcal{G}^A(R)$). The idea underlying this condition is to guarantee that if the location of $\Omega_A = \Omega_{A\cup\{i\}}$ that is situated further away of agent $i$ was selected at $R'$, then this same location is selected at $R$. Second, if agent $i$ is situated strictly between the two preselected alternatives, the set of coalitions that can impose the left point of the range has to be invariant to the type of preferences of agent $i$ (*i.e.*, $\mathcal{G}^{A\cup\{i\}}(R') = \mathcal{G}^A(R)$) whenever agent $i$ is not indifferent between the two preselected alternatives (*i.e.*, $i \notin I_A(R) \cup I_{A\cup\{i\}}(R')$).

**Proposition 6** *Suppose that $f$ is strategy-proof. Then, for all $A \subset N$, all $i \in N \setminus A$, and all profiles $R \in \mathcal{R}^A$ and $R' = (R'_i, R_{-i}) \in \mathcal{R}^{A\cup\{i\}}$ such that $\Omega_A = \Omega_{A\cup\{i\}}$,*

- *$\mathcal{G}^A(R) \subseteq \mathcal{G}^{A\cup\{i\}}(R')$ whenever $i \leq l_A$.*
- *$\mathcal{G}^{A\cup\{i\}}(R') \subseteq \mathcal{G}^A(R)$ whenever $i \geq h_A$.*
- *$\mathcal{G}^A(R) = \mathcal{G}^{A\cup\{i\}}(R')$ whenever $i \in N_A$ and $i \notin I_A(R) \cup I_{A\cup\{i\}}(R')$.*

Next, we analyze what happens if agent $i$ is situated weakly to the left or weakly to the right of the alternatives of $\Omega_A$ and at least one of these locations changes when agent $i$ changes her type of preferences from single-dipped to single-peaked (*i.e.*, $\Omega_A \neq \Omega_{A\cup\{i\}}$) in such a way that $(l_A, h_A) \cap (l_{A\cup\{i\}}, h_{A\cup\{i\}})$ is non-empty. We then find that if agent $i$ is situated weakly to the right (respectively, left) of the alternatives of $\Omega_A$ and a coalition $B$ is decisive to implement the left preselected alternative when agent $i$ declares to have single-peaked (respectively, single-dipped) preferences, then the intersection between $B$ and all agents with single-dipped preferences that are situated strictly between the two preselected alternatives is a decisive set once agent $i$ changes her type of preferences. That is, if, for example, $i$ is situated weakly to the right of the alternatives of $\Omega_A$ (*i.e.*, $i \geq h_A$), then the unique possibility that $(l_A, h_A) \cap (l_{A\cup\{i\}}, h_{A\cup\{i\}}) \neq \emptyset$ and that $\Omega_A \neq \Omega_{A\cup\{i\}}$ is by Proposition 4 (first point) that $i \geq h_{A\cup\{i\}} \geq h_A > l_{A\cup\{i\}} \geq l_A$. For those cases, the proposition implies that if a set of agents $B$ is $l_{A\cup\{i\}}$–decisive at $R'$ (*i.e.*, $B \in \mathcal{G}^{A\cup\{i\}}(R')$), then the intersection of $B$ and the set of agents with single-dipped preferences situated strictly between the preselected alternatives of $\Omega_A$ (*i.e.*, $B \cap (N_A \setminus A)$) is $l_A$–decisive at $R$ (*i.e.*, $B \cap (N_A \setminus A) \in \mathcal{G}^A(R)$). The idea underlying this condition is to guarantee that if $l_{A\cup\{i\}}$ is selected at $R'$, then $l_A$ is selected at $R$.

**Proposition 7** *Suppose that $f$ is strategy-proof. Then, for all $A \subset N$, all $i \in N \setminus A$, and all profiles $R \in \mathcal{R}^A$ and $R' = (R'_i, R_{-i}) \in \mathcal{R}^{A\cup\{i\}}$ such that $\Omega_A \neq \Omega_{A\cup\{i\}}$,*

- *if $i \geq h_{A\cup\{i\}} \geq h_A > l_{A\cup\{i\}} \geq l_A$ and $B \in \mathcal{G}^{A\cup\{i\}}(R')$, then $B \cap (N_A \setminus A) \in \mathcal{G}^A(R)$.*
- *if $i \leq l_{A\cup\{i\}} \leq l_A < h_{A\cup\{i\}} \leq h_A$ and $B \in \mathcal{G}^A(R)$, then $B \cap (N_{A\cup\{i\}} \setminus A) \in \mathcal{G}^{A\cup\{i\}}(R')$.*

Finally, in the last case $i$ is situated strictly between the two preselected alternatives of $\Omega_A$ but not strictly between the ones of $\Omega_{A\cup\{i\}}$ (*i.e.*, $i \in N_A \setminus N_{A\cup\{i\}}$). Then, we know from Proposition 4 (fourth point) that $i \in \Omega_{A\cup\{i\}}$. Given that $|\Omega_{A\cup\{i\}}| = 2$, the other location preselected at $\Omega_{A\cup\{i\}}$ is one of the locations of $\Omega_A$ by Proposition 4. We then find that if a coalition is able to impose the point different from $i$ when agent $i$ has single-peaked preferences, the single-peaked agents of this coalition can impose the very same point if agent $i$ has single-dipped preferences and is not indifferent between $l_A$ and $h_A$ (*i.e.*, $i \notin I_A(R)$). That is, if, for example, $\Omega_{A\cup\{i\}} = \{l_A, i\}$ and a set of agents $B$ is $l_{A\cup\{i\}}$–decisive at $R'$ (*i.e.*, $B \in \mathcal{G}^{A\cup\{i\}}(R')$), the intersection of $B$ and the set of agents with single-peaked preferences (*i.e.*, $B \cap A$) is $l_A$–decisive at $R$ (*i.e.*, $B \cap A \in \mathcal{G}^A(R)$). The idea underlying this condition is to guarantee that if $l_{A\cup\{i\}}$ is selected at $R'$, then this same alternative is selected at $R$.

**Proposition 8** *Suppose that $f$ is strategy-proof. Then, for all $A \subset N$, all $i \in N \setminus A$, and all profiles $R \in \mathcal{R}^A$ and $R' = (R'_i, R_{-i}) \in \mathcal{R}^{A\cup\{i\}}$ such that $i \in N_A \setminus N_{A\cup\{i\}}$ and $i \notin I_A(R)$,*

- *if $l_A = l_{A\cup\{i\}}$ and $B \in \mathcal{G}^{A\cup\{i\}}(R')$, then $B \cap A \in \mathcal{G}^A(R)$.*
- *if $h_A = h_{A\cup\{i\}}$ and $B \in \mathcal{G}^A(R)$, then $B \cap N_{A\cup\{i\}} \cap A \in \mathcal{G}^{A\cup\{i\}}(R')$.*

To summarize this set of four propositions, we provide an overview of how the cases of Proposition 4 are covered. To do so, suppose that $i = 3$ and consider any set $A \subseteq (N \setminus \{3\})$ such that both $\Omega_A$ and $\Omega_{A\cup\{3\}}$ include two preselected alternatives. There are three cases.

1. Let $3 \in N_A$, for example $\Omega_A = \{1, 5\}$. Then, by the third and the fourth point of Proposition 4, we have the following three possibilities:

    i. Suppose that $\Omega_{A\cup\{3\}} = \Omega_A = \{1, 5\}$. The third point of Proposition 6 applies and the decisive coalitions do not change between $R \in \mathcal{R}^A$ and $R' = (R'_3, R_{-3}) \in \mathcal{R}^{A\cup\{3\}}$ whenever $3 \notin I_A(R) \cup I_{A\cup\{3\}}(R')$.

    ii. Suppose that $\Omega_{A\cup\{3\}} \neq \Omega_A$ and $3 \in N_{A\cup\{3\}}$, for example $\Omega_{A\cup\{3\}} = \{1, 4\}$. Proposition 5 applies and agent 3 is a dictator both at $\Omega_A$ and at $\Omega_{A\cup\{3\}}$.

iii. Suppose that $3 \in \Omega_{A\cup\{3\}}$ and $\Omega_A \cap \Omega_{A\cup\{3\}} \neq \emptyset$, for example $\Omega_{A\cup\{3\}} = \{1,3\}$. Proposition 8 (first or second point depending on whether 3 is $h_{A\cup\{3\}}$ or $l_{A\cup\{3\}}$) applies and the decisive coalitions are so that if 1 is selected at $R' = (R'_3, R_{-3}) \in \mathcal{R}^{A\cup\{3\}}$, then 1 is also selected at $R \in \mathcal{R}^A$.

2. Let $3 \leq l_A$, for example, $\Omega_A = \{5,7\}$. Then, by the second point of Proposition 4, we have the following three possibilities:

   i. Suppose that $\Omega_{A\cup\{3\}} = \Omega_A = \{5,7\}$. The first point of Proposition 6 applies and the decisive coalitions that can implement 5 at $R' = (R'_3, R_{-3}) \in \mathcal{R}^{A\cup\{3\}}$ are a weak superset of the ones that can do that at $R \in \mathcal{R}^A$.

   ii. Suppose that $h_{A\cup\{3\}} \leq l_A$, for example $\Omega_{A\cup\{3\}} = \{4,5\}$. None of the propositions applies and no restrictions on the decisive coalitions are imposed.

   iii. Suppose that $h_{A\cup\{3\}} > l_A$, for example $\Omega_{A\cup\{3\}} = \{4,6\}$. The second point of Proposition 7 applies and the decisive coalitions are such that if 5 is selected at $R \in \mathcal{R}^A$, then 4 is selected at $R' = (R'_3, R_{-3}) \in \mathcal{R}^{A\cup\{3\}}$.

3. Let $3 \geq h_A$. The possibilities implied by the first point of Proposition 4 are similar to the case when $3 \leq l_A$. Moreover, the propositions applied in each case are exactly the same as before, but using the second point of Proposition 6 and the first point of Proposition 7.

### The characterization result

So far, we have established a set of necessary conditions for a social choice rule $f$ to be strategy-proof. Our main theorem shows that the union of these conditions is also sufficient.

**Theorem 1** *A social choice rule $f$ is strategy-proof if and only if there is a function $\omega : 2^N \to T^2$ satisfying Proposition 4 and a family $\{f_A : \mathcal{R}^A \to \Omega_A\}_{A\subseteq N}$ of votings by collections of $l_A$–decisive sets satisfying Propositions 5 to 8 such that for all $A \subseteq N$ and all preference profiles $R \in \mathcal{R}^A$, $\omega(A) = \Omega_A$ and $f(R) = f_A(R)$.*

## 5 Additional characterizations

The structure of the characterized family and in particular the conditions imposed by Propositions 4 to 8 depend on the relation between $N$ and $T$. In fact, there are situations, such as Example 1, in which all strategy-proof rules with at least three alternatives in the range are dictatorial, leading again to the impossibility of combining the axioms of the Gibbard-Satterthwaite theorem. Fortunately, Examples 2 and 3 indicate that this is generally not the case. To characterize all situations in which we can escape from this impossibility, we need to introduce the following notation: a set $S \subseteq T$ is said to be *full at* $N$ if $N \cap S = \emptyset$ and $N \subset (\min S, \max S)$.

**Theorem 2** *There is a non-dictatorial strategy-proof social choice rule* $f$ *with* $|\Omega| > 2$ *if and only if there is a set* $S \subseteq T$ *of size* $|S| = 3$ *that is not full at* $N$.

A practical way of checking the condition of Theorem 2 is the following.

**Corollary 2** *The following statements hold.*

- *If* $|T| \geq 5$, *there is a non-dictatorial strategy-proof social choice rule* $f$ *with* $|\Omega| > 2$.
- *If* $|T| = 4$, *there is a non-dictatorial strategy-proof social choice rule* $f$ *with* $|\Omega| > 2$ *if and only if* $T$ *cannot be partitioned into* $T_1$ *and* $T_2$ *such that* $|T_1| = |T_2| = 2$, $\max T_1 < \min N$, *and* $\max N < \min T_2$.
- *If* $|T| = 3$, *there is a non-dictatorial strategy-proof social choice rule* $f$ *with* $|\Omega| > 2$ *if and only if* $N \cap T \neq \emptyset$ *or* $N \not\subset (\min T, \max T)$.

One can see that if there are more than 4 alternatives, then it is always possible to find non-dictatorial strategy-proof social choice rules that have at least three alternatives in the range. Otherwise, it is sufficient to have an agent that is situated at a feasible location or to the left (or right) of at least three alternatives.[9] However, even if the condition in Theorem 2 is met and there is a triple that is not full at $N$, it is not guaranteed that we can construct *desirable* rules, because it is possible that for some $N$ and $T$ the addition of Pareto efficiency to the condition of strategy-proofness leads to the dictatorial rules. Examples 2 and 3 indicate again that this is generally not the case. Our next proposition presents a necessary condition on $T$ for the existence of Pareto efficient rules.

**Proposition 9** *Suppose* $T$ *is such that there is a Pareto efficient social choice rule* $f$. *Then,* $\min T$ *and* $\max T$ *exist.*

[9] If $|T| = 3$ and none of these conditions is satisfied, the set of strict preferences of $\mathcal{R}$ coincides with the universal strict preference domain (see Example 1). Thus, $\mathcal{R}$ restricts the universal strict preference domain but does not allow us to escape from the Gibbard-Satterthwaite impossibility only if $|T| = 4$ and the relation between $N$ and $T$ is the one described in the second point of Corollary 2.

Proposition 9 shows that independently of the locations of the agents, it is necessary that $T$ has a minimum and a maximum. So, we assume this from now on. Our next result characterizes the additional conditions the relation between $N$ and $T$ has to satisfy in order to be able to combine Pareto efficiency and strategy-proofness without arriving at the dictatorial rules.[10]

**Theorem 3** *Suppose $T$ is such that $|T| > 2$ and both $\min T$ and $\max T$ exist. There is a strategy-proof social choice rule $f$ that is non-dictatorial and Pareto efficient if and only if at least one of the following conditions hold:*

- $N \not\subset (\min T, \max T)$.
- *There are two agents $i, j \in N$ such that $i \in T$, and both $\max\{x \in T : x \leq j\}$ and $\min\{x \in T : x \geq j\}$ exist.*

According to Theorem 3, there are two possibilities. The first is that at least one agent is situated weakly to the left or weakly to the right of all feasible locations. The second is that there are two agents $i$ and $j$, one situated at a feasible location ($i \in T$), and the other having defined nearest feasible points weakly to her left ($\max\{x \in T : x \leq j\}$) and weakly to her right ($\min\{x \in T : x \geq j\}$).[11] Since there is no closed-form representation of all Pareto efficient and strategy-proof rules valid for all sets $N$ and $T$ that satisfy at least one of these possibilities, we consider now two focal cases that reflect two conditions that are sufficient to guarantee possibility results.

## Agents situated outside (min T, max T)

The first condition that guarantees on its own the possibility of combining Pareto efficiency and strategy-proofness without arriving at dictatorial rules is that at least one agent is not situated in $(\min T, \max T)$. We are going to provide the characterization for the focal case when $N \cap (\min T, \max T) = \emptyset$. Observe that this framework covers situations like the one in Example 2. Let $N_l = \{i \in N : i \leq \min T\}$ and $N_h = \{i \in N : i \geq \max T\}$ be the sets of agents situated weakly to the left and weakly to the right of all feasible locations, respectively. Obviously, $N_l \cup N_h = N$.

[10] The result is only presented for the cases when $|T| > 2$ because, as it is well-known, there always exist non-dictatorial strategy-proof rules that are Pareto efficient when $|T| \leq 2$.

[11] Observe that this includes the possibility that $j \in T$ because in that case $\max\{x \in T : x \leq j\} = \min\{x \in T : x \geq j\} = j$.

Since no agent is situated strictly between alternatives of $T$, only one alternative gets preselected in the first step of the two-step rules characterized in Theorem 1. Therefore, the range of $\omega$ becomes $T$ instead of $T^2$. This implies that the characterized rules are types-only (they only depend on the type of preferences, single-peaked or single-dipped, of each agent). One can also observe that the preference of any agent of $N_l$ (respectively, $N_h$) with a single-dipped preference can be seen as a single-peaked preference *on the set of feasible locations* with the peak at $\max T$ (respectively, $\min T$). Hence, each agent only has two admissible preferences and both are single-peaked on the set of feasible locations with the peak in $\min T$ or in $\max T$. Consequently, $\mathcal{R}$ corresponds with a subdomain of the single-peaked preference domain. Since all generalized median voter rules are strategy-proof on the domain of single-peaked preferences (Moulin [17]), we know that they are strategy-proof here as well.[12] We introduce a definition of these rules in our subdomain.

**Definition 2** *The social choice rule $f$ is said to be a generalized median voter rule if there is a function $\pi : 2^N \to T$ such that for all $S \subseteq S' \subseteq N$, $\pi(S) \geq \pi(S')$, and for all preference profiles $R \in \mathcal{R}$, $f(R) = \pi(S)$, where $S = \{i \in N : \min T \; P_i \; \max T\}$.*[13]

The following theorem shows that the generalized median voter rules are the unique strategy-proof rules whenever each agent is situated weakly to the left or weakly to the right of all feasible alternatives. The theorem also shows that to additionally obtain Pareto efficiency, it is necessary to require that $\pi(N) = \min T$ and $\pi(\emptyset) = \max T$.[14]

**Theorem 4** *Let $T$ be such that both $\min T$ and $\max T$ exist and that $N \cap (\min T, \max T) = \emptyset$. A social choice rule $f$ is strategy-proof if and only if it is a generalized median voter rule. A social choice rule $f$ is strategy-proof and Pareto efficient if and only if it is a generalized median voter rule with $\pi(N) = \min T$ and $\pi(\emptyset) = \max T$.*

Observe that the rule included in Example 2 belongs to the characterized family.

[12] We are very grateful to two anonymous referees who have pointed out this fact to us.

[13] An alternative definition of these rules would be based on Theorem 1 and, in particular, on the function $\omega$. A social choice rule $f$ is a generalized median voter rule if there is a function $\omega : 2^N \to T$ such that for all $S \subset N$ and all $i \in N \setminus S$, $\omega(S \cup \{i\}) \geq \omega(S)$ (respectively, $\omega(S \cup \{i\}) \leq \omega(S)$) whenever $i \in N_h$ (respectively, $i \in N_l$) and that for all $A \subseteq N$ and all preference profiles $R \in \mathcal{R}^A$, $f(R) = \omega(A)$.

[14] These conditions can be expressed also in terms of the function $\omega$ of Footnote 13 by saying that $\omega(N_l) = \min T$ and $\omega(N_h) = \max T$.

### Agents situated at feasible locations

The second condition that guarantees on its own the possibility of combining Pareto efficiency and strategy-proofness without arriving at dictatorial rules is that there are at least two agents situated at feasible locations. We are thus interested in the focal case when $N \subseteq T$. Since the cases with agents situated at the extreme locations have been studied in the previous subsection, we assume for the sake of simplicity that $N \subset (\min T, \max T)$. This setting includes many of the natural frameworks studied in the literature, for example, when, like in Example 3, $T$ is a closed interval and all agents are situated within $T$. Since the description of the Pareto efficient and strategy-proof rules is still quite complex, we impose tops-onliness as an additional condition. Formally, a social choice rule $f$ is *tops-only* if for all preference profiles $R, R' \in \mathcal{R}$ such that $t(R_i) \equiv \{x \in T : x\, R_i\, y \text{ for all } y \in T\} = \{x \in T : x\, R'_i\, y \text{ for all } y \in T\} \equiv t(R'_i)$ for all $i \in N$, then $f(R) = f(R')$. Given that $N \subset T \cap (\min T, \max T)$, it is easy to see that for all $i \in N$ and all preferences $R_i \in \mathcal{R}_i^+$ and $R'_i \in \mathcal{R}_i^-$, we have that $t(R_i) = \{i\}$ and $t(R'_i) \in \{\{\min T\}, \{\max T\}, \{\min T, \max T\}\}$.

Before providing a formal definition of the non-dictatorial rules that are strategy-proof and Pareto efficient, we are going to describe their core structure intuitively. All rules of the family ask the agents first about their type of preferences (single-peaked or single-dipped). Then, two possibilities appear. On the one hand, if there are agents that report to have single-peaked preferences, the selected point —which is determined via an aggregator $\omega$ that considers only the information about the location of these agents with single-peaked preferences— lies between the leftmost and the rightmost location of these agents. So, for example, if the set of agents that report to have single-peaked preferences is $A \in 2^N \setminus \emptyset$, the rule selects a location $\omega(A) \in [\min A, \max A]$. On the other hand, if all agents report to have single-dipped preferences, the outcome is one of the two extreme points of $T$. To take a decision between them, agents are asked about their top alternatives (i.e., their ordinal preferences between $\min T$ and $\max T$). In particular, each rule defines a set of pairs of coalitions $\mathcal{G} \subseteq 2^N \times 2^N$ and operates as follows. If the pair of sets formed by those agents weakly preferring $\min T$ to $\max T$ and those strictly preferring $\min T$ to $\max T$ belongs to $\mathcal{G}$, then $\min T$ is selected. Otherwise, $\max T$ is selected.

Up to this core structure, it is necessary to incorporate further conditions. First, the aggregator $\omega$ has to satisfy the following *monotonicity* property: for all $A \in 2^N \setminus \emptyset$ and all $i \in N \setminus A$, $\omega(A) \geq \omega(A \cup \{i\}) \geq i$ (respectively, $\omega(A) \leq \omega(A \cup \{i\}) \leq i$) whenever $\omega(A) \geq i$ (respectively, $\omega(A) \leq i$). Second, the set of pairs of coalitions $\mathcal{G}$ has to satisfy the following *monotonicity* property: for all $A, A', B, B' \subseteq N$ with $A \subseteq A'$ and $B \subseteq B'$, [if $(A, B) \in \mathcal{G}$, then $(A', B') \in \mathcal{G}$] and [if $(A, B') \notin \mathcal{G}$, then $(A \cup (B' \setminus B), B) \notin \mathcal{G}$]. Third, $\mathcal{G}$ has to satisfy also the following *efficiency* property: $(N, B) \in \mathcal{G}$ for all $B \neq \emptyset$ and $(A, \emptyset) \notin \mathcal{G}$ for all $A \neq N$.[15] The formal definition of the rules is then as follows.

**Definition 3** *The social choice rule $f$ is said to be a conditional two-step rule if there is a monotone function $\omega : 2^N \backslash \emptyset \to (\min T, \max T)$, and a set of pairs of coalitions $\mathcal{G} \subset 2^N \times 2^N$ satisfying monotonicity and efficiency such that for all $A \subseteq N$ and all preference profiles $R \in \mathcal{R}^A$,*

$$f(R) = \begin{cases} \omega(A) \in [\min A, \max A] & \text{if } A \neq \emptyset \\ \min T & \text{if } A = \emptyset \text{ and } (\{i \in N : \min T \in t(R_i)\}, \{i \in N : \min T = t(R_i)\}) \in \mathcal{G} \\ \max T & \text{otherwise.} \end{cases}$$

The following theorem states the characterization result.

**Theorem 5** *Suppose that $T$ is such that both* $\min T$ *and* $\max T$ *exist and that $N \subset T \cap (\min T, \max T)$. A social choice rule $f$ is strategy-proof, Pareto efficient, and tops-only if and only if it is a conditional two-step rule.*

We can see that any conditional two-step rule depends on two characteristics: the aggregator $\omega$ used to choose an interior point, and the pairs of coalitions of agents $\mathcal{G}$ used to choose one of the extremes. For instance, the rule proposed in Example 3 is obtained if $\omega$ is the mean and $\mathcal{G} = \{(A, B) \in N^2 : |A| > |N \setminus B|\}$. Finally, note that one particularity of the characterized rules is that as soon as one agent has single-peaked preferences, then all agents with single-dipped preferences are treated as dummies.

# 6 Concluding remarks

In this paper, we have studied the problem of locating a public facility when the general sentiment of each agent towards the facility (it could be considered a good or a bad) is unknown to the planner. Since it is well-known that the preference domain cannot accommodate simultaneously all single-peaked and all single-dipped preferences without arriving at the Gibbard-Satterthwaite impossibility, there emerges the necessity to further

[15]Observe that the set $\mathcal{G}$ introduced here and the correspondences $\mathcal{G}^A$ for each $A \subseteq N$ defined in Section 3 are different concepts. However, as it can be seen in the proof of Theorem 5, $\mathcal{G}$ is deduced from $\mathcal{G}^\emptyset$. Similarly, although the aggregator $\omega$ in Definition 3 is not specified for the $\emptyset$ (contrary to Corollary 1), one can observe that the preselected alternatives for that case are $\min T$ and $\max T$.

restrict preferences in order to obtain a strategy-proof preference revelation mechanism. It turns out that the public decision makers tend to know the location of the agents in many real life applications and that it is often reasonable to assume that the peaks/dips of the preferences are closely related with the agents' own locations. Consequently, we consider here the domain $\mathcal{R}$ when agents can have any single-peaked and any single-dipped preference with the sole limitation that the peaks/dips correspond to the agents' locations. We have shown that this domain $\mathcal{R}$ allows for the construction of non-dictatorial strategy-proof rules with $|\Omega| > 2$ in most cases and we have characterized them.

The model assumes that the exact location of each agent is publicly known, yet there can be some imprecision in the determination of agents' locations in real-life applications. If such an imprecision was in place and we only knew that each agent is located in a point of an interval, we would have to work with a domain bigger than $\mathcal{R}$ and, therefore, we would obtain a smaller set of strategy-proof rules. The results will depend on the quantity of the imprecision. Two extreme cases are the following. On the one hand, if it is only known that each agent is situated at a point of an interval that includes $[\min T, \max T]$, then the domain would include all single-peaked and all single-dipped preferences on the set of feasible alternatives and, therefore, we would have the Gibbard-Satterthwaite impossibility. On the other hand, if for each agent for whom we have imprecise information about her location, there is no point of $T$ inside her interval of possible locations, then all rules that are strategy-proof on $\mathcal{R}$ would be strategy-proof on that domain too.

The domain $\mathcal{R}$, and the associated strategy-proof rules characterized in this paper, complements the existing results for the domains of only single-peaked and of only single-dipped preferences. The decision which of the existing domain restrictions is the appropriate and, as a consequence, which are the correct strategy-proof rules to implement depends crucially on the particular facility analyzed. If the planner expects that the facility provokes unanimous opinions in the society, then the facility would be a public good or a public bad and the rules characterized in the classical domains would be optimal. However, if the public decision maker suspects that the facility can produce different opinions, the rules derived in this paper seem to be more appropriate.

# Appendix

The following concepts are used in the course of our proofs. A non-ordered pair of alternatives $\{x, y\}$ is said to be a fixed pair for agent $i$ if for all $a, b \in \{x, y\}$ such that $a < b$, we cannot have that $a < i < b$. Observe that given any type of preferences of agent $i$

(single-peaked or single-dipped), if $\{x, y\}$ is a fixed pair for $i$, then this agent has always the same ordinal preferences over $\{x, y\}$. Similarly, the ordered pair of alternatives $(x, y)$ is said to be a fixed pair for agent $i$ at $\mathcal{R}_i^A$ if for all $R_i \in \mathcal{R}_i^A$, $x \, P_i \, y$. Or, to say it differently, $(x, y)$ is a fixed pair for $i$ at $\mathcal{R}_i^A$ if $i \in A$ and $[y < x \leq i$ or $y > x \geq i]$, or if $i \notin A$ and $[i \leq y < x$ or $i \geq y > x]$. We can then see that if $\{x, y\}$ is a fixed pair for agent $i$, then $(x, y)$ is a fixed pair for one type of preferences and $(y, x)$ is a fixed pair for the other type. Finally, let $O_i(A, R_{-i})$ be the option set of agent $i$ given the preferences $R_{-i}$ of the other agents and given that the set of agents with single-peaked preferences at $R$ is equal to $A \subseteq N$. That is, $x \in O_i(A, R_{-i})$ if there is a preference profile $R = (R_i, R_{-i}) \in \mathcal{R}^A$ such that $f(R) = x$.

## Proof of Proposition 1

Our first lemma shows that if $x$ and $y$ belong to an option set of agent $i$, then $i$ is situated strictly between $x$ and $y$.

**Lemma 1** *Suppose that $f$ is strategy-proof. Then, for all $A \subseteq N$, all agents $i \in N$, all profiles $R \in \mathcal{R}^A$, and all $x, y \in O_i(A, R_{-i})$ such that $x < y$, $x < i < y$.*

Proof: Since $x, y \in O_i(A, R_{-i})$ by assumption, there are two preferences $R'_i, R''_i \in \mathcal{R}_i^A$ for agent $i$ such that $f(R'_i, R_{-i}) = x$ and $f(R''_i, R_{-i}) = y$. If $i \leq x$, then agent $i$ can manipulate $f$ at $(R''_i, R_{-i})$ via $R'_i$ whenever $i \in A$ and at $(R'_i, R_{-i})$ via $R''_i$ whenever $i \notin A$. Similarly, if $y \leq i$, then agent $i$ can manipulate $f$ at $(R'_i, R_{-i})$ via $R''_i$ whenever $i \in A$ and at $(R''_i, R_{-i})$ via $R'_i$ whenever $i \notin A$. Hence, $x < i < y$. □

Lemma 1 directly implies that $O_i(A, R_{-i})$ contains at most two alternatives.

**Corollary 3** *Suppose that $f$ is strategy-proof. Then, for all $A \subseteq N$, all agents $i \in N$, and all profiles $R \in \mathcal{R}^A$, $|O_i(A, R_{-i})| \leq 2$.*

Next, we show that $f$ always selects a maximal alternative of an agent's option set.

**Lemma 2** *Suppose that $f$ is strategy-proof. Then, for all $A \subseteq N$, all agents $i \in N$, all $x, y \in T$, and all profiles $R \in \mathcal{R}^A$ such that $O_i(A, R_{-i}) = \{x, y\}$ and $f(R) = x$, $x \, R_i \, y$.*

Proof: Suppose otherwise, that is, there is a set $A \subseteq N$, a profile $R \in \mathcal{R}^A$, two alternatives $x, y \in T$, and an agent $i \in N$ such that $O_i(A, R_{-i}) = \{x, y\}$, $f(R) = x$, and $y \, P_i \, x$. Since $y \in O_i(A, R_{-i})$ by assumption, there is a preference $R'_i \in \mathcal{R}_i^A$ such that $f(R'_i, R_{-i}) = y$. Then, agent $i$ manipulates $f$ at $R$ via $R'_i$. □

An essential part in the proof of the proposition is to show that if the proposition was wrong (*i.e.*, $f$ is strategy-proof and $|\Omega_A| > 2$ for some $A \subseteq N$), then there would be a profile $R \in \mathcal{R}^A$, and two agents $i, j \in N$ with other preferences $R'_i \in \mathcal{R}^A_i$, $R'_j \in \mathcal{R}^A_j$ so that the outcomes at $R$, $(R'_i, R_{-i})$, and $(R'_j, R_{-j})$ differ. Since the formal argument is rather lengthy, we divide this part in four lemmata. Lemma 3 establishes the desired implication, but substituting agent $j$ for some set of agents $S$, not necessarily of cardinality 1.

**Lemma 3** *Suppose that $f$ is strategy-proof. Then, for all $A \subseteq N$ such that $|\Omega_A| > 2$, there is an agent $i \in N$, a set of agents $S \subseteq N \setminus \{i\}$, and three profiles $R, (R'_i, R_{-i}), (R'_S, R_{-S}) \in \mathcal{R}^A$ such that $f(R) \neq f(R'_i, R_{-i}) \neq f(R'_S, R_{-S}) \neq f(R)$.*

Proof: Consider any $A \subseteq N$ and suppose that $|\Omega_A| > 2$. Then, there are three preference profiles $\bar{R}, \bar{R}', \bar{R}'' \in \mathcal{R}^A$ such that $f(\bar{R}) \neq f(\bar{R}') \neq f(\bar{R}'') \neq f(\bar{R})$. Let $f(\bar{R}) = x$. Starting at $\bar{R}$, construct the sequence of profiles in which the preferences of all agents $j \in N$ are changed one-by-one from $\bar{R}_j$ to $\bar{R}'_j$ so that the final profile is $\bar{R}'$. Since $f(\bar{R}) \neq f(\bar{R}')$, the outcome of the function must have changed along this sequence. Let $\hat{R}$ be the profile when the outcome is $x$ for the last time in the sequence and let $i$ be the next agent changing preferences in this sequence. Then, $f(\bar{R}'_i, \hat{R}_{-i}) \neq x$ and denote this distinct alternative by $y$. That is, $f(\hat{R}) = x$ and $f(\bar{R}'_i, \hat{R}_{-i}) = y$. We consider now two cases.

- Suppose that $f(\bar{R}'') \neq y$. If $f(\bar{R}''_i, \hat{R}_{-i}) \notin \{x, y\}$, then $|O_i(A, \hat{R}_{-i})| > 2$ contradicting Corollary 3. Hence, $f(\bar{R}''_i, \hat{R}_{-i}) \in \{x, y\}$ and $f(\bar{R}'') \notin \{x, y\}$. Denote $S = N \setminus \{i\}$. If $f(\bar{R}''_i, \hat{R}_{-i}) = x$, the result is obtained by defining $R = (\bar{R}''_i, \hat{R}_{-i})$, $R'_i = \bar{R}'_i$, and $R'_S = \bar{R}''_S$. On the other hand, if $f(\bar{R}''_i, \hat{R}_{-i}) = y$, the result is obtained by defining $R = (\bar{R}''_i, \hat{R}_{-i})$, $R'_i = \hat{R}_i$, and $R'_S = \bar{R}''_S$.
- Suppose that $f(\bar{R}'') = y$. Then, $f(\bar{R}') \notin \{x, y\}$. Let $S = N \setminus \{i\}$. Then, the result is obtained by defining $R = (\bar{R}'_i, \hat{R}_{-i})$, $R'_i = \hat{R}_i$, and $R'_S = \bar{R}'_S$.

This completes the proof of the lemma. □

The next lemma is needed later on.

**Lemma 4** *Given any $A \subseteq N$, suppose that there are sets $V \subset S \subseteq N$ and three profiles $R, (R'_V, R_{-V}), (R'_S, R_{-S}) \in \mathcal{R}^A$ such that $f(R) \neq f(R'_V, R_{-V}) \neq f(R'_S, R_{-S}) \neq f(R)$. Then, there are two subsets $W_1 \subset W_2 \subseteq S$ and an agent $i \in W_2 \setminus W_1$ such that $f(R'_{W_1}, R_{-W_1}) \neq f(R'_{W_1 \cup \{i\}}, R_{-(W_1 \cup \{i\})}) \neq f(R'_{W_2}, R_{-W_2}) \neq f(R'_{W_1}, R_{-W_1})$.*

Proof: Take a profile $R \in \mathcal{R}^A$, a set of agents $S \subseteq N$ with the alternative preferences $R'_S$, and a set $V \subset S$ such that $(R'_S, R_{-S}) \in \mathcal{R}^A$ and $f(R) \neq f(R'_V, R_{-V}) \neq f(R'_S, R_{-S}) \neq f(R)$. Starting at $R \in \mathcal{R}^A$, construct the sequence of profiles in which the preferences of all agents $j \in V$ are changed one-by-one from $R_j$ to $R'_j$ so that the sequence ends at $(R'_V, R_{-V})$. Since $f(R) \neq f(R'_V, R_{-V})$ by assumption, the outcome must have changed along this sequence. So, let $W_1 \subset V$ be the set of agents that have changed preferences in the sequence the last time the rule selects $f(R)$, and let $i$ be the next agent changing preferences in the sequence. Then, $f(R) = f(R'_{W_1}, R_{-W_1}) \neq f(R'_{W_1 \cup \{i\}}, R_{-(W_1 \cup \{i\})})$. If $f(R'_{W_1 \cup \{i\}}, R_{-(W_1 \cup \{i\})}) \neq f(R'_V, R_{-V})$, the result follows from setting $W_2$ equal to $V$. If, on the other hand, $f(R'_{W_1 \cup \{i\}}, R_{-(W_1 \cup \{i\})}) = f(R'_V, R_{-V})$, the result follows from setting $W_2$ equal to $S$. □

We show that whenever the set of agents $S$ defined in Lemma 3 is such that $|S| > 1$, then we can replicate this result with another set of agents $M$ with lower cardinality.

**Lemma 5** *Suppose that $f$ is strategy-proof and that for some $A \subseteq N$, there is an agent $i \in N$, a set of agents $S \subseteq N \backslash \{i\}$, with $|S| > 1$, and three profiles $R, (R'_i, R_{-i}), (R'_S, R_{-S}) \in \mathcal{R}^A$ such that $f(R) \neq f(R'_i, R_{-i}) \neq f(R'_S, R_{-S}) \neq f(R)$. Then, there exists an agent $j \in N$, a set of agents $M \subseteq N \backslash \{j\}$, with $|M| < |S|$, and three profiles $\bar{R}, (\bar{R}'_j, \bar{R}_{-j}), (\bar{R}'_M, \bar{R}_{-M}) \in \mathcal{R}^A$ such that $f(\bar{R}) \neq f(\bar{R}'_j, \bar{R}_{-j}) \neq f(\bar{R}'_M, \bar{R}_{-M}) \neq f(\bar{R})$.*

Proof: Let $f(R) = x$, $f(R'_i, R_{-i}) = y$, and $f(R'_S, R_{-S}) = z$. The proof is divided into five steps that exhaust all possibilities.

*Step 1: We prove the lemma if there is a set $V \subset S \cup \{i\}$, with $V \notin \{S, \{i\}\}$, such that $f(R'_V, R_{-V}) \notin \{x, y\}$.*

Consider a set $V \subset S \cup \{i\}$, with $V \notin \{S, \{i\}\}$, such that $f(R'_V, R_{-V}) \notin \{x, y\}$. First, if $i \notin V$, observe that $f(R) \neq f(R'_i, R_{-i}) \neq f(R'_V, R_{-V}) \neq f(R)$ and that $|V| < |S|$. Denote profile $R$ by $\bar{R}$, preference $R'_i$ by $\bar{R}'_i$, and preferences $R'_V$ by $\bar{R}'_V$ to obtain the desired result (with $i$ playing the role of $j$ and $V$ that of $M$). Second, if $i \in V$, denote profile $(R'_i, R_{-i})$ by $\bar{R}$, preference $R_i$ by $\bar{R}'_i$, and preferences $R'_{V \setminus \{i\}}$ by $\bar{R}'_{V \setminus \{i\}}$. Since $|V \setminus \{i\}| < |S|$, we have the desired result (with $i$ playing the role of $j$ and $V \setminus \{i\}$ that of $M$).

*Step 2: We prove the lemma if there is a set $V \subset S$ such that $f(R'_V, R_{-V}) = y$.*

Consider a set $V \subset S$ such that $f(R'_V, R_{-V}) = y$. Observe that $f(R) \neq f(R'_V, R_{-V}) \neq f(R'_S, R_{-S}) \neq f(R)$. By Lemma 4, there are two sets $W_1 \subset W_2 \subseteq S$ and an agent $k \in W_2 \setminus W_1$ such that $f(R'_{W_1}, R_{-W_1}) \neq f(R'_{W_1 \cup \{k\}}, R_{-(W_1 \cup \{k\})}) \neq f(R'_{W_2}, R_{-W_2}) \neq$

$f(R'_{W_1}, R_{-W_1})$. Denote profile $(R'_{W_1 \cup \{k\}}, R_{-(W_1 \cup \{k\})})$ by $\bar{R}$, preference $R_k$ by $\bar{R}'_k$, and preferences $R'_{(W_2 \setminus W_1) \setminus \{k\}}$ by $\bar{R}'_{(W_2 \setminus W_1) \setminus \{k\}}$. Since $|(W_2 \setminus W_1) \setminus \{k\}| < |S|$, we have the desired result (with $k$ playing the role of $j$ and $(W_2 \setminus W_1) \setminus \{k\}$ that of $M$).

*Step 3: We prove the lemma if $f(R'_V, R_{-V}) = x$ for all sets $V \subset S$ and there is some set $W \subset S \cup \{i\}$, with $i \in W \neq \{i\}$, such that $f(R'_W, R_{-W}) = y$.*

Consider a set $W \subset S \cup \{i\}$, where $i \in W \neq \{i\}$, such that $f(R'_W, R_{-W}) = y$. By assumption, $f(R'_V, R_{-V}) = x$ for all $V \subset S$. Then, by setting $V = W \setminus \{i\}$, we can see that $f(R'_{W \setminus \{i\}}, R_{-(W \setminus \{i\})}) = x$. Denote $(R'_{W \setminus \{i\}}, R_{-(W \setminus \{i\})})$ by $\bar{R}$, $R'_i$ by $\bar{R}'_i$ and $R'_{S \setminus (W \setminus \{i\})}$ by $\bar{R}'_{S \setminus (W \setminus \{i\})}$. Since $|S \setminus (W \setminus \{i\})| < |S|$, we obtain the desired result (with $i$ playing the role of $j$ and $S \setminus (W \setminus \{i\})$ that of $M$).

*Step 4: We prove the lemma if $f(R'_V, R_{-V}) = x$ for all $V \subset S \cup \{i\}$, with $V \notin \{S, \{i\}\}$, and $f(R'_{S \cup \{i\}}, R_{-(S \cup \{i\})}) \neq x$.*

Consider any agent $j \in S$. By assumption, $f(R'_V, R_{-V}) = x$ for all $V \subset S \cup \{i\}$ with $V \notin \{S, \{i\}\}$. By setting $V = (S \cup \{i\}) \setminus \{j\}$, we can see that $f(R'_{(S \cup \{i\}) \setminus \{j\}}, R_{-((S \cup \{i\}) \setminus \{j\})}) = x$. Similarly, by setting $V = \{i, j\}$, we can see that $f(R'_{\{i,j\}}, R_{-\{i,j\}}) = x$. First, if $f(R'_{S \cup \{i\}}, R_{-(S \cup \{i\})}) \notin \{x, z\}$, denote $(R'_{S \cup \{i\}}, R_{-(S \cup \{i\})})$ by $\bar{R}$, $R_j$ by $\bar{R}'_j$, and $R_i$ by $\bar{R}'_i$ to establish the result (with $\{i\}$ playing the role of $M$). Second, if $f(R'_{S \cup \{i\}}, R_{-(S \cup \{i\})}) = z$, denote $(R'_{\{i,j\}}, R_{-\{i,j\}})$ by $\bar{R}$, $R_j$ by $\bar{R}'_j$, and $R'_{S \setminus \{j\}}$ by $\bar{R}'_{S \setminus \{j\}}$. Since $|S \setminus \{j\}| < |S|$, we obtain the desired result (with $S \setminus \{j\}$ playing the role of $M$).

*Step 5: We prove the lemma if $f(R'_V, R_{-V}) = x$ for all $V \subseteq S \cup \{i\}$, with $V \notin \{S, \{i\}\}$.*

*Step 5.a: We show that for all $j \in S$, $O_j(A, (R'_i, R_{-\{i,j\}})) = \{x, y\}$ and $O_j(A, (R'_{S \setminus \{j\}}, R_{-S})) = \{x, z\}$.*

Consider any agent $j \in S$. By assumption, $f(R'_V, R_{-V}) = x$ for all $V \subseteq S \cup \{i\}$ with $V \notin \{S, \{i\}\}$. It follows thus from setting $V = \{i, j\}$ that $f(R'_{\{i,j\}}, R_{-\{i,j\}}) = x$. This, together with $f(R'_i, R_{-i}) = y$, implies that $\{x, y\} \subseteq O_j(A, (R'_i, R_{-\{i,j\}}))$. Then, by Corollary 3, $O_j(A, (R'_i, R_{-\{i,j\}})) = \{x, y\}$. It also follows from setting $V = S \setminus \{j\}$ that $f(R'_{S \setminus \{j\}}, R_{-(S \setminus \{j\})}) = x$. This, together with $f(R'_S, R_{-S}) = z$, implies that $\{x, z\} \subseteq O_j(A, (R'_{S \setminus \{j\}}, R_{-S}))$. Again, by Corollary 3, $O_j(A, (R'_{S \setminus \{j\}}, R_{-S})) = \{x, z\}$.

*Step 5.b: We show that $O_i(A, R_{-i}) = \{x, y\}$ and $O_i(A, (R'_S, R_{-(S \cup \{i\})})) = \{x, z\}$.*

Since $f(R) = x$ and $f(R'_i, R_{-i}) = y$, we have that $\{x, y\} \subseteq O_i(A, R_{-i})$. And, by Corollary 3, $O_i(A, R_{-i}) = \{x, y\}$. By assumption, $f(R'_V, R_{-V}) = x$ for all $V \subseteq S \cup \{i\}$ with $V \notin \{S, \{i\}\}$. Then, it follows from setting $V = S \cup \{i\}$ that $f(R'_{S \cup \{i\}}, R_{-(S \cup \{i\})}) = x$.

This, together with $f(R'_S, R_{-S}) = z$, implies that $\{x, z\} \subseteq O_i(A, (R'_S, R_{-(S\cup\{i\})}))$. Hence, $O_i(A, (R'_S, R_{-(S\cup\{i\})})) = \{x, z\}$ again by Corollary 3.

*Step 5.c: We complete the proof of Step 5.*

Consider any agent $j \in S$ and any preference $R''_j \in \mathcal{R}^A_j$ such that $y\, P''_j\, x$ and $z\, P''_j\, x$, which is possible by Step 5.a and Lemma 1. By Step 5.a, $O_j(A, (R'_i, R_{-\{i,j\}})) = \{x, y\}$ and $O_j(A, (R'_{S\setminus\{j\}}, R_{-S})) = \{x, z\}$. Then, by Lemma 2, $f(R''_j, R'_i, R_{-\{i,j\}}) = y$ and $f(R''_j, R'_{S\setminus\{j\}}, R_{-S}) = z$. Starting with $(R''_j, R'_i, R_{-\{i,j\}})$, construct the sequence of profiles in which the preferences of all agents $l \in (S \setminus \{j\}) \cup \{i\}$ are changed one-by-one (from $R_l$ to $R'_l$ for all agents but $i$ and from $R'_i$ to $R_i$) so that the sequence ends at $(R''_j, R'_{S\setminus\{j\}}, R_{-S})$. Denote profile $(R''_j, R'_i, R_{-\{i,j\}})$ by $\hat{R}$ and profile $(R''_j, R'_{S\setminus\{j\}}, R_{-S})$ by $(\hat{R}'_{(S\setminus\{j\})\cup\{i\}}, \hat{R}_{-((S\setminus\{j\})\cup\{i\})})$.

First, if only $y$ and $z$ are chosen by $f$ along this sequence, then let $W \subset (S \setminus \{j\}) \cup \{i\}$ be the set of agents that have changed preferences in the sequence the last time the rule selects $y$, and let $k \in (S \cup \{i\}) \setminus (W \cup \{j\})$ be the next agent changing preferences in the sequence. Then, $f(\hat{R}'_W, \hat{R}_{-W}) = y \neq z = f(\hat{R}'_{W\cup\{k\}}, \hat{R}_{-(W\cup\{k\})})$. Thus, $\{y, z\} \subseteq O_k(A, (\hat{R}'_W, \hat{R}_{-(W\cup\{k\})}))$. Then, by Lemma 1, we have that $k$ is situated strictly between $y$ and $z$. However, if $k \neq i$, by Step 5.a and Lemma 1, we also have that $k$ is situated strictly between $x$ and $y$, and between $x$ and $z$. Clearly, this is not possible. Similarly, if $k = i$, by Step 5.b and Lemma 1, we also have that $i$ is situated strictly between $x$ and $y$, and between $x$ and $z$, which again is impossible. Second, if there are at least three different outcomes chosen by $f$ along this sequence, then, by Lemma 4, there are two sets $W_1 \subset W_2 \subseteq (S\setminus\{j\})\cup\{i\}$ and an agent $k \in W_2\setminus W_1$ such that $f(\hat{R}'_{W_1}, \hat{R}_{-W_1}) \neq f(\hat{R}'_{W_1\cup\{k\}}, \hat{R}_{-(W_1\cup\{k\})}) \neq f(\hat{R}'_{W_2}, \hat{R}_{-W_2}) \neq f(\hat{R}'_{W_1}, \hat{R}_{-W_1})$. Denote profile $(\hat{R}'_{W_1\cup\{k\}}, \hat{R}_{-(W_1\cup\{k\})})$ by $\bar{R}$, preference $\hat{R}_k$ by $\bar{R}'_k$, and preferences $\hat{R}'_{W_2\setminus(W_1\cup\{k\})}$ by $\bar{R}'_{W_2\setminus(W_1\cup\{k\})}$. Since $|W_2 \setminus (W_1 \cup \{k\})| < |S|$, we obtain the desired result (with $k$ playing the role of $j$ and $W_2 \setminus (W_1 \cup \{k\})$ that of $M$). $\square$

Finally, we prove the desired result.

**Lemma 6** *Suppose that $f$ is strategy-proof. Then, for all $A \subseteq N$ such that $|\Omega_A| > 2$, there are two agents $i, j \in N$ and three profiles $R, (R'_i, R_{-i}), (R'_j, R_{-j}) \in \mathcal{R}^A$ such that $f(R) \neq f(R'_i, R_{-i}) \neq f(R'_j, R_{-j}) \neq f(R)$.*

<u>Proof</u>: Take any $A \subseteq N$ such that $|\Omega_A| > 2$. By Lemma 3, there is an agent $i \in N$, a set of agents $S \subseteq N \setminus \{i\}$, and three profiles $R, (R'_i, R_{-i}), (R'_S, R_{-S}) \in \mathcal{R}^A$ such that $f(R) \neq f(R'_i, R_{-i}) \neq f(R'_S, R_{-S}) \neq f(R)$. If $|S| = 1$, the proof is concluded. If $|S| > 1$, apply Lemma 5 iteratively to reduce the cardinality of $S$ to 1. $\square$

Now, we are ready to prove Proposition 1.

Proof of Proposition 1: Suppose that $|\Omega_A| > 2$. Then, by Lemma 6, there are three profiles $R, (R'_i, R_{-i})$, $(R'_j, R_{-j}) \in \mathcal{R}^A$ such that $f(R) \neq f(R'_i, R_{-i}) \neq f(R'_j, R_{-j}) \neq f(R)$. Let $f(R) = x$, $f(R'_i, R_{-i}) = y$, $f(R'_j, R_{-j}) = z$, and $f(R'_{\{i,j\}}, R_{-\{i,j\}}) = w$. Observe that, although $x$, $y$ and $z$ are different, it could be that $w \in \{x, y, z\}$. Also, assume without loss of generality that $x < y$. Next, we study the implications of the different option sets.

(1) Since $\{x, y\} \subseteq O_i(A, R_{-i})$, $O_i(A, R_{-i}) = \{x, y\}$ by Corollary 3. Given that $x < y$ by assumption, Lemma 1 implies that $x < i < y$.
(2) Observe that $\{z, w\} \subseteq O_i(A, (R'_j, R_{-\{i,j\}}))$. So, if $w \neq z$, then $O_i(A, (R'_j, R_{-\{i,j\}})) = \{w, z\}$ by Corollary 3 and $w < i < z$ or $z < i < w$ by Lemma 1.
(3) Since $\{x, z\} \subseteq O_j(A, R_{-j})$, $O_j(A, R_{-j}) = \{x, z\}$ by Corollary 3. Then, Lemma 1 implies that $x < j < z$ or $z < j < x$.
(4) Observe that $\{w, y\} \subseteq O_j(A, (R'_i, R_{-\{i,j\}}))$. So, if $w \neq y$, then $O_j(A, (R'_i, R_{-\{i,j\}})) = \{w, y\}$ by Corollary 3 and $w < j < y$ or $y < j < w$ by Lemma 1.

We show that $f$ is not strategy-proof by constructing manipulations depending on how $w$ relates to the other alternatives.

Case 1: Suppose that $w = y$. Then, $y < i < z$ or $z < i < y$ by (2). Suppose that (2) states that $y < i < z$. Since $x < i < y$ by (1), we have $i < y < i$, which is impossible. So, we must have that (2) states that $z < i < y$. By (3) we have $x < j < z$ or $z < j < x$. Suppose that (3) states that $z < j < x$ (the other case is similar and thus omitted). Then, we have that $z < j < x < i < y$. The remainder of this case is divided into two parts:

- If $i \in A$, consider any preference $R''_i \in \mathcal{R}^+_i$ such that $x \, P''_i \, y \, P''_i \, z$. Since $O_i(A, R_{-i}) = \{x, y\}$ by (1) and $O_i(A, (R'_j, R_{-\{i,j\}})) = \{y, z\}$ by (2), it follows from Lemma 2 that $f(R''_i, R_{-i}) = x$ and $f(R''_i, R'_j, R_{-\{i,j\}}) = y$. Thus, $\{x, y\} \subseteq O_j(A, (R''_i, R_{-\{i,j\}}))$ and, by Lemma 1, $j$ is situated strictly between $x$ and $y$. This is a contradiction because we have already seen before that $j < x < y$.

- If $i \notin A$, consider any preference $R''_i \in \mathcal{R}^-_i$ such that $z \, P''_i \, y \, P''_i \, x$. Since $O_i(A, R_{-i}) = \{x, y\}$ by (1) and $O_i(A, (R'_j, R_{-\{i,j\}})) = \{y, z\}$ by (2), it follows from Lemma 2 that $f(R''_i, R_{-i}) = y$ and $f(R''_i, R'_j, R_{-\{i,j\}}) = z$. Thus, $\{y, z\} \subseteq O_j(A, (R''_i, R_{-\{i,j\}}))$ and, by Corollary 3, $O_j(A, (R''_i, R_{-\{i,j\}})) = \{y, z\}$. We separate the proof depending whether $j$ belongs to $A$ or not.

If $j \notin A$, consider any preference $R''_j \in \mathcal{R}^-_j$ such that $y \, P''_j \, z \, P''_j \, x$. Since $O_j(A, R_{-j}) = \{x, z\}$ by (3), it follows from Lemma 2 that $f(R''_j, R_{-j}) = z$. Since we also know that $O_j(A, (R''_i, R_{-\{i,j\}})) = \{y, z\}$, Lemma 2 also implies that $f(R''_{\{i,j\}}, R_{-\{i,j\}}) = y$. Agent $i$ then manipulates $f$ at this profile via $R_i$ to obtain $z$.

If $j \in A$, consider any preference $R''_j \in \mathcal{R}^+_j$ such that $x \, P''_j \, z \, P''_j \, y$. Since $O_j(A, R_{-j}) = \{x, z\}$ by (3), Lemma 2 implies that $f(R''_j, R_{-j}) = x$. Given that $O_j(A, (R''_i, R_{-\{i,j\}})) = \{y, z\}$, Lemma 2 also implies that $f(R''_{\{i,j\}}, R_{-\{i,j\}}) = z$. Given that $(z, x)$ is a fixed pair for $i$ at $\mathcal{R}^A_i$, agent $i$ then manipulates $f$ at $(R''_j, R_{-j})$ via $R''_i$.

<u>Case 2</u>: Suppose that $w = z$. The proof is similar to the one above and is thus omitted.

<u>Case 3</u>: Suppose that $w \notin \{y, z\}$. By (2), we have that $z < i < w$ or $w < i < z$. Suppose that (2) states that $z < i < w$. Then, since $j$ is situated strictly between $x$ and $z$ by (3), and $x < i$ by (1), we conclude that $j < i$. Also, observe that $j$ is situated strictly between $y$ and $w$ by (4) and that both $y$ and $w$ are greater than $i$ by (1) and the assumption on (2). Consequently, $j > i$, which cannot be. Thus, (2) must state that $w < i < z$. By (4), we have that $y < j < w$ or $w < j < y$. Suppose that (4) states that $y < j < w$. Given this assumption on (4) and that $x < i < y$ by (1), we have that $x < i < y < j < w$. So, $i < w$, which contradicts that $w < i$ by (2). Consequently, (4) must state that $w < j < y$. By (3), we have that $z < j < x$ or $x < j < z$. Suppose that (3) states that $z < j < x$. Then, $w < i < z < j < x$ by (2) and the assumption on (3), which contradicts that $x < i$ by (1). Consequently, (3) must state that $x < j < z$.

At this point, we can see that the four conditions $x < i < y$, $w < i < z$, $w < j < y$, and $x < j < z$ are indeed compatible for the moment. In fact, it turns out that $x$ and $w$ are both smaller than each $i$ and $j$, which are in turn both smaller than each $y$ and $z$. This implies that both $\{w, x\}$ and $\{y, z\}$ are fixed pairs for both agents $i$ and $j$. Finally, consider any preference $\hat{R}_j \in \mathcal{R}^A_j$ such that $z \, \hat{P}_j \, x$ and $y \, \hat{P}_j \, w$. Since $O_j(A, R_{-j}) = \{x, z\}$ by (3) and $O_j(A, (R'_i, R_{-\{i,j\}})) = \{w, y\}$ by (4), Lemma 2 implies that $f(\hat{R}_j, R_{-j}) = z$ and $f(R'_i, \hat{R}_j, R_{-\{i,j\}}) = y$. Agent $i$ then manipulates $f$ at $(\hat{R}_j, R_{-j})$ via $R'_i$ when her fixed pair at $\mathcal{R}^A_i$ is $(y, z)$ and at $(R'_i, \hat{R}_j, R_{-\{i,j\}})$ via $R_i$ when her fixed pair at $\mathcal{R}^A_i$ is $(z, y)$. $\square$

## Proof of Proposition 2

We first establish the following lemma.

**Lemma 7** *Suppose that* $f$ *is strategy-proof. Then, for all* $A, S \subseteq N$ *and all profiles* $R, (R'_S, R_{-S}) \in \mathcal{R}^A$ *such that* $f(R) \neq f(R'_S, R_{-S})$*, there is a set* $D \subset S$ *and an agent* $i \in S \setminus D$ *such that* $f(R) = f(R'_D, R_{-D}) \neq f(R'_{D\cup\{i\}}, R_{-(D\cup\{i\})}) = f(R'_S, R_{-S})$.

Proof: Starting at $R$, construct the sequence of profiles in which the preferences of all agents $j \in S$ are changed one-by-one from $R_j$ to $R'_j$ so that the final profile is $(R'_S, R_{-S})$. Since $f(R) \neq f(R'_S, R_{-S})$ by assumption, the outcome of the function must have changed along this sequence. Let $D \subset S$ be the set of agents that have changed preferences in the sequence the last time the rule selects $f(R)$. That is, $f(R'_D, R_{-D}) = f(R)$. Let $i \in S \setminus D$ be the next agent changing preferences in the sequence. Then, by construction, $f(R'_{D\cup\{i\}}, R_{-(D\cup\{i\})}) \neq f(R'_D, R_{-D})$. Since $|\Omega_A| \leq 2$ by Proposition 1, it follows that $f(R) = f(R'_D, R_{-D}) \neq f(R'_{D\cup\{i\}}, R_{-(D\cup\{i\})}) = f(R'_S, R_{-S})$. □

We are ready to prove the proposition.

Proof of Proposition 2: Take any $A \subseteq N$ and suppose by contradiction that there are two profiles $R, R' \in \mathcal{R}^A$ such that $R_{N_A} = R'_{N_A}$ and $f(R) \neq f(R')$. By Lemma 7, there is a set $D \subset N \setminus N_A$ and an agent $i \in (N \setminus N_A) \setminus D$ such that $f(R) = f(R'_D, R_{-D}) \neq f(R'_{D\cup\{i\}}, R_{-(D\cup\{i\})}) = f(R')$. Thus, $O_i(A, (R'_D, R_{-(D\cup\{i\})})) = \Omega_A$ by Corollary 3. Since $i \notin N_A$ by construction, this contradicts Lemma 1. □

## Proof of Proposition 3

Take any $A \subseteq N$ and define, associated to $f$, for each profile $R \in \mathcal{R}^A$ a set $\mathcal{G}^A(R) \subseteq 2^{N_A \setminus I_A(R)}$ of $l_A$–decisive sets in the following way: $B \in \mathcal{G}^A(R)$ if there is a profile $R' \in \mathcal{R}^A$ such that $R'_{I_A} = R_{I_A}$, $L_A(R') = B$, and $f(R') = l_A$. By definition, the correspondence $\mathcal{G}^A$ satisfies the first point of Definition 1.

*Step 1: We show that for all* $R, R' \in \mathcal{R}^A$ *such that* $R_{I_A} = R'_{I_A}$ *and* $L_A(R) = L_A(R')$*, then* $f(R) = f(R')$.

Suppose by contradiction that for some $R, R' \in \mathcal{R}^A$ such that $R_{I_A} = R'_{I_A}$ and $L_A(R) = L_A(R')$, $f(R) \neq f(R')$. Assume without loss of generality that $f(R) = h_A \neq l_A = f(R')$. By Proposition 2, $f(R') = f(R'_{N_A}, R_{-N_A})$. Thus, $f(R) \neq f(R'_{N_A}, R_{-N_A})$. It follows then from Lemma 7 that there is a set $D \subset N_A$ and an agent $j \in N_A \setminus D$ such that $f(R) = f(R'_D, R_{-D}) = h_A \neq l_A = f(R'_{D\cup\{j\}}, R_{-(D\cup\{j\})}) = f(R'_{N_A}, R_{-N_A})$. Since $R'_{I_A} = R_{I_A}$ by assumption and $f(R'_D, R_{-D}) \neq f(R'_{D\cup\{j\}}, R_{-(D\cup\{j\})})$, we conclude that $j \notin I_A(R) \cup I_A(R')$. If $j \in L_A(R)$, agent $j$ manipulates $f$ at $(R'_D, R_{-D})$ via $R'_j$. If, however, $j \in H_A(R)$, we deduce from $L_A(R) = L_A(R')$ that $j \in H_A(R')$ and $j$ manipulates $f$ at $(R'_{D\cup\{j\}}, R_{-(D\cup\{j\})})$ via $R_j$. This concludes Step 1.

Step 1 guarantees that for all $R, R' \in \mathcal{R}^A$ such that $R_{I_A} = R'_{I_A}$, $\mathcal{G}^A(R) = \mathcal{G}^A(R')$. Thus, $\mathcal{G}^A$ satisfies the second point of Definition 1. Similarly, Step 1 implies that for all $R \in \mathcal{R}^A$, $f(R) = l_A$ if and only if $L_A(R) \in \mathcal{G}^A(R)$. Then, it only remains to be shown that $\mathcal{G}^A$ satisfies the last three points of Definition 1.

*Step 2: If $B \in \mathcal{G}^A(R)$ for some $R \in \mathcal{R}^A$, then for all $j \in N_A\backslash(I_A(R)\cup B)$, $B\cup\{j\} \in \mathcal{G}^A(R)$.*
Assume that $B \in \mathcal{G}^A(R)$ for some $R \in \mathcal{R}^A$ and consider any $j \in N_A \setminus (I_A(R) \cup B)$. Suppose by contradiction that $B \cup \{j\} \notin \mathcal{G}^A(R)$, that is, there is no profile $\bar{R} \in \mathcal{R}^A$ such that $\bar{R}_{I_A} = R_{I_A}$, $L_A(\bar{R}) = B \cup \{j\}$, and $f(\bar{R}) = l_A$. That is, for all $\bar{R} \in \mathcal{R}^A$ such that $\bar{R}_{I_A} = R_{I_A}$ and $L_A(\bar{R}) = B \cup \{j\}$, $f(\bar{R}) = h_A$. Consider one such profile $\bar{R} \in \mathcal{R}^A$, together with a preference $R'_j \in \mathcal{R}^A_j$ such that $h_A\, P'_j\, l_A$. Since $(R'_j, \bar{R}_{-j}) \in \mathcal{R}^A$ and $(R'_j, \bar{R}_{-j})_{I_A} = R_{I_A}$, we have by the second point of Definition 1 (implied by Step 1) that $\mathcal{G}^A(R'_j, \bar{R}_{-j}) = \mathcal{G}^A(R)$. Then, $B \in \mathcal{G}^A(R'_j, \bar{R}_{-j})$. Since $L_A(R'_j, \bar{R}_{-j}) = B$, we obtain that $f(R'_j, \bar{R}_{-j}) = l_A$. Then, agent $j$ manipulates $f$ at $\bar{R}$ via $R'_j$. This concludes Step 2.

The successive application of Step 2 implies that if $B \in \mathcal{G}^A(R)$ and $B \subseteq C \in 2^{N_A\backslash I_A(R)}$, then $C \in \mathcal{G}^A(R)$. Then, $\mathcal{G}^A$ satisfies the third point of Definition 1.

*Step 3: If $I_A(R) = \emptyset$, then $\emptyset \notin \mathcal{G}^A(R) \neq \emptyset$.*
We will only show that if $I_A(R) = \emptyset$, then $N_A \in \mathcal{G}^A(R)$ and, thus, $\mathcal{G}^A(R) \neq \emptyset$. The proof that if $I_A(R) = \emptyset$, then $\emptyset \notin \mathcal{G}^A(R)$ follows a similar reasoning.

Suppose on the contrary that $N_A \notin \mathcal{G}^A(R)$, that is, there is no profile $\bar{R} \in \mathcal{R}^A$ such that $\bar{R}_{I_A} = R_{I_A}$, $L_A(\bar{R}) = N_A$, and $f(\bar{R}) = l_A$. That is, for all $\bar{R} \in \mathcal{R}^A$ such that $\bar{R}_{I_A} = R_{I_A}$ and $L_A(\bar{R}) = N_A$, $f(\bar{R}) = h_A$. Consider one such profile $\bar{R} \in \mathcal{R}^A$. Since $l_A \in \Omega_A$, there is a profile $R' \in \mathcal{R}^A$ such that $f(R') = l_A$. It follows from Proposition 2 that $f(\bar{R}) = f(\bar{R}_{N_A}, R'_{-N_A}) = h_A$. Then, considering profiles $R'$ and $(\bar{R}_{N_A}, R'_{-N_A})$, and applying Lemma 7, we can see that there is a set $D \subset N_A$ and an agent $i \in N_A \setminus D$ such that $f(\bar{R}_D, R'_{-D}) = l_A \neq h_A = f(\bar{R}_{D\cup\{i\}}, R'_{-(D\cup\{i\})})$. Then, agent $i$ manipulates $f$ at $(\bar{R}_{D\cup\{i\}}, R'_{-(D\cup\{i\})})$ via $R'_i$. This concludes Step 3 and proves that $\mathcal{G}^A$ satisfies the last point of Definition 1.

*Step 4: We show that for all profiles $R, R' \in \mathcal{R}^A$ such that $R'_{I_A(R)} = R_{I_A}$ and $I_A(R') = I_A(R) \cup \{j\}$ for some $j \in N_A \setminus I_A(R)$, [if $B \in \mathcal{G}^A(R)$ and $j \notin B$, then $B \in \mathcal{G}^A(R')$] and [if $j \notin B$ and $B \cup \{j\} \notin \mathcal{G}^A(R)$, then $B \notin \mathcal{G}^A(R')$].*

We will only show the first implication because the other is similar. So, suppose that $R \in \mathcal{R}^A$ and $j \in N_A$ such that $B \in \mathcal{G}^A(R)$ and $j \notin B \cup I_A(R)$. We show that for all profiles $R' \in \mathcal{R}^A$ such that $R'_{I_A(R)} = R_{I_A}$ and $I_A(R') = I_A(R) \cup \{j\}$, $B \in \mathcal{G}^A(R')$. Suppose on the contrary that there is a profile $R' \in \mathcal{R}^A$ such that $R'_{I_A(R)} = R_{I_A}$, $I_A(R') = I_A(R) \cup \{j\}$, and $B \notin \mathcal{G}^A(R')$. Then, there is no profile $\bar{R} \in \mathcal{R}^A$ such that $\bar{R}_{I_A} = R'_{I_A}$, $L_A(\bar{R}) = B$, and $f(\bar{R}) = l_A$. That is, for all $\bar{R} \in \mathcal{R}^A$ such that $\bar{R}_{I_A} = R'_{I_A}$ and $L_A(\bar{R}) = B$, $f(\bar{R}) = h_A$. Consider one such profile $\bar{R} \in \mathcal{R}^A$, together with a preference $R''_j \in \mathcal{R}^A_j$ such that $h_A \, P''_j \, l_A$. Since $(R''_j, \bar{R}_{-j}) \in \mathcal{R}^A$ and $(R''_j, \bar{R}_{-j})_{I_A} = R_{I_A}$, we have by the second point of Definition 1 (implied by Step 1) that $\mathcal{G}^A(R''_j, \bar{R}_{-j}) = \mathcal{G}^A(R)$. Then, $B \in \mathcal{G}^A(R''_j, \bar{R}_{-j})$. Since $L_A(R''_j, \bar{R}_{-j}) = B$, we obtain that $f(R''_j, \bar{R}_{-j}) = l_A$. Then, agent $j$ manipulates $f$ at $(R''_j, \bar{R}_{-j})$ via $\bar{R}_j$. This completes Step 4.

The successive application of Step 4 implies that for all $R, R' \in \mathcal{R}^A$ such that $R'_{I_A(R)} = R_{I_A}$ and $I_A(R') = I_A(R) \cup C$ for some $C \subseteq N_A \setminus I_A(R)$:

- If $B \cap C = \emptyset$ and $B \cup C \notin \mathcal{G}^A(R)$, then $B \notin \mathcal{G}^A(R')$.
- If $B \in \mathcal{G}^A(R)$ and $B \cap C = \emptyset$, then $B \in \mathcal{G}^A(R')$.

Then, $\mathcal{G}^A$ satisfies the fourth point of Definition 1. Therefore, $f_A$ is a voting by collections of $l_A$–decisive sets, which concludes the proof.

## Proof of Proposition 4

We first establish the following lemma.

**Lemma 8** *Suppose that $f$ is strategy-proof. For all $A \subseteq N$ and all $R \in \mathcal{R}^A$, if $H_A(R) = N_A$ (respectively, $L_A(R) = N_A$), then $f(R) = h_A$ (respectively, $f(R) = l_A$).*

Proof: Consider any $R \in \mathcal{R}^A$ for some $A \subseteq N$. We only show that if $H_A(R) = N_A$, then $f(R) = h_A$ (the proof of the other implication is similar). Since $H_A(R) = N_A$, we have that $L_A(R) = I_A(R) = \emptyset$. Then, Proposition 3 (exactly the last point of Definition 1) implies that $\emptyset \notin \mathcal{G}^A(R)$. Therefore, $f(R) = h_A$. $\square$

We make use of the following notation. For all $A \subseteq N$ and $R \in \mathcal{R}$, $L_A(R) = \{i \in N_A : l_A \, P_i \, h_A\}$, $H_A(R) = \{i \in N_A : h_A \, P_i \, l_A\}$ and $I_A(R) = \{i \in N_A : h_A \, R_i \, l_A \text{ and } l_A \, R_i \, h_A\}$. Note that the difference with the corresponding definitions in the main text are that are applied from now on to any profile $R \in \mathcal{R}$ (not necessarily $R \in \mathcal{R}^A$).

Consider any $A \subset N$ and any $i \in N \setminus A$.

*Step 1: If $h_A \leq i$ (respectively, $l_A \geq i$), then $h_{A\cup\{i\}} \leq i$ (respectively, $l_{A\cup\{i\}} \geq i$).*

We will only consider the case when $h_A \leq i$ because the other is similar. Suppose by contradiction that $h_{A\cup\{i\}} > i$. Consider a profile $R \in \mathcal{R}^A$ such that $H_{A\cup\{i\}}(R) = N_{A\cup\{i\}}$ and $h_{A\cup\{i\}}\, P_i\, l_A$. Observe then that $h_{A\cup\{i\}}\, P_i\, x$ for all $x \in \Omega_A$. If $i \in N_{A\cup\{i\}}$, consider any $R'_i \in \mathcal{R}_i^+$ with $h_{A\cup\{i\}}\, P'_i\, l_{A\cup\{i\}}$. Then, $(R'_i, R_{-i}) \in \mathcal{R}^{A\cup\{i\}}$ and $H_{A\cup\{i\}}(R'_i, R_{-i}) = N_{A\cup\{i\}}$. Then, Lemma 8 implies that $f(R'_i, R_{-i}) = h_{A\cup\{i\}}$ and agent $i$ manipulates $f$ at $R$ via $R'_i$ in order to obtain $h_{A\cup\{i\}}$ instead of any element of $\Omega_A$. If $i \notin N_{A\cup\{i\}}$, consider any $R'_i \in \mathcal{R}_i^+$. Again, $(R'_i, R_{-i}) \in \mathcal{R}^{A\cup\{i\}}$ and $H_{A\cup\{i\}}(R'_i, R_{-i}) = N_{A\cup\{i\}}$. Then, Lemma 8 implies that $f(R'_i, R_{-i}) = h_{A\cup\{i\}}$ and agent $i$ manipulates $f$ at $R$ via $R'_i$ in order to obtain $h_{A\cup\{i\}}$ instead of any element of $\Omega_A$.

*Step 2: If $h_A \leq i$ (respectively, $l_A \geq i$), then $h_{A\cup\{i\}} \geq h_A$ (respectively, $l_{A\cup\{i\}} \leq l_A$).*

We will only consider the case when $h_A \leq i$ because the other is similar. Suppose by contradiction that $h_{A\cup\{i\}} < h_A$ and consider the profile $R \in \mathcal{R}^{A\cup\{i\}}$ such that $H_A(R) = N_A$ and $H_{A\cup\{i\}}(R) = N_{A\cup\{i\}}$. Consider any $R'_i \in \mathcal{R}_i^-$ and observe that $(R'_i, R_{-i}) \in \mathcal{R}^A$ and $H_A(R'_i, R_{-i}) = N_A$. By Lemma 8, $f(R) = h_{A\cup\{i\}}$ and $f(R'_i, R_{-i}) = h_A$. Thus, agent $i$ manipulates $f$ at $R$ via $R'_i$.

*Step 3: If $i \in N_A$, then $i \in [l_{A\cup\{i\}}, h_{A\cup\{i\}}]$.*

We will only show that $i \leq h_{A\cup\{i\}}$ because the other implication is similar. Suppose by contradiction that $i > h_{A\cup\{i\}}$ and consider a profile $R \in \mathcal{R}^{A\cup\{i\}}$ such that $H_A(R) = N_A$ and $h_A\, P_i\, h_{A\cup\{i\}}$. Observe then that $h_A\, P_i\, x$ for all $x \in \Omega_{A\cup\{i\}}$. Consider any $R'_i \in \mathcal{R}_i^-$ with $h_A\, P'_i\, l_A$. Then, $(R'_i, R_{-i}) \in \mathcal{R}^A$ and $H_A(R'_i, R_{-i}) = N_A$. Then, Lemma 8 implies that $f(R'_i, R_{-i}) = h_A$. Thus, agent $i$ manipulates $f$ at $R$ via $R'_i$ in order to obtain $h_A$ instead of any element of $\Omega_{A\cup\{i\}}$.

*Step 4: If $i \in N_A$, then $l_{A\cup\{i\}} \geq l_A$ and $h_{A\cup\{i\}} \leq h_A$.*

We will only show that $l_{A\cup\{i\}} \geq l_A$ because the proof that $h_{A\cup\{i\}} \leq h_A$ is similar. Suppose otherwise, that is, $l_{A\cup\{i\}} < l_A$. Consider a profile $R \in \mathcal{R}^{A\cup\{i\}}$ such that $L_{A\cup\{i\}}(R) = N_{A\cup\{i\}}$ and $L_A(R) = N_A$. Consider any $R'_i \in \mathcal{R}_i^-$ with $l_A\, P'_i\, h_A$. Then, $(R'_i, R_{-i}) \in \mathcal{R}^A$ and $L_A(R'_i, R_{-i}) = N_A$. Then, Lemma 8 implies that $f(R) = l_{A\cup\{i\}}$ and $f(R'_i, R_{-i}) = l_A$. Thus, agent $i$ manipulates $f$ at $R$ via $R'_i$.

*Step 5: If $i \in N_A$ and $i \in \Omega_{A\cup\{i\}}$, then $\Omega_{A\cup\{i\}} = \{i\}$ or $\Omega_A \cap \Omega_{A\cup\{i\}} \neq \emptyset$.*

We assume without loss of generality that $l_{A\cup\{i\}} = i$ and show that $h_{A\cup\{i\}} \in \{i, h_A\}$. Suppose otherwise, that is, $h_{A\cup\{i\}} \notin \{i, h_A\}$. Then, by Steps 3 and 4, $h_{A\cup\{i\}} \in (i, h_A)$. Now, consider a profile $R \in \mathcal{R}^{A\cup\{i\}}$ such that $H_{A\cup\{i\}} = N_{A\cup\{i\}}$, $L_A(R) = N_A$ and $l_A \, P_i \, h_{A\cup\{i\}}$. Consider any $R_i' \in \mathcal{R}_i^-$ with $l_A \, P_i' \, h_A$. Then, $(R_i', R_{-i}) \in \mathcal{R}^A$ and $L_A(R_i', R_{-i}) = N_A$. Then, Lemma 8 implies that $f(R) = h_{A\cup\{i\}}$ and $f(R_i', R_{-i}) = l_A$. Thus, agent $i$ manipulates $f$ at $R$ via $R_i'$.

## Proof of Proposition 5

Consider any $A \subset N$ and any $i \in N \setminus A$ such that $i \in N_A \cap N_{A\cup\{i\}}$ and $\Omega_A \neq \Omega_{A\cup\{i\}}$.

*Step 1: We prove the proposition if $\Omega_A \cap \Omega_{A\cup\{i\}} = \emptyset$.*

Observe that $l_A < l_{A\cup\{i\}} < i < h_{A\cup\{i\}} < h_A$ by Proposition 4 (third point). We will only show that $i$ is a dictator at $\Omega_A$ because the other implication is similar. Suppose otherwise, that is, there is some profile $R \in \mathcal{R}^A$ such that $f(R) = x$ and $y \, P_i \, x$, where $x, y \in \Omega_A$. Assume without loss of generality that $x = h_A$. Then, $L_A(R) \notin \mathcal{G}^A(R)$. Consider a profile $R' \in \mathcal{R}^A$ such that $H_A(R') = N_A \setminus \{i\}$, $L_{A\cup\{i\}}(R') = N_{A\cup\{i\}}$, and $l_{A\cup\{i\}} \, P_i' \, h_A$. We are going to show that $f(R') = h_A$. Since $I_A(R') = \emptyset$, $I_A(R) = I_A(R') \cup I_A(R)$ and $I_A(R') \cap I_A(R) = \emptyset$. Then, given that $L_A(R) \notin \mathcal{G}^A(R)$, Proposition 3 (exactly the second part of the fourth point of Definition 1)[16] implies that $L_A(R) \notin \mathcal{G}^A(R')$. Note also that $L_A(R') \subseteq L_A(R)$. Then, applying Proposition 3 (exactly the third point of Definition 1) we obtain that $L_A(R') \notin \mathcal{G}^A(R')$ and, thus, $f(R') = h_A$. Consider now any $R_i'' \in \mathcal{R}_i^+$ such that $l_{A\cup\{i\}} \, P_i'' \, h_{A\cup\{i\}}$. Then, $(R_i'', R_{-i}') \in \mathcal{R}^{A\cup\{i\}}$ and $L_{A\cup\{i\}}(R_i'', R_{-i}') = N_{A\cup\{i\}}$. Then, Lemma 8 implies that $f(R_i'', R_{-i}') = l_{A\cup\{i\}}$ and agent $i$ manipulates $f$ at $R'$ via $R_i''$.

*Step 2: We prove the proposition if $|\Omega_A \cap \Omega_{A\cup\{i\}}| = 1$.*

We assume without loss of generality that $\Omega_A \cap \Omega_{A\cup\{i\}} = h_A = h_{A\cup\{i\}}$. Observe that $l_A < l_{A\cup\{i\}} < i < h_{A\cup\{i\}} = h_A$ by Proposition 4 (third point).

*Step 2.a: We show that agent $i$ is a dictator at $\Omega_A$ if and only if agent $i$ is a dictator at $\Omega_{A\cup\{i\}}$.*

We only show that if $i$ is a dictator at $\Omega_{A\cup\{i\}}$, then $i$ is also a dictator at $\Omega_A$ (the proof of the other implication is similar). So, suppose that $i$ is a dictator at $\Omega_{A\cup\{i\}}$ but, by contradiction, that there is some $R \in \mathcal{R}^A$ such that $f(R) = x$ and $y \, P_i \, x$, where $x, y \in \Omega_A$. Let $x = h_A$ — the case when $x = l_A$ is similar and thus omitted— and consider a preference $R_i' \in \mathcal{R}_i^-$ such

[16] Note that with respect to Definition 1, the roles of $R$ and $R'$ are here exchanged, and that here the role of $B$ is played by $L_A(R)$ and that of $C$ is played by $I_A(R') = \emptyset$. Observe also that Definition 1 literally says that if $L_A(R) \in \mathcal{G}^A(R')$, then $L_A(R) \in \mathcal{G}^A(R)$. But since $L_A(R) \notin \mathcal{G}^A(R)$, we can deduce that $L_A(R) \notin \mathcal{G}^A(R')$.

that $l_{A\cup\{i\}}\, P'_i\, h_A$. If it was the case that $f(R'_i, R_{-i}) = l_A$, then agent $i$ would manipulate $f$ at $R$ via $R'_i$. Hence, $f(R'_i, R_{-i}) = h_A$. Since $i$ is a dictator at $\Omega_{A\cup\{i\}}$ by assumption, we also have that for all $R''_i \in \mathcal{R}^+_i$ with $l_{A\cup\{i\}}\, P''_i\, h_A$, $i \in L_{A\cup\{i\}}(R''_i, R_{-i}) \in \mathcal{G}^{A\cup\{i\}}(R''_i, R_{-i})$. Therefore, $f(R''_i, R_{-i}) = l_{A\cup\{i\}}$. Agent $i$ then manipulates $f$ at $(R'_i, R_{-i})$ via any $R''_i \in \mathcal{R}^+_i$ with $l_{A\cup\{i\}}\, P''_i\, h_A$.

*Step 2.b: We show that for all $R \in \mathcal{R}^A$ with $I_A(R) = \emptyset$, all $R'_i \in \mathcal{R}^+_i$, and all $B \in \mathcal{G}^A(R)$ with $i \notin B$, $B \cap N_{A\cup\{i\}} \cap A \in \mathcal{G}^{A\cup\{i\}}(R'_i, R_{-i})$.*

Suppose by contradiction that there is a profile $R \in \mathcal{R}^A$, a preference $R'_i \in \mathcal{R}^+_i$, and a set $B \subseteq \mathcal{G}^A(R)$ such that $I_A(R) = \emptyset$, $i \notin B$, and $B \cap N_{A\cup\{i\}} \cap A \notin \mathcal{G}^{A\cup\{i\}}(R'_i, R_{-i})$. Observe that for any $j \in N_{A\cup\{i\}} \cap A$ such that $l_A\, P_j\, h_A$ we also have that $l_{A\cup\{i\}}\, P_j\, h_A$. Consider then a profile $\bar{R} \in \mathcal{R}^A$ such that $I_A(\bar{R}) = \emptyset$, $L_A(\bar{R}) = B$, and $L_{A\cup\{i\}}(\bar{R}) = B \cap N_{A\cup\{i\}} \cap A$. Since $i \notin B = L_A(\bar{R})$ and $I_A(\bar{R}) = \emptyset$, $h_A\, \bar{P}_i\, l_A$. Also, $\bar{R}_{I_A} = R_{I_A}$ and $(R'_i, \bar{R}_{-i})_{I_{A\cup\{i\}}} = (R'_i, R_{-i})_{I_{A\cup\{i\}}}$. Therefore, by Proposition 3 (exactly the second point of Definition 1), we have that $\mathcal{G}^A(\bar{R}) = \mathcal{G}^A(R)$ and $\mathcal{G}^{A\cup\{i\}}(R'_i, \bar{R}_{-i}) = \mathcal{G}^{A\cup\{i\}}(R'_i, R_{-i})$. Thus, $B \in \mathcal{G}^A(\bar{R})$ and $B \cap N_{A\cup\{i\}} \cap A \notin \mathcal{G}^{A\cup\{i\}}(R'_i, \bar{R}_{-i})$. Given that $L_A(\bar{R}) = B$ and $L_{A\cup\{i\}}(R'_i, \bar{R}_{-i}) = B \cap N_{A\cup\{i\}} \cap A$, we have that $f(\bar{R}) = l_A$ and $f(R'_i, \bar{R}_{-i}) = h_A$. Then, agent $i$ manipulates $f$ at $\bar{R}$ via $R'_i$.

*Step 2.c: We show that for all $R \in \mathcal{R}^A$ with $I_A(R) = \emptyset$, all $R'_i \in \mathcal{R}^+_i$ with $h_A\, P'_i\, l_{A\cup\{i\}}$, and all $D \in \mathcal{G}^{A\cup\{i\}}(R'_i, R_{-i})$ with $i \notin D$, $D \cap (N \setminus A) \in \mathcal{G}^A(R)$.*

Suppose by contradiction that there is a profile $R \in \mathcal{R}^A$, a preference $R'_i \in \mathcal{R}^+_i$, and a set $D \in \mathcal{G}^{A\cup\{i\}}(R'_i, R_{-i})$ such that $I_A(R) = \emptyset$, $h_A\, P'_i\, l_{A\cup\{i\}}$, $i \notin D$, and $D \cap (N \setminus A) \notin \mathcal{G}^A(R)$. Observe that for any $j \in N_{A\cup\{i\}} \cap (N \setminus A)$ such that $l_{A\cup\{i\}}\, P_j\, h_A$ we also have that $l_A\, P_j\, h_A$. Consider then a profile $\bar{R} \in \mathcal{R}^A$ such that $I_A(\bar{R}) = \emptyset$, $L_{A\cup\{i\}}(\bar{R}) = D$, and $L_A(\bar{R}) = D \cap (N \setminus A)$. Then, $\bar{R}_{I_A} = R_{I_A}$ and $(R'_i, \bar{R}_{-i})_{I_{A\cup\{i\}}} = (R'_i, R_{-i})_{I_{A\cup\{i\}}}$. Therefore, by Proposition 3 (exactly the second point of Definition 1) we have that $\mathcal{G}^A(\bar{R}) = \mathcal{G}^A(R)$ and $\mathcal{G}^{A\cup\{i\}}(R'_i, \bar{R}_{-i}) = \mathcal{G}^{A\cup\{i\}}(R'_i, R_{-i})$. Thus, $D \in \mathcal{G}^{A\cup\{i\}}(R'_i, \bar{R}_{-i})$ and $D \cap (N \setminus A) \notin \mathcal{G}^A(\bar{R})$. Given that $L_A(\bar{R}) = D \cap (N \setminus A)$ and $L_{A\cup\{i\}}(R'_i, \bar{R}_{-i}) = D$, we have that $f(\bar{R}) = h_A$ and $f(R'_i, \bar{R}_{-i}) = l_{A\cup\{i\}}$. Then, agent $i$ manipulates $f$ at $(R'_i, \bar{R}_{-i})$ via $\bar{R}_i$.

*Step 2.d: We show that for all $R \in \mathcal{R}^A$ with $I_A(R) = \emptyset$, $B \in \mathcal{G}^A(R)$ if and only if $i \in B$.*

Consider any $R \in \mathcal{R}^A$ such that $I_A(R) = \emptyset$. We only prove that if $i \notin B$, then $B \notin \mathcal{G}^A(R)$ (the converse part can be proved with similar arguments, but adapting Steps 2.b and 2.c). Suppose by contradiction that there is $B \in \mathcal{G}^A(R)$ such that $i \notin B$. Then, by Step 2.b, for all $R'_i \in \mathcal{R}^+_i$, $B \cap N_{A\cup\{i\}} \cap A \in \mathcal{G}^{A\cup\{i\}}(R'_i, R_{-i})$. Then, by Step 2.c (setting $D = B \cap N_{A\cup\{i\}} \cap A$

and $R'_i \in \mathcal{R}_i^+$ such that $h_A \, P'_i \, l_{A\cup\{i\}}$), we obtain that $(B \cap N_{A\cup\{i\}} \cap A) \cap (N \setminus A) \in \mathcal{G}^A(R)$. However, $(B \cap N_{A\cup\{i\}} \cap A) \cap (N \setminus A) = \emptyset$, contradicting Proposition 3 (exactly the last point of Definition 1).

*Step 2.e: We complete the proof of Step 2.*

Consider first any $R \in \mathcal{R}^A$ such that $I_A(R) = \emptyset$. By Step 2.d, for all $D \subseteq N_A$, $D \in \mathcal{G}^A(R)$ if and only if $i \in D$. Consider now any $R' \in \mathcal{R}^A$ with $i \notin I_A(R')$. Since $I_A(R) = \emptyset$, $I_A(R') = I_A(R) \cup I_A(R')$ and $I_A(R) \cap I_A(R') = \emptyset$. Take any $B \in 2^{N_A \setminus I_A(R')}$ with $i \in B$. Then, $B \in 2^{N_A \setminus I_A(R)}$ and, since $i \in B$, $B \in \mathcal{G}^A(R)$. Then, Proposition 3 (exactly the second part of the fourth point of Definition 1)[17] implies that $B \in \mathcal{G}^A(R')$. Therefore, $B \in \mathcal{G}^A(R')$ if and only if $i \in B$. Thus, agent $i$ is a dictator at $\Omega_A$. Finally, by Step 2.a, agent $i$ is also a dictator at $\Omega_{A\cup\{i\}}$.

## Proof of Proposition 6

Consider any $A \subset N$ and any $i \in N \setminus A$ such that $\Omega_A = \Omega_{A\cup\{i\}}$. Consider also any two preference profiles $R \in \mathcal{R}^A$ and $R' = (R'_i, R_{-i}) \in \mathcal{R}^{A\cup\{i\}}$ such that $i \notin I_A(R) \cup I_{A\cup\{i\}}(R')$.[18] = Then, we can prove the lemma by showing that $\mathcal{G}^A(R) \subseteq \mathcal{G}^{A\cup\{i\}}(R')$ whenever $i < h_A$ and that $\mathcal{G}^A(R) \supseteq \mathcal{G}^{A\cup\{i\}}(R')$ whenever $i > l_A$. We only show the first implication because the other is similar. Suppose by contradiction that $i < h_A$, but there is a set $C$ such that $C \in \mathcal{G}^A(R) \setminus \mathcal{G}^{A\cup\{i\}}(R')$.

First, if $i \notin C$, consider a profile $R'' \in \mathcal{R}^A$ such that $R''_{I_A} = R_{I_A}$ and $L_A(R'') = C$. Then, by Proposition 3 (exactly by the second point of Definition 1), $\mathcal{G}^A(R'') = \mathcal{G}^A(R)$ and we obtain that $f(R'') = l_A$. If, on the one hand, $i \leq l_A$, then, $(R'_i, R''_{-i})_{I_{A\cup\{i\}}} = R'_{I_{A\cup\{i\}}}$. Therefore, by Proposition 3 (exactly the second point of Definition 1), $\mathcal{G}^{A\cup\{i\}}(R'_i, R''_{-i}) = \mathcal{G}^{A\cup\{i\}}(R')$. Since $L_{A\cup\{i\}}(R'_i, R''_{-i}) = C$, we have that $f(R'_i, R''_{-i}) = h_{A\cup\{i\}} = h_A$. Thus, agent $i$ manipulates $f$ at $(R'_i, R''_{-i})$ via $R''_i$. If, on the other hand, $i \in N_A$, consider a preference $\tilde{R}_i \in \mathcal{R}_i^+$ such that $h_A \, \tilde{P}_i \, l_A$. Then, $(\tilde{R}_i, R''_{-i})_{I_{A\cup\{i\}}} = R'_{I_{A\cup\{i\}}}$ and, by Proposition 3 (exactly the second point of Definition 1), $\mathcal{G}^{A\cup\{i\}}(\tilde{R}_i, R''_{-i}) = \mathcal{G}^{A\cup\{i\}}(R')$. Since $L_{A\cup\{i\}}(\tilde{R}_i, R''_{-i}) = C$, we have that $f(\tilde{R}_i, R''_{-i}) = h_{A\cup\{i\}} = h_A$. Thus, agent $i$ manipulates $f$ at $R''$ via $\tilde{R}_i$.

Second, if $i \in C$ (and, then, $i \in N_A$), consider a profile $\bar{R} \in \mathcal{R}^{A\cup\{i\}}$ such that $\bar{R}_{I_{A\cup\{i\}}} = R'_{I_{A\cup\{i\}}}$ and $L_{A\cup\{i\}}(\bar{R}) = C$. Consider also a preference $\hat{R}_i \in \mathcal{R}_i^-$ such that $l_A \, \hat{P}_i \, h_A$. Then, $L_A(\hat{R}_i, \bar{R}_{-i}) = C$ and $R_{I_A} = (\hat{R}_i, \bar{R}_{-i})_{I_A}$. Then, by Proposition 3 (exactly the second

[17] Note that with respect to Definition 1, the role of $C$ is played here by $I_A(R')$.

[18] Observe that this condition is obviously satisfied if $i \notin N_A$.

point of Definition 1), $\mathcal{G}^{A\cup\{i\}}(\bar{R}) = \mathcal{G}^{A\cup\{i\}}(R')$ and $\mathcal{G}^{A}(\hat{R}_i, \bar{R}_{-i}) = \mathcal{G}^{A}(R)$. Therefore, $f(\bar{R}) = h_{A\cup\{i\}} = h_A$ and $f(\hat{R}_i, \bar{R}_{-i}) = l_A$. Then, agent $i$ manipulates $f$ at $\bar{R}$ via $\hat{R}_i$.

## Proof of Proposition 7

Consider any $A \subset N$ and any $i \in N \setminus A$ such that $\Omega_A \neq \Omega_{A\cup\{i\}}$. Consider also any two preference profiles $R \in \mathcal{R}^A$ and $R' = (R'_i, R_{-i}) \in \mathcal{R}^{A\cup\{i\}}$. We will only consider the case when $i \geq h_{A\cup\{i\}} \geq h_A > l_{A\cup\{i\}} \geq l_A$ because the other is similar. Then, suppose that $B \in \mathcal{G}^{A\cup\{i\}}(R')$, but assume by contradiction that $B \cap (N_A \setminus A) \notin \mathcal{G}^A(R)$. Observe that for any $j \in N_A \cap N_{A\cup\{i\}} \cap (N \setminus A)$ such that $l_{A\cup\{i\}}\, P_j\, h_{A\cup\{i\}}$ we also have that $l_A\, P_j\, h_A$. Then, we can construct a profile $R'' \in \mathcal{R}^{A\cup\{i\}}$ such that $R''_{I_{A\cup\{i\}}} = R'_{I_{A\cup\{i\}}}$, $L_{A\cup\{i\}}(R'') = B$, and $L_A(R'') = B \cap (N_A \setminus A)$. Since $(R_i, R''_{-i})_{I_A} = R_{I_A}$, we have by Proposition 3 (exactly the second point of Definition 1) that $\mathcal{G}^{A\cup\{i\}}(R'') = \mathcal{G}^{A\cup\{i\}}(R')$ and $\mathcal{G}^A(R_i, R''_{-i}) = \mathcal{G}^A(R)$. Observe that $L_A(R_i, R''_{-i}) = B \cap (N_A \setminus A)$. Then, $f(R'') = l_{A\cup\{i\}}$ and $f(R_i, R''_{-i}) = h_A$. Given that $(h_A, l_{A\cup\{i\}})$ is a fixed pair for $i$ at $\mathcal{R}_i^+$, agent $i$ manipulates $f$ at $R''$ via $R_i$.

## Proof of Proposition 8

Consider any $A \subset N$ and any $i \in N \setminus A$ such that $i \in N_A \setminus N_{A\cup\{i\}}$. Consider also any two preference profiles $R \in \mathcal{R}^A$ and $R' = (R'_i, R_{-i}) \in \mathcal{R}^{A\cup\{i\}}$ such that $i \notin I_A(R)$. We will only consider the case when $l_A = l_{A\cup\{i\}}$ because the other is similar. Suppose by contradiction that there is some $B \in \mathcal{G}^{A\cup\{i\}}(R')$, but $B \cap A \notin \mathcal{G}^A(R)$. Observe that for any $j \in N_A \cap N_{A\cup\{i\}} \cap A$ such that $l_{A\cup\{i\}}\, P_j\, h_{A\cup\{i\}} = i$ we also have that $l_A\, P_j\, h_A$. Then, we can construct a profile $\bar{R} \in \mathcal{R}^{A\cup\{i\}}$ such that $\bar{R}_{I_{A\cup\{i\}}} = R'_{I_{A\cup\{i\}}}$, $L_{A\cup\{i\}}(\bar{R}) = B$, and $L_A(\bar{R}) = B \cap A$. Then, $h_A\, \bar{P}_i\, l_A$. Consider also a preference $\hat{R}_i \in \mathcal{R}_i^-$ such that $h_A\, \hat{P}_i\, l_A$ and observe that $(\hat{R}_i, \bar{R}_{-i})_{I_A} = R_{I_A}$. Then, by Proposition 3 (exactly the second point of Definition 1), $\mathcal{G}^{A\cup\{i\}}(\bar{R}) = \mathcal{G}^{A\cup\{i\}}(R')$ and $\mathcal{G}^A(\hat{R}_i, \bar{R}_{-i}) = \mathcal{G}^A(R)$. Given that $L_{A\cup\{i\}}(\bar{R}) = B$ and $L_A(\hat{R}_i, \bar{R}_{-i}) = B \cap A$, we obtain that $f(\bar{R}) = l_{A\cup\{i\}} = l_A$ and $f(\hat{R}_i, \bar{R}_{-i}) = h_A$. Thus, agent $i$ manipulates $f$ at $\bar{R}$ via $\hat{R}_i$.

## Proof of Theorem 1

Necessity follows from the arguments in Sections 3 and 4. For sufficiency consider any rule $f$ such that there is a function $\omega : 2^N \to T^2$ satisfying Proposition 4 and a family $\{f_A : \mathcal{R}^A \to \Omega_A\}_{A\subseteq N}$ of votings by collections of $l_A$–decisive sets satisfying Propositions 5 to 8 such that for all $A \subseteq N$ and all preference profiles $R \in \mathcal{R}^A$, $\omega(A) = \Omega_A$ and $f(R) = f_A(R)$. Suppose by contradiction that there is a set $A \subseteq N$, a preference profile $R \in \mathcal{R}^A$, and an agent $i \in N$ with the alternative preference $R'_i \in \mathcal{R}_i$ such that $f(R')\, P_i\, f(R)$, where $R' \equiv (R'_i, R_{-i})$. Let $B \in \{A, A \setminus \{i\}, A \cup \{i\}\}$ be such that $R' \in \mathcal{R}^B$. Suppose without loss of generality that $f(R) < f(R')$. We divide the proof into three cases:

<u>Case 1</u>: *Let* $B = A$.

Observe that the assumption $f(R') > f(R)$ implies that $f(R') = h_A\, P_i\, l_A = f(R)$. Then, $i \notin L_A(R) \subseteq L_A(R')$.

First, if $i \notin N_A$, Proposition 2 implies that $f(R) = f(R')$ contradicting $f(R')\, P_i\, f(R)$. Second, if $i \in N_A$, by Proposition 3 (exactly by Definition 1), $L_A(R) \in \mathcal{G}^A(R)$ and $L_A(R') \notin \mathcal{G}^A(R')$. If $R_{I_A} = R'_{I_A}$, then we have by Proposition 3 (exactly the second point of Definition 1) that $\mathcal{G}^A(R) = \mathcal{G}^A(R')$. Thus, $L_A(R) \in \mathcal{G}^A(R')$. Given that $L_A(R) \subseteq L_A(R')$, we can apply Proposition 3 (exactly the third point of Definition 1) to obtain $L_A(R') \in \mathcal{G}^A(R')$. This is a contradiction. If, however, $R_{I_A} \neq R'_{I_A}$, then $R'_{I_A(R)} = R_{I_A}$ and $I_A(R') = I_A(R) \cup \{i\}$. Then, since $L_A(R) \in \mathcal{G}^A(R)$ we obtain by Proposition 3 (exactly the second part of the fourth point of Definition 1)[19] that $L_A(R) \in \mathcal{G}^A(R')$. Then, given that $L_A(R) \subseteq L_A(R')$, again by Proposition 3 (exactly the third point of Definition 1), we have that $L_A(R') \in \mathcal{G}^A(R')$. This is again a contradiction.

<u>Case 2</u>: *Let* $i \in A$ *and* $B = A \setminus \{i\}$.

Suppose first that $i \in N_A$. By Proposition 4 (third point), $h_B \geq h_A$ and $l_B \leq l_A$. Hence, $i \in N_B$. Also, the assumption $f(R') > f(R)$ implies that $f(R') = h_B$.

If, on the one hand, $\Omega_B \neq \Omega_A$, then, by Proposition 5, $i$ is a dictator at both $\Omega_A$ and $\Omega_B$. Since $i$ is a dictator at $\Omega_A$, $f(R)\, R_i\, h_A$. Given that $R_i \in \mathcal{R}_i^+$, we also have that $h_A\, R_i\, h_B = f(R')$. Therefore, $f(R)\, R_i\, f(R')$, which contradicts that $f(R')\, P_i\, f(R)$.

If, on the other hand, $\Omega_B = \Omega_A$, then it follows from $f(R') > f(R)$ that $f(R) = l_A$. Consequently, by Proposition 3 (exactly by Definition 1), $L_A(R) \in \mathcal{G}^A(R)$ and $L_B(R') \notin \mathcal{G}^B(R')$. Also, since $f(R) = l_A$, $f(R') = h_A$, and $f(R')\, P_i\, f(R)$, we must have that $i \in H_A(R)$. Thus, $L_A(R) \subseteq L_B(R')$. Observe that by Proposition 6 (third point), $\mathcal{G}^A(R) = \mathcal{G}^B(R')$ whenever $i \notin I_A(R) \cup I_B(R')$. By construction, $i \notin I_A(R)$. If $i \notin I_B(R')$, then the fact that $L_A(R) \in \mathcal{G}^A(R)$ implies that $L_A(R) \in \mathcal{G}^B(R')$. Then, by Proposition 3 (exactly the third point of Definition 1), $L_B(R') \in \mathcal{G}^B(R')$. This is a contradiction. Consequently, we have shown that for all $R''_i \in \mathcal{R}_i^-$ such that $i \notin I_B(R''_i, R_{-i})$, $f(R''_i, R_{-i}) = l_B$. Suppose now that $i \in I_B(R')$. Consider $R''_i \in \mathcal{R}_i^-$ such that $h_B\, P''_i\, l_B$. Then, $i \notin I_B(R'')$, where $R'' = (R''_i, R_{-i}) \in \mathcal{R}^B$. By our previous finding, $f(R'') = l_B$. Observe also that $L_B(R'') =$

[19]Note that with respect to Definition 1, the roles of $B$ and $C$ are played here by $L_A(R)$ and $\{i\}$, respectively.

$L_B(R')$. Then, by Proposition 3 (exactly by Definition 1), $L_B(R'') \in \mathcal{G}^B(R'')$. Then, applying Proposition 3 (exactly the second part of the fourth point of Definition 1)[20], we obtain that $L_B(R'') = L_B(R') \in \mathcal{G}^B(R')$. This is a contradiction.

Suppose next that $i \notin N_A$. Since $R_i \in \mathcal{R}_i^+$, it follows from $f(R') > f(R)$ and $f(R')\, P_i\, f(R)$ that $i \geq h_A$. Then, by Proposition 4 (first point), $l_B \leq l_A \leq i$ and $h_B \leq h_A \leq i$. Observe that all this is only possible if $f(R) = l_A$ and $f(R') = h_B$. Consequently, by Proposition 3 (exactly by Definition 1), $L_A(R) \in \mathcal{G}^A(R)$ and $L_B(R') \notin \mathcal{G}^B(R')$.
If, on the one hand, $\Omega_A = \Omega_B$, observe that, since $i \notin N_A$, $L_A(R) = L_B(R')$. We also know from Proposition 6 (second point) that $\mathcal{G}^A(R) \subseteq \mathcal{G}^B(R')$. Then, the fact that $L_A(R) \in \mathcal{G}^A(R)$ implies that $L_A(R) = L_B(R') \in \mathcal{G}^B(R')$ and this is a contradiction.
If, on the other hand, $\Omega_A \neq \Omega_B$, then, by Proposition 7 (first point), $L_A(R) \in \mathcal{G}^A(R)$ implies that $L_A(R) \cap (N_B \setminus A) \in \mathcal{G}^B(R')$. Given that $L_A(R) \cap (N_B \setminus A) \subseteq L_B(R')$, we obtain by Proposition 3 (exactly the third point of Definition 1) that $L_B(R') \in \mathcal{G}^B(R')$. This is a contradiction.

*Case 3: Let $i \notin A$ and $B = A \cup \{i\}$.*

Suppose first that $i \in N_A$. By Proposition 4 (third point), $i \leq h_B \leq h_A$ and $i \geq l_B \geq l_A$. Also, the assumption $f(R') > f(R)$ implies that $f(R) = l_A$.
First, if $i \in N_B$ and $\Omega_B \neq \Omega_A$, then, by Proposition 5, $i$ is a dictator at both $\Omega_A$ and $\Omega_B$. Since $i$ is a dictator at $\Omega_A$ and $R_i \in \mathcal{R}_i^-$, $f(R) = l_A\, R_i\, x$ for all $x \in \Omega_B$. Then, $f(R)\, R_i\, f(R')$, and this is a contradiction.
Second, if $i \in N_B$ and $\Omega_B = \Omega_A$, then the fact that $f(R') > f(R)$ implies that $f(R') = h_B$. Then, by Proposition 3 (exactly by Definition 1), $L_A(R) \in \mathcal{G}^A(R)$ and $L_B(R') \notin \mathcal{G}^B(R')$. Also, since $f(R) = l_A$, $f(R') = h_A$, and $f(R')\, P_i\, f(R)$, we must have that $i \in H_A(R)$. Thus, $L_A(R) \subseteq L_B(R')$. Observe that by Proposition 6 (third point), $\mathcal{G}^A(R) = \mathcal{G}^B(R')$ if $i \notin I_A(R) \cup I_B(R')$. By construction, $i \notin I_A(R)$. If $i \notin I_B(R')$, then the fact that $L_A(R) \in \mathcal{G}^A(R)$ implies that $L_A(R) \in \mathcal{G}^B(R')$. Then, by Proposition 3 (exactly the third point of Definition 1), $L_B(R') \in \mathcal{G}^B(R')$. This is a contradiction. Consequently, we have shown that for all $R_i'' \in \mathcal{R}_i^+$ such that $i \notin I_B(R_i'', R_{-i})$, $f(R_i'', R_{-i}) = l_B$. Suppose now that $i \in I_B(R')$. Consider $R_i'' \in \mathcal{R}_i^+$ such that $h_B\, P_i''\, l_B$. Then, $i \notin I_B(R'')$, where $R'' = (R_i'', R_{-i}) \in \mathcal{R}^B$. By our previous finding, $f(R'') = l_B$. Observe also that $L_B(R'') = L_B(R')$. Then, by Proposition 3 (exactly by Definition 1), $L_B(R'') \in \mathcal{G}^B(R'')$. Then,

[20] Note that with respect to Definition 1, the roles of $B$ and $C$ are played here by $L_B(R'') = L_B(R')$ and $\{i\}$, respectively.

applying Proposition 3 (exactly the second part of the fourth point of Definition 1)[21], we obtain that $L_B(R'') = L_B(R') \in \mathcal{G}^B(R')$. This is a contradiction.
Third, if $i \notin N_B$, then, by Proposition 4 (fourth point), $\Omega_B = \{\{i\}, \{l_A, i\}, \{i, h_A\}\}$. Given that $f(R')\, P_i\, f(R) = l_A$ and that $R_i \in \mathcal{R}_i^-$, the only possibility is that $\Omega_B = \{i, h_A\}$ and $f(R') = h_A$. Then, by Proposition 3 (exactly by Definition 1), $L_A(R) \in \mathcal{G}^A(R)$ and $L_B(R') \notin \mathcal{G}^B(R')$. Also, since $f(R) = l_A$, $f(R') = h_A$, and $f(R')\, P_i\, f(R)$, we must have that $i \in H_A(R)$. Thus, $L_A(R) \subseteq L_B(R')$. Given that $L_A(R) \in \mathcal{G}^A(R)$, we have by Proposition 8 (second point) that $L_A(R) \cap N_B \cap A \in \mathcal{G}^B(R')$. Since $L_A(R) \cap N_B \cap A \subseteq L_B(R')$, we obtain by Proposition 3 (exactly the third point of Definition 1) that $L_B(R') \in \mathcal{G}^B(R')$. This is a contradiction.

Suppose next that $i \notin N_A$. Since $R_i \in \mathcal{R}_i^-$, it follows from $f(R') > f(R)$ and $f(R')\, P_i\, f(R)$ that $i \leq l_A$. Then, by Proposition 4 (second point), $i \leq l_B \leq l_A$ and $i \leq h_B \leq h_A$. Observe that all this is only possible if $f(R) = l_A$ and $f(R') = h_B$. Consequently, by Proposition 3 (exactly by Definition 1), $L_A(R) \in \mathcal{G}^A(R)$ and $L_B(R') \notin \mathcal{G}^B(R')$.
If, on the one hand, $\Omega_A = \Omega_B$, observe that, since $i \notin N_A$, $L_A(R) = L_B(R')$. We also know from Proposition 6 (first point) that $\mathcal{G}^A(R) \subseteq \mathcal{G}^B(R')$. Then, the fact that $L_A(R) \in \mathcal{G}^A(R)$ implies that $L_A(R) = L_B(R') \in \mathcal{G}^B(R')$ and this is a contradiction.
If, on the other hand, $\Omega_A \neq \Omega_B$, then, by Proposition 7 (second point), $L_A(R) \in \mathcal{G}^A(R)$ implies that $L_A(R) \cap (N_B \setminus A) \in \mathcal{G}^B(R')$. Given that $L_A(R) \cap (N_B \setminus A) \subseteq L_B(R')$, we obtain by Proposition 3 (exactly the third point of Definition 1) that $L_B(R') \in \mathcal{G}^B(R')$. This is a contradiction.

## Proof of Theorem 2

We first establish a set of lemmas that will help us in the proofs of Theorems 2 and 3.

**Lemma 9** *Suppose that $f$ is strategy-proof. For all $A \subseteq N_\emptyset$ such that $\Omega \cap A = \emptyset$ and $\Omega_{\{i\}} = \Omega_\emptyset$ for all $i \in A$, then $\Omega_A = \Omega_\emptyset$.*

Proof: The proof is by induction on the size of $A$. First, the result for $|A| \leq 1$ holds by assumption. So, consider now any $A \subseteq N_\emptyset$ such that $|A| > 1$, $\Omega_{\{i\}} = \Omega_\emptyset$ for all $i \in A$, and $\Omega \cap A = \emptyset$. Suppose that the lemma holds for all $D \subseteq N_\emptyset$ such that $|D| < |A|$ but, by contradiction, $\Omega_A \neq \Omega_\emptyset$. Take any $i, j \in A$. Since $A \subseteq N_\emptyset$, we have that $A \setminus \{i\} \subseteq N_\emptyset$ and $A \setminus \{j\} \subseteq N_\emptyset$. Similarly, given that $\Omega \cap A = \emptyset$, we have that $\Omega \cap (A \setminus \{i\}) = \Omega \cap (A \setminus \{j\}) = \emptyset$.

[21] Note that with respect to Definition 1, the roles of $B$ and $C$ are played here by $L_B(R'') = L_B(R')$ and $\{i\}$, respectively.

Then, by setting $D = A \setminus \{i\}$ and $D = A \setminus \{j\}$, we can apply the induction hypothesis and it follows that $\Omega_{A\setminus\{i\}} = \Omega_{A\setminus\{j\}} = \Omega_\emptyset \neq \Omega_A$. Since $A \subseteq N_{A\setminus\{i\}} = N_{A\setminus\{j\}}$ and $i, j \in A$, we have that $i, j \in N_{A\setminus\{i\}} = N_{A\setminus\{j\}}$. Given that $\Omega \cap A = \emptyset$ by assumption, we can apply Proposition 4 (third point) on the one hand to $\Omega_{A\setminus\{i\}}$ and $\Omega_A$ and, on the other hand, to $\Omega_{A\setminus\{j\}}$ and $\Omega_A$ in order to see that $i \in N_A$ and $j \in N_A$. Thus, $i \in N_{A\setminus\{i\}} \cap N_A$ and $j \in N_{A\setminus\{j\}} \cap N_A$. Next, apply Proposition 5 first to $\Omega_{A\setminus\{i\}}$ and $\Omega_A$ and afterwards to $\Omega_{A\setminus\{j\}}$ and $\Omega_A$ in order to obtain that both $i$ and $j$ are dictators at $\Omega_A$. This is a contradiction.
□

**Lemma 10** *Suppose that $f$ is strategy-proof. For all $A \subseteq N$, all $i \in N_A \cap A$, and all $j \in N \setminus (N_A \cup A)$ such that $i$ is a dictator at $\Omega_A$ and $i \in N_{A\cup\{j\}\setminus\{i\}}$, then $\Omega_{A\cup\{j\}} = \Omega_A$.*

Proof: Suppose by contradiction that $\Omega_{A\cup\{j\}} \neq \Omega_A$. Since $j \notin N_A$, we can assume without loss of generality that $j \leq l_A$. First, if $h_{A\cup\{j\}} > l_A$, consider a profile $R \in \mathcal{R}^A$ such that $I_A(R) = I_{A\cup\{j\}}(R) = \emptyset$ and a preference $R'_j \in \mathcal{R}^+_j$. By Proposition 4 (second point), $j \notin N_{A\cup\{j\}}$. Given that $i$ is a dictator at $\Omega_A$, $\{i\} \in \mathcal{G}^A(R)$. It follows then from Proposition 7 (second point) that $\{i\} \cap (N_{A\cup\{j\}} \setminus A) \in \mathcal{G}^{A\cup\{j\}}(R'_j, R_{-j})$. However, since $i \in A$, $\{i\} \cap (N_{A\cup\{j\}} \setminus A) = \emptyset$ and, therefore, $\emptyset \in \mathcal{G}^{A\cup\{j\}}(R'_j, R_{-j})$. Since $I_{A\cup\{j\}}(R'_j, R_{-j}) = \emptyset$, this violates Proposition 3 (exactly the last point of Definition 1). Second, if $h_{A\cup\{j\}} \leq l_A$, it follows from $i \in N_A$ that $l_A < i$ and from Proposition 4 (first point) applied to $\Omega_{A\cup\{j\}\setminus\{i\}}$ and $\Omega_{A\cup\{j\}}$ that $h_{A\cup\{j\}\setminus\{i\}} \leq h_{A\cup\{j\}}$. Thus, $h_{A\cup\{j\}\setminus\{i\}} < i$, which contradicts that $i \in N_{A\cup\{j\}\setminus\{i\}}$. □

**Lemma 11** *Suppose that $f$ is strategy-proof. For all $A \subset N$, all $i, j \in N_A$ such that $i \neq j \notin A$ and $\Omega_A = \Omega_{A\cup\{j\}}$, $i$ is a dictator at $\Omega_A$ if and only if $i$ is a dictator at $\Omega_{A\cup\{j\}}$.*

Proof: We show that if $i$ is a dictator at $\Omega_A$, then $i$ is also a dictator at $\Omega_{A\cup\{j\}}$ (the proof of the other implication is similar). Suppose by contradiction that $i$ is not a dictator at $\Omega_{A\cup\{j\}}$. Consider any $R \in \mathcal{R}^{A\cup\{j\}}$ such that $i \notin I_{A\cup\{j\}}(R)$ and any $B \in \mathcal{G}^{A\cup\{j\}}(R)$ such that $i \notin B$ (the case where $i \in B \notin \mathcal{G}^{A\cup\{j\}}(R)$ is similar). Also, let $R'_j \in \mathcal{R}^-_j$ and $R''_j \in \mathcal{R}^+_j$ be such that $j \notin I_A(R'_j, R_{-j}) \cup I_{A\cup\{j\}}(R''_j, R_{-j})$. Given that $\Omega_{A\cup\{j\}} = \Omega_A$ and $j \in N_A$, we obtain by Proposition 6 (third point) that $\mathcal{G}^A(R'_j, R_{-j}) = \mathcal{G}^{A\cup\{j\}}(R''_j, R_{-j})$. Since $i$ is a dictator at $\Omega_A$, $B \notin \mathcal{G}^A(R'_j, R_{-j})$ and $B \cup \{j\} \notin \mathcal{G}^A(R'_j, R_{-j})$. Thus, $B \notin \mathcal{G}^{A\cup\{j\}}(R''_j, R_{-j})$ and $B \cup \{j\} \notin \mathcal{G}^{A\cup\{j\}}(R''_j, R_{-j})$. If $j \notin I_{A\cup\{j\}}(R)$, then by Proposition 3 (exactly the second point of Definition 1), $\mathcal{G}^{A\cup\{j\}}(R''_j, R_{-j}) = \mathcal{G}^{A\cup\{j\}}(R)$. Thus, $B \notin \mathcal{G}^{A\cup\{j\}}(R)$, which is a contradiction. If, however, $j \in I_{A\cup\{j\}}(R)$, it follows from $B \cup \{j\} \notin \mathcal{G}^{A\cup\{j\}}(R''_j, R_{-j})$

and the application of Proposition 3 (exactly the first part of the fourth point of Definition 1)[22] that $B \notin \mathcal{G}^{A\cup\{j\}}(R)$, which is a contradiction. $\square$

**Lemma 12** *Suppose that $f$ is strategy-proof. For all $A, B \subseteq N$ and all $i \in N_A \setminus [(A \setminus B) \cup (B \setminus A)]$ such that $i$ is a dictator at $\Omega_A$, $[(A\setminus B)\cup(B\setminus A)]\cap\Omega = \emptyset$, and $[(A\setminus B)\cup(B\setminus A)] \subseteq N_A$, then $\Omega_B = \Omega_A$ and $i$ is a dictator at $\Omega_B$.*

Proof:

*Step 1: We show that $\Omega_{A\cap B} = \Omega_A$ and that $i$ is a dictator at $\Omega_{A\cap B}$.*

If $A \setminus B = \emptyset$, then $A \cap B = A$. Thus, $\Omega_{A\cap B} = \Omega_A$ and agent $i$ is a dictator at $\Omega_{A\cap B}$. If $A \setminus B \neq \emptyset$, consider any $j \in A \setminus B$. We show that $\Omega_A = \Omega_{A\setminus\{j\}}$ and that $i$ is a dictator at $\Omega_{A\setminus\{j\}}$.

First, suppose by contradiction that $\Omega_A \neq \Omega_{A\setminus\{j\}}$. Given that $[(A \setminus B) \cup (B \setminus A)] \subseteq N_A$ by assumption, $j \in N_A$. We can thus apply Proposition 4 (third point) to $\Omega_{A\setminus\{j\}}$ and $\Omega_A$ to see that $j \in N_{A\setminus\{j\}}$. By Proposition 5, applied to $\Omega_{A\setminus\{j\}}$ and $\Omega_A$, $j$ is a dictator at $\Omega_A$. This contradicts that $i$ is a dictator at $\Omega_A$. Then, $\Omega_A = \Omega_{A\setminus\{j\}}$. Finally, note that (a) $i, j \in N_{A\setminus\{j\}}$ because $i, j \in N_A$ and $\Omega_A = \Omega_{A\setminus\{j\}}$, (b) $j \notin A \setminus \{j\}$, (c) $\Omega_A = \Omega_{A\setminus\{j\}}$, and (d) $i$ is a dictator at $\Omega_A$. Then, it follows from Lemma 11 that $i$ is a dictator at $\Omega_{A\setminus\{j\}}$.[23]

The successive application of the arguments of the previous paragraph to all agents of $A\setminus B$ yields that $\Omega_{A\cap B} = \Omega_A$ and that $i$ is a dictator at $\Omega_{A\cap B}$.

*Step 2: We show that $\Omega_B = \Omega_{A\cap B}$ and that $i$ is a dictator at $\Omega_B$.*

If $B \setminus A = \emptyset$, then $A \cap B = B$. Thus, $\Omega_B = \Omega_{A\cap B}$ and, by Step 1, agent $i$ is a dictator at $\Omega_B$. If $B \setminus A \neq \emptyset$, consider any $j \in B \setminus A$. We show that $\Omega_{(A\cap B)\cup\{j\}} = \Omega_{A\cap B}$ and that $i$ is a dictator at $\Omega_{(A\cap B)\cup\{j\}}$.

First, suppose by contradiction that $\Omega_{(A\cap B)\cup\{j\}} \neq \Omega_{A\cap B}$. Since $[(A \setminus B) \cup (B \setminus A)] \cap \Omega = \emptyset$, we have that $j \notin \Omega$. Given that $[(A \setminus B) \cup (B \setminus A)] \subseteq N_A$ by assumption, $j \in N_A$. We can thus apply Proposition 4 (third point) to $\Omega_{A\cap B}$ and $\Omega_{(A\cap B)\cup\{j\}}$ to see that $j \in N_{(A\cap B)\cup\{j\}}$. By Proposition 5, applied to $\Omega_{A\cap B}$ and $\Omega_{(A\cap B)\cup\{j\}}$, $j$ is a dictator at $\Omega_{A\cap B}$. This contradicts that $i$ is a dictator at $\Omega_{A\cap B}$. Then, $\Omega_{(A\cap B)\cup\{j\}} = \Omega_{A\cap B}$. Finally, note that (a) $i, j \in N_{A\cap B}$ because $i, j \in N_A$ and $\Omega_A = \Omega_{A\cap B}$ by Step 1, (b) $j \notin A \cap B$ because $j \in B \setminus A$, (c) $\Omega_{(A\cap B)\cup\{j\}} = \Omega_{A\cap B}$, and (d) $i$ is a dictator at $\Omega_{A\cap B}$. Then, it follows from Lemma 11 that $i$ is a dictator at $\Omega_{(A\cap B)\cup\{j\}}$.[24]

[22] Note that with respect to Definition 1, the roles of profiles $R$ and $R'$ are played here by $(R''_j, R_{-j})$ and $R$, respectively. Also, the role of $C$ is played here by $\{j\}$.

[23] Note that with respect to Lemma 11, the role of $A$ is played here by $A \setminus \{j\}$.

[24] Note that with respect to Lemma 11, the role of $A$ is played here by $A \cap B$.

The successive application of the arguments of the previous paragraph to all agents of $B \setminus A$ yields that $\Omega_B = \Omega_{A \cap B}$ and that $i$ is a dictator at $\Omega_B$. This concludes Step 2.

The joint application of Steps 1 and 2 implies that $\Omega_A = \Omega_B$. This concludes the proof. $\square$

**Lemma 13** *Suppose that $f$ is strategy-proof. If there exists an agent $i \in N_{\{i\}} \subseteq N_\emptyset = N$ such that $\Omega_{\{i\}} \neq \Omega_\emptyset$ and $(N \setminus \{i\}) \cap \Omega = \emptyset$, then $f$ is dictatorial.*

Proof: Observe that since $\Omega_\emptyset \neq \Omega_{\{i\}}$ and $i \in N_\emptyset \cap N_{\{i\}}$, we have by Proposition 5, applied to $\Omega_\emptyset$ and $\Omega_{\{i\}}$, that $i$ is a dictator at $\Omega_\emptyset$ and at $\Omega_{\{i\}}$.

*Step 1: We show that for all $A \subseteq N \setminus \{i\}$, $\Omega_A = \Omega_\emptyset$ and $i$ is a dictator at $\Omega_A$.*

Consider any $A \subseteq N \setminus \{i\}$. Note that (a) $i \in N_\emptyset \setminus A$ because by assumption $N_\emptyset = N$ and $i \notin A$, (b) $i$ is a dictator at $\Omega_\emptyset$, (c) $A \cap \Omega = \emptyset$ because by assumption $i \notin A$ and $(N \setminus \{i\}) \cap \Omega = \emptyset$, and (d) $A \subseteq N_\emptyset$ because $N_\emptyset = N$ by assumption. Then, it follows from Lemma 12 that $\Omega_A = \Omega_\emptyset$ and that $i$ is a dictator at $\Omega_A$.[25]

*Step 2: We show that for all $A \subseteq N$ with $i \in A$, $\Omega_A = \Omega_{\{i\}}$ and $i$ is a dictator at $\Omega_A$.*

The proof is by induction on the size of $A$. First, let $|A| = 1$. Then, $A = \{i\}$. By definition, $\Omega_A = \Omega_{\{i\}}$ and we already know that $i$ is a dictator at $\Omega_{\{i\}}$. So, consider now any $A \subseteq N$ such that $i \in A$ and $|A| > 1$ and suppose that for all $D \subset A$ such that $i \in D$, $\Omega_D = \Omega_{\{i\}}$ and $i$ is a dictator at $\Omega_D$. We have to prove that $\Omega_A = \Omega_{\{i\}}$ and that $i$ is a dictator at $\Omega_A$.

We show first that $\Omega_A = \Omega_{\{i\}}$. Suppose otherwise and consider any $j \in A \setminus \{i\}$. Setting $D = A \setminus \{j\}$ in the induction hypothesis, we obtain that $\Omega_{A \setminus \{j\}} = \Omega_{\{i\}}$ and that $i$ is a dictator at $\Omega_{A \setminus \{j\}}$. First, if $j \in N_A$, then applying Proposition 4 (third point) to $\Omega_{A \setminus \{j\}}$ and $\Omega_A$ we obtain that $j \in N_{A \setminus \{j\}}$. Given that $\Omega_A \neq \Omega_{A \setminus \{j\}}$, applying Proposition 5 to $\Omega_{A \setminus \{j\}}$ and $\Omega_A$ we obtain that $j$ is a dictator at $\Omega_{A \setminus \{j\}}$. This contradicts that $i$ is a dictator at $\Omega_{A \setminus \{j\}}$. Second, if $j \notin N_A$, note that (a) $i \in N_{A \setminus \{j\}} \cap (A \setminus \{j\})$ because $i \in A$, $i \in N_{\{i\}}$ and $\Omega_{A \setminus \{j\}} = \Omega_{\{i\}}$ by the induction hypothesis, (b) $j \notin N_{A \setminus \{j\}} \cup (A \setminus \{j\})$ because by the application of Proposition 4 (first or second point depending on whether or not $j \geq h_A$) to $\Omega_{A \setminus \{j\}}$ and $\Omega_A$ we obtain from $j \notin N_A$ that $j \notin N_{A \setminus \{j\}}$, (c) $i$ is a dictator at $\Omega_{A \setminus \{j\}}$ by the induction hypothesis, and (d) $i \in N_{A \setminus \{i\}}$ because $\Omega_{A \setminus \{i\}} = \Omega_\emptyset$ by Step 1 and $N_\emptyset = N$. It follows from Lemma 10 that $\Omega_A = \Omega_{A \setminus \{j\}}$.[26] Then, since $\Omega_{A \setminus \{j\}} = \Omega_{\{i\}}$, $\Omega_A = \Omega_{\{i\}}$.

[25] Note that with respect to Lemma 12, the roles of $A$ and $B$ are played here by $\emptyset$ and $A$, respectively. Observe also that $[(\emptyset \setminus A) \cup (A \setminus \emptyset)] = A$.

[26] Note that the role of $A$ in Lemma 10 is played here by $A \setminus \{j\}$.

We now show that $i$ is a dictator at $\Omega_A$. Observe that since $\Omega_{A\setminus\{i\}} = \Omega_\emptyset$ by Step 1, $\Omega_{A\setminus\{i\}} \neq \Omega_A$. Since $\Omega_A = \Omega_{A\setminus\{j\}}$ and $i$ is a dictator at $\Omega_{A\setminus\{j\}}$ by the induction hypothesis, $i \in N_A$. Given that $i$ is a dictator at $\Omega_{A\setminus\{i\}}$ by Step 1, $i \in N_{A\setminus\{i\}}$. It follows then from Proposition 5 (applied to $\Omega_{A\setminus\{i\}}$ and $\Omega_A$) that $i$ is a dictator at $\Omega_A$.

*Step 3: We show that $f$ is a dictatorship of agent $i$.*

We have that for all $A \subseteq N$, $\Omega_A = \Omega_{\{i\}}$ whenever $i \in A$ by Step 2 and $\Omega_A = \Omega_\emptyset$ whenever $i \notin A$ by Step 1. Then, $\Omega = \{l_\emptyset, l_{\{i\}}, h_{\{i\}}, h_\emptyset\}$. Also, by Steps 1 and 2, agent $i$ is a dictator at $\Omega_A$ for all $A \subseteq N$. Thus, for any $A \subseteq N$, agent $i$ obtains one of her maximal alternatives of the set $\Omega_A$ at any profile $R \in \mathcal{R}^A$. It remains to be shown that this maximal alternative of the set $\Omega_A$ is also one of her maximal alternatives of the entire range $\Omega$. The application of Proposition 4 (third point) to $\Omega_\emptyset$ and $\Omega_{\{i\}}$ implies that $l_\emptyset \leq l_{\{i\}} < i < h_{\{i\}} \leq h_\emptyset$. Then, agent $i$ always prefers her best alternative of $\Omega_{\{i\}}$ (respectively, $\Omega_\emptyset$) to any other alternative of $\Omega$ when she has single-peaked (respectively, single-dipped) preferences. Then, $f$ is dictatorial, being $i$ the dictator. $\square$

**Lemma 14** *There is no strategy-proof and non-dictatorial rule $f$ with $|\Omega| > 2$ and $\Omega$ full at $N$.*

Proof: Suppose by contradiction that there is a non-dictatorial strategy-proof rule with $|\Omega| > 2$ and $\Omega$ full at $N$. Then, $N \subset (\min\Omega, \max\Omega) \setminus \Omega$.

*Step 1: We show that $\Omega_\emptyset = \{\min\Omega, \max\Omega\}$.*

We only prove that $l_\emptyset = \min\Omega$ (the proof that $h_\emptyset = \max\Omega$ is similar). Suppose by contradiction that $l_\emptyset \neq \min\Omega$. Since $l_\emptyset \in \Omega$, $l_\emptyset > \min\Omega$. And since $\min\Omega \in \Omega$, there exists $A \subseteq N$ such that $\min\Omega = l_A$. It follows then from $N \subset (\min\Omega, \max\Omega) \setminus \Omega$ that for all $j \in A$, $j > \min\Omega$. Then, consider any $j \in A$ and apply Proposition 4 (first or third point depending on whether or not $j \geq h_A$) to $\Omega_{A\setminus\{j\}}$ and $\Omega_A$ to see that $l_{A\setminus\{j\}} \leq l_A$. First, if $A \setminus \{j\} = \emptyset$, we obtain that $l_\emptyset \leq \min\Omega$, which is a contradiction. Second, if $A \setminus \{j\} \neq \emptyset$, consider any $k \in A \setminus \{j\}$ and note that, since $l_{A\setminus\{j\}} \leq l_A = \min\Omega$ and $k > \min\Omega$, $k > l_{A\setminus\{j\}}$. Then, apply Proposition 4 (first or third point depending on whether or not $k \geq h_{A\setminus\{j\}}$) to $\Omega_{A\setminus\{j,k\}}$ and $\Omega_{A\setminus\{j\}}$ to see that $l_{A\setminus\{j,k\}} \leq l_{A\setminus\{j\}}$. Thus, $l_{A\setminus\{j,k\}} \leq \min\Omega$. Proceeding iteratively in this way with all agents of $A$, we arrive at $l_\emptyset \leq \min\Omega$. This is a contradiction.

*Step 2: We reach a contradiction.*

Since $N \subset (\min \Omega, \max \Omega) \setminus \Omega$ and, by Step 1, $\Omega_\emptyset = \{\min T, \max T\}$, we have that $N_\emptyset = N$. Suppose first that for all $i \in N$, $\Omega_{\{i\}} = \Omega_\emptyset$. Consider any $A \subseteq N$. Note that (a) $A \subseteq N_\emptyset$ because $N_\emptyset = N$, and (b) $\Omega \cap A = \emptyset$ because $N \subset (\min \Omega, \max \Omega) \setminus \Omega$. By Lemma 9, $\Omega_A = \Omega_\emptyset$ for all $A \subseteq N$. Thus, $|\Omega| = 2$, which is a contradiction. Suppose next that there is an agent $i \in N$ such that $\Omega_{\{i\}} \neq \Omega_\emptyset$. Note that (a) $i \in N_{\{i\}} \subseteq N_\emptyset = N$ because it follows from $N \subset (\min \Omega, \max \Omega) \setminus \Omega$ and, by Step 1, $\Omega_\emptyset = \{\min \Omega, \max \Omega\}$ that $N_\emptyset = N$ and because the assumption $i \notin \Omega$ implies that we can apply Proposition 4 (third point) to $\Omega_\emptyset$ and $\Omega_{\{i\}}$ to obtain that $i \in N_{\{i\}}$, and (b) $(N \setminus \{i\}) \cap \Omega = \emptyset$ because $N \subset (\min \Omega, \max \Omega) \setminus \Omega$. It follows then from Lemma 13 that $f$ is dictatorial, which is a contradiction. □

We are ready to prove the theorem.

$\Rightarrow$]: We show that if all triples of $T$ are full at $N$, then $\Omega$, $|\Omega| > 2$, is also full at $N$. First, since $\Omega \subseteq T$, we have that for all $x \in \Omega \setminus \{\min \Omega, \max \Omega\}$, $\{\min \Omega, x, \max \Omega\}$ is full at $N$. Then, $N \subset (\min\{\min \Omega, x, \max \Omega\}, \max\{\min \Omega, x, \max \Omega\})$. Since $\min\{\min \Omega, x, \max \Omega\} = \min \Omega$ and $\max\{\min \Omega, x, \max \Omega\} = \max \Omega$, we obtain that $N \subset (\min \Omega, \max \Omega)$. Second, consider any $x \in \Omega$ and note that for all $y, z \in T$, $\{x, y, z\}$ is full at $N$. Hence, $x \notin N$. Thus, $N \cap \Omega = \emptyset$. Therefore, $\Omega$ is full at $N$. The result follows finally from Lemma 14.

$\Leftarrow$]: Suppose that there is a triple $\{x, y, z\} \subseteq T$ that is not full at $N$. We show that there is a non-dictatorial strategy-proof rule $f$ with $|\Omega| > 2$.

*Step 1: We prove the statement if $N \cap \{x, y, z\} \neq \emptyset$.*

Suppose without loss of generality that $x \in N$ and $y < z$. Consider any $j \in N \setminus \{x\}$. First, if $j \leq y$, then the rule $f$ such that $\Omega_A = x$ whenever $x \in A$, $\Omega_A = y$ whenever $A \cap \{x, j\} = \{j\}$, and $\Omega_A = z$ otherwise, has three alternatives in its range and is both non-dictatorial and strategy-proof. Second, if $j \geq z$, then the rule $f$ such that $\Omega_A = x$ whenever $x \in A$, $\Omega_A = y$ whenever $A \cap \{x, j\} = \emptyset$, and $\Omega_A = z$ otherwise, has three alternatives in its range and is both non-dictatorial and strategy-proof. Third, if $j \in (y, z)$, then the rule $f$ such that $\Omega_A = x$ whenever $x \in A$, and $\Omega_A = \{y, z\}$ otherwise, where $\mathcal{G}^A(R)$ contains all non-empty coalitions of $2^{N_A \setminus I_A(R)}$ for all $R \in \mathcal{R}^A$ with $x \notin A$, has three alternatives in its range and is both non-dictatorial and strategy-proof.

*Step 2: We prove the statement if $N \cap \{x, y, z\} = \emptyset$ and $N \not\subset (\min\{x, y, z\}, \max\{x, y, z\})$.*

Suppose without loss of generality that $x < y < z$. Then, by assumption, there is an agent $i \in N$ such that $i \notin [x, z]$. We only show the case when $i < x$ because the case when $i > z$ is similar. Consider any $j \in N \setminus \{i\}$. First, if $j < y$, then the rule $f$ such that

$\Omega_A = x$ whenever $i \in A$, $\Omega_A = y$ whenever $A \cap \{i,j\} = \{j\}$, and $\Omega_A = z$ otherwise, has three alternatives in its range and is both non-dictatorial and strategy-proof. Second, if $j > z$, then the rule $f$ such that $\Omega_A = x$ whenever $i \in A$, $\Omega_A = y$ whenever $A \cap \{i,j\} = \emptyset$ and $\Omega_A = z$ otherwise, has three alternatives in its range and is both non-dictatorial and strategy-proof. Third, if $j \in (y,z)$, then the rule $f$ such that $\Omega_A = x$ whenever $i \in A$, and $\Omega_A = \{y,z\}$ otherwise, where $\mathcal{G}^A(R)$ contains all non-empty coalitions of $2^{N_A \setminus I_A(R)}$ for all $R \in \mathcal{R}^A$ with $i \notin A$, has three alternatives in its range and is both non-dictatorial and strategy-proof.

## Proof of Proposition 9

Let $N_l = \{i \in N : i \leq \inf T\}$, $N_h = \{i \in N : i \geq \sup T\}$, and $N_c = N \setminus (N_l \cup N_h)$.[27] We show that $\min T$ exists (the proof with $\max T$ is similar). Suppose by contradiction that $\min T$ does not exist. Consider a profile $R \in \mathcal{R}^{N_l}$ such that for all $i \in N_c$ and all $y \in T$, with $y > i$, there is an alternative $x \in T$, with $x \leq i$, such that $x \, P_i \, y$. Since there is no Pareto efficient alternative in this profile, it is not possible to construct a Pareto efficient rule.

## Proof of Theorem 3

We first start with a lemma.

**Lemma 15** *Suppose that $f$ is strategy-proof and Pareto efficient. If $N \subset (\min T, \max T)$, then $\Omega_\emptyset = \{\min T, \max T\}$.*

Proof: Consider any two profiles $R, R' \in \mathcal{R}^\emptyset$ such that for all $i \in N$, $\min T \, P_i \max T$ and $\max T \, P'_i \min T$. Then, by Pareto efficiency, $f(R) = \min T$ and $f(R') = \max T$. Since $|\Omega_\emptyset| \leq 2$ by Proposition 1, $\Omega_\emptyset = \{\min T, \max T\}$. □

We are now ready to prove the theorem. Consider a set $T$ such that $|T| > 2$ and both $\min T$ and $\max T$ exist. Then, there is an alternative $y \in T \setminus \{\min T, \max T\}$.

$\Leftarrow$]: Suppose that at least one of the conditions of the theorem holds. We show that there is a non-dictatorial, strategy-proof, and Pareto efficient rule $f$.

*Step 1: We prove the statement if $N \not\subset (\min T, \max T)$.*

[27] If $\inf T$ or $\sup T$ does not exist, the corresponding sets are empty.

Suppose that $i \in N$ such that $i \notin (\min T, \max T)$ and consider any $j \in N \setminus \{i\}$. Assume without loss of generality that $i \leq \min T$. First, if $j \in (\min T, y]$, then the rule $f$ such that $\Omega_A = \max T$ whenever $A \cap \{i,j\} = \emptyset$, $\Omega_A = \min T$ whenever $i \in A$, and $\Omega_A = y$ otherwise, is non-dictatorial, strategy-proof, and Pareto efficient. Second, if $j \in (y, \max T)$, then the rule $f$ such that $\Omega_A = \max T$ whenever $i \notin A$, $\Omega_A = \min T$ whenever $A \cap \{i,j\} = \{i\}$, and $\Omega_A = y$ otherwise, is non-dictatorial, strategy-proof, and Pareto efficient. Third, if $j \leq \min T$, the rule $f$ such that $\Omega_A = \max T$ whenever $A \cap \{i,j\} = \emptyset$, $\Omega_A = \min T$ whenever $\{i,j\} \subseteq A$, and $\Omega_A = y$ otherwise, is non-dictatorial, strategy-proof, and Pareto efficient. Fourth, if $j \geq \max T$, the rule $f$ such that $\Omega_A = \max T$ whenever $A \cap \{i,j\} = \{j\}$, $\Omega_A = \min T$ whenever $A \cap \{i,j\} = \{i\}$, and $\Omega_A = y$ otherwise, is non-dictatorial, strategy-proof, and Pareto efficient.

*Step 2: We prove the statement if* $N \subset (\min T, \max T)$ *and there are two agents* $i, j \in N$ *such that* $i \in T$*, and both* $\max\{x \in T : x \leq j\}$ *and* $\min\{x \in T : x \geq j\}$ *exist.*

Consider the rule $f$ such that $\Omega_A = i$ whenever $i \in A$, $\Omega_A = \{\max\{x \in T : x \leq j\}, \min\{x \in T : x \geq j\}\}$ whenever $A \cap \{i,j\} = \{j\}$, and $\Omega_A = \{\min T, \max T\}$ otherwise, where $j$ is a dictator at all $\Omega_A$ such that $|\Omega_A| = 2$ (and choosing $l_A$ if $j$ is indifferent between $l_A$ and $h_A$). This rule is non-dictatorial, strategy-proof, and Pareto efficient.

$\Rightarrow$]: Suppose that none of the conditions of the theorem holds. We show that there is no non-dictatorial, strategy-proof, and Pareto efficient rule. Suppose by contradiction that there is such a rule. We divide the proof into two steps. The first step reaches a contradiction if neither the first condition ($N \not\subset (\min T, \max T)$) nor the first part of the second condition (there is an agent $i \in N$ such that $i \in T$) is satisfied. That is, we reach a contradiction if $N \subset (\min T, \max T)$ and $N \cap T = \emptyset$. The second step reaches a contradiction when, although the first part of the second condition holds, neither the first condition nor the second part of the second condition (there is an agent $j \in N \setminus \{i\}$ such that $\max\{x \in T : x \leq j\}$ and $\min\{x \in T : x \geq j\}$ exist) is satisfied. That is, we reach a contradiction whenever $N \subset (\min T, \max T)$, $|N \cap T| = 1$,[28] and for and for all $j \in N \setminus T$, $\max\{x \in T : x \leq j\}$ or $\min\{x \in T : x \geq j\}$ does not exist.

*Step 1: We reach a contradiction if* $N \subset (\min T, \max T)$ *and* $N \cap T = \emptyset$*.*

Since $N \subset (\min T, \max T)$, we have by Lemma 15 that $\Omega_\emptyset = \{\min T, \max T\}$. Consider now a profile $R \in \mathcal{R}^N$ such that for all $i \in N$, $y \, P_i \min T \, P_i \max T$ (note that $y$ has

[28]Note that we restrict that only one agent is situated at a feasible point since if there are more than one, the second part of the second condition will also be satisfied.

been defined at the beginning of the proof as an alternative $y \in T \setminus \{\min T, \max T\}$). Then, $f(R) \notin \{\min T, \max T\}$ by Pareto efficiency. Thus, $|\Omega| > 2$. Note also that since $N \subset (\min T, \max T)$ and $\{\min T, \max T\} \subset \Omega$, $N \subset (\min \Omega, \max \Omega)$. Similarly, since $N \cap T = \emptyset$ and $\Omega \subseteq T$, $N \cap \Omega = \emptyset$. Then, $\Omega$ is full at $N$. Finally, apply Lemma 14 to reach a contradiction.

*Step 2: We reach a contradiction if* $N \subset (\min T, \max T)$, $|N \cap T| = 1$, *and for all* $j \in N \setminus T$, $\max\{x \in T : x \leq j\}$ *or* $\min\{x \in T : x \geq j\}$ *does not exist.*

Let $N \cap T = \{i\}$. Since $N \subset (\min T, \max T)$, $\Omega_\emptyset = \{\min T, \max T\}$ by Lemma 15. Then, $N_\emptyset = N$.

*Step 2.a: We show that for all* $S \subset N$ *such that* $\min N \in S$, $N \cap [\min S, \max S] = S$ *and* $\max S < i$, $\Omega_S = \{\min T, \max T\}$.

The proof is by induction on the size of $S$. Suppose first that $|S| = 1$, which implies that $S = \{\min N\}$. Assume by contradiction that $\Omega_S \neq \{\min T, \max T\}$. Since $N_\emptyset = N$, $\min N \in N_\emptyset$. Given that $\min N = \max S < i$ and $N \cap T = \{i\}$, $\min N \notin T$. Apply Proposition 4 (third point) to $\Omega_\emptyset$ and $\Omega_S$ to see that $\min N \in N_S$. The application of Proposition 5 to $\Omega_\emptyset$ and $\Omega_S$ implies then that $\min N$ is a dictator at $\Omega_S$.

By assumption, either $\max\{x \in T : x \leq \min N\}$ or $\min\{x \in T : x \geq \min N\}$ does not exist. If $\max\{x \in T : x \leq \min N\}$ does not exist, consider any $z \in T \cap (l_S, \min N)$ and denote $S^* = N \cap [z, \min N]$. If $\min\{x \in T : x \geq \min N\}$ does not exist, consider any $z \in T \cap (\min N, h_S)$ such that $z < i$ and denote $S^* = N \cap [\min N, z]$. We are going to show that $\Omega_{S^*} = \Omega_S$ and that $\min N$ is a dictator at $\Omega_{S^*}$. If $\max\{x \in T : x \leq \min N\}$ does not exist, then $S^* = S$. Consequently, $\Omega_{S^*} = \Omega_S$ and $\min N$ is a dictator at $\Omega_{S^*}$. If $\min\{x \in T : x \geq \min N\}$ does not exist, note that (a) $\min N \in N_S \setminus (S^* \setminus \{\min N\})$ because $\min N \in N_S$, (b) $\min N$ is a dictator at $\Omega_S$, (c) $(S^* \setminus \{\min N\}) \cap \Omega = \emptyset$ because $N \cap T = \{i\}$, $\Omega \subseteq T$ and it follows from $z < i$ that $i \notin S^*$, and (d) $(S^* \setminus \{\min N\}) \subseteq N_S$ because $\min N \in N_S$ and $z < h_S$. Then, it follows from Lemma 12 that $\Omega_{S^*} = \Omega_S$ and that $\min N$ is a dictator at $\Omega_{S^*}$.[29] Finally, consider a profile $R \in \mathcal{R}^{S^*}$ such that $h_S \, P_{\min N} \, l_S$ and $z \, P_k \, h_S$ for all $k \in N_{S^*}$. Observe that, for all $l \notin N_{S^*}$, $z \, P_l \, h_S$ because $l \geq h_S > z$ and $R_l \in \mathcal{R}_l^-$. Then, since $\min N$ is a dictator at $\Omega_{S^*}$, $f(R) = h_S$, but $z$ Pareto dominates $h_S$. This is a contradiction.

[29] Note that with respect to Lemma 12, the roles of $A$, $B$ and $i$ are played here by $S$, $S^*$ and $\min N$, respectively. Observe also that $[(S \setminus S^*) \cup (S^* \setminus S)] = (S^* \setminus \{\min N\}) = N \cap (\min N, z]$.

Consider now any $S \subset N$ such that $|S| > 1$, $\min N \in S$, $N \cap [\min S, \max S] = S$, and $\max S < i$. Suppose that for all $S' \subseteq S$ such that $\min N \in S'$ and $N \cap [\min S', \max S'] = S'$, $\Omega_{S'} = \Omega_\emptyset$, but, by contradiction, $\Omega_S \neq \{\min T, \max T\}$. By the induction hypothesis (setting $S' = S \setminus \{\max S\}$), $\Omega_{S\setminus\{\max S\}} = \{\min T, \max T\}$. Given that $\max S < i$ and $N \cap T = \{i\}$, $\max S \notin T$. Apply Proposition 4 (third point) to $\Omega_{S\setminus\{\max S\}}$ and $\Omega_S$ to see that $\max S \in N_S$. The application of Proposition 5 to $\Omega_{S\setminus\{\max S\}}$ and $\Omega_S$ implies then that $\max S$ is a dictator at $\Omega_S$.

By assumption, either $\max\{x \in T : x \leq \max S\}$ or $\min\{x \in T : x \geq \max S\}$ does not exist. If $\max\{x \in T : x \leq \max S\}$ does not exist, consider any $z \in T \cap (l_S, \max S)$ and denote $S^* = S$. If $\min\{x \in T : x \geq \max S\}$ does not exist, consider any $z \in T \cap (\max S, h_S)$ such that $z < i$ and denote $S^* = N \cap [\min N, z]$. We are going to show that $\Omega_{S^*} = \Omega_S$ and that $\max S$ is a dictator at $\Omega_{S^*}$. If $\max\{x \in T : x \leq \max S\}$ does not exist, then $S^* = S$. Consequently, $\Omega_{S^*} = \Omega_S$ and $\max S$ is a dictator at $\Omega_{S^*}$. If $\min\{x \in T : x \geq \max S\}$ does not exist, note that (a) $\max S \in N_S \setminus (S^* \setminus S)$ because $\max S \in N_S \cap S$, (b) $\max S$ is a dictator at $\Omega_S$, (c) $(S^* \setminus S) \cap \Omega = \emptyset$ because $N \cap T = \{i\}$, $\Omega \subseteq T$, and it follows from $z < i$ that $i \notin S^*$, and (d) $(S^* \setminus S) \subseteq N_S$ because $\max S \in N_S$ and $z < h_S$. Then, it follows from Lemma 12 that $\Omega_{S^*} = \Omega_S$ and that $\max S$ is a dictator at $\Omega_{S^*}$.[30] Consider finally a profile $R \in \mathcal{R}^{S^*}$ such that $h_S \, P_{\max S} \, l_S$ and $z \, P_k \, h_S$ for all $k \in N_{S^*}$. Observe that, for all $l \notin N_{S^*}$, $z \, P_l \, h_S$ because $l \geq h_S > z$ and $R_l \in \mathcal{R}_l^-$. Then, since $\max S$ is a dictator at $\Omega_{S^*}$, $f(R) = h_S$, but $z$ Pareto dominates $h_S$. This is a contradiction.

*Step 2.b: We show that for all $j \in N \setminus \{i\}$, $\Omega_{\{j\}} = \{\min T, \max T\}$.*

Consider any $j < i$.[31] Denote $S = [\min N, j] \cap N$. By Step 2.a, $\Omega_S = \{\min T, \max T\}$. If $|S| = 1$, then $S = \{j\}$ and $\Omega_{\{j\}} = \{\min T, \max T\}$. If $|S| > 1$, consider any agent $k \in S \setminus \{j\}$. Since $\Omega_S = \{\min T, \max T\}$ and $N \subset (\min T, \max T)$, $k \in N_S$. Apply Proposition 4 (third point) to $\Omega_{S\setminus\{k\}}$ and $\Omega_S$ to see that $\Omega_{S\setminus\{k\}} = \{\min T, \max T\}$. If $|S \setminus \{k\}| = 1$, then $S \setminus \{k\} = \{j\}$ and, therefore, $\Omega_{\{j\}} = \{\min T, \max T\}$. If $|S \setminus \{k\}| > 1$, proceed iteratively in this way with all agents of $S \setminus \{j\}$ to arrive at $\Omega_{\{j\}} = \{\min T, \max T\}$.

*Step 2.c: We show that for all $A \subseteq N \setminus \{i\}$, $\Omega_A = \{\min T, \max T\}$.*

Consider any $A \subseteq N \setminus \{i\}$. Note that (a) for all $j \in A$, $\Omega_{\{j\}} = \Omega_\emptyset$ by Step 2.b, (b) $A \subseteq N_\emptyset$ because it follows from $\Omega_\emptyset = \{\min T, \max T\}$ and $N \subset (\min T, \max T)$ that $N_\emptyset = N$, and

[30]Note that with respect to Lemma 12, the roles of $A$, $B$ and $i$ are played here by $S$, $S^*$ and $\max S$, respectively. Observe also that $[(S \setminus S^*) \cup (S^* \setminus S)] = (S^* \setminus S) = N \cap (\max S, z]$.

[31]If $j > i$, the proof is similar, but with an adapted version of Step 2.a that proves (in a similar way to Step 2.a) that for all $S \subset N$ such that $\max N \in S$, $N \cap [\min S, \max S] = S$ and $\min S > i$, then $\Omega_S = \{\min T, \max T\}$.

(c) $A \cap \Omega = \emptyset$ because $(N \setminus \{i\}) \cap T = \emptyset$, $A \subseteq N \setminus \{i\}$, and $\Omega \subseteq T$. Then, it follows from Lemma 9 that $\Omega_A = \Omega_\emptyset$. Hence, $\Omega_A = \{\min T, \max T\}$.

*Step 2.d: We show that* $\Omega_{\{i\}} \neq \{\min T, \max T\}$.

Suppose by contradiction that $\Omega_{\{i\}} = \{\min T, \max T\}$. We show that $\Omega_A = \{\min T, \max T\}$ for all $A \subseteq N$ by induction on the size of $A$. If $|A| = 0$, $A = \emptyset$ and we already know that $\Omega_\emptyset = \{\min T, \max T\}$. If $|A| = 1$ and $A = \{i\}$, then $\Omega_{\{i\}} = \{\min T, \max T\}$ by assumption. If $|A| = 1$ and $A \neq \{i\}$, then $\Omega_A = \{\min T, \max T\}$ by Step 2.b. So, consider any $A \subseteq N$ such that $|A| > 1$ and suppose that for all $B \subset A$, $\Omega_B = \{\min T, \max T\}$, but, by contradiction, $\Omega_A \neq \{\min T, \max T\}$. If $i \notin A$, then $\Omega_A = \{\min T, \max T\}$ by Step 2.c. Hence, $i \in A$. Consider any agent $j \in A \setminus \{i\}$ and suppose without loss of generality that $j > i$. By the induction hypothesis (setting $B = A \setminus \{j\}$ and $B = A \setminus \{i\}$), $\Omega_{A\setminus\{j\}} = \Omega_{A\setminus\{i\}} = \{\min T, \max T\}$. Since $N \cap T = \{i\}$ and $\Omega \subseteq T$, $j \notin \Omega$. Also, since $\Omega_{A\setminus\{j\}} = \{\min T, \max T\}$ and $N \subset (\min T, \max T)$, $j \in N_{A\setminus\{j\}}$. Apply then Proposition 4 (third point) to $\Omega_{A\setminus\{j\}}$ and $\Omega_A$ to see that $j \in N_A$. The application of Proposition 5 to $\Omega_{A\setminus\{j\}}$ and $\Omega_A$ implies then that $j$ is a dictator at $\Omega_A$ and at $\Omega_{A\setminus\{j\}}$. Since $\Omega_{A\setminus\{i\}} = \{\min T, \max T\}$ and $N \subset (\min T, \max T)$, $i \in N_{A\setminus\{i\}}$. Apply then Proposition 4 (third and fourth point) to $\Omega_{A\setminus\{i\}}$ and $\Omega_A$ to see that $l_A \in [\min T, i]$, $h_A \in [i, \max T]$, and $h_A \in \{i, \max T\}$ whenever $l_A = i$.

Suppose first that $l_A < i$. Since $j \in N_A$ and $i < j$, $h_A > i$. Thus, $i \in N_A$. Apply Proposition 5 to $\Omega_{A\setminus\{i\}}$ and $\Omega_A$ to see that $i$ is a dictator at $\Omega_A$. This contradicts that $j$ is a dictator at $\Omega_A$. Suppose next that $l_A = i$. Since $j \in N_A$, $h_A = \max T$. Therefore, $\Omega_A = \{i, \max T\}$. Note that (a) $j \in N_A \setminus (N_A \setminus A)$ because $j \in N_A \cap A$, (b) $j$ is a dictator at $\Omega_A$, (c) $(N_A \setminus A) \cap \Omega = \emptyset$ because $N \cap T = \{i\}$, $\Omega \subseteq T$, and because it follows from $\Omega_A = \{i, \max T\}$ that $i \notin N_A$, and (d) $N_A \setminus A \subseteq N_A$. It follows then from Lemma 12 that $\Omega_{A\cup N_A} = \Omega_A = \{i, \max T\}$ and that $j$ is a dictator at $\Omega_{A\cup N_A}$.[32] It can be shown in a similar way that $\Omega_{A\cup N_A\setminus\{j\}} = \Omega_{A\setminus\{j\}} = \{\min T, \max T\}$ and that $j$ is a dictator at $\Omega_{A\cup N_A\setminus\{j\}}$.

We now show that $\Omega_N = \Omega_{A\cup N_A}$. If $N = A \cup N_A$, then $\Omega_N = \Omega_{A\cup N_A}$. If $N \neq A \cup N_A$, then $A \cup N_A \subset N$. Consider any $k \in N \setminus (N_A \cup A)$ and note that (a) $j \in N_{A\cup N_A} \cap (A \cup N_A)$ because $j$ is a dictator at $\Omega_A$ and at $\Omega_{A\cup N_A}$, (b) $k \notin N_{A\cup N_A} \cup (A \cup N_A)$ by definition of $k$ and the fact that $\Omega_{A\cup N_A} = \Omega_A$, (c) $j$ is a dictator at $\Omega_{A\cup N_A}$ and (d) $j \in N_{A\cup N_A\cup\{k\}\setminus\{j\}}$ because $N \subset (\min T, \max T)$ and $\Omega_{A\cup N_A\cup\{k\}\setminus\{j\}} = \{\min T, \max T\}$.[33] Then, it follows from

[32]Note that with respect to Lemma 12, the roles of $B$ and $i$ are played here by $A\cup N_A$ and $j$, respectively. Observe also that $[((A \cup N_A) \setminus A) \cup (A \setminus (A \cup N_A))] = N_A \setminus A$.

[33]Suppose that $\Omega_{A\cup N_A\cup\{k\}\setminus\{j\}} \neq \{\min T, \max T\}$. Since $\Omega_{A\cup N_A\setminus\{j\}} = \{\min T, \max T\}$, $k \in N_{A\cup N_A\setminus\{j\}}$.

Lemma 10 that $\Omega_{A\cup N_A\cup\{k\}} = \Omega_{A\cup N_A}$.[34] Proceeding iteratively in this way with all agents of $N \setminus (N_A \cup A)$ we arrive at $\Omega_N = \Omega_{A\cup N_A}$. Thus, $\Omega_N = \{i, \max T\}$. Let $z \in T$ be such that $z \in (i, j)$ when $\max\{x \in T : x \leq j\}$ does not exist and $z \in (j, \max T)$ when $\min\{x \in T : x \geq j\}$ does not exist. Consider a profile $R \in \mathcal{R}^N$ such that for all $k \in N_N$, $z\, P_k\, \max T\, P_k\, i$. Observe that, for all $l \notin N_N$, $z\, P_l\, \max T$ because $l \leq i < z < \max T$ and $R_l \in \mathcal{R}_l^+$. Then, $H_N(R) = N_N$. By Lemma 8, $f(R) = \max T$. This is a contradiction because $z$ Pareto dominates $\max T$.

We have thus shown that for all $A \subseteq N$, $\Omega_A = \{\min T, \max T\}$. Consequently, $\Omega_N = \{\min T, \max T\}$. Consider a profile $R \in \mathcal{R}^N$ such that for all $i \in N$, $y\, P_i\, \min T\, P_i\, \max T$ (note that $y$ has been defined at the beginning of the proof as an alternative $y \in T \setminus \{\min T, \max T\}$). By Pareto efficiency, $f(R) \notin \{\min T, \max T\}$, which is a contradiction.

*Step 2.e: We show that $\Omega_{\{i\}} = \{i\}$.*

Suppose by contradiction that $\Omega_{\{i\}} \neq \{i\}$. Since $N_\emptyset = N$, $i \in N_\emptyset$. The application of Proposition 4 (third and fourth point) to $\Omega_\emptyset$ and $\Omega_{\{i\}}$ gives us then three possibilities for $\Omega_{\{i\}}$. First, if $\Omega_{\{i\}} = \{\min T, \max T\}$, we reach a contradiction by Step 2.d. Second, if $\Omega_{\{i\}} \in \{\{i, \max T\}, \{\min T, i\}\}$, consider a profile $R \in \mathcal{R}^\emptyset$ such that $I_\emptyset(R) = I_{\{i\}}(R) = \emptyset$ and any preference $R'_i \in \mathcal{R}_i^+$. Then, by Proposition 3 (exactly the last point of Definition 1), $\mathcal{G}^\emptyset(R) \neq \emptyset$. Consider a coalition $B \subseteq N_\emptyset$ such that $B \in \mathcal{G}^\emptyset(R)$ and observe that by Proposition 8 (first or second point depending whether $\Omega_{\{i\}} \in \{\min T, i\}$ or $\{i, \max T\}$), $B \cap N_{\{i\}} \cap \{i\} \in \mathcal{G}^{\{i\}}(R'_i, R_{-i})$. Since $B \cap N_{\{i\}} \cap \{i\} = \emptyset$, we have that $\emptyset \in \mathcal{G}^{\{i\}}(R'_i, R_{-i})$. Since $I_{\{i\}}(R'_i, R_{-i}) = \emptyset$, this contradicts Proposition 3 (exactly the last point of Definition 1). Third, if $\Omega_{\{i\}} = \{l_{\{i\}}, h_{\{i\}}\} \neq \{\min T, \max T\}$, with $i \in N_{\{i\}} \subseteq N_\emptyset$, then note that (a) $(N \setminus \{i\}) \cap \Omega = \emptyset$ because $(N \setminus \{i\}) \cap T = \emptyset$ and $\Omega \subseteq T$, and (b) $i \in N_{\{i\}} \subseteq N_\emptyset = N$ because $N_\emptyset = N$. We can thus apply Lemma 13 to see that $f$ is dictatorial, which is a contradiction.

*Step 2.f: We show that for all $A \subseteq N$, $\Omega_A = \{i\}$ whenever $i \in A$.*

Consider any $A \subseteq N$ such that $i \in A$. The proof is by induction on the size of $A$. If $|A| = 1$, the result follows from Step 2.e. So, consider now any $A \subseteq N$ such that $i \in A$ and $|A| > 1$ and suppose that for all $B \subset A$, with $i \in B$, $\Omega_B = \{i\}$. Consider any $j \in A \setminus \{i\}$.

Given that $k \notin T$, we have by Proposition 4 (third point) that $k \in N_{A\cup N_A\cup\{k\}\setminus\{j\}}$. By Proposition 5 applied to $\Omega_{A\cup N_A\setminus\{j\}}$ and $\Omega_{A\cup N_A\cup\{k\}\setminus\{j\}}$, $k$ is a dictator at $\Omega_{A\cup N_A\setminus\{j\}}$. This contradicts that $j$ is a dictator at $\Omega_{A\cup N_A\setminus\{j\}}$.

[34] Note that with respect to Lemma 10, the roles of $A$, $i$ and $j$ are played here by $A \cup N_A$, $j$ and $k$, respectively.

It follows from the induction hypothesis (by setting $B = A\setminus\{j\}$) that $\Omega_{A\setminus\{j\}} = \{i\}$. Apply Proposition 4 (first or second point depending on whether or not $j > i$) to $\Omega_{A\setminus\{j\}}$ and $\Omega_A$ to see that $l_A$ and $h_A$ are both weakly between $i$ and $j$. We also have by Step 2.c that $\Omega_{A\setminus\{i\}} = \{\min T, \max T\}$. Then, since $i \in N_{A\setminus\{i\}}$, we can apply Proposition 4 (third and fourth point) to $\Omega_{A\setminus\{i\}}$ and $\Omega_A$ to obtain that $l_A \in [\min T, i]$, $h_A \in [i, \max T]$, and if $i \in \Omega_A$, then $\Omega_A = \{\{i\}, \{\min T, i\}, \{i, \max T\}\}$. Since $N \subset (\min T, \max T)$, $j \notin \{\min T, \max T\}$. Then, the only possibility is $l_A = h_A = \{i\}$.

*Step 2.g: We show that $i$ is a dictator at $\Omega_{N\setminus\{i\}}$.*

By Step 2.c, $\Omega_{N\setminus\{i\}} = \{\min T, \max T\}$. Suppose by contradiction that $i$ is not a dictator at $\Omega_{N\setminus\{i\}}$. Then, there exists a profile $R \in \mathcal{R}^{N\setminus\{i\}}$ , with $i \notin I_{N\setminus\{i\}}(R)$, and a coalition $B \subseteq N_{N\setminus\{i\}}$, with $i \notin B$, such that $B \in \mathcal{G}^{N\setminus\{i\}}(R)$ (the proof that it is not possible that $i \in B \notin \mathcal{G}^{N\setminus\{i\}}(R)$ is similar). Then, since $B \subseteq (N\setminus\{i\})\setminus I_{N\setminus\{i\}}(R)$, we have by Proposition 3 (exactly the third point of Definition 1) that $(N \setminus \{i\}) \setminus I_{N\setminus\{i\}}(R) \in \mathcal{G}^{N\setminus\{i\}}(R)$. Observe, on the other hand, that by assumption, there is an alternative $z \in T \setminus \{\min T, \max T, i\}$. Assume that $z > i$ (the proof when $z < i$ is similar) and consider a profile $R' \in \mathcal{R}^{N\setminus\{i\}}$ such that $R'_{I_{N\setminus\{i\}}} = R_{I_{N\setminus\{i\}}}$, $z \, P'_j \, \min T \, P'_j \, \max T$ for all $j \in (N \setminus \{i\}) \setminus I_{N\setminus\{i\}}(R)$, and $\max T \, P'_i \, z \, P'_i \, \min T$. Observe that for all $k \in I_{N\setminus\{i\}}(R)$, $z \, P'_k \, \min T$. By Proposition 3 (exactly the second point of Definition 1), $\mathcal{G}^{N\setminus\{i\}}(R') = \mathcal{G}^{N\setminus\{i\}}(R)$ and, thus, $(N \setminus \{i\}) \setminus I_{N\setminus\{i\}}(R) \in \mathcal{G}^{N\setminus\{i\}}(R')$. Since $L_{N\setminus\{i\}}(R') = (N \setminus \{i\}) \setminus I_{N\setminus\{i\}}(R)$, $f(R') = \min T$, but $z$ Pareto dominates $\min T$. This is a contradiction.

*Step 2.h: We show that $i$ is a dictator at $\Omega_A$ for all $A \subseteq N \setminus \{i\}$.*

Consider any $A \subseteq N \setminus \{i\}$. Note that (a) $i \in N_{N\setminus\{i\}} \setminus ((N \setminus \{i\}) \setminus A)$ because it follows from the fact that $i$ is a dictator at $\Omega_{N\setminus\{i\}}$ by Step 2.g that $i \in N_{N\setminus\{i\}}$, (b) $i$ is a dictator at $\Omega_{N\setminus\{i\}}$ by Step 2.g, (c) $((N \setminus \{i\}) \setminus A) \cap \Omega = \emptyset$ because $N \cap T = \{i\}$ and $\Omega \subseteq T$, and (d) $(N \setminus \{i\}) \setminus A \subseteq N_{N\setminus\{i\}}$ because it follows from the fact that $\Omega_{N\setminus\{i\}} = \{\min T, \max T\}$ by Step 2.c and $N \subset (\min T, \max T)$ that $N_{N\setminus\{i\}} = N$. Therefore, by Lemma 12, $i$ is a dictator at $\Omega_A$.[35]

*Step 2.i: We reach a contradiction.*

By Step 2.h, agent $i$ is for all $A \subseteq N\setminus\{i\}$ a dictator at $\Omega_A$. This, together with the fact that $\Omega_A = \{\min T, \max T\}$ for all $A \subseteq N \setminus \{i\}$ by Step 2.c, implies that if $i$ has single-dipped preferences, she obtains one of her maximal alternatives. Given that, by Step 2.f, she

[35] Note that with respect to Lemma 12, the roles of $A$ and $B$ are played here by $N\setminus\{i\}$ and $A$, respectively. Observe also that $[((N \setminus \{i\}) \setminus A) \cup (A \setminus (N \setminus \{i\}))] = (N \setminus \{i\}) \setminus A$.

also obtains her most preferred alternative (her own location) when she has single-peaked preferences, the rule is dictatorial, being $i$ the dictator. This is a contradiction.

## Proof of Theorem 4

Let $T$ be such that both $\min T$ and $\max T$ exist and that $N \cap (\min T, \max T) = \emptyset$. The proof considers jointly the definition of the generalized median voter rules from footnote 13 and the extra condition for Pareto efficiency from footnote 14.

$\Leftarrow$]: Take any generalized median voter rule $f$. Since all generalized median voters rules are strategy-proof on the domain of single-peaked preferences and since $\mathcal{R}$ is a subdomain of the domain of single-peaked preferences, $f$ is strategy-proof.

Suppose additionally that $\Omega_{N_l} = \min T$ and $\Omega_{N_h} = \max T$. We have to show that $f$ is Pareto efficient. Take any profile $R \in \mathcal{R}^A$. If $A = N_l$ (respectively, $A = N_h$), all agents prefer $\min T$ (respectively, $\max T$) to any other alternative, and precisely this unanimously best alternative is chosen. In all other cases, for all $x, y \in T$, there are two agents $i, j \in N$ such that $x \, P_i \, y$ and $y \, P_j \, x$. So, any alternative is Pareto efficient. This implies that the outcome $f(R)$ is Pareto efficient in these cases as well.

$\Rightarrow$]: Consider any strategy-proof rule $f$. We show that there is a function $\omega : 2^N \to T$, with $\omega(A) = \Omega_A$, satisfying the condition of the generalized median voter rules such that for all $A \subseteq N$ and all $R \in \mathcal{R}^A$, $f(R) = \Omega_A$. By strategy-proofness, $f$ belongs to the family characterized in Theorem 1. First, we prove that the range of $\omega$ is $T$ instead of $T^2$. Suppose otherwise, that is, there is a set $A \subseteq N$ such that $l_A \neq h_A$. Since $N \cap (\min T, \max T) = \emptyset$ by assumption, $N_A = \emptyset$. This contradicts Proposition 2. Second, the first point of Proposition 4 implies that for all $S \subset N$ and all $i \in N \setminus S$, $\omega(S \cup \{i\}) \geq \omega(S)$ whenever $i \in N_h$. Similarly, the second point of Proposition 4 implies that for all $S \subset N$ and all $i \in N \setminus S$, $\omega(S \cup \{i\}) \leq \omega(S)$ whenever $i \in N_l$. Then, $f$ is a generalized median voter rule. Suppose finally that $f$ is also Pareto efficient. Then, observe that for any $R \in \mathcal{R}^{N_l}$ (respectively, $R \in \mathcal{R}^{N_h}$), $\min T$ (respectively, $\max T$) Pareto dominates all other alternatives. Then, $\Omega_{N_l} = \min T$ and $\Omega_{N_h} = \max T$.

## Proof of Theorem 5

Let $T$ be such that both $\min T$ and $\max T$ exist and that $N \subset T \cap (\min T, \max T)$.

$\Leftarrow$] Consider any conditional two-step rule $f$. We show that $f$ is strategy-proof, Pareto efficient, and tops-only. Note that $\Omega_\emptyset = \{\min T, \max T\}$ and $|\Omega_A| = 1$ whenever $A \neq \emptyset$.

First, we show that $f$ is tops-only. Consider any $R \in \mathcal{R}^A$ for some $A \subseteq N$. If $A \neq \emptyset$, $|\Omega_A| = 1$ and this outcome $f(R) = \Omega_A$ depends on $A$, the locations of the agents with single-peaked preferences. Since the top alternative of any agent with single-peaked preferences is her location, the outcome of the rule in these profiles only depends on the top alternatives of the agents with single-peaked preferences. Hence, $f$ is tops-only in these profiles. If, however, $A = \emptyset$, $\Omega_A = \{\min T, \max T\}$ and $f(R)$ depends on the sets $\{i \in N : \min T \in t(R_i)\}$ and $\{i \in N : \min T = t(R_i)\}$. That is, it depends on the top alternatives of all agents. Hence, $f$ is tops-only in these profiles as well.

Next, we show that $f$ is strategy-proof. To see that $f$ is one of the rules defined in Corollary 1, define from $\omega$ the function $\omega^* : 2^N \to T^2$ in such a way that $\omega^*(A) = \omega(A)$ whenever $A \neq \emptyset$ and $\omega^*(\emptyset) = \{\min T, \max T\}$. Then, $\omega^*$ satisfies the properties of the function $\omega$ in Corollary 1. Note that $\omega^*$ describes the preselected alternatives in the first step of the rule. Second, since $|\omega^*(A)| = 2$ only if $A = \emptyset$, we have to define only the correspondence $\mathcal{G}^\emptyset$. We do that using $\mathcal{G}$ in the following way: for any $R \in \mathcal{R}^\emptyset$ and any $B \in 2^{N \setminus I_\emptyset(R)}$, $B \in \mathcal{G}^\emptyset(R)$ whenever $(B \cup I_\emptyset(R), B) \in \mathcal{G}$. Observe then that the first two conditions of Definition 1 are satisfied by construction, that the third and the fourth condition of Definition 1 are implied by the monotonicity property of $\mathcal{G}$, and that the last condition of Definition 1 is implied by the efficiency condition of $\mathcal{G}$. We can thus conclude that $f$ is one of the rules defined in Corollary 1. To further see that $f$ satisfies Proposition 4, note on the one hand that if $A \neq \emptyset$, then only the first and the second point of Proposition 4 apply. The monotonicity property ensures that these two points are satisfied. On the other hand, if $A = \emptyset$, then only the third and the fourth point of Proposition 4 apply. Since $\Omega_{\{i\}} = i$ for all $i \in N$, these two points are also satisfied. We can thus conclude that $f$ satisfies Proposition 4. Finally, Propositions 5 to 8 never apply because there is no $A \subset N$ and $i \in N \setminus A$ such that $|\omega^*(A)| = |\omega^*(A \cup \{i\})| = 2$. Consequently, $f$ belongs to the family characterized in Theorem 1. Hence, it is strategy-proof.

Finally, we show that $f$ is Pareto-efficient. Consider any $R \in \mathcal{R}^A$ for some $A \subseteq N$. If $A = \emptyset$, then $f(R) \in \{\min T, \max T\}$. Suppose that $f(R) = \min T$ (the case when $f(R) = \max T$ is similar). Then, $(\{i \in N : \min T \in t(R_i)\}, \{i \in N : t(R_i) = \min T\}) \in \mathcal{G}$. Then, by the monotonicity and the efficiency conditions of $\mathcal{G}$, we have that $\{i \in N : \min T \in t(R_i)\} = N$ or $\{i \in N : t(R_i) = \min T\} \neq \emptyset$. In any case, $\min T$ is Pareto efficient. If $A \neq \emptyset$, then $f(R) \in [\min A, \max A]$. Observe that for all $x, y, z, w \in T$ such that $x, y \in [\min A, \max A]$, with $x < y$, $z < \min A$ and $w > \max A$, we have that $x \, P_{\min A} \, y \, P_{\min A} \, w$ and $y \, P_{\max A} \, x \, P_{\max A} \, z$. Then, no alternative inside the interval $[\min A, \max A]$ is Pareto dominated by any other alternative. This guarantees that any choice in this interval is

Pareto efficient.

$\Rightarrow$]: Consider any tops-only, strategy-proof and Pareto efficient rule $f$. We show that $f$ is a conditional two-step rule.

By strategy-proofness, the rule belongs to the family of Theorem 1. By Lemma 15, $\Omega_\emptyset = \{\min T, \max T\}$.

*Step 1: We prove that for all $A \subseteq N$ such that $|\Omega_A| = 2$, $\Omega_A = \{\min T, \max T\}$.*

Consider any $A \subseteq N$ such that $|\Omega_A| = 2$. Suppose by contradiction that $l_A \neq \min T$ or $h_A \neq \max T$. Let $l_A \neq \min T$ (the proof when $h_A \neq \max T$ is similar and thus omitted). Since $l_A \in \Omega_A$, there is a profile $R \in \mathcal{R}^A$ such that $f(R) = l_A$. Consider now the profile $R' \in \mathcal{R}^A$ such that for all $i \in N$, $t(R'_i) = t(R_i)$, and $H_A(R') = N_A$.

We show that such a profile $R'$ exists. First, consider any $i \in N_A \cap A$. Then, for all $\bar{R}_i \in \mathcal{R}_i^+$, $t(\bar{R}_i) = t(R_i) = i \notin \Omega_A$. Hence, there is a preference $R'_i \in \mathcal{R}_i^+$ such that $h_A \, P'_i \, l_A$. Second, consider any $j \in N_A \cap (N \setminus A)$. If $t(R_j) = \max T$, then there is a preference $R'_j \in \mathcal{R}_j^-$ with $t(R'_j) = \max T$ such that $\max T \, R'_j \, h_A \, P'_j \, l_A$. If, on the other hand, $t(R_j) = \min T$, then there is a preference $R'_j \in \mathcal{R}_j^-$ with $t(R'_j) = \min T$ such that $\min T \, P'_j \, h_A \, P'_j \, l_A$ (this is because $l_A > \min T$ by assumption). Finally, if $t(R_j) = \{\min T, \max T\}$, then there is a preference $R'_j \in \mathcal{R}_j^-$ with $t(R'_j) = \{\min T, \max T\}$ such that $\min T \, R'_j \, h_A \, P'_j \, l_A$ (this is again because $l_A > \min T$ by assumption). Consequently, there is a profile $R' \in \mathcal{R}^A$ such that for all $i \in N$, $t(R'_i) = t(R_i)$, and $H_A(R') = N_A$.

Next, apply tops-onlyness to obtain that $f(R') = f(R) = l_A$. Then, $\emptyset \in \mathcal{G}^A(R')$ and $I_A(R') = \emptyset$, which contradicts Proposition 3 (exactly the last point of Definition 1).

*Step 2: We prove that if there is an agent $i \in N$ such that $\Omega_{\{i\}} \neq \{\min T, \max T\}$, then $\Omega_{\{i\}} = i$.*

Consider any $i \in N$ such that $\Omega_{\{i\}} \neq \{\min T, \max T\}$. Then, by Step 1, $|\Omega_{\{i\}}| = 1$. Since $N \subset (\min T, \max T)$ and $\Omega_\emptyset = \{\min T, \max T\}$, $i \in N_\emptyset$. Apply Proposition 4 (third point) to $\Omega_\emptyset$ and $\Omega_{\{i\}}$ to see that $\Omega_{\{i\}} = i$. This concludes Step 2.

Let $D = \{i \in N : \Omega_{\{i\}} = i\}$.

*Step 3: We prove that if $A \cap D = \emptyset$, then $\Omega_A = \{\min T, \max T\}$.*

Consider any $A \subseteq N$ such that $A \cap D = \emptyset$. The proof is by induction on the size of $A$. Suppose that $|A| \leq 1$. If $|A| = 0$, we already know that $\Omega_\emptyset = \{\min T, \max T\}$. If $|A| = 1$, then, by the definition of the set $D$, $\Omega_A = \Omega_\emptyset$. So, consider now any $A \subseteq N$ such that $A \cap D = \emptyset$ and $|A| > 1$. Suppose that for all $B \subset A$, $\Omega_B = \{\min T, \max T\}$, but, by contradiction, $\Omega_A \neq \{\min T, \max T\}$. Then, by Step 1, $|\Omega_A| = 1$. Consider any two agents $i, j \in A$. By setting $B = A \setminus \{i\}$ and, alternatively, $B = A \setminus \{j\}$, it follows from the induction hypothesis that $\Omega_{A\setminus\{i\}} = \Omega_{A\setminus\{j\}} = \{\min T, \max T\}$. Thus, $\Omega_{A\setminus\{i\}} \neq \Omega_A \neq \Omega_{A\setminus\{j\}}$. Since $N \subset (\min T, \max T)$, we have that $i, j \in N_\emptyset = N_{A\setminus\{i\}} = N_{A\setminus\{j\}}$. Apply Proposition 4 (third point) on the one hand to $\Omega_{A\setminus\{i\}}$ and $\Omega_A$ and, on the other hand, to $\Omega_{A\setminus\{j\}}$ and $\Omega_A$ in order to see that $\Omega_A = i$ and $\Omega_A = j$. This is a contradiction.

*Step 4: We prove that $D = N$.*

Suppose by contradiction that $D \subset N$. Then, by Step 3, $\Omega_{N\setminus D} = \{\min T, \max T\}$. Let $R, R' \in \mathcal{R}^{N\setminus D}$ be such that for all $i \in D$, $t(R_i) = t(R'_i) = \{\min T, \max T\}$, while for all $j \in N \setminus D$, $\min T\, P_j \max T$ and $\max T\, P'_j \min T$. By tops-onliness, $f(R) = f(R')$. If $f(R) = f(R') = \min T$, this violates Pareto efficiency since $\max T$ Pareto dominates $\min T$ at $R'$. If $f(R) = f(R') = \max T$, this violates Pareto efficiency since $\min T$ Pareto dominates $\max T$ at $R$.

*Step 5: We prove that if $A \neq \emptyset$, then $|\Omega_A| = 1$ and $\Omega_A \in [\min A, \max A]$.*

The proof proceeds by induction on the size of $A$. If $|A| = 1$, the result follows from Step 4 and the definition of the set $D$. Suppose now that $|A| > 1$. The induction hypothesis states that for all $B \subset A$, with $B \neq \emptyset$, $|\Omega_B| = 1$ and $\Omega_B \in [\min B, \max B]$. Consider any $j \in A$. By setting $B = A \setminus \{j\}$, it follows from the induction hypothesis that $|\Omega_{A\setminus\{j\}}| = 1$ and $\Omega_{A\setminus\{j\}} \in [\min(A \setminus \{j\}), \max(A \setminus \{j\})]$. Since $A \setminus \{j\} \subset A$, we have that $\Omega_{A\setminus\{j\}} \in [\min A, \max A]$. Then, applying Proposition 4 (third point) to $\Omega_{A\setminus\{j\}}$ and $\Omega_A$, we obtain that $l_A$ is weakly between $l_{A\setminus\{j\}}$ and $j$. Given that $l_{A\setminus\{j\}} \geq \min A$ and $\min A > \min T$, we have that $l_{A\setminus\{j\}} > \min T$ and, thus, $\Omega_A \neq \{\min T, \max T\}$. Then, by Step 1 we have that $|\Omega_A| = 1$. Since $j, l_{A\setminus\{j\}} \in [\min A, \max A]$, we have that $l_A \in [\min A, \max A]$ and, thus, $\Omega_A \in [\min A, \max A]$.

*Step 6: We prove that the rule is a conditional two-step rule.*

By Step 1, $\Omega_\emptyset = \{\min T, \max T\}$ and, by Step 5, $|\Omega_A| = 1$, where $\Omega_A \in [\min A, \max A]$, whenever $A \neq \emptyset$.

Consider any $A \subseteq N$. First, if $A \neq \emptyset$, consider a function $\omega : 2^N \setminus \emptyset \to (\min T, \max T)$ such that for all $A \subseteq N$ with $A \neq \emptyset$, $\omega(A) = \Omega_A$. By Proposition 4 (first and second point), $\omega$ is monotone. Second, if $A = \emptyset$, then $N_\emptyset = N$. By Proposition 3 (exactly by Definition 1) we have for all $R \in \mathcal{R}^\emptyset$ that $f(R) = \min T$ whenever $L_\emptyset(R) = \{i \in N : t(R_i) = \min T\} \in \mathcal{G}^\emptyset(R)$, and $f(R) = \max T$ otherwise. By tops-onliness, for all $R' \in \mathcal{R}^\emptyset$ such that

$t(R_i) = t(R_i')$ for all $i \in N$, $f(R') = f(R)$. Thus, we can fix $\mathcal{G}^{\emptyset}(R') = \mathcal{G}^{\emptyset}(R)$. That is, the $l_{\emptyset}$–decisive sets depend only on the set of agents indifferent between $l_{\emptyset} = \min T$ and $h_{\emptyset} = \max T$ and not on their preferences. Consequently, the second point of Definition 1 becomes now the following stronger condition: for all $R' \in \mathcal{R}^{\emptyset}$ such that $I_{\emptyset}(R) = I_{\emptyset}(R')$, $\mathcal{G}^{\emptyset}(R) = \mathcal{G}^{\emptyset}(R')$. So, the choice at any profile $R \in \mathcal{R}^{\emptyset}$ between $\min T$ and $\max T$ depends only on $L_{\emptyset}(R)$ and $I_{\emptyset}(R)$. Then, we can define, using the correspondence $\mathcal{G}^{\emptyset}$, a set of pairs of coalitions $\mathcal{G} \subseteq 2^N \times 2^N$ in the following way: $\mathcal{G} = \{(A, B) \in 2^N \times 2^N :$ there exists $R \in \mathcal{R}^{\emptyset}$ such that $B \in \mathcal{G}^{\emptyset}(R)$ and $I_{\emptyset}(R) = A \setminus B\} = \{(A, B) \in 2^N \times 2^N :$ there exists $R \in \mathcal{R}^{\emptyset}$ such that $\min T \in t(R_i)$ for all $i \in A$, $t(R_j) = \min T$ for all $j \in B$, and $f(R) = \min T\}$.

Next, we have to see how the other conditions on $\mathcal{G}^{\emptyset}$ translate to $\mathcal{G}$. The third point of Definition 1 implies that if $(A, B) \in \mathcal{G}$ and $B \subset B'$, then $(A, B') \in \mathcal{G}$. The first part of the fourth point of Definition 1 implies that if $(A, B') \notin \mathcal{G}$ and $B \subset B'$, then $(A \cup (B' \setminus B), B) \notin \mathcal{G}$. The second part of the fourth point of Definition 1 implies that if $(A, B) \in \mathcal{G}$ and $A \subset A'$, then $(A', B) \in \mathcal{G}$. Therefore, the third and the fourth point of Definition 1 imply that $\mathcal{G}$ is monotone. By Pareto efficiency, we have the efficiency condition of $\mathcal{G}$: $(N, B) \in \mathcal{G}$ for all $B \neq \emptyset$ and $(A, \emptyset) \notin \mathcal{G}$ for all $A \neq N$. Finally, the last point of Definition 1 implies that $(\emptyset, \emptyset) \notin \mathcal{G}$ and that there exists $B \subseteq N$ such that $(B, B) \in \mathcal{G}$. These conditions are always satisfied because $\mathcal{G}$ is efficient.

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