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Generalized forms of monotonicity in the data aggregation framework

Mikel Sesma Sara

TESIS DOCTORAL

Pamplona, abril de 2019

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Generalized forms of monotonicity in the data aggregation framework

Memoria presentada por

Mikel Sesma Sara

para optar al grado de Doctor por

la Universidad Pública de Navarra

Pamplona, abril de 2019

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Autorización

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HACEN CONSTAR que el presente trabajo titulado "*Generalized forms of monotonicity in the data aggregation framework*" ha sido realizado bajo su dirección por D. Mikel Sesma Sara.

Autorizándole a presentarlo como Memoria para optar al grado de Doctor por la Universidad Pública de Navarra.

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Pamplona, abril de 2019

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A Esther, a Jon
y a mis padres.

Acknowledgements

Esta memoria concluye una etapa muy importante de mi vida y no habría podido llevar a cabo este proyecto sin la ayuda y el apoyo de muchas personas.

En primer lugar, me gustaría agradecer a la Universidad Pública de Navarra por posibilitar que yo haya hecho esta tesis y, en particular, a mis directores. A Humberto por la acogida en el grupo, por la confianza que ha depositado en mí y por abrirme las puertas al mundo de la investigación. To Radko, thank you for your continuous support, for welcoming me into your university in Bratislava and for your invaluable guidance.

I would also like to express my gratitude to all the people that I met during the three months that I stayed in Slovakia. To Tomáš and Mária Ždímalová for their disposition to help me whenever I needed it. To Mária Minárová for the numerous lunch breaks with me and for taking me to play basketball and tennis. To Adam, Dominika and Pavol for their lessons on slovak food and language. Ďakujem šetkým!

Por supuesto, tengo que agradecer a la gente con la que comparto el día a día en la universidad, a toda la gente que forma el grupo de investigación GIARA. Particularmente, a Laura por las confidencias del despacho y por algún que otro contraejemplo, a Cédric por el ISCAMI 2016, mi primer congreso, a Carlos por los ratos de café en el despacho, a Javi por su criterio matemático, a Edurne por su inestimable ayuda, a Miguel por sus consejos, a Aránzazu por "las risas" y el buen humor, a Dani por la natación en Liptovský Ján y las enseñanzas del refranero, a Josean por los paseos de vuelta del café, a Mikel por los nachos que aún me debe, a Elkano por los OVO de basket y a Uriz por los ataques de risa de Granada. Los lunes no lo son tanto gracias a vosotros.

También me gustaría agradecer a *la kuadrilla*. Porque, aunque los kilómetros y la vida hagan que nos juntemos esporádicamente, hay cosas que nunca cambian. Por ejemplo, la *qualité* en el volley.

Agradezco a mi familia. Especialmente a mis padres, a los que debo todo y cuyo ejemplo me ha traído hasta aquí. Quiero agradecer también a mi hermano Jon, que es un crack, y a mis abuelos, que siempre me han apoyado en la consecución de mi objetivo de ser, en sus palabras, "un hombre ilustre". También agradezco a mi familia de Sangüesa, Begoña, César y Sergio, por muchos paseos por —casi siempre— sendas o caminos y por llevarme a escalar y no dejarme arriba.

Por último, a ti, Esther. No imagino mejor compañera de viaje para muchas otras aventuras como lo ha sido esta.

¡MUCHAS GRACIAS! MILA ESKER!

Contents

Abstract

The process of aggregation addresses the problem of combining a collection of numerical values into a single representative number and the functions that perform this operation are called aggregation functions. Aggregation functions are required to satisfy two boundary conditions and to be monotone with respect to all their arguments.

One of the trends in the research area of aggregation functions is the relaxation of the condition of monotonicity. In that attempt, various relaxed forms of monotonicity have been introduced. This is the case of weak, directional and cone monotonicity.

However, all these relaxed forms of monotonicity are based on the idea of increasing, or decreasing, along a fixed ray defined by a real vector. There is no notion of monotonicity allowing the direction of increasingness to depend on the specific values to aggregate, nor there exists any other notion that considers increasingness along more general paths, such as curves. Additionally, another trend in the theory of aggregation is the extension to handle more general scales than real numbers and there is no relaxation of monotonicity available in that general context.

In this dissertation, we propose a collection of new relaxed forms of monotonicity for which the directions of monotonicity may vary from one point of the domain to another. Specifically, we introduce the concepts of ordered directional monotonicity, strengthened ordered directional monotonicity and pointwise directional monotonicity. Based on the concept of ordered directionally monotone functions, we propose an edge detection algorithm that justifies the applicability of these concepts. Furthermore, we generalize the concept of directional monotonicity so that, instead of directions in $\mathbb{R}^n$, more general paths are considered: we define curve-based monotonicity. Finally, combining both trends in the theory of aggregation, we generalize the concept of directional monotonicity to functions that are defined on more general scales than real numbers.

Structure of the dissertation

This dissertation is divided in two parts:

- Part I: we describe the concept of directional monotonicity as a relaxation of standard monotonicity in the aggregation setting and we motivate and discuss our research findings.

- Part II: we present a collection of published, accepted and submitted works related to this dissertation.

Part I starts with an introductory section (Section 1) regarding aggregation functions, the relaxation of the monotonicity axiom and the notions of weak and directional monotonicity, which have lead to the introduction of the concept of a pre-aggregation function. In Section 2 we motivate and justify the relevance of the subject of study (Section 2.1) and, we set the objectives of the current dissertation (Section 2.2). In Section 3 we discuss briefly our results and, lastly, we expose the conclusions of our findings, along with some directions of future research, in Sections 4 and 5.

Part II is comprised by the following publications:

- H. Bustince, E. Barrenechea, M. Sesma-Sara, J. Lafuente, G.P. Dimuro, R. Mesiar, A. Kolesárová. Ordered directionally monotone functions: Justification and application, *IEEE Transactions on Fuzzy Systems*, 26 (4): 2237-2250, 2018.
DOI: 10.1109/TFUZZ.2017.2769486.

- M. Sesma-Sara, J. Lafuente, A. Roldán, R. Mesiar, H. Bustince. Strengthened ordered directionally monotone functions. Links between the different notions of monotonicity, *Fuzzy Sets and Systems*, 357: 151-172, 2019.
DOI: 10.1016/j.fss.2018.07.007.

- M. Sesma-Sara, L. De Miguel, A. F. Roldán López de Hierro, J. Lafuente, R. Mesiar, H. Bustince. Pointwise directional increasingness and geometric interpretation of directionally monotone functions, *Information Sciences*, Submitted.

- A. F. Roldán López de Hierro, M. Sesma-Sara, J. Špirková, J. Lafuente, A. Pradera, R. Mesiar, H. Bustince. Curve-based monotonicity: a generalization of directional monotonicity, *International Journal of General Systems*.
DOI: 10.1080/03081079.2019.1586684.

- M. Sesma-Sara, R. Mesiar, H. Bustince. Weak and directional monotonicity of functions on Riesz spaces to fuse uncertain data, *Fuzzy Sets and Systems*.
DOI: 10.1016/j.fss.2019.01.019.

- M. Sesma-Sara, L. De Miguel, M. Pagola, A. Burusco, R. Mesiar, H. Bustince. New measures for comparing matrices and their application to image processing, *Applied Mathematical Modelling*, 61: 498-520, 2018.
DOI: 10.1016/j.apm.2018.05.006.

Part I.
Thesis

1 Introduction

At this time, on the *Information Age*, adequate data fusion methods are essential. It is not hard to find several examples of information systems that combine data from numerous sources of information: multi-criteria and group decision making problems, research evaluation metrics, economic indices and indicators, fuzzy connectives, rule based systems, ensemble models and social and political surveys, to name a few. The field of information fusion studies techniques to combine data from multiple sources and aggregation functions, also called aggregation operators, are some of the functions that can be used for combining data.

The theory of aggregation addresses the problem of combining a set of numerical values to find a single representative number. This idea of aggregation is probably as old as counting, however, not until the 1980s did the theory aggregation become an independent research field. Prior to that date, specific classes of aggregation functions were independently studied within the domain of different areas of knowledge. For instance, the theory of means dates back to 1930 [50, 67], triangular norms were introduced in the framework of probabilistic metric spaces [62] and copulas were studied in probability theory and statistics [79]. The cited examples and their duals conform the three main classes in the classification of aggregation functions [37]:

- averaging aggregation functions, whose values lie within the minimum and the maximum of the inputs (e.g., monotone means);
- conjunctive aggregation functions, whose values are bounded by the minimum of the inputs (e.g., triangular norms and copulas);
- disjunctive aggregation functions, whose values are above the maximum of the inputs (e.g., triangular conorms and co-copulas).

Among the class of averaging aggregation functions, integrals with respect to not necessarily additive measures stand out, particularly Choquet [26] and Sugeno integrals [82].

Although the mentioned classes of aggregation functions have continued to be studied independently, as can be derived from the existence of specific monographs [17, 48, 49], the theory of aggregation as a genuine theory has attracted a lot of attention. There exist several monographs on the topic [9, 12, 38, 42, 83], as well as research articles, both from the theoretical [8, 20, 24, 32, 65] and applied points of view [31, 39, 40, 69, 85].

Formally, an aggregation function is a function whose domain is the unit hypercube $[0,1]^n$, for $n \in \mathbb{N}$, and outputs a number in the unit interval, $[0,1]$. The choice of the interval $[0,1]$ is made by convention, being all the developments in the theory of aggregation readily extensible to any arbitrary closed real interval $[a,b]$.

Definition 1.1. A function $A : [0,1]^n \to [0,1]$ is said to be an aggregation function if

(i) $A(0,\dots,0) = 0$;

(ii) $A(1,\dots,1) = 1$;

(iii) A is increasing[1] with respect to all its arguments, i.e., if $\mathbf{x}, \mathbf{y} \in [0,1]^n$ such that $x_i \leq y_i$ for all $i \in \{1,\dots,n\}$, then $A(\mathbf{x}) \leq A(\mathbf{y})$.

The first and second conditions are natural boundary conditions for a function whose objective is that of aggregating n numbers between 0 and 1. The third condition, which regards monotonicity, seems natural as well since, *a priori*, any increase in the inputs should result in the increase of the output. Specifically, for decision making problems, the increase of one criterion should result in the increase, or at least not decrease, of the overall score.

However, in real world problems, data may be perverted by noise or there may be outliers among data. In these situations, the monotonicity condition causes that a great deal of aggregation functions behave poorly as they are influenced by single input values. For example, in image processing, if we were to find a representative value for a region of pixels, an abrupt variation in a certain pixel with respect to its neighbours may be probably a noise corruption and it would affect negatively in the computation of such representative value by means of an aggregation function. A concrete example in which this situation occurs is image reduction [69, 86]. In this setting, non-monotone means are often employed, e.g., the mode, Gini means, Lehmer means, Bajraktarevic means [13, 17] and mixture functions [11, 70, 81]. Robust estimators of location, used in statistical analysis, are generally not monotone either [72].

Nevertheless, although monotonicity with respect to all arguments may be too restrictive for certain applications, our expectation with respect to the behaviour of means requires that some monotonicity-like condition is satisfied, e.g., in the case of robust estimators of location shift-invariance is required.

With the objective of providing a theoretical framework of functions that are valid to fuse data, although in some cases monotonicity is not satisfied, and having in mind that some sort

[1] In this dissertation we convene to use the terms *increasing* and *decreasing* functions to refer to monotone non-decreasing and non-increasing functions, respectively. In the cases where strict monotonicity is required, we use the terms *strictly increasing* and *strictly decreasing*.

of monotonicity is necessary, Wilkin and Beliakov proposed in [87] a relaxed form of monotonicity: weak monotonicity. Weak monotonicity permits the aggregate value to decrease even if one of the inputs increases, but requires that if all the inputs increase by the same amount, then the aggregate value must increase, not decrease, as well.

Definition 1.2. A function $f : [0,1]^n \to [0,1]$ is said to be weakly increasing if for all $\mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n$ and $c > 0$ such that $(x_1 + c, \ldots, x_n + c) \in [0,1]^n$, it holds that $f(x_1, \ldots, x_n) \leq f(x_1 + c, \ldots, x_n + c)$. Similarly, f is said to be weakly decreasing if for all $\mathbf{x} \in [0,1]^n$ and $c > 0$ such that $(x_1 + c, \ldots, x_n + c) \in [0,1]^n$, it holds that $f(x_1, \ldots, x_n) \geq f(x_1 + c, \ldots, x_n + c)$. If a function f is both weakly increasing and weakly decreasing, then f is said to be weakly monotone constant.

Remark 1.3. Shift-invariance implies weak monotonicity.

Note that weak monotonicity can be seen as monotonicity along the ray $(1, \ldots, 1)$. This consideration led to the generalization of weak monotonicity to directional monotonicity in [21], a relaxation of monotonicity that considers increasingness or decreasingness of functions along general real directions $\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\}$.

Definition 1.4. Let $\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\}$. A function $f : [0,1]^n \to [0,1]$ is said to be $\vec{r}$-increasing, if for all $c > 0$ and $\mathbf{x} \in [0,1]^n$ such that $\mathbf{x} + c\vec{r} \in [0,1]^n$, it holds that $f(\mathbf{x}) \leq f(\mathbf{x} + c\vec{r})$. Similarly, f is said to be $\vec{r}$-decreasing, if for all $c > 0$ and $\mathbf{x} \in [0,1]^n$ such that $\mathbf{x} + c\vec{r} \in [0,1]^n$, it holds that $f(\mathbf{x}) \geq f(\mathbf{x} + c\vec{r})$. If f is both $\vec{r}$-increasing and $\vec{r}$-decreasing, then f is said to be $\vec{r}$-constant.

Monotonicity of derivable functions is linked to the sign of partial derivatives. Similarly, directional monotonicity is linked to directional derivatives, a concept that has been largely studied and applied [76, 77, 92]. However, the concept of directional monotonicity had been only scarcely studied in rather specific settings, e.g., [46].

The introduction of directional monotonicity has broaden the framework of data fusion functions, allowing other functions, such as implication operators [4, 5], to be also considered as valid functions to fuse information.

Subsequently, the notions of weak and directional monotonicity have been studied and, parallel to the development of this dissertation, new relaxed forms of monotonicity have arisen. A relevant one is cone-monotonicity, which was defined in [10] with respect to positive cones $C \subset (\mathbb{R}_+)^n$. We present the definition for any cone $C \subset \mathbb{R}^n$.

Definition 1.5. Let $C \subset \mathbb{R}^n$ be a nonempty cone. A function $f : [0,1]^n \to [0,1]$ is said to be cone increasing with respect to C if f is $\vec{r}$-increasing for any direction $\vec{r} \in C$. Similarly, f is said to be cone decreasing with respect to C if it is $\vec{r}$-decreasing for any direction $\vec{r} \in C$.

In the same manner, several of the aforementioned families of means have been studied from the perspectives of weak and directional monotonicity, giving rise to many works [10, 11, 13, 21, 81, 87].

Furthermore, the aim of relaxing the monotonicity condition of aggregation functions to create a framework in which a greater deal of functions is contemplated has been explicitly generated with the introduction of the so-called pre-aggregation functions [58].

Definition 1.6. A function $A : [0,1]^n \to [0,1]$ is said to be a pre-aggregation function if

(i) $A(0, \dots, 0) = 0$;

(ii) $A(1, \dots, 1) = 1$;

(iii) A is $\vec{r}$-increasing for some $\vec{r} \in [0,1]^n$.

It is worth to point out that the direction of increasingness of a function must be positive so that it can be consider a pre-aggregation function. This fact concurs with the necessity of maintaining some sort of monotonicity.

Pre-aggregation functions have been employed in numerous applications and, particularly, the case of fuzzy rule-based classification problems stands out [36, 56, 57, 58].

2 Motivation and objectives

2.1 Motivation

The introduced relaxed forms of monotonicity are based on monotonicity of a function f along a direction, or a set of directions, that is common to all the points in the domain of f. For weak monotonicity the direction is given by $(1, \ldots, 1)$, for directional monotonicity by a general vector $\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\}$ and for cone-monotonicity for all the vectors $\vec{r}$ that form a nonempty cone $C \subset \mathbb{R}^n$.

However, on the one hand, there is no relaxed form of monotonicity that permits the direction of monotonicity to vary from one point of the domain to another. The interest of this sort of property is readily justified, as it allows to model certain relations among data that directionally monotone functions are not able to model. For example, in some cases, it might be desirable that an increase of some already high inputs does not yield an increase in the aggregate value, but that increases of low inputs result in high increases of the aggregate value. Hence, there is a necessity to introduce a notion of monotonicity that takes into account the specific values to aggregate in order to set the direction of monotonicity.

An example of an application field in which the mentioned developments are potentially useful is computer vision. Specifically, edge detection. The task of detecting edges in an image is not explicitly defined since the concept of edge itself is not clearly stated. We understand that an edge is a *big enough jump between the intensity of a pixel and the intensity of its neighbours*. Some edge dectection algorithms are based on intensity changes between a pixel and its neighbours with no additional information [6, 19, 41], whereas other edge detectors include the information provided by the directions in which those intensities change for each pixel [25, 55]. Thus, in the process of edge detection, the use of a function that fuses information and increases or decreases along a certain direction depending on the values to fuse might be useful.

Furthermore, there is no work in the literature regarding the geometrical aspects of directionally monotone functions. Studying the classes of functions that satisfy different relaxations of monotonicity helps furthering the theoretical understanding of such concepts. Besides, looking at these properties from a different perspective might help arising ideas of their applicability.

On the other hand, all the proposed forms of monotonicity are based on the idea of monotonicity along a direction defined by a ray in $\mathbb{R}^n$. Regardless the fact that the direction might change from certain points to others, the idea of monotonicity along a linear direction is fixed.

This is a limitation in order to model more complex relations among data. The introduction of a relaxed form of monotonicity that is not based on rays in $\mathbb{R}^n$ is due. The consideration of more general paths, such as curves $\alpha : [0,1] \to \mathbb{R}^n$, could lead to a more general relaxed form of monotonicity.

Finally, in various works regarding the state-of-the-art of the theory aggregation [38, 63] two major trends are cited.

- The relaxation of the monotonicity constraints defining new forms of monotonicity.
- The extension of the concept of aggregation functions to more general scales than real numbers.

The first item has already been discussed. With respect to the second item, there exist works dealing with aggregation functions defined on posets [33, 51], functions intended to aggregate information from graphs [84], to aggregate infinite sequences [64, 71] and intervals [22, 29], among others. One of the reasons of the existence of works regarding different scales is that in real applications there is a degree of uncertainty in the data (missing inputs, measurement errors, etc.). As a consequence, many general forms of aggregation functions have been proposed in order to fuse uncertain data coming from different extensions of fuzzy sets, e.g., type-2 fuzzy sets [43, 54], n-dimensional fuzzy sets [31], Atanassov intuitionistic fuzzy sets [28] and interval-valued fuzzy sets [15, 19, 22], among others.

Consequently, it would be interesting to join both trends in the theory of aggregation and define a relaxed form of monotonicity for functions that are able to handle the uncertainty resulting from different extensions of fuzzy sets.

2.2 Objectives

The main objective of this dissertation is:

To develop new relaxed forms of monotonicity, admitting different directions of increasingness and more general paths than rays, and to extend these concepts to more general frameworks than real functions.

For its achievement, we set the following particular objectives:

- To introduce a relaxed form of monotonicity that, resembling OWA operators, is dependent of the relative size of the inputs.
- To study the behaviour of such notions of monotonicity in the task of edge detection.

- To define a local notion of relaxed monotonicity.
- To study the introduced concepts from a geometrical perspective.
- To generalize directional monotonicity by considering general curves in $\mathbb{R}^n$ instead of rays.
- To generalize directional monotonicity for functions that handle various types of uncertain data coming from different extensions of fuzzy sets, particularly interval-valued functions.

3 Discussion of research findings

3.1 Ordered Directionally Monotone Functions: Justification and Application

In this work we introduce the concept of an ordered directionally monotone function, or OD monotone function. These functions satisfy a directional monotonicity condition with the particularity that the direction is different from one point of the domain to another.

Specifically, the direction of monotonicity depends on the relative size of the inputs. Inspired by the ideas in [88], to define the concept of OD monotonicity of a function $f : [0,1]^n \to [0,1]$, we need to consider a decreasing permutation of the inputs. We denote by $\mathcal{S}_n$ the set of all permutations of n elements.

Definition 3.1. Let $\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\}$. A function $f : [0,1]^n \to [0,1]$ is said to be ordered directionally (OD) $\vec{r}$-increasing if for all $c > 0$, all permutation $\sigma \in \mathcal{S}_n$ and $\mathbf{x} \in [0,1]^n$ with $x_{\sigma(1)} \geq \ldots \geq x_{\sigma(n)}$ such that

$$1 \geq x_{\sigma(1)} + cr_1 \geq \ldots \geq x_{\sigma(n)} + cr_n \geq 0, \tag{1}$$

it holds that

$$f(\mathbf{x}) \leq f(\mathbf{x} + c\vec{r}_{\sigma^{-1}}), \tag{2}$$

where σ^{-1} is the inverse permutation of σ. Similarly, f is said to be OD $\vec{r}$-decreasing if for all $c > 0$, $\sigma \in \mathcal{S}_n$ and $\mathbf{x} \in [0,1]^n$ with $x_{\sigma(1)} \geq \ldots \geq x_{\sigma(n)}$ such that

$$1 \geq x_{\sigma(1)} + cr_1 \geq \ldots \geq x_{\sigma(n)} + cr_n \geq 0,$$

it holds that $f(\mathbf{x}) \geq f(\mathbf{x} + c\vec{r}_{\sigma^{-1}})$. If f is both OD $\vec{r}$-increasing and OD $\vec{r}$-decreasing, f is said to be OD $\vec{r}$-constant.

The main theoretical results of this work are the characterization of OD monotonicity for dual functions, the characterization of standard monotonicity in terms of OD monotonicity and the construction method of OD monotone functions by means of Choquet integrals. These results correspond to Corollay 3.8, Proposition 3.12 and Theorem 4.7 of our work, respectively.

Additionally, in this work we present an edge detection algorithm that is based on OD monotone functions. Our method follows the edge detection scheme proposed by Bezdek et al. [14], which is composed by four phases: conditioning, feature extracting, blending and scaling.

(ED1) Smoothening the original image with a Gaussian filter;

(ED2) Obtaining the feature image using ordered directionally monotone functions;

(ED3) Thinning the feature image using the Non-Maximum Suppression (NMS) procedure;

(ED4) Binarizing the feature image applying hysteresis to obtain the black and white edge image.

Our proposal focuses on step (ED2); we establish a methodology to obtain a feature image based on OD monotone functions. The construction of the feature image results from filtering the image by means of an OD $\vec{r}$-increasing function, where the direction $\vec{r}$ varies from pixel to pixel depending on local variations of intensity.

With respect to the remaining steps, for (ED1), we use a Gaussian filter with deviation $\sigma = 1$. For (ED3), Kovesis' implementation of the NMS algorithm is used [52] and, for (ED4), we obtain a binary image with the hysteresis algorithm proposed in [59].

Finally, we test our algorithm using images from BSDS500 dataset [1] and comparing our results with the edge detection methodologies proposed in [25], [41], [55] and [80].

After the completion of this work, we arrive at the following conclusions:

- The notion of OD monotonicity allows to define a family of functions that satisfy a directional monotonicity condition where the direction varies depending on the relative size of the inputs.
- It is possible to characterize standard monotonicity of a function $f : [0,1]^n \to [0,1]$ by OD monotonicity along n appropriate directions.
- The OD monotone functions used in the proposed edge detection algorithm are capable of fusing information taking into account the most influential directions for each point.
- The use of OD monotone functions enable to construct a simple edge detection algorithm, which does not overcome, but yields competitive results with respect to those by Canny's method [25].

This section of the thesis is associated with the following publication:

- H. Bustince, E. Barrenechea, M. Sesma-Sara, J. Lafuente, G.P. Dimuro, R. Mesiar, A. Kolesárová. Ordered directionally monotone functions: Justification and application, *IEEE Transactions on Fuzzy Systems*, 26 (4): 2237-2250, 2018.
  DOI: 10.1109/TFUZZ.2017.2769486.

3.2 Strengthened ordered directionally monotone functions. Links between the different notions of monotonicity

Based on the concept of ordered directional (OD) monotonicity and the results presented in Section 3.1, in this work we propose a new notion of monotonicity: strengthened ordered directional (SOD) monotonicity.

The concept of SOD monotonicity is intimately related to OD monotonicity, with the sole difference that condition (1) is not required for a function to verify inequality (2).

Definition 3.2. Let $\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\}$. A function $f : [0,1]^n \to [0,1]$ is said to be strengthened ordered directionally (SOD) $\vec{r}$-increasing if for all $c > 0$, all permutation $\sigma \in \mathcal{S}_n$ and $\mathbf{x} \in [0,1]^n$ such that $x_{\sigma(1)} \geq \ldots \geq x_{\sigma(n)}$ and $\mathbf{x} + c\vec{r}_{\sigma^{-1}} \in [0,1]^n$, it holds that $f(\mathbf{x}) \leq f(\mathbf{x} + c\vec{r}_{\sigma^{-1}})$, where σ^{-1} is the inverse permutation of σ. Similarly, f is said to be SOD $\vec{r}$-decreasing if for all $c > 0$, $\sigma \in \mathcal{S}_n$ and $\mathbf{x} \in [0,1]^n$ such that $x_{\sigma(1)} \geq \ldots \geq x_{\sigma(n)}$ and $\mathbf{x} + c\vec{r}_{\sigma^{-1}} \in [0,1]^n$, it holds that $f(\mathbf{x}) \geq f(\mathbf{x} + c\vec{r}_{\sigma^{-1}})$. If f is both SOD $\vec{r}$-increasing and SOD $\vec{r}$-decreasing, f is said to be SOD $\vec{r}$-constant.

From the definition, it follows that, given a vector $\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\}$, if a function f is SOD $\vec{r}$-increasing, then f is OD $\vec{r}$-increasing. However, the converse statement does not hold.

In our work, we develop a profound theoretical study about the main properties of SOD monotone functions, as well as about how this concept relates to the previously introduced relaxations of monotonicity, i.e., weak, directional and ordered directional monotonicity. Among the main results, Proposition 3.3 asserts that for specific vectors $\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\}$, namely, the ones such that $r_1 \geq \ldots \geq r_n$, the notions of OD and SOD monotonicity are equivalent and Proposition 3.7 characterizes directional, OD and SOD monotonicity for the dual f^d of a function f. On the other hand, Proposition 3.4 shows that it is equivalent for a function to be $\vec{r}$-increasing and $(-\vec{r})$-decreasing and, similarly, that it is equivalent to be OD $\vec{r}$-increasing and OD $(-\vec{r})$-decreasing. However, this is not so for the case of SOD monotonicity, as shown in Remark 5.10.

Additionally, in this work we define two families of functions:

- Linear fusion functions: Given $\mu \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^n \setminus \{\vec{0}\}$, we say that $(\mu, \vec{v})$ generates the linear fusion function $L[\mu, \vec{v}]$, given by

$$L[\mu, \vec{v}]\,(\mathbf{x}) = \mu + \mathbf{x} \cdot \vec{v} = \mu + \sum_{i=1}^{n} x_i v_i,$$

  if $\mu + \mathbf{x} \cdot \vec{v} \in [0,1]$ for all $\mathbf{x} \in [0,1]^n$.

- Ordered linear fusion functions: Given $\mu \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^n \setminus \{\vec{0}\}$, we say that $(\mu, \vec{v})$ generates the ordered linear fusion function $OL[\mu, \vec{v}]$, given by

$$OL[\mu, \vec{v}]\,(\mathbf{x}) = \mu + \mathbf{x}_\sigma \cdot \vec{v} = \mu + \sum_{i=1}^{n} x_{\sigma(i)} v_i,$$

if $\mu + \mathbf{x}_\sigma \cdot \vec{v} \in [0,1]$ for all $\mathbf{x} \in [0,1]^n$ and all $\sigma \in \mathcal{S}_n$ such that $x_{\sigma(1)} \geq \ldots \geq x_{\sigma(n)}$.

In Proposition 4.6 we characterize all the directions $\vec{r}$ for which a linear fusion function is $\vec{r}$-increasing, OD $\vec{r}$-increasing and SOD $\vec{r}$-increasing. Similarly, in Proposition 4.15 we characterize the direction $\vec{r}$ for which an ordered linear fusion function is OD $\vec{r}$-increasing. The characterization of directions with respect to directional and SOD monotonicity is achieved for the 2-dimensional case in Proposition 4.17.

Then, we introduce an operation $*$ between functions f_1 and f_2 from $[0,1]^2$ to $[0,1]$ such that $f_1(x,x) = f_2(x,x)$ for all $x \in [0,1]$,

$$(f_1 * f_2)(x,y) = \begin{cases} f_1(x,y) \text{ if } x \geq y\,, \\ f_2(x,y) \text{ if } x \leq y\,. \end{cases}$$

As a consequence of Proposition 5.4, when applied to ordered linear fusion functions ($n = 2$), this operation generalizes the Łukasiewicz implication (Example 5.5) and the Choquet integral (Example 5.9).

We end this work showing a characterization of standard and weak monotonicity in terms of directional, OD and SOD monotonicity with respect to an appropriate set of a finitely many directions (Propositions 6.1 and 6.3). Consequently, directional, OD and SOD monotonicity are valid tools to check whether a function is monotone in the regular sense. Additionally, in the final part of this work, we also present an aggregation-based construction method for OD and SOD functions.

After the completion of this work, we arrive at the following conclusions:

- SOD monotonicity resembles OD monotonicity in the sense that it allows to define a family of functions that satisfy a directional monotonicity condition where the direction varies depending on the relative size of the inputs, but it is more restrictive as every SOD monotone function is OD monotone but not vice-versa.
- The directions along which linear and ordered linear fusion functions increase, in the directional, OD and SOD sense, are characterized and these two families of functions make it possible to generalize well-known functions such as the Łukasiewicz implication and the Choquet integral.

- It is possible to characterize standard and weak monotonicity of a function $f : [0,1]^n \to [0,1]$ by directional, OD and SOD monotonicity along n appropriate directions.

This section of the thesis is associated with the following publication:

- M. Sesma-Sara, J. Lafuente, A. Roldán, R. Mesiar, H. Bustince. Strengthened ordered directionally monotone functions. Links between the different notions of monotonicity, *Fuzzy Sets and Systems*, 357: 151-172, 2019.
  DOI: 10.1016/j.fss.2018.07.007.

3.3 Pointwise directional increasingness and geometric interpretation of directionally monotone functions

The goal of relaxing the monotonicity condition of aggregation functions has led to the introduction of a variety of relaxations of monotonicity, all of which are global properties, i.e., demanded to the whole domain of functions. In this work we aim at studying the similarities and particularities of the main relaxations of monotonicity from a geometrical point of view. Specifically, we study some geometric aspects of weak, directional, cone, OD and SOD monotonicity. For this purpose, we introduce the concept of pointwise directional increasingness (or directional increasingness at a point), a local condition that focuses on the monotonicity of a function $f : [0,1]^n \to [0,1]$ at a point $\mathbf{x} \in [0,1]^n$ along a direction $\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\}$.

Definition 3.3. Let $\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\}$. A function $f : [0,1]^n \to [0,1]$ is said to be $\vec{r}$-increasing at a point $\mathbf{x} \in [0,1]^n$ if for any $c > 0$ such that $\mathbf{x} + c\vec{r} \in [0,1]^n$, it holds that $f(\mathbf{x}) \leq f(\mathbf{x} + c\vec{r})$. Similarly, f is said to be $\vec{r}$-decreasing at a point $\mathbf{x} \in [0,1]^n$ if for any $c > 0$ such that $\mathbf{x} + c\vec{r} \in [0,1]^n$, it holds that $f(\mathbf{x}) \geq f(\mathbf{x} + c\vec{r})$. If f is both $\vec{r}$-increasing and $\vec{r}$-decreasing at $\mathbf{x}$, then f is said to be $\vec{r}$-constant at $\mathbf{x}$.

This is a local property that is studied at an specific point $\mathbf{x} \in [0,1]^n$ and at the points $\mathbf{x} + c\vec{r}$. In consequence, there exist properties that directionally monotone functions satisfy but not pointwise directionally monotone functions. For instance, if a function $f : [0,1]^n \to [0,1]$ is $\vec{r}$-increasing at $\mathbf{x} \in [0,1]^n$ for some $\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\}$, then f is not necessarily $(-\vec{r})$-decreasing at $\mathbf{x}$.

Pointwise directional monotonicity enables a better understanding of the effect of each of the relaxations of monotonicity in different points of the domain. In this work, we characterize each of the cited notions of monotonicity in terms of pointwise directional monotonicity.

Particularly, Remark 3.3 shows that a function $f : [0,1]^n \to [0,1]$ is weakly increasing if and only if it is $\vec{1}$-increasing at $\mathbf{x}$, for all $\mathbf{x} \in [0,1]^n$. Similarly, given $\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\}$, Remark 3.3 shows that f is $\vec{r}$-increasing if and only if it is $\vec{r}$-increasing at $\mathbf{x}$ for all $\mathbf{x} \in [0,1]^n$. The case of cone monotonicity with respect to a cone $C \subset \mathbb{R}^n$ is equivalent to that of directional monotonicity considering all directions $\vec{r} \in C$.

The characterization of SOD monotonicity and OD monotonicity in terms of pointwise directional monotonicity is shown in Theorems 3.6 and 3.9, respectively. In order to state these theorems, it is necessary to define a subset of $[0,1]^n$ that is dependent of a permutation $\sigma \in \mathcal{S}_n$: $\Omega_\sigma \subset [0,1]^n$, as the set that contains all the elements of the hypercube which are decreasingly ordered through the permutation σ, i.e.,

$$\Omega_\sigma = \{\mathbf{x} \in [0,1]^n \mid x_{\sigma(1)} \geq x_{\sigma(2)} \geq \ldots \geq x_{\sigma(n)}\}.$$

Theorem 3.4 (Theorem 3.6 of our work)**.** *Let $\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\}$. A function $f : [0,1]^n \to [0,1]$ is SOD $\vec{r}$-increasing if and only if it is $\vec{r}_{\sigma^{-1}}$-increasing at $\mathbf{x}$, for all $\mathbf{x} \in \Omega_\sigma$ and for all $\sigma \in \mathcal{S}_n$.*

Theorem 3.5 (Theorem 3.9 of our work)**.** *Let $\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\}$. A function $f : [0,1]^n \to [0,1]$ is OD $\vec{r}$-increasing if and only if it is $\vec{r}_{\sigma^{-1}}$-increasing at $\mathbf{x}$ within[2] the region Ω_σ, for all $\mathbf{x} \in \Omega_\sigma$ and for all $\sigma \in \mathcal{S}_n$.*

Thus, for a function $f : [0,1]^n \to [0,1]$ and a point $\mathbf{x} \in [0,1]^n$, we define the set of all directions $\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\}$ for which f is pointwise $\vec{r}$-increasing at $\mathbf{x}$:

$$D_f(\mathbf{x}) = \left\{\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\} \mid f \text{ is } \vec{r}\text{-increasing at } \mathbf{x}\right\}.$$

Moreover, we define $D_f(H)$ as the set of all directions $\vec{r}$ for which a function f is $\vec{r}$-increasing at $\mathbf{x}$, for all $\mathbf{x} \in H$. Then, it holds that

$$D_f(H) = \bigcap_{\mathbf{x} \in H} D_f(\mathbf{x}).$$

Similarly, we can define the set D_f of directions $\vec{r}$ for which a function f is $\vec{r}$-increasing, and, in the same manner, we can define the sets OD_f and SOD_f:

$$\begin{aligned} D_f &= \left\{\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\} \mid f \text{ is } \vec{r}\text{-increasing}\right\}, \\ OD_f &= \left\{\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\} \mid f \text{ is OD } \vec{r}\text{-increasing}\right\}, \\ SOD_f &= \left\{\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\} \mid f \text{ is SOD } \vec{r}\text{-increasing}\right\}. \end{aligned}$$

According to the characterizations of each notion of monotonicity in terms of pointwise directional monotonicity, we obtain the following equalities:

$$D_f = \bigcap_{\mathbf{x} \in [0,1]^n} D_f(\mathbf{x});$$

$$SOD_f = \left\{\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\} \mid \vec{r}_{\sigma^{-1}} \in D_f(\Omega_\sigma)\ \forall \sigma \in \mathcal{S}_n\right\};$$

$$OD_f = \left\{\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\} \mid \vec{r}_{\sigma^{-1}} \in D_{f|_{\Omega_\sigma}}(\Omega_\sigma)\ \forall \sigma \in \mathcal{S}_n\right\}.$$

[2] Pointwise directional increasingness of a function $f : [0,1]^n \to [0,1]$ within a region Ω coincides with pointwise directional increasingness of the function $f|_\Omega : \Omega \to [0,1]$.

Based on these considerations, we remark some geometric aspects about each relaxation of monotonicity. We focus on some special points of the domain. On the one hand, the points for which a function trivially satisfies each monotonicity condition can be seen in Table 1. On the other hand, it is worth highlighting Proposition 4.6 and Theorem 4.8, which deal with pointwise directional increasingness of an SOD monotone function $f : [0,1]^n \to [0,1]$ at points $\mathbf{x} \in \bigcap_{j=1}^{k} \Omega_{\sigma_j}$, for $\Omega_{\sigma_1}, \ldots, \Omega_{\sigma_k} \subset [0,1]^n$.

After the completion of this work, we arrive at the following conclusions:

- Pointwise directional monotonicity is a local condition that allows to characterize the global notions of weak, directional, cone, OD and SOD monotonicity.
- Each monotonicity condition has a distinctive impact in different regions of the domain. In particular, there exist points that trivially satisfy the different conditions of monotonicity.
- The behaviour of a function $f : [0,1]^n \to [0,1]$ at points of equal components, i.e., $(x, \ldots, x) \in [0,1]^n$, is relevant in terms of SOD monotonicity.

This section of the thesis is associated with the following publication:

- M. Sesma-Sara, L. De Miguel, A. F. Roldán López de Hierro, J. Lafuente, R. Mesiar, H. Bustince. Pointwise directional increasingness and geometric interpretation of directionally monotone functions, *Information Sciences*, Submitted.

3.4 Curve-based monotonicity: a generalization of directional monotonicity

In this work we propose a new relaxed form of monotonicity that generalizes directional monotonicity: curve-based monotonicity. This notion is able to model monotonicity relations among the inputs of a function $f : [0,1]^n \to [0,1]$ that might be more complex than those considered by directional monotonicity. Instead of studying monotonicity along rays in $\mathbb{R}^n$, curve-based monotonicity examines the value of a function along a general path in $\mathbb{R}^n$ that is given by a curve α.

A curve on $\mathbb{R}^n$ is a function $\alpha : I_\alpha \to \mathbb{R}^n$, where the domain I_α is a nonempty interval in $\mathbb{R}$. In this work, we assume that I_α is positive, i.e., $I_\alpha \subset \mathbb{R}_+$, and that $0 \in I_\alpha$. Without loss of generality, we set $\alpha(0) = \mathbf{0}$. The nature of interval I_α, whether it is bounded/unbounded or open/closed, is unimportant for most of the properties studied in this work. In fact, in Section 5.1 we demonstrate that the character of the domain I_α of the curves that we study in this work can be reduced to two cases: curves defined on $[0,1]$ and curves defined on $[0,1)$.

Thus, for a given curve $\alpha : I_\alpha \to \mathbb{R}^n$, we define the concept of curve-based monotonicity.

Definition 3.6. Let $f : [0,1]^n \to [0,1]$ and let $\alpha : I_\alpha \to \mathbb{R}^n$ be a curve such that $\alpha(0) = \mathbf{0}$. We say that f is α-increasing if

$$f(\mathbf{x} + \alpha(t)) \geq f(\mathbf{x}) \qquad \text{for all } \mathbf{x} \in [0,1]^n \text{ and all } t \in I_\alpha \setminus \{0\}$$
$$\qquad\qquad\qquad\qquad \text{such that } \mathbf{x} + \alpha(s) \in [0,1]^n \text{ for all } 0 < s \leq t.$$

Similarly, f is α-decreasing if

$$f(\mathbf{x} + \alpha(t)) \leq f(\mathbf{x})$$

for all $\mathbf{x} \in [0,1]^n$ and all $t \in I_\alpha \setminus \{0\}$ such that $\mathbf{x} + \alpha(s) \in [0,1]^n$ for all $0 < s \leq t$.

If f is both α-increasing and α-decreasing, then f is said to be α-constant.

Straight lines (or segments of straight lines) are a particular instance of curve. Hence, curve-based monotonicity is a generalization of directional monotonicity.

Note that for a function f to be α-increasing, once the curve leaves the unit hypercube $[0,1]^n$, it has no influence in the property of α-monotonicity of f. Indeed, the condition that must hold is $f(\mathbf{x}) \leq f(\mathbf{x} + \alpha(t))$ provided that all the points $\mathbf{x} + \alpha(s) \in [0,1]^n$ for all $0 < s \leq t$. Therefore, the points $\mathbf{x} + \alpha(t) \notin [0,1]^n$, even in the case that the curve eventually returns to take values within $[0,1]^n$, do not influence the condition of α-monotonicity. This is shown in Figure **1**.

Furthermore, if a function $f : [0,1]^n \to [0,1]$ is α-increasing for a certain curve α, it does not necessarily mean that f increases along the graph of such curve. Instead, α-increasingness

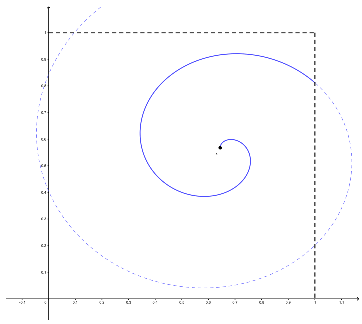

Figure 1. Example of points $\mathbf{x} + \alpha(t)$ that have influence on the α-monotonicity of functions (in solid blue) and points that do not (dashed blue).

of f refers to the property of non-decreasingness in the values of f when evaluating the points across which, starting from a fixed $\mathbf{x} \in [0,1]^n$, the curve α goes. This fact is explicitly discussed in Section 3.3.

Standard monotonicity of a function $f : [0,1]^n \to [0,1]$ can be characterized in terms of curve-based monotonicity for appropriate curves α (Lemma 4.2) and so happens with weak monotonicity (Lemma 4.3). Another relevant feature of this notion is that curve based monotonicity of a function $f : [0,1]^n \to [0,1]$ is entirely determined by a local property of f (Theorem 4.9).

We also study the conditions of curve-based monotonicity of functions with respect to the composition of two, or more, curves. By composition of two curves, we refer to the curve whose graph goes through the first curve and, then, through the second. Formally, given two curves $\alpha, \beta : [0,1] \to \mathbb{R}^n$, we define their composition $\alpha \star \beta : [0,1] \to \mathbb{R}^n$ by

$$\alpha \star \beta(t) = \begin{cases} \alpha(2t), & \text{if } 0 \le t \le 0.5, \\ \alpha(1) + \beta(2t-1), & \text{if } 0.5 < t \le 1. \end{cases}$$

If a function $f : [0,1]^n \to [0,1]$ is α-increasing and β-increasing, then f is also $(\alpha \star \beta)$-increasing (Lemma 5.6).

Finally, in Section 6, we study idempotent and averaging behaviour of curve-based monotone functions.

After the completion of this work, we arrive at the following conclusions:

- The notion of curve-based monotonicity is a relaxation of monotonicity that generalizes directional monotonicity for functions $f : [0,1]^n \to [0,1]$.

- Standard monotonicity can be characterized in terms of curve based monotonicity. However, this characterization is not as practical as the one based on directional monotonicity.
- The domain of the curves can be reduced to two cases: $[0,1]$ and $[0,1)$.
- The conditions of idempotency and averaging behaviour of a function $f : [0,1]^n \to [0,1]$ are characterized in terms of conditions related to α-monotonicity of f.

This section of the thesis is associated with the following publication:

- A. F. Roldán López de Hierro, M. Sesma-Sara, J. Špirková, J. Lafuente, A. Pradera, R. Mesiar, H. Bustince. Curve-based monotonicity: a generalization of directional monotonicity, *International Journal of General Systems*.
  DOI: 10.1080/03081079.2019.1586684.

3.5 Weak and directional monotonicity of functions on Riesz spaces to fuse uncertain data

Following both trends in the theory aggregation, the one towards the relaxation of the monotonicity condition and the one towards the extension to other domains besides real numbers, in this work we introduce the concept of directional monotonicity for functions that take values in Riesz spaces, also known as vector lattices [90].

A vector space V endowed with a partial order relation $\leq_V$ is said to be a partially ordered vector space if the order structure and the vector space structure are compatible, i.e., if the following conditions hold for any $u, v \in V$:

- If $u \leq_V v$, then $u + w \leq_V v + w$ for every $w \in V$;
- If $u \leq_V v$, then $\alpha u \leq_V \alpha v$ for every real $\alpha \geq 0$.

If, additionally, $(V, \leq_V)$ forms a lattice, then V is said to be a Riesz space.

The Cartesian product $V^n = V \times \ldots \times V$ is a Riesz space with respect to the product order $\leq_{V^n}$, which results from considering $\leq_V$ component-wise.

We define directional monotonicity of functions $F : V^n \to V$, understanding that the directions are non-zero vectors from V^n.

Definition 3.7. Let $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$ and let $\mathbf{v} = (v_1, \ldots, v_n) \in V^n$ such that $v_i \neq \vec{0}$ for some $1 \leq i \leq n$. We say that a function $F : V^n \to V$ is $\mathbf{v}$-increasing if for all $\mathbf{x} \in V^n$ and $c > 0$, it holds that $F(\mathbf{x} + c\mathbf{v}) \geq_V F(\mathbf{x})$. Similarly, we say that F is $\mathbf{v}$-decreasing if for all $\mathbf{x} \in V^n$ and $c > 0$, it holds that $F(\mathbf{x} + c\mathbf{v}) \leq_V F(\mathbf{x})$. If F is both $\mathbf{v}$-increasing and $\mathbf{v}$-decreasing, we say that F is $\mathbf{v}$-constant.

We also extend the concept of weak monotonicity to w-weak monotonicity, focusing on a fixed $\vec{0} \neq w \in V$.

Definition 3.8. Let $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$ and let $\vec{0} \neq w \in V$. We say that a function $F : V^n \to V$ is w-weakly increasing if for all $\mathbf{x} \in V^n$ and $c > 0$, it holds that $F(\mathbf{x} + c(w, \ldots, w)) \geq_V F(\mathbf{x})$. Similarly, we say that F is w-weakly decreasing if for all $\mathbf{x} \in V^n$ and $c > 0$, it holds that $F(\mathbf{x} + c(w, \ldots, w)) \leq_V F(\mathbf{x})$.

In our work, we show that functions that satisfy these broader notions of monotonicity meet the most relevant properties of standard directional monotonicity. Among them, the main properties are that it is equivalent for a function F to be $\mathbf{v}$-increasing and $(-\mathbf{v})$-decreasing (Proposition 4.6), the set of directions $\mathbf{v}$ for which a function F is $\mathbf{v}$-increasing is closed under convex combination (Theorem 4.7) and that standard monotonicity for functions $F : V^n \to V$ can be characterized in terms of directional monotonicity along appropriate directions (Theorem 4.10 and Corollary 4.11).

The structure of a Riesz space, and the adaptation of Definition 3.7 to certain convex sublattices, permits to introduce a framework to define directional monotonicity for functions that handle various types of uncertain data coming from different extensions of fuzzy sets. Specifically, we provide a definition of directional monotonicity that is extensible for functions that fuse type-2 fuzzy values [91], fuzzy multiset values [89], n-dimensional fuzzy values [7], Atanassov intuitionistic fuzzy values [2] and interval-valued fuzzy values [91].

Particularly, we focus on the interval-valued (IV) setting, i.e., functions $F : L([0,1])^n \to [0,1]$, where

$$L([0,1]) = \{[x,y] \mid x, y \in [0,1], x \leq y\}.$$

The order relation in this setting is given by

$$[a,b] \leq_L [c,d] \text{ if and only if } a \leq c \text{ and } b \leq d.$$

Definition 3.9. Let $\mathbf{v} = (a_1, b_1, \ldots, a_n, b_n) \in (\mathbb{R}^2)^n \setminus \{\mathbf{0}\}$. A function $F : L([0,1])^n \to L([0,1])$ is said to be $\mathbf{v}$-increasing if for all $\mathbf{x} \in L([0,1])^n$ and $c > 0$ such that $\mathbf{x} + c\mathbf{v} \in L([0,1])^n$, it holds that $F(\mathbf{x}) \leq_L F(\mathbf{x} + c\mathbf{v})$. Similarly, F is $\mathbf{v}$-decreasing if for all $\mathbf{x} \in L([0,1])^n$ and $c > 0$ such that $\mathbf{x} + c\mathbf{v} \in L([0,1])^n$, it holds that $F(\mathbf{x}) \geq_L F(\mathbf{x} + c\mathbf{v})$. In the case that F is simultaneously $\mathbf{v}$-increasing and $\mathbf{v}$-decreasing, F is said to be $\mathbf{v}$-constant.

Note that since $L([0,1])^n$ is a bounded convex sublattice of the Riesz space $(\mathbb{R}^2)^n$, then, for functions $F : L([0,1])^n \to L([0,1])$, the directions along which directional monotonicity is considered come from $(\mathbb{R}^2)^n$.

Similarly, we extend the concept of weak monotonicity to the interval-valued setting.

Definition 3.10. Let $(a,b) \in \mathbb{R}^2 \setminus \{\vec{0}\}$. A function $F : L([0,1])^n \to L([0,1])$ is said to be (a,b)-weakly increasing if for all $\mathbf{x} \in L([0,1])^n$ and $c > 0$ such that $\mathbf{x} + c(a,b,\ldots,a,b) \in L([0,1])^n$, it holds that $F(\mathbf{x}) \leq_L F(\mathbf{x} + c(a,b,\ldots,a,b))$. Similarly, F is said to be (a,b)-weakly decreasing if for all $\mathbf{x} \in L([0,1])^n$ and $c > 0$ such that $\mathbf{x} + c(a,b,\ldots,a,b) \in L([0,1])^n$, it holds that $F(\mathbf{x}) \geq_L F(\mathbf{x} + c(a,b,\ldots,a,b))$.

Besides the rest of discussed properties, in the setting of IV functions, the characterization of standard monotonicity by means of directional monotonicity along a set of finitely many directions is feasible (Theorem 6.5).

A relevant family of IV functions is that of IV representable functions [35]. A function $F : L([0,1])^n \to L([0,1])$ is said to be representable if it satisfies

$$F([\underline{x_1}, \overline{x_1}], \ldots, [\underline{x_n}, \overline{x_n}]) = [f(\underline{x_1}, \ldots, \underline{x_n}), g(\overline{x_1}, \ldots, \overline{x_n})],$$

for some functions $f, g : [0,1]^n \to [0,1]$ such that $f(x_1, \ldots, x_n) \leq g(y_1, \ldots, y_n)$ whenever $x_i \leq y_i$ for all $1 \leq i \leq n$.

Referring to functions f and g as the component functions of F, in our work we show that the directions of monotonicity of IV representable functions are totally determined by the directions of monotonicity of each of the component functions (Theorem 6.6).

Thus, in order to construct directionally monotone IV representable functions, it suffices to make use of the numerous examples of directionally monotone functions that can be found in [10, 11, 13, 21, 87]. In our work, we study the directions of increasingness of the discrete IV Choquet integral [3], due to its relevance in diverse applications [47] (Example 6.9).

In the final part of this work, we study the particular case of directional monotonicity for functions $F : L([0,1])^n \to L([0,1])$ that increase along directions formed by closed real intervals, i.e., the cases in which F is $\mathbf{v}$-increasing for $\mathbf{v} = ([a_1, b_1], \ldots, [a_n, b_n]) \in L(\mathbb{R})^n$. We refer to this notion as interval directional monotonicity (IDM) and, unlike the case of directional monotonicity, standard monotonicity cannot be characterized by means of IDM (Remark 6.12).

Although not present in our work, these findings lead us to introduce the concept of an IV pre-aggregation function.

Definition 3.11. A function $F : L([0,1])^n \to L([0,1])$ is said to be an IV pre-aggregation function if it satisfies the following conditions.

1. $F([0,0], \ldots, [0,0]) = [0,0]$;
2. $F([1,1], \ldots, [1,1]) = [1,1]$;
3. There exists a vector $\mathbf{v} = ((a_1, b_1), \ldots, (a_n, b_n)) \in ((\mathbb{R}_+)^2)^n$ such that F is $\mathbf{v}$-increasing.

After the completion of this work, we arrive at the following conclusions:

- Riesz spaces provide a framework in which it is possible to generalize the notion of directional monotonicity to functions that fuse data from the type-2 fuzzy, fuzzy multiset, n-dimensional fuzzy, Atanassov intuitionistic fuzzy and interval-valued fuzzy settings.
- The notion of directional monotonicity presented in this works recovers completely, as a particular instance, the usual form of directional monotonicity for functions $f : [0,1]^n \to [0,1]$.

- Directionally monotone functions in this general setting satisfy all the relevant properties of standard directionally monotone functions.
- The directions of monotonicity of IV representable functions are determined by the directions of monotonicity of the component functions.
- In the case that the directions of monotonicity are formed by intervals (IDM), it is not possible to characterize standard monotonicity in terms of directional monotonicity.

This section of the thesis is associated with the following publication:

- M. Sesma-Sara, R. Mesiar, H. Bustince. Weak and directional monotonicity of functions on Riesz spaces to fuse uncertain data, *Fuzzy Sets and Systems*.
  DOI: 10.1016/j.fss.2019.01.019.

3.6 New measures for comparing matrices and their application to image processing

This chapter can be seen as an appendix as it does not deal with any relaxation of monotonicity per se. However, we include it in this dissertation as a study that has been developed concurrently with the rest of discussed works.

In this work we introduce matrix resemblance functions, a class of functions to measure the level of resemblance between two matrices. We study the conditions under which this class of functions satisfy a set of properties that are favourable for image comparison operators and we present a neighbourhood-based image comparison algorithm that makes use of the class of matrix resemblance functions. We conclude this work with three instances of potential applications for this proposal: tamper and detection of image damaged areas, defect detection in industrial processes and video motion detection and object tracking.

The neighbourhood of a pixel can be defined as the square set of pixels surrounding it, i.e., a square matrix with values in the unit interval. We denote the set of such $n \times n$ matrices by $\mathcal{M}_n([0,1])$.

Definition 3.12. A function $\Psi : \mathcal{M}_n([0,1]) \times \mathcal{M}_n([0,1]) \to [0,1]$ is called a matrix resemblance function if it satisfies the following properties:

(MRF1) $\Psi(A,B) = 1$ if and only if $A = B$;

(MRF2) $\Psi(A,B) = 0$ if and only if there exist i and j such that $\{a_{ij}, b_{ij}\} = \{0,1\}$;

(MRF3) $\Psi(A,B) = \Psi(B,A)$ for all $A, B \in \mathcal{M}_n([0,1])$.

Conditions (MRF1) and (MRF3) are natural for matrix comparison operators. Condition (MRF2) is based on the erosion operator of mathematical morphology [16, 66] (see Section 5 of our work for the relation between matrix resemblance functions and fuzzy mathematical morphology).

We present two construction methods that provide an algebraic expression for this class of functions. The first one is based in the concept of restricted equivalence functions [18] (Theorem 4.4) and the second in the concept of inclusion grades for fuzzy sets in the sense of Sinha and Dougherty [78] (Theorem 4.7). These construction methods are related and, in our work, we analyze their relation (Section 4.4) and the cases in which both construction methods are equivalent (Theorems 4.13 and 4.16).

Comparison measures are expected to verify some specific properties [23, 34]. In our work we study the conditions under which matrix resemblance functions satisfy the following properties:

- Symmetry;
- Reflexivity;
- The comparison of an image and its complement should yield the minimum possible number, in this case 0;
- Invariance under permutation;
- Monotonicity;
- Shift invariance.

We also show that matrix resemblance functions do not meet the conditions to be homogeneous, migrative or additive.

After this theoretical study, we propose an image comparison algorithm that is underpinned on matrix resemblance functions. The proposed algorithm consists in comparing the neighbourhood of a pixel from the first image with the neighbourhood from the pixel in the same position of the second image using a matrix resemblance function. Then, the output image is built by setting the pixel in that position with the result of the comparison of neighbourhoods.

An advantage of the proposed algorithm is that, while matrix resemblance functions produce a number, the output of the global comparison of two images is a third image, enabling to distinguish different zones where both images are more (or less) similar. Therefore, in the final step of our algorithm we apply an image segmentation technique to extract the different areas in which the images are equally similar (or dissimilar).

We conclude this work with three possible applications of our proposal that illustrate its applicability. The first one is detection of tampered areas in images [44, 45]. The second is defect detection in industrial processes, such as in the manufacturing production of PCBs [27]. The third is video motion detection using a video from a human motion database[3] that is described in [53].

After the completion of this work, we arrive at the following conclusions:

[3] http://serre-lab.clps.brown.edu/resource/hmdb-a-large-human-motion-database/

- Matrix resemblance functions can be constructed by means of restricted equivalence functions and Sinha and Dougherty's inclusion grades.
- The second construction method enables to find a relation between the class of matrix resemblance functions and the erosion operator from fuzzy mathematical morphology.
- Matrix resemblance functions meet the criteria usually demanded to image comparison measures.
- The proposed image comparison algorithm reflects the level of similarity of the images in different regions.
- The proposed algorithm can be applied to tamper detection, defect detection and video motion detection.

This section of the thesis is associated with the following publication:

- M. Sesma-Sara, L. De Miguel, M. Pagola, A. Burusco, R. Mesiar, H. Bustince. New measures for comparing matrices and their application to image processing, *Applied Mathematical Modelling*, 61: 498-520, 2018.
DOI: 10.1016/j.apm.2018.05.006.

4 Conclusions

In this dissertation we have proposed several relaxed forms of monotonicity, all of which are to some extent related to the notions of weak and directional monotonicity. The first two forms of monotonicity, OD and SOD monotonicity, are based on the fact that the direction of monotonicity depends on the relative size of the inputs. Hence, the direction varies from one point of the domain to another and, in order to facilitate dealing with this fact, we introduce the third notion of monotonicity: pointwise directional monotonicity or directional monotonicity at a point, a local condition that studies the monotonicity of a function $f : [0,1]^n \to [0,1]$ at a point $\mathbf{x} \in [0,1]^n$ along a direction $\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\}$. The fourth notion of monotonicity is curve-based monotoniciy that extends directional monotonicity of functions $f : [0,1]^n \to [0,1]$ considering general curves, instead of rays, in $\mathbb{R}^n$. The fifth notion of monotonicity is a generalization of directional monotonicity for functions that are defined on a Cartesian product of Riesz spaces and that output values in a Riesz space.

The main conclusions are:

- OD and SOD monotonicity are relaxed forms of monotonicity that define a family of functions satisfying a directional monotonicity condition where the direction of increasingness, or decreasingness, is determined by the relative size of the inputs of each function.
- The use of OD monotone functions in a local feature extraction process permits to design an edge detection algorithm that obtains competitive results while standing out for its simplicity.
- Although the notions of OD and SOD monotonicity are essentially different, under the right circumstances they coincide. Moreover, standard monotonicity of a function $f : [0,1]^n \to [0,1]$ is characterized by directional, OD and SOD monotonicity of f with respect to a set of n directions.
- Pointwise directional monotonicity is a local form of monotonicity, which is able to characterize the global notions of weak, directional, cone, OD and SOD monotonicity. This notion permits to study the effect of each of the global forms of monotonicity in different regions of the domain and, additionally, shows some revelling geometric aspects about them.
- Curve-based monotonicity is a relaxed form of monotonicity that generalizes directional monotonicity for functions $f : [0,1]^n \to [0,1]$ considering curves $\alpha : I_\alpha \to \mathbb{R}^n$ instead of

rays $\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\}$. The domain I_α of the curves can be reduced to two cases: $[0,1]$ and $[0,1)$.

- Riesz spaces provide a framework in which it is possible to generalize the notion of directional monotonicity to functions that fuse data from the type-2 fuzzy, fuzzy multiset, n-dimensional fuzzy, Atanassov intuitionistic fuzzy and interval-valued fuzzy settings. This generalization recovers the standard case of directional monotonicity for functions $f : [0,1]^n \to [0,1]$. With respect to the interval-valued case, the fact that the directions of monotonicity of IV representable functions are determined by the directions of monotonicity of the component functions is relevant. Additionally, we conclude that it is not possible to characterize standard monotonicity when the directions are formed by intervals (IDM).

5 Future research lines

With the exception of the OD monotone function based edge detection algorithm and the image comparison algorithm discussed in Sections 3.1 and 3.6, respectively, the findings of this dissertation are fundamentally theoretical. Hence, our perspectives with respect to future research lines are principally practical.

Edge detection based on SOD monotone functions

We have proposed an edge detection algorithm based on OD monotone functions in Section 3.1 and we have introduced the notion of SOD monotonicity in Section 3.2, which is more restrictive than OD monotonicity. In fact, every SOD monotone function is also OD monotone, but the converse does not hold. We think that it would be interesting to study the influence of each type of monotonicity in the task of detecting edges. Our intention is to construct an edge detection algorithm based on the notion of SOD monotonicity and to study the performance of SOD monotone functions in comparison to the performance of OD monotone functions that are not SOD monotone.

Extension of OD and SOD monotonicity to a more general framework

As presented in Section 3.5 with directional monotonicity, the concepts of OD and SOD monotonicity can also be extended to the framework of Riesz spaces. In fact, interval-valued fusion techniques have been applied to detect edges in images [19, 60, 61]. We intend to join both approaches to study the possibility of constructing an edge detector based on interval-valued OD and SOD monotone functions.

Use of interval-valued pre-aggregation functions in fuzzy rule-based classification problems

Pre-aggregation functions have been employed in fuzzy rule-based classification problems with high accuracy rates [36, 56, 57, 58]. Similarly, interval-valued fuzzy sets have also succeeded in many classification problems [73, 74, 75]. Thus, we have set as an intention for future research joining both approaches and applying interval-valued pre-aggregation functions in fuzzy rule-based classification problems.

Directional monotonicity of interval-valued functions with respect to an admissible order

The generalization of directional monotonicity to the interval-valued setting discussed in Section 3.5 is made with respect the partial order in $L([0,1])$. However, there exist several proposals of linear orders for the set of closed intervals, being the so-called admissible orders a remarkable family of such order relations [20, 30]. In fact, many applications that make use of interval-valued functions are based on this type of order relations [22, 31, 68] and, therefore, in the future we aim at generalizing the concept of directional monotonicity to interval-valued functions with respect to admissible orders.

6 Castellano: resumen y conclusiones

6.1 Resumen

El proceso de agregación trata el problema de combinar una colección de valores numéricos en un único valor que los represente y las funciones encargadas de esta operación se denominan funciones de agregación. A las funciones de agregación se les exige que cumplan dos condiciones de contorno y, además, han de ser monótonas con respecto a todos sus argumentos.

Una de las tendencias en el área de investigación de las funciones de agregación es la relajación de la condición de monotonía. En este respecto, se han introducido varias formas de monotonía relajada. Tal es el caso de la monotonía débil, la monotonía direccional y la monotonía respecto a un cono.

Sin embargo, todas estas relajaciones de monotonía están basadas en la idea de crecer, o decrecer, a lo largo de un rayo definido por un vector real. No existe noción de monotonía que permita que la dirección de crecimiento dependa de los valores a fusionar, ni tampoco existe noción de monotonía que considere el crecimiento a lo largo de caminos más generales, como son las curvas. Además, otra de las tendencias en la teoría de la agregación es la extensión a escalas más generales que la de los números reales y no existe relajación de monotonía disponible para este contexto general.

En esta tesis, proponemos una colección de nuevas formas de monotonía relajada para las cuales las direcciones de monotonía pueden variar dependiendo del punto del dominio. En concreto, introducimos los conceptos de monotonía direccional ordenada, monotonía direccional ordenada reforzada y monotonía direccional punto a punto. Basándonos en funciones que cumplan las propiedades de monotonía direccional ordenada, proponemos un algoritmo de detección de bordes que justifica la aplicabilidad de estos conceptos. Por otro lado, generalizamos el concepto de monotonía direccional tomando, en lugar de direcciones en $\mathbb{R}^n$, caminos más generales: definimos el concepto de monotonía basado en curvas. Por último, combinando ambas tendencias en la teoría de la agregación, generalizamos el concepto de monotonía direccional a funciones definidas en escalas más generales que la de los números reales.

6.2 Conclusiones

En esta tesis hemos propuesto varias formas de monotonía relajada, todas las cuales están de algún modo relacionadas con las nociones de monotonía débil y monotonía direccional. Las primeras dos formas de monotonía, la monotonía direccional ordenada y la direccional ordenada reforzada, están basadas en el hecho de que la dirección de monotonía es dependiente de el tamaño relativo de los valores de entrada. Por lo tanto, dicha dirección varía de un punto del dominio a otro y, en aras de facilitar el manejo de esta situación, introducimos el concepto de monotonía direccional punto a punto, una condición local que estudia la monotonía de una función $f : [0,1]^n \to [0,1]$ en un punto $\mathbf{x} \in [0,1]^n$ a lo largo de la dirección $\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\}$. La cuarta noción de monotonía es la denominada monotonía basada en curvas, que extiende la monotonía direccional de funciones $f : [0,1]^n \to [0,1]$ considerando curvas generales, en lugar de rayos, en $\mathbb{R}^n$. La quinta noción de monotonía es una generalización de la monotonía direccional para funciones que están definidas en el producto cartesiano de espacios de Riesz y toman valores en espacios de Riesz.

Las principales conclusiones son:

- La monotonía direccional ordenada y la monotonía direccional ordenada reforzada son formas de monotonía relajada que definen una familia de funciones que satisfacen una condición de monotonía en la cual la dirección de crecimiento, o decrecimiento, está determinada por el tamaño relativo de sus valores de entrada.

- El uso de funciones monótonas direccionales ordenadas en procesos de extracción de características locales permite diseñar un algoritmo de detección de bordes que obtiene resultados competitivos y, a la vez, destaca por su sencillez.

- Aunque las nociones de monotonía direccional ordenada y monotonía direccional ordenada reforzada son fundamentalmente diferentes, bajo ciertas condiciones adecuadas son equivalentes. Además, la monotonía estándar de una función $f : [0,1]^n \to [0,1]$ puede caracterizarse mediante monotonía direccional, direccional ordenada y direccional ordenada reforzada con respecto a un conjunto determinado de n direcciones.

- La monotonía direccional punto a punto es una forma local de monotonía que permite caracterizar las nociones globales de monotonía débil, direccional, cónica, direccional ordenada y direccional ordenada reforzada. Este concepto permite estudiar el efecto de cada una de las formas globales de monotonía en diferentes zonas del dominio y, además, mostrar varios aspectos geométricos sobre las mismas.

- La monotonía basada en curvas es una forma de monotonía relajada que generaliza la monotonía direccional de funciones $f : [0,1]^n \to [0,1]$ considerando curvas $\alpha : I_\alpha \to \mathbb{R}^n$ en lugar de rayos $\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\}$. El dominio I_α de las curvas se reduce a dos casos: $[0,1]$ y $[0,1)$.

- Los espacios de Riesz presentan un marco en el que es posible generalizar el concepto de monotonía direccional para funciones que fusionan datos provenientes conjuntos difusos tipo-2, multisets difusos, conjuntos difusos n-dimensionales, conjuntos Atanassov intuicionistas difusos y conjuntos difusos intervalo-valorados. Esta generalización recupera el caso estándar de monotonía direccional para funciones $f : [0,1]^n \to [0,1]$. Con respecto al caso intervalo-valorado, el hecho de que la dirección de crecimiento de las funciones intervalo-valoradas representables esté determinada por las direcciones de crecimiento de cada una de las funciones componentes es relevante. Además, concluimos que no es posible caracterizar la monotonía estándar cuando las direcciones están formadas por intervalos (caso IDM).

Part II.
Publications: Published, accepted and submitted works

1 Ordered Directionally Monotone Functions: Justification and Application

Associated publication:

- H. Bustince, E. Barrenechea, M. Sesma-Sara, J. Lafuente, G.P. Dimuro, R. Mesiar, A. Kolesárová. Ordered directionally monotone functions: Justification and application, *IEEE Transactions on Fuzzy Systems*, 26 (4): 2237-2250, 2018.
  DOI: 10.1109/TFUZZ.2017.2769486.
  - Status: Published.
  - Impact factor (JCR 2017): 8.415.
  - Categories:
    - Computer Science, Artificial Intelligence. Ranking 4/132 (Q1).
    - Engineering, Electrical & Electronic. Ranking 7/260 (Q1).

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2 Strengthened ordered directionally monotone functions. Links between the different notions of monotonicity

Associated publication:

- M. Sesma-Sara, J. Lafuente, A. Roldán, R. Mesiar, H. Bustince. Strengthened ordered directionally monotone functions. Links between the different notions of monotonicity, *Fuzzy Sets and Systems*, 357: 151-172, 2019.
  DOI: 10.1016/j.fss.2018.07.007.
  - Status: Published.
  - Impact factor (JCR 2017): 2.675.
  - Categories:
    - Computer Science, Theory & Methods. Ranking 18/103 (Q1).
    - Mathematics, applied. Ranking 12/252 (Q1).
    - Statistics & Probability. Ranking 8/123 (Q1).

Available online at www.sciencedirect.com
ScienceDirect
Fuzzy Sets and Systems 357 (2019) 151–172
FUZZY sets and systems
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Strengthened ordered directionally monotone functions. Links between the different notions of monotonicity

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Received 23 October 2017; received in revised form 17 July 2018; accepted 20 July 2018
Available online 24 July 2018

Abstract

In this work, we propose a new notion of monotonicity: *strengthened ordered directional monotonicity*. This generalization of monotonicity is based on directional monotonicity and ordered directional monotonicity, two recent weaker forms of monotonicity. We discuss the relation between those different notions of monotonicity from a theoretical point of view. Additionally, along with the introduction of two families of functions and a study of their connection to the considered monotonicity notions, we define an operation between functions that generalizes the Choquet integral and the Łukasiewicz implication.

Keywords: Strengthened ordered directional monotonicity; Directional monotonicity; Generalizations of monotonicity

1. Introduction

Monotonicity with respect to each argument is one of the axioms around which the concept of aggregation function is built (see [1,9]). Aggregation functions' goal is to fuse information, generating a representative value from a number of inputs, and these functions are used in a very vast and diverse field of applications [7,8,10,14]. However, the mentioned condition of monotonicity with respect to every argument is sometimes excessively restrictive, which causes to drop from the theoretical framework functions that otherwise are sound for certain applications, such as fuzzy implication operators, the mode, the Gini and Lehmer means, etc. (see [3]).

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https://doi.org/10.1016/j.fss.2018.07.007
0165-0114/

With the purpose of creating a wider framework of functions that are valid to fuse information, recently some weaker forms of monotonicity have arised (see [2]). In [16], Wilkin and Beliakov introduced the notion of weak monotonicity, which can be seen as monotonicity along the ray $(1, 1, \ldots, 1)$. This interpretation led to a generalization of that notion of monotonicity, considering monotonicity along any ray in $\mathbb{R}^n$ and originating directional monotonicity [6]. The possibility of choosing any vector $\vec{r}$ allows to pick a function that increases accordingly to the needs of a certain application, with no need of it being increasing with respect to each of its arguments.

However, both of the aforementioned notions require that the direction of increasingness or decreasingness is fixed beforehand and does not vary according to the point of the domain that is being considered. Based on Yager's ideas ([17]), in [5] ordered directionally monotone functions were introduced. This notion of monotonicity enables the direction of increasingness (or decreasingness) to vary from one point to another. Specifically, the direction of monotonicity depends on the relative size of the inputs, provided that a certain comonotonicity condition is fulfilled.

The relaxation of the monotonicity condition for aggregation functions is listed as a recent trend in Aggregation Theory [13]. One of the main advances of the introduction of directional monotonicity is the formation of the so called pre-aggregation functions [12], which have been successfully applied in fuzzy rule-based classification problems [11]. Furthermore, ordered directionally monotone functions have been used in the field of computer vision, see [5,15] for an application of ordered directionally monotone functions in edge detection.

The restriction that comes from the comonotonicity condition in the definition of ordered directional monotonicity makes us limit to the cases in which the input vector and the result of increasing it along a direction are comonotone, making the family of ordered directionally monotone functions larger than if the condition were removed. This work attempts to achieve the following goals:

- To introduce a new generalization of monotonicity based on ordered directional monotonicity but with no comonotonicity condition.
- To study the properties and relations between the different notions of monotonicity.
- To define two classes of functions and an operation between them that enable to generalize the Choquet integral and the Łukasiewicz implication.

We call the new notion of monotonicity *strengthened ordered directional monotonicity*. This generalization of monotonicity is based on that of ordered directional monotonicity, but removing the comonotonicity condition from the definition. The family of strengthened ordered directionally monotone functions is embedded in that of ordered directionally monotone functions, i.e., every strengthened directionally monotone function is ordered directionally monotone, but not the other way around.

Moreover, we carry out a deep study of the properties that the different families of functions satisfy, as well as the relations among them. We show the conditions for which it is equivalent for a function to be increasing with respect to all its arguments and to be increasing in the sense of the discussed different notions of monotonicity.

As to the third goal, we also present two classes of functions - linear fusion functions and ordered linear fusion functions - and show their main properties in terms of the distinct types of monotonicity. Additionally, we introduce an operation between functions from $[0, 1]^2$ to $[0, 1]$ that, when applied to ordered linear fusion functions, generalizes the Choquet integral and the Łukasiewicz implication.

This work is organized as follows. In the next section we recall some preliminary notions that are used throughout the paper, including the definition of strengthened ordered directionally monotone functions and some introductory properties. In Section 3 we study a set of properties about the three different notions of monotonicity – directional monotonicity, ordered directional monotonicity and strengthened ordered directional monotonicity – and we show how these concepts are related. In Section 4 we introduce the family of linear fusion functions and the family of ordered linear fusion functions and we show the behaviour of these families of functions in terms of the discussed notions of directional monotonicity. In Section 5 we introduce the operation $*$ between functions for $n = 2$ and show how Choquet integrals and the Łukasiewicz implication can be derived from this operation. In Section 6 we present the relation of every notion of monotonicity that is considered throughout the paper and we finish the work with some concluding remarks.

Table 1
Conditions that characterize the specials points $\mathbf{x} \in [0,1]^n$ for each type of increasingness.

F	if $0 < c \in \mathbb{R}$, $\sigma \in \mathcal{S}_n$, then
$\vec{r}$-increasing	$\mathbf{x} + c\vec{r} \notin [0,1]^n$
SOD $\vec{r}$-increasing	$\mathbf{x}_\sigma \in [0,1]^n_{(\geq)} \implies \mathbf{x}_\sigma + c\vec{r} \notin [0,1]^n$
OD $\vec{r}$-increasing	$\mathbf{x}_\sigma \in [0,1]^n_{(\geq)} \implies \mathbf{x}_\sigma + c\vec{r} \notin [0,1]^n_{(\geq)}$

2. Preliminaries

Let $n \in \mathbb{N}$, $n > 1$. We use an arrow to refer to vectors of $\mathbb{R}^n$, $\vec{r} = (r_1, \dots, r_n) \in \mathbb{R}^n$ and we set $\vec{r}^{\,d} = (r_n, \dots, r_1)$.

We use bold letters to specify points of the hypercube $[0,1]^n$, so we set $\mathbf{x} = (x_1, \dots, x_n) \in [0,1]^n$. In particular, we write $\mathbf{0} = (0, \dots, 0)$ and $\mathbf{1} = (1, \dots, 1)$. If $\mathbf{x}, \mathbf{y} \in [0,1]^n$, we set $\mathbf{x} \leq \mathbf{y}$ if $x_i \leq y_i$ for each $i \in \{1, \dots, n\}$.

$\mathcal{S}_n$ denotes the symmetrical group of degree n. Given a permutation $\sigma \in \mathcal{S}_n$ we denote the inverse permutation by σ^{-1}, i.e., $\sigma\sigma^{-1} = \mathrm{id}$, and if $\mathbf{a} = (a_1, \dots, a_n)$ and $\sigma \in \mathcal{S}_n$, we set $\mathbf{a}_\sigma = (a_{\sigma(1)}, \dots, a_{\sigma(n)})$.

If $H \subseteq \mathbb{R}^n$, we set $H_{(\geq)} = \{(h_1, \dots, h_n) \in H \mid h_1 \geq \dots \geq h_n\}$ (we occasionally refer to $H_{(\leq)}$, $H_{(>)}$, $H_{(<)}$, $H_{(=)}$ with the obvious meanings).

Let us recall the concepts of directional monotonicity and ordered directional monotonicity, which were introduced in [6] and [5], respectively.

Definition 2.1 *([6]).* Let $F\colon [0,1]^n \to [0,1]$ and $\vec{r} \in \mathbb{R}^n$, we say that F is $\vec{r}$-increasing (resp. $\vec{r}$-decreasing), if for all $c > 0$ and $\mathbf{x} \in [0,1]^n$ such that $\mathbf{x} + c\vec{r} \in [0,1]^n$ it holds that $F(\mathbf{x} + c\vec{r}) \geq F(\mathbf{x})$ (resp. $F(\mathbf{x} + c\vec{r}) \leq F(\mathbf{x})$). If $F(\mathbf{x} + c\vec{r}) = F(\mathbf{x})$, then we say that F is $\vec{r}$-constant.

Definition 2.2 *([5]).* Let $F\colon [0,1]^n \to [0,1]$ and $\vec{r} \in \mathbb{R}^n$, we say that F is ordered directionally (OD) $\vec{r}$-increasing (resp. OD $\vec{r}$-decreasing) if for all $\mathbf{x} \in [0,1]^n$, $\sigma \in \mathcal{S}_n$ and $c > 0$ such that $\mathbf{x}_\sigma \in [0,1]^n_{(\geq)}$ and $\mathbf{x}_\sigma + c\vec{r} \in [0,1]^n_{(\geq)}$, it holds that $F(\mathbf{x} + c\vec{r}_{\sigma^{-1}}) \geq F(\mathbf{x})$ (resp. $F(\mathbf{x} + c\vec{r}_{\sigma^{-1}}) \leq F(\mathbf{x})$). If $F(\mathbf{x} + c\vec{r}_{\sigma^{-1}}) = F(\mathbf{x})$, then we say that F is OD $\vec{r}$-constant.

We now introduce the central concept of this work, strengthened ordered directional monotonicity.

Definition 2.3. Let $F\colon [0,1]^n \to [0,1]$ and $\vec{r} \in \mathbb{R}^n$, we say that F is strengthened ordered directionally (SOD) $\vec{r}$-increasing (resp. SOD $\vec{r}$-decreasing) if for all $\mathbf{x} \in [0,1]^n$, $\sigma \in \mathcal{S}_n$ and $c > 0$ such that $\mathbf{x}_\sigma \in [0,1]^n_{(\geq)}$ and $\mathbf{x}_\sigma + c\vec{r} \in [0,1]^n$, it holds that $F(\mathbf{x} + c\vec{r}_{\sigma^{-1}}) \geq F(\mathbf{x})$ (resp. $F(\mathbf{x} + c\vec{r}_{\sigma^{-1}}) \leq F(\mathbf{x})$). If $F(\mathbf{x} + c\vec{r}_{\sigma^{-1}}) = F(\mathbf{x})$, then we say that F is SOD $\vec{r}$-constant.

For brevity, to refer to a function F that is monotone according to each of the defined types, we say that F is T $\vec{r}$-increasing for T $\in \{\emptyset, \text{ SOD, OD}\}$. Moreover, we say that F is T $\vec{r}$-monotone if it is either T $\vec{r}$-increasing or T $\vec{r}$-decreasing for T $\in \{\emptyset, \text{ SOD, OD}\}$.

Note that the case in which $\vec{r} = \vec{0}$ is trivial, as every function is T $\vec{0}$-increasing, T $\vec{0}$-decreasing and T $\vec{0}$-constant, for T $\in \{\emptyset, \text{ SOD, OD}\}$.

Of course SOD $\vec{r}$-increasingness implies OD $\vec{r}$-increasingness and we will see that the reciprocal statement is not true in general. A first flash of the differences between these classes of monotonicity can be done by the observation of the points in $[0,1]^n$ for which a function $F\colon [0,1]^n \to [0,1]$ trivially satisfies the conditions from the different classes of monotonicity. We refer to these points as *special*.

For instance, for a function $F\colon [0,1]^n \to [0,1]$ to be increasing (if $\mathbf{x}, \mathbf{y} \in [0,1]^n$ such that $\mathbf{x} \leq \mathbf{y}$, then $F(\mathbf{x}) \leq F(\mathbf{y})$), the unique special point is $\mathbf{1}$, as with $\mathbf{x} = \mathbf{1}$ only the trivial situation $\mathbf{y} = \mathbf{1}$ is to be considered (and if F is in fact increasing, then $F(\mathbf{1}) = \max\{F(\mathbf{x}) \mid \mathbf{x} \in [0,1]^n\}$).

The conditions that characterize the special points $\mathbf{x} \in [0,1]^n$ for the considered types of increasingness, associated to vectors $\vec{r} \in \mathbb{R}^n$, appear in Table 1.

Set $O = (0,0)$, $X_1 = (1,0)$, $X_2 = (0,1)$ and $U = (1,1)$. In Fig. 1 we see an example of a special point $\mathbf{x}$ for $\vec{r}$-increasingness.

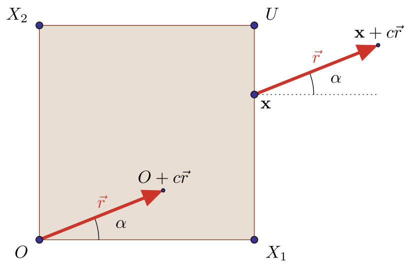

Fig. 1. Example of special point $\mathbf{x} \in [0, 1]^2$ for a direction $\vec{r} \in \mathbb{R}^2$.

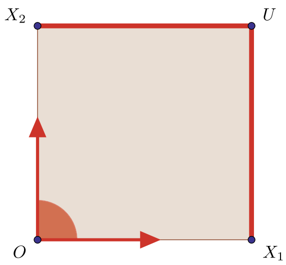

Fig. 2. Directions $\vec{0} \neq \vec{r} \in \mathbb{R}^2$ whose angle with OX_1 satisfies $\alpha \in (0, \pi/2)$ and their corresponding special points; the join of the closed segments X_2U and UX_1.

It is clear that for each $\vec{0} \neq \vec{r} \in \mathbb{R}^n$ whose angle α with OX_1 satisfies $\alpha \in (0, \pi/2)$ the corresponding special points form the join of the closed segments X_2U and UX_1. We illustrate this situation in Fig. 2.

If $\alpha = 0$ however only the points of the closed segment UX_1 are special, and for $\alpha \in (3\pi/2, 2\pi)$, the special points form the joint of the closed segments OX_1 and UX_1. In this sense, $\alpha = 0$ marks a transition (as also do $\alpha = \pi/2$, $\alpha = \pi$ and $\alpha = 3\pi/2$).

We depict the different situations by means of schemes which associate, through the use of colors, vectors of transition and corresponding special points in the case $n = 2$. For a vector $\vec{r}$ lying between two consecutive transition vectors, the set of special points is the join of the special points corresponding to the consecutive transition vectors. Fig. 3 shows the transition vectors and sets of special points for the case of directional monotonicity and Figs. 4 and 5 for the cases of SOD monotonicity and OD monotonicity, respectively.

Observe, for instance, the case of Fig. 4 in which F is SOD $\vec{r}$-increasing. For $\alpha = \pi/2$ the only special point is U and for $\alpha = \pi$ the only one is O. If $\alpha \in (3\pi/2, 2\pi)$, then the special points draw the perimeter of the square OX_1UX_2. If $\alpha \in (\pi/2, \pi)$, only the points O and U are special.

In the case of Fig. 5, in which F is OD $\vec{r}$-increasing, for $\pi/2 \leq \alpha \leq \pi$ the special points form the diagonal OU.

Remark 2.4. An obvious generalization of the introduced concepts of (S)OD $\vec{r}$-increasingness appears changing in the definitions the triangle $[0, 1]^n_{(\geq)}$ for any subset $\mathbf{S}$ of $[0, 1]^n$.

3. Basic facts

We begin by developing some basic properties on T $\vec{r}$-monotonicity for T $\in \{\emptyset,$ SOD, OD$\}$.

First of all we remark that in the definitions corresponding to SOD and OD, instead of having required that $\mathbf{x}_\sigma \in [0, 1]^n_{(\geq)}$, we could set $\mathbf{x}_\sigma \in [0, 1]^n_{(\leq)}$. The corresponding developments would be equivalent, as the following remark states for the OD case (the SOD case is similar).

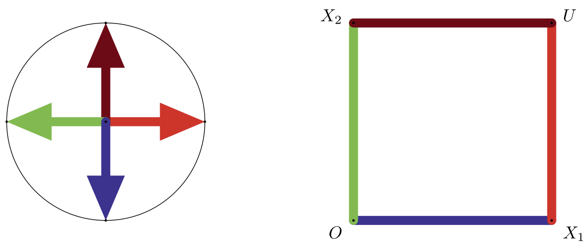

Fig. 3. *Left:* Transition vectors for directional monotonicity. *Right:* Sets of special points in $[0, 1]^2$ for directional monotonicity. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

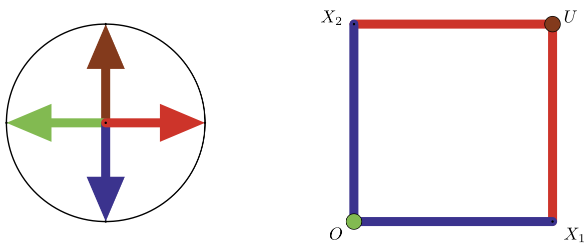

Fig. 4. *Left:* Transition vectors for SOD monotonicity. *Right:* Sets of special points in $[0, 1]^2$ for SOD monotonicity.

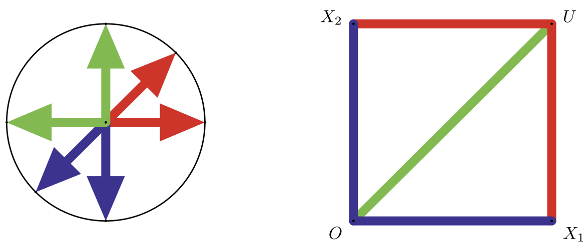

Fig. 5. *Left:* Transition vectors for OD monotonicity. *Right:* Sets of special points in $[0, 1]^2$ for OD monotonicity.

Proposition 3.1. *For all* $\mathbf{x} \in [0,1]^n$, $\vec{r} \in \mathbb{R}^n$, $0 < c \in \mathbb{R}$ *and* $\sigma \in \mathcal{S}_n$ *the following assertions are equivalent:*

(1) $\mathbf{x}_\sigma, \mathbf{x}_\sigma + c\vec{r} \in [0,1]^n_{(\geq)} \implies F(\mathbf{x} + c\vec{r}_{\sigma^{-1}}) \geq F(\mathbf{x})$.
(2) $\mathbf{x}_\sigma, \mathbf{x}_\sigma + c\vec{r}^{\,d} \in [0,1]^n_{(\leq)} \implies F(\mathbf{x} + c(\vec{r}^{\,d})_{\sigma^{-1}}) \geq F(\mathbf{x})$.

Proof. We must deal with permutations and in this case it is useful to handle permutation matrices. For $\sigma \in \mathcal{S}_n$, the permutation matrix P_σ denotes the $n \times n$ matrix resulting from the application of the permutation σ to the indices of the rows of the identity matrix I_n. With this it is immediate that

$$\mathbf{x}_\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = (x_1, \ldots, x_n)\, P_\sigma = \mathbf{x} P_\sigma \,.$$

With the permutation matrix $D = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}$, we have $\vec{r}^{\,d} = \vec{r} D$.

Let us show that (1) $\Longrightarrow$ (2). Assume that $\mathbf{x}_\sigma$, $\mathbf{x}_\sigma + c\vec{r}^{\,d} \in [0,1]^n_{(\leq)}$, that is, $\mathbf{x}P_\sigma$, $\mathbf{x}P_\sigma + c\vec{r}D \in [0,1]^n_{(\leq)}$. Then obviously $\mathbf{x}P_\sigma D$, $\mathbf{x}P_\sigma D + c\vec{r} \in [0,1]^n_{(\geq)}$, as $D^2 = I_n$. Then, by (1), $F(\mathbf{x} + c\vec{r}(P_\sigma D)^{-1}) \geq F(\mathbf{x})$. As $(P_\sigma D)^{-1} = D^{-1}P_\sigma^{-1} = DP_{\sigma^{-1}}$, and $\vec{r}DP_{\sigma^{-1}} = \vec{r}^{\,d}P_{\sigma^{-1}} = (\vec{r}^{\,d})_{\sigma^{-1}}$, we have the thesis. Analogously one shows that (2) $\Longrightarrow$ (1). $\square$

Let $F\colon [0,1]^n \to [0,1]$ and let us set, for $\mathrm{T} \in \{\emptyset,\ \mathrm{SOD}, \mathrm{OD}\}$, the notation

$$\mathcal{C}_{\mathrm{T}}(F) = \{\vec{r} \in \mathbb{R}^n \mid F \text{ is T } \vec{r}\text{-constant}\}$$
$$\mathcal{D}^{\uparrow}_{\mathrm{T}}(F) = \{\vec{r} \in \mathbb{R}^n \mid F \text{ is T } \vec{r}\text{-increasing}\}$$

particularizing $\mathcal{C}(F) = \mathcal{C}_\emptyset(F)$ and $\mathcal{D}^{\uparrow}(F) = \mathcal{D}^{\uparrow}_\emptyset(F)$.

We next result follows immediately.

Proposition 3.2. *Let $F\colon [0,1]^n \to [0,1]$ be a function. Then*

(1) $\mathcal{C}_T(F) \subseteq \mathcal{D}^{\uparrow}_T(F)$ for $T \in \{\emptyset,\ SOD, OD\}$.
(2) $\mathcal{C}_{SOD}(F) \subseteq \mathcal{C}_{OD}(F)$.
(3) $\mathcal{D}^{\uparrow}_{SOD}(F) \subseteq \mathcal{D}^{\uparrow}_{OD}(F)$.

Proposition 3.3. *If $\vec{r} \in \mathbb{R}^n_{(\geq)}$, then F is SOD $\vec{r}$-increasing if and only if F is OD $\vec{r}$-increasing, and F is SOD $\vec{r}$-constant if and only if F is OD $\vec{r}$-constant.*

Observe that the statements of Propositions 3.2 and 3.3 changing increasingness by decreasingness are valid.

Proposition 3.4. *Let $F\colon [0,1]^n \to [0,1]$ be a function and $k > 0$.*

(1) For $T \in \{\emptyset,\ SOD, OD\}$, F is simultaneously $T\,\vec{r}$-increasing and $T\,\vec{r}$-decreasing if and only if F is $T\,\vec{r}$-constant.
(2) For $T \in \{\emptyset,\ SOD, OD\}$, F is $T\,\vec{r}$-increasing (resp. decreasing) if and only if F is $T\,(k\vec{r})$-increasing (resp. decreasing).
(3) For $T \in \{\emptyset,\ OD\}$, F is $T\,\vec{r}$-increasing if and only if F is $T\,(-k\vec{r})$-decreasing.

Proof. (1) and (2) They are immediate.

(3) By (2) we may assume that $k = 1$. Suppose that F is OD $\vec{r}$-increasing. Let $\mathbf{x} \in [0,1]^n$, $k > 0$ and $\sigma \in \mathcal{S}_n$ such that $\mathbf{x}_\sigma$, $\mathbf{x}_\sigma + k(-\vec{r}) \in [0,1]^n_{(\geq)}$. Set $\mathbf{y} = \mathbf{x} + k(-\vec{r})_{\sigma^{-1}}$. We have that $\mathbf{y}_\sigma = \mathbf{x}_\sigma + k(-\vec{r}) \in [0,1]^n_{(\geq)}$ (hence $\mathbf{y} \in [0,1]^n$) and $\mathbf{y}_\sigma + k\vec{r} = \mathbf{x}_\sigma \in [0,1]^n_{(\geq)}$. As F is OD $\vec{r}$-increasing, we have $F(\mathbf{y}) \leq F(\mathbf{y} + k\vec{r}_{\sigma^{-1}})$, that is $F(\mathbf{x} + k(-\vec{r})_{\sigma^{-1}}) \leq F(\mathbf{x})$, hence F is OD $(-\vec{r})$-decreasing. Analogously one has the converse statement.

Similarly one shows that F is $\vec{r}$-increasing if and only if it is $(-k\vec{r})$-decreasing. $\square$

Item (2) of Proposition 3.4 shows that the vectors used for determining directions in all considered cases $\mathrm{T} \in \{\emptyset,\ \mathrm{SOD}, \mathrm{OD}\}$ can be normalized. Item 3 shows that for $\mathrm{T} \in \{\emptyset,\ \mathrm{OD}\}$, the developments which result by considering T increasingness and decreasingness are equivalent. However, in Section 5 we show that, in general, this is false for $T = SOD$.

Proposition 3.5. *Let $F\colon [0,1]^n \to [0,1]$ and $\vec{r} \in \mathbb{R}^n$. Let $F^c\colon [0,1]^n \to [0,1]$ be defined by $F^c(\mathbf{x}) = 1 - F(\mathbf{x})$. Then*

(1) For $T \in \{\emptyset,\ SOD, OD\}$:
  (a) F is $T\,\vec{r}$-increasing if and only if F^c is $T\,\vec{r}$-decreasing.
  (b) $\mathcal{C}_T(F) = \mathcal{C}_T(F^c)$.
  (c) F is $T\,\vec{r}$-constant if and only if both F and F^c are $T\,\vec{r}$-increasing.
(2) For $T \in \{\emptyset,\ OD\}$, F is $T\,\vec{r}$-increasing if and only if F^c is $T\,(-\vec{r})$-increasing.

Proof. (1) The claims in (a) are direct. For instance, for $\mathrm{T} = \emptyset$, if $c \in \mathbb{R}^+$ and $\mathbf{x}$, $\mathbf{x} + c\vec{r} \in [0,1]^n$, then $F(\mathbf{x}) \leq F(\mathbf{x} + c\vec{r})$ if and only if $F^c(\mathbf{x}) = 1 - F(\mathbf{x}) \geq 1 - F(\mathbf{x} + c\vec{r}) = F^c(\mathbf{x} + c\vec{r})$.

(b) and (c) follow from (a) as $(F^c)^c = F$.

(2) F is T $\vec{r}$-increasing if and only if F is T $(-\vec{r})$-decreasing by Proposition 3.4 and this is equivalent to F^c being T $(-\vec{r})$-increasing by (1). □

In Section 5 we show that Proposition 3.5(2) is in general not true for $T = SOD$.

Proposition 3.6. *Let $F: [0,1]^n \to [0,1]$ and $\vec{r} \in \mathbb{R}^n$. Let $G: [0,1]^n \to [0,1]$ defined by $G(\mathbf{x}) = F(\mathbf{1} - \mathbf{x})$.*

(1) F is $\vec{r}$-increasing if and only if G is $(-\vec{r})$-increasing.
(2) For $T \in \{SOD, OD\}$, F is $T\,\vec{r}$-increasing if and only if G is $T\,(-\vec{r})^d$-increasing.

Proof. (1) It is straightforward.

(2) Let F be OD $\vec{r}$-increasing and let $\mathbf{x} \in [0,1]^n$. Consider $\sigma \in \mathcal{S}_n$ such that $\mathbf{x}_\sigma \in [0,1]^n_{(\geq)}$ and $c > 0$ such that $\mathbf{x}_\sigma - c\vec{r}^{\,d} \in [0,1]^n_{(\geq)}$.

Put $\mathbf{y} = \mathbf{1} - \mathbf{x}$. Then $\sigma^d \in \mathcal{S}_n$, given by $\sigma^d(i) = \sigma(n - i + 1)$ for all $i = 1, \ldots, n$, is a permutation such that $\mathbf{y}_{\sigma^d} \in [0,1]^n_{(\geq)}$ and $\mathbf{y}_{\sigma^d} + c\vec{r} \in [0,1]^n_{(\geq)}$. Due to the OD $\vec{r}$-increasingness of F, we have $F(\mathbf{y} + c\vec{r}_{(\sigma^d)^{-1}}) \geq F(\mathbf{y})$, and from $\sigma^d(i) = \sigma(n - i + 1) = k$ we have $(\sigma^d)^{-1}(k) = i$ and $\sigma^{-1}(k) = n - i + 1$, hence $(\sigma^d)^{-1}(k) = n - \sigma^{-1}(k) + 1$, and finally, $(\vec{r}^{\,d})_{\sigma^{-1}} = \vec{r}_{(\sigma^d)^{-1}}$.

Therefore, we get

$$
\begin{aligned}
G\left(\mathbf{x} + c\left(-\vec{r}^{\,d}\right)_{\sigma^{-1}}\right) &= F\left(\mathbf{1} - \mathbf{x} + c\vec{r}^{\,d}_{\sigma^{-1}}\right) \\
&= F\left(\mathbf{y} + c\vec{r}_{(\sigma^d)^{-1}}\right) \\
&\geq F(\mathbf{y}) = F(\mathbf{1} - \mathbf{x}) = G(\mathbf{x}),
\end{aligned}
$$

which means that G is OD $(-\vec{r})^d$-increasing.

The converse follows from the fact that $-\left(-\vec{r}^{\,d}\right)^d = \vec{r}$.

The case of $\mathrm{T} = \mathrm{SOD}$ is analogous. □

The dual function F^d of a function $F: [0,1]^n \to [0,1]$ is defined for each $\mathbf{x} \in [0,1]^n$ by $F^d(\mathbf{x}) = 1 - F(\mathbf{1} - \mathbf{x})$. Propositions 3.5 and 3.6 arise the following result.

Proposition 3.7. *Let $F: [0,1]^n \to [0,1]$, $\vec{r} \in \mathbb{R}^n$ and $F^d: [0,1]^n \to [0,1]$ be the dual function of F defined by $F^d(\mathbf{x}) = 1 - F(\mathbf{1} - \mathbf{x})$.*

(1) F is $\vec{r}$-increasing if and only if F^d is $\vec{r}$-increasing.
(2) F is OD $\vec{r}$-increasing if and only if F^d is OD $\vec{r}^{\,d}$-increasing.
(3) F is SOD $\vec{r}$-increasing if and only if F^d is SOD $-\vec{r}^{\,d}$-decreasing.

The next result follows from the definition of each notion of monotonicity.

Proposition 3.8. *Let $F: [0,1]^n \to [0,1]$ be a $T\,\vec{r}$-monotone function for $T \in \{\emptyset,\ SOD,\ OD\}$. Then, if $\varphi: [0,1] \to [0,1]$ is an increasing (resp. decreasing) function, then the function $\varphi \circ F: [0,1]^n \to [0,1]$ is an $T\,\vec{r}$-monotone function of the same (resp. reversed) type as F.*

Lemma 3.9. *Assume that the function $F: [0,1]^n \to [0,1]$ satisfies $F(\mathbf{x}_\sigma) = F(\mathbf{x})$ for all $\mathbf{x} \in [0,1]^n$ and $\sigma \in \mathcal{S}_n$. Then*

(1) $\mathcal{D}^\uparrow(F) \subseteq \mathcal{D}^\uparrow_{SOD}(F)$ and $\mathcal{C}(F) \subseteq \mathcal{C}_{SOD}(F)$.
(2) $\vec{r} \in \mathcal{D}^\uparrow(F)$ if and only if $\vec{r}_\sigma \in \mathcal{D}^\uparrow(F)\ \forall \sigma \in \mathcal{S}_n$. Analogously for $\mathcal{C}(F)$.

Proof. (1) Let $\vec{r} \in \mathbb{R}^n$, $\mathbf{x} \in [0,1]^n$, $\sigma \in \mathcal{S}_n$, $c \in \mathbb{R}^+$ with $\mathbf{x}_\sigma \in [0,1]^n_{(\geq)}$, $\mathbf{x}_\sigma + c\vec{r} \in [0,1]^n$. Assume that F is $\vec{r}$-increasing. Then

$$\begin{aligned} F(\mathbf{x} + c\vec{r}_{\sigma^{-1}}) &= F(\mathbf{x}_\sigma + c\vec{r}) \text{ (as } (\mathbf{x} + c\vec{r}_{\sigma^{-1}})_\sigma = \mathbf{x}_\sigma + c\vec{r}) \\ &\geq F(\mathbf{x}_\sigma) \text{ (as } F \text{ is } \vec{r}\text{-increasing)} \\ &= F(\mathbf{x}), \end{aligned}$$

hence F is SOD $\vec{r}$-increasing. Analogously one has $\mathcal{C}(F) \subseteq \mathcal{C}_{\mathrm{SOD}}(F)$.

(2) Let $\vec{r} \in \mathcal{D}^{\uparrow}(F)$. If $\mathbf{x}$, $\mathbf{x}+c\vec{r}_\sigma \in [0,1]^n$, then $\mathbf{x}_{\sigma^{-1}}$, $\mathbf{x}_{\sigma^{-1}} + c\vec{r} \in [0,1]^n$, hence $F(\mathbf{x}) = F(\mathbf{x}_{\sigma^{-1}}) \leq F(\mathbf{x}_{\sigma^{-1}} + c\vec{r}) = F(\mathbf{x} + c\vec{r}_\sigma)$ and so $\vec{r}_\sigma \in \mathcal{D}^{\uparrow}(F)$. Analogously for $\mathcal{C}(F)$. □

Proposition 3.10. *Let $F\colon [0,1]^n \to [0,1]$ be a function and define the function $\widehat{F}\colon [0,1]^n \to [0,1]$ as follows: if $\mathbf{x} \in [0,1]^n$, take $\sigma \in \mathcal{S}_n$ such that $\mathbf{x}_\sigma \in [0,1]^n_{(\geq)}$ and put $\widehat{F}(\mathbf{x}) = F(\mathbf{x}_\sigma)$. If $\vec{r} \in \mathbb{R}^n$ is such that F is $\vec{r}$-increasing, then $\widehat{F}$ is OD $\vec{r}$-increasing.*

Proof. Let $\mathbf{x} \in [0,1]^n$, $\sigma \in \mathcal{S}_n$ and $c \in \mathbb{R}^+$ such that $\mathbf{x}_\sigma$, $\mathbf{x}_\sigma + c\vec{r} \in [0,1]^n_{(\geq)}$. Then, with $\mathbf{y} = \mathbf{x} + c\vec{r}_{\sigma^{-1}}$, we have that $\mathbf{y}_\sigma = \mathbf{x}_\sigma + c\vec{r} \in [0,1]^n_{(\geq)}$. So, by definition of $\widehat{F}$,

$$\widehat{F}(\mathbf{x} + c\vec{r}_{\sigma^{-1}}) = \widehat{F}(\mathbf{y}) = F(\mathbf{y}_\sigma) = F(\mathbf{x}_\sigma + c\vec{r}) \geq F(\mathbf{x}_\sigma) = \widehat{F}(\mathbf{x}),$$

as F is $\vec{r}$-increasing. □

In Section 5 we show that in the hypothesis of Proposition 3.10, $\widehat{F}$ is not necessarily SOD $\vec{r}$-increasing.

Remark 3.11. Consider a function $F\colon [0,1]^n \to [0,1]$. Then

a. $\widehat{(\widehat{F})} = \widehat{F}$.
b. $\widehat{F} = F$ if and only if $F(\mathbf{x}_\sigma) = F(\mathbf{x})$ for all $\mathbf{x} \in [0,1]^n$ and $\sigma \in \mathcal{S}_n$.
c. $(\widehat{F})^c = \widehat{F^c}$.
For $(\widehat{F})^c(\mathbf{x}) = 1 - \widehat{F}(\mathbf{x}) = 1 - F(\mathbf{x}_\sigma) = F^c(\mathbf{x}_\sigma) = \widehat{F^c}(\mathbf{x})$ if $\mathbf{x} \in [0,1]^n$ and $\sigma \in \mathcal{S}_n$ is such that $\mathbf{x}_\sigma \in [0,1]^n_{(\geq)}$.

4. Linearity and ordered linearity

In this section we study a relevant class of functions that are useful in order to introduce some examples and to describe some well-known functions (like the Łukasiewicz implication and the discrete Choquet integral).

Given $\mu \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^n$, let us consider the function $F: \mathbb{R}^n \to \mathbb{R}$ given by $F(\mathbf{x}) = \mu + \mathbf{x} \cdot \vec{v}$ for all $\mathbf{x} \in \mathbb{R}^n$, where $\cdot$ denotes the usual scalar product, i.e., $\mathbf{x} \cdot \vec{v} = \sum_{i=1}^n x_i v_i$. The restriction of F to $[0,1]^n$ is a function when $F([0,1]^n) \subseteq [0,1]$.

Definition 4.1. We say that a pair $(\mu, \vec{v}) \in \mathbb{R} \times \mathbb{R}^n$ generates a *linear fusion function* if $\mu + \mathbf{x} \cdot \vec{v} \in [0,1]$ for all $\mathbf{x} \in [0,1]^n$. In such a case, we denote by $L[\mu, \vec{v}]$ the function

$$L[\mu, \vec{v}](\mathbf{x}) = \mu + \mathbf{x} \cdot \vec{v} \qquad \text{for all } \mathbf{x} \in [0,1]^n,$$

which we call the $[\mu, \vec{v}]$-*linear fusion function* (or the *fusion function generated by the pair* $(\mu, \vec{v}) \in \mathbb{R} \times \mathbb{R}^n$).

Remark 4.2. We use the term *fusion function* to explicitly distinguish this class of functions from that of linear functions.

Lemma 4.3. *Let $\vec{v} \in \mathbb{R}^n$ and consider the map $F\colon [0,1]^n \to \mathbb{R}$ given by $F(\mathbf{x}) = \mathbf{x} \cdot \vec{v}$. Set*

$$\mathcal{M} = \{(x_1, \dots, x_n) \in [0,1]^n \mid x_i \in \{0,1\},\ i = 1, \dots, n\}.$$

Then, there exist $\mathbf{a}, \mathbf{b} \in \mathcal{M}$ such that $F(\mathbf{a}) = \max F$ and $F(\mathbf{b}) = \min F$.

Proof. Set $\mathcal{P} = \{\lambda_i \mid \lambda_i > 0,\ 1 \le i \le n\}$, $\mathcal{N} = \{\lambda_i \mid \lambda_i < 0,\ 1 \le i \le n\}$, where $\vec{v} = (\lambda_1, \dots, \lambda_n)$. The claim follows from the obvious facts that $\max F = 0$ if $\mathcal{P} = \emptyset$, $\max F = \sum_{\lambda \in \mathcal{P}} \lambda$ if $\mathcal{P} \neq \emptyset$ and $\min F = 0$ if $\mathcal{N} = \emptyset$ and $\min F = \sum_{\lambda \in \mathcal{N}} \lambda$ if $\mathcal{N} \neq \emptyset$. □

Proposition 4.4. *The pair $[\mu, \vec{v}] \in \mathbb{R} \times \mathbb{R}^n$ defines a linear fusion function if and only if $0 \le \mu + \sum_{i \in S} \lambda_i \le 1$ for all $S \subseteq \{1, \dots, n\}$, where $\vec{v} = (\lambda_1, \dots, \lambda_n)$.*

Proof. It follows immediately from Lemma 4.3. □

Corollary 4.5. *If $[\mu, \vec{v}] \in \mathbb{R} \times \mathbb{R}^n$ satisfies the conditions in Proposition 4.4, then also $[1-\mu, -\vec{v}]$ satisfies them and $F = \mathrm{L}[\mu, \vec{v}]$ if and only if $F^c = \mathrm{L}[1-\mu, -\vec{v}]$.*

Proof. Note that $0 \le \mu + \sum_{i \in S} \lambda_i \le 1$ if and only if $1 \ge (1-\mu) + \sum_{i \in S} (-\lambda_i) \ge 0$, where $\vec{v} = (\lambda_1, \dots, \lambda_n)$ and $S \subseteq \{1, \dots, n\}$. □

Proposition 4.6. *Assume that $[\mu, \vec{v}] \in \mathbb{R} \times \mathbb{R}^n$ defines the $[\mu, \vec{v}]$*-linear fusion function $F\colon [0,1]^n \to [0,1]$. *Then*

(1) $\mathcal{D}^{\uparrow}(F) = \{\vec{r} \in \mathbb{R}^n \mid \vec{r} \cdot \vec{v} \ge 0\}$,
(2) $\mathcal{C}(F) = \{\vec{r} \in \mathbb{R}^n \mid \vec{r} \cdot \vec{v} = 0\}$,
(3) $\mathcal{D}^{\uparrow}_{OD}(F) = \mathcal{D}^{\uparrow}_{SOD}(F) = \{\vec{r} \in \mathbb{R}^n \mid \vec{r}_\sigma \cdot \vec{v} \ge 0$ for all $\sigma \in \mathcal{S}_n\}$,
(4) $\mathcal{C}_{OD}(F) = \mathcal{C}_{SOD}(F) = \{\vec{r} \in \mathbb{R}^n \mid \vec{r}_\sigma \cdot \vec{v} = 0$ for all $\sigma \in \mathcal{S}_n\}$.

Proof. (1) and (2) follow from the fact that $F(\mathbf{x} + c\vec{r}) - F(\mathbf{x}) = c\vec{r} \cdot \vec{v}$ and (3) and (4) from $F(\mathbf{x} + c\vec{r}_{\sigma^{-1}}) - F(\mathbf{x}) = c\vec{r}_{\sigma^{-1}} \cdot \vec{v}$, if $\mathbf{x} \in [0,1]^n$, $c > 0$, $\sigma \in \mathcal{S}_n$ and $\mathbf{x}_\sigma + c\vec{r} \in [0,1]^n$. □

Example 4.7.

(1) The constant function $F\colon [0,1]^n \to [0,1]$, given by $F(\mathbf{x}) = k$ for all $\mathbf{x} \in [0,1]^n$, where $k \in [0,1]$, is the $[k, \mathbf{0}]$-linear fusion function.
(2) If $\mathbf{w} = (w_1, \dots, w_n) \in [0,1]^n$ satisfies $\sum_{i=1}^n w_i = 1$, the corresponding weighted average $F\colon [0,1]^n \to [0,1]$ given by $F(\mathbf{x}) = \mathbf{x} \cdot \mathbf{w}$ if $\mathbf{x} \in [0,1]^n$ is the $[0, \mathbf{w}]$-linear fusion function.

Definition 4.8. We say that a pair $(\mu, \vec{v}) \in \mathbb{R} \times \mathbb{R}^n$ generates a *ordered (O) linear fusion function* if $\mu + \mathbf{x}_\sigma \cdot \vec{v} \in [0,1]$ for all $\mathbf{x} \in [0,1]^n$ and $\sigma \in \mathcal{S}_n$ such that $\mathbf{x}_\sigma \in [0,1]^n_{(\ge)}$. In such a case, we denote by $OL[\mu, \vec{v}]$ the function

$$OL[\mu, \vec{v}]\,(\mathbf{x}) = \mu + \mathbf{x}_\sigma \cdot \vec{v} \qquad \text{for all } \mathbf{x} \in [0,1]^n,$$

which we call the *ordered* $[\mu, \vec{v}]$*-linear fusion function*.

Remark 4.9. Note that if $F = OL[\mu, \vec{v}]$, then $F(\mathbf{x}_\sigma) = F(\mathbf{x})$ for all $\sigma \in \mathcal{S}_n$. In particular, $F([0,1]^n) = F([0,1]^n_{(\ge)})$.

Lemma 4.10. *Let $\vec{v} = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ and consider the function $F : [0,1]^n \to \mathbb{R}$ given by $F(\mathbf{x}) = \mathbf{x}_\sigma \cdot \vec{v}$ if $\mathbf{x} \in [0,1]^n$, where $\sigma \in \mathcal{S}_n$ satisfies $\mathbf{x}_\sigma \in [0,1]^n_{(\ge)}$. Set*

$$\mathbf{u}_0 = \mathbf{0}, \quad \mathbf{u}_r = (1, \dots, \underset{r}{\underset{\frown}{1}}, 0, \dots, 0) \in [0,1]^n, \ \text{for } r \in \{1, \dots, n\}.$$

Then there exist $j, k \in \{0, 1, \dots, n\}$ such that $\max_{\mathbf{x}} F(\mathbf{x}) = F(\mathbf{u}_j)$ and $\min_{\mathbf{x}} F(\mathbf{x}) = F(\mathbf{u}_k)$.

Proof. Set $M = \max_{0 \le i \le n} F(\mathbf{u}_i)$ and $m = \min_{0 \le i \le n} F(\mathbf{u}_i)$. Proceed by induction on n. Let $n = 1$. If $\lambda_1 \ge 0$, then $M = F(1) = \max_x F(x)$ and $m = F(0) = \min_x F(x)$. If $\lambda_1 \le 0$, then $M = F(0) = \max_x F(x)$ and $m = F(1) = \min_x F(x)$.

Let now $n > 1$ and consider the case of the maximum. Let us suppose that there exists $\mathbf{x} \in [0,1]^n$ such that $M < F(\mathbf{x})$. Since $F(\mathbf{u}_0) = 0$, we have that $F(\mathbf{x}) > 0$. We may assume, without loss of generality, that $x_1 \ge \ldots \ge x_n$, where $=(x_1, \dots, x_n)$. As $F(\mathbf{x}) > 0$, then $x_1 > 0$, and

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$$
\begin{aligned}
\mathbf{x}\cdot\vec{v} &= x_1\lambda_1+\ldots+x_n\lambda_n\\
&= x_1\left(\lambda_1+\frac{x_2}{x_1}\lambda_2+\ldots+\frac{x_n}{x_1}\lambda_n\right)\\
&\le \lambda_1+\frac{x_2}{x_1}\lambda_2+\ldots+\frac{x_n}{x_1}\lambda_n\\
&= \frac{1}{x_1}\mathbf{x}\cdot\vec{v},
\end{aligned}
$$

as $\lambda_1+\frac{x_2}{x_1}\lambda_2+\ldots+\frac{x_n}{x_1}\lambda_n>0$ and $x_1\in[0,1]$. Set now $\mathbf{y}=(y_1,\ldots,y_n)=\frac{1}{x_1}\mathbf{x}$, and observe that $1=y_1\ge\ldots\ge y_n$.

If we consider $\mathbf{z}=(z_2,\ldots,z_n)\in[0,1]^{n-1}$, then by the induction hypothesis, the scalar product $(z_2,\ldots,z_n)\cdot(\lambda_2,\ldots,\lambda_n)$ reaches an absolute maximum for some

$$\mathbf{t}=(t_2,\ldots,t_n)\in\{(1,\ldots,\underset{r}{\underset{\frown}{1}},0,\ldots,0)\in[0,1]^{n-1}\mid 0\le r\le n-1\}.$$

Thus,

$$
\begin{aligned}
M &< F(\mathbf{x})=\lambda_1+(y_2,\ldots,y_n)\cdot(\lambda_2,\ldots,\lambda_n)\\
&\le \lambda_1+\mathbf{t}\cdot(\lambda_2,\ldots,\lambda_n)=(1,t_2,\ldots,t_n)\cdot(\lambda_1,\ldots,\lambda_n),
\end{aligned}
$$

where $(1,t_2,\ldots,t_n)\in\{\mathbf{u}_0,\mathbf{u}_1,\ldots,\mathbf{u}_n\}$, which contradicts the election of M.
Observe that $(-F)(\mathbf{x})=\mathbf{x}_\sigma\cdot(-\vec{v})$. So $\min F=\max(-F)$. □

Remark 4.11. In the conditions of Lemma 4.10, as F is a continuous function on a compact set of $\mathbb{R}^n$, we know of the existence of a maximum and a minimum, but this fact is not used in the proof of Lemma 4.10. (A similar remark can be made about Proposition 4.4.).

Proposition 4.12. *The pair $[\mu,\vec{v}]\in\mathbb{R}\times\mathbb{R}^n$, where $\vec{v}=(\lambda_1,\ldots,\lambda_n)$, defines an O linear fusion function if and only if $0\le\mu\le1$ and $0\le\mu+\sum_{i=1}^{r}\lambda_i\le1$ for all $r\in\{1,\ldots,n\}$.*

Proof. It is an immediate consequence of Lemma 4.10. □

Corollary 4.13. *If $F:[0,1]^n\to[0,1]$ is the $[\mu,\vec{v}]$-linear fusion function, then $\widehat{F}$ is the O $[\mu,\vec{v}]$-linear fusion function.*

Analogously as for Corollary 4.5 we have the following result.

Corollary 4.14. *Let $[\mu,\vec{v}]\in\mathbb{R}\times\mathbb{R}^n$ satisfying the conditions of Corollary 4.12. Then, the pair $[1-\mu,-\vec{v}]$ also satisfies the conditions of Corollary 4.12 and $F=\mathrm{OL}[\mu,\vec{v}]$ if and only if $F^c=\mathrm{OL}[1-\mu,-\vec{v}]$.*

Proposition 4.15. *Let $[\mu,\vec{v}]\in\mathbb{R}\times\mathbb{R}^n$ satisfying the conditions of Corollary 4.12 and $F=\mathrm{OL}[\mu,\vec{v}]$. Then*

(1) $\mathcal{D}^{\uparrow}_{OD}(F)=\{\vec{r}\in\mathbb{R}^n\mid\vec{r}\cdot\vec{v}\ge0\}$.
(2) $\mathcal{C}_{OD}(F)=\{\vec{r}\in\mathbb{R}^n\mid\vec{r}\cdot\vec{v}=0\}$.

Proof. Let $\mathbf{x}\in[0,1]^n$, $c>0$ and $\sigma\in\mathcal{S}_n$ such that $\mathbf{x}_\sigma$, $\mathbf{x}_\sigma+c\vec{r}\in[0,1]^n_{(\ge)}$. Then $F(\mathbf{x}+c\vec{r}_{\sigma^{-1}})=\mu+(\mathbf{x}_\sigma+c\vec{r})\cdot\vec{v}=F(\mathbf{x})+c\vec{r}\cdot\vec{v}$ and the thesis follows. □

In several occasions we focus on the particular case $n=2$. Observe that if $\sigma\in\mathcal{S}_2$, we have that $\sigma^{-1}=\sigma$.
The following auxiliary result follows immediately from a simple geometric approach.

Lemma 4.16. *Let $\vec{r}=(r_1,r_2)\in\mathbb{R}^2$.*

(1) There exist $\mathbf{y}\in[0,1]^2_{(\ge)}$ and $c>0$ such that $\mathbf{y}+c\vec{r}\in[0,1]^2_{(\ge)}$.

(2) There exist $\mathbf{y} \in [0,1]^2_{(\le)}$ *and* $c > 0$ *such that* $\mathbf{y} + c\vec{r} \in [0,1]^2_{(\le)}$.
(3) If $\vec{r} \in \mathbb{R}^2_{(>)}$ *there exist* $\mathbf{y} \in [0,1]^2_{(<)}$ *and* $c > c' > 0$ *such that* $\mathbf{y} + c\vec{r} \in [0,1]^2_{(>)}$ *and* $\mathbf{y} + c'\vec{r} \in [0,1]^2_{(=)}$.
(4) If $\vec{r} \in \mathbb{R}^2_{(<)}$ *there exist* $\mathbf{y} \in [0,1]^2_{(>)}$ *and* $c > c' > 0$ *such that* $\mathbf{y} + c\vec{r} \in [0,1]^2_{(<)}$ *and* $\mathbf{y} + c'\vec{r} \in [0,1]^2_{(=)}$.

Proposition 4.17. *Let* $[\mu, \vec{v}] \in \mathbb{R} \times \mathbb{R}^2$ *satisfying the conditions of Corollary 4.12 and* $F = \mathrm{OL}[\mu, \vec{v}]$. *Then*

(1) $\mathcal{D}^{\uparrow}_{OD}(F) = \{\vec{r} \in \mathbb{R}^2 \mid \vec{r} \cdot \vec{v} \ge 0\}$.
(2) $\mathcal{C}_{OD}(F) = \{\vec{r} \in \mathbb{R}^2 \mid \vec{r} \cdot \vec{v} = 0\}$.
(3) $\mathcal{D}^{\uparrow}_{SOD}(F) = \{\vec{r} \in \mathbb{R}^2_{(\ge)} \mid \vec{r} \cdot \vec{v} \ge 0\} \cup \{\vec{r} \in \mathbb{R}^2_{(\le)} \mid \vec{r} \cdot \vec{v} \ge 0 \text{ and } \vec{r} \cdot \vec{v}^{\,d} \ge 0\}$.
(4) $\mathcal{C}_{SOD}(F) = \{\vec{r} \in \mathbb{R}^2_{(\ge)} \mid \vec{r} \cdot \vec{v} = 0\} \cup \{\vec{r} \in \mathbb{R}^2_{(\le)} \mid \vec{r} \cdot \vec{v} = \vec{r} \cdot \vec{v}^{\,d} = 0\}$.
(5) $\mathcal{D}^{\uparrow}(F) = \{\vec{r} \in \mathbb{R}^2 \mid \vec{r} \cdot \vec{v} \ge 0 \text{ and } \vec{r} \cdot \vec{v}^{\,d} \ge 0\}$.
(6) $\mathcal{C}(F) = \{\vec{r} \in \mathbb{R}^2 \mid \vec{r} \cdot \vec{v} = \vec{r} \cdot \vec{v}^{\,d} = 0\}$.

Proof. (1) and (2) are particular cases of Proposition 4.15.

Let $\mathbf{x} \in [0,1]^2$, $c > 0$ and $\sigma \in \mathcal{S}_2$ such that $\mathbf{x}_\sigma \in [0,1]^2_{(\ge)}$ and $\mathbf{x}_\sigma + c\vec{r} \in [0,1]^2$. If actually $\mathbf{x}_\sigma + c\vec{r} \in [0,1]^2_{(\ge)}$, then we have

$$F(\mathbf{x} + c\vec{r}_\sigma) - F(\mathbf{x}) = c\vec{r} \cdot \vec{v}\,. \tag{A}$$

Let us assume that $x_{\sigma(1)} + cr_1 < x_{\sigma(2)} + cr_2$ (then necessarily is $r_1 < r_2$). Then $F(\mathbf{x} + c\vec{r}_\sigma) = \mu + (x_{\sigma(2)} + cr_2)\lambda_1 + (x_{\sigma(1)} + cr_1)\lambda_2$. Thus

$$F(\mathbf{x} + c\vec{r}_\sigma) - F(\mathbf{x}) = (x_{\sigma(1)} - x_{\sigma(2)})(\lambda_2 - \lambda_1) + c\vec{r} \cdot \vec{v}^{\,d}\,. \tag{B}$$

Observe that

$$\vec{r} \cdot \vec{v}^{\,d} = \vec{r} \cdot \vec{v} + (r_2 - r_1)(\lambda_1 - \lambda_2)\,. \tag{C}$$

(3) Let us assume that F is SOD $\vec{r}$-increasing. If $r_1 \ge r_2$, then we are in (A). By Lemma 4.16, if $r_1 < r_2$ both situations (A) and (B), in the last case also with $x_{\sigma(1)} = x_{\sigma(2)}$, can occur. Therefore

$$\mathcal{D}^{\uparrow}_{\mathrm{SOD}}(F) \subseteq \{\vec{r} \in \mathbb{R}^2_{(\ge)} \mid \vec{r} \cdot \vec{v} \ge 0\} \cup \{\vec{r} \in \mathbb{R}^2_{(\le)} \mid \vec{r} \cdot \vec{v} \ge 0 \text{ and } \vec{r} \cdot \vec{v}^{\,d} \ge 0\}\,.$$

Conversely, if $r_1 \ge r_2$ and $\vec{r} \cdot \vec{v} \ge 0$ then (see A) $F(\mathbf{x} + c\vec{r}_\sigma) \ge F(\mathbf{x})$.

Let $r_1 < r_2$, $\vec{r} \cdot \vec{v} \ge 0$, $\vec{r} \cdot \vec{v}^{\,d} \ge 0$.

If $\lambda_1 \le \lambda_2$, then (see (A) and (B)) $F(\mathbf{x} + c\vec{r}_\sigma) \ge F(\mathbf{x})$.

Suppose that $\lambda_1 > \lambda_2$. If $x_{\sigma(1)} = x_{\sigma(2)}$, then $F(\mathbf{x} + c\vec{r}_\sigma) \ge F(\mathbf{x})$. Suppose further that $x_{\sigma(1)} > x_{\sigma(2)}$. With $x_{\sigma(1)} - x_{\sigma(2)} = x$, $c(r_2 - r_1) = r$, we have $0 < x < r$. So $c\vec{r} \cdot \vec{v}^{\,d} = c\vec{r} \cdot \vec{v} + r(\lambda_1 - \lambda_2)$ (see (C)) and

$$\begin{aligned} F(\mathbf{x} + c\vec{r}_\sigma) - F(\mathbf{x}) &= x(\lambda_2 - \lambda_1) + c\vec{r} \cdot \vec{v} + r(\lambda_1 - \lambda_2) \\ &= (r - x)(\lambda_1 - \lambda_2) + c\vec{r} \cdot \vec{v} > 0\,, \end{aligned}$$

hence F is SOD $\vec{r}$-increasing.

(4) Arguing as in (3) we deduce that

$$\mathcal{C}_{\mathrm{SOD}}(F) \subseteq \{\vec{r} \in \mathbb{R}^2_{(\ge)} \mid \vec{r} \cdot \vec{v} = 0\} \cup \{\vec{r} \in \mathbb{R}^2_{(\le)} \mid \vec{r} \cdot \vec{v} = \vec{r} \cdot \vec{v}^{\,d} = 0\}\,.$$

If $r_1 \ge r_2$ and $\vec{r} \cdot \vec{v} = 0$, then $F(\mathbf{x} + c\vec{r}_\sigma) = F(\mathbf{x})$ (see (A)). Assume now that $r_1 < r_2$, $\vec{r} \cdot \vec{v} = \vec{r} \cdot \vec{v}^{\,d} = 0$. We have so $(r_2 - r_1)(\lambda_1 - \lambda_2) = 0$ (see (C)) and therefore it must be $\lambda_1 = \lambda_2$ and so $F(\mathbf{x} + c\vec{r}_\sigma) = F(\mathbf{x})$ (see (B)) also in this case.

Let now $\mathbf{x} \in [0,1]^2$ and $c > 0$ such that $\mathbf{x} + c\vec{r} \in [0,1]^2$. We have the following possibilities (set $d = F(\mathbf{x} + c\vec{r}) - F(\mathbf{x})$):

- $\mathbf{x} \in [0,1]^2_{(\ge)}$ and
  (a) $\mathbf{x} + c\vec{r} \in [0,1]^2_{(\ge)}$, when $d = c\vec{r} \cdot v$, or

(b) $x_1 + cr_1 \leq x_2 + cr_2$, when $d = (x_1 - x_2)(\lambda_2 - \lambda_1) + c\vec{r} \cdot \vec{v}^{\,d}$ (and necessarily $r_1 \leq r_2$), or

- $\mathbf{x} \in [0,1]^2_{(\leq)}$ and
  (c) $x_1 + cr_1 \geq x_2 + cr_2$, when $d = (x_2 - x_1)(\lambda_2 - \lambda_1) + c\vec{r} \cdot \vec{v}$ (and necessarily $r_1 \geq r_2$), or
  (d) $\mathbf{x} + c\vec{r} \in [0,1]^2_{(\leq)}$, when $d = c\vec{r} \cdot \vec{v}^{\,d}$.

(5) Therefore if F is $\vec{r}$-increasing, necessarily $\vec{r} \cdot \vec{v} \geq 0$ and $\vec{r} \cdot \vec{v}^{\,d} \geq 0$. Let us assume that this happens. If we are in the case (a) or (d), then $d \geq 0$. If we are in the case (b) and $\lambda_2 \geq \lambda_1$, then $d \geq 0$. If $\lambda_2 < \lambda_1$ then (see (C))

$$d = [c(r_2 - r_1) - (x_1 - x_2)](\lambda_1 - \lambda_2) + c\vec{r} \cdot \vec{v} \geq 0$$

since $c(r_2 - r_1) \geq x_1 - x_2$ because $x_2 + cr_2 \geq x_1 + cr_1$. Proceed analogously in the case (c).

(6) Proceed analogously as in (4). □

Remark 4.18. Let $[\mu, (\lambda_1, \lambda_2)] \in \mathbb{R} \times \mathbb{R}^2$ satisfying the conditions of Corollary 4.12 and let F the corresponding O linear fusion function. We may simplify some expressions of Proposition 4.17 in some cases. From (C) in Proposition 4.17 we deduce that if $\lambda_1 \geq \lambda_2$ then $r_1 \leq r_2$ implies $\vec{r} \cdot \vec{v}^{\,d} \geq \vec{r} \cdot \vec{v}$. So if $\lambda_1 \geq \lambda_2$, then

$$\mathcal{D}^{\uparrow}_{\mathrm{SOD}}(F) = \mathcal{D}^{\uparrow}_{\mathrm{OD}}(F) = \{\vec{r} \in \mathbb{R}^2 \mid \vec{r} \cdot \vec{v} \geq 0\}\,.$$

Analogously, if $\vec{r} \cdot \vec{v} = 0$, then $\vec{r} \cdot \vec{v}^{\,d} = (r_2 - r_1)(\lambda_1 - \lambda_2)$. So, in this case, $\vec{r} \cdot \vec{v}^{\,d} = 0$ if and only if either $r_1 = r_2$ or $\lambda_1 = \lambda_2$. Therefore

- If $\lambda_1 \neq \lambda_2$ then $\mathcal{C}_{\mathrm{SOD}}(F) = \mathcal{C}(F) = \{\vec{r} \in \mathbb{R}^2_{(\geq)} \mid \vec{r} \cdot \vec{v} = 0\}$.
- If $\lambda_1 = \lambda_2$ then $\mathcal{C}_{\mathrm{SOD}}(F) = \mathcal{C}_{\mathrm{OD}}(F) = \mathcal{C}(F) = \{\vec{r} \in \mathbb{R}^2 \mid \vec{r} \cdot \vec{v} = 0\}$.

Example 4.19. Let us present some examples of O linear fusion functions. We omit the mentions $\mathbf{x} \in [0,1]^n$ and $\sigma \in \mathcal{S}_n$ is such that $\mathbf{x}_\sigma \in [0,1]^n_{(\geq)}$.

(1) If $\mathbf{a} = (a_1, \ldots, a_n) \in [0,1]^n_{(\geq)}$, put $\widetilde{\mathbf{a}} = (a_1, -a_2, \ldots, (-1)^{n-1}a_n)$. Then we have the O $[0, \widetilde{\mathbf{a}}]$-linear fusion function $F_{\mathbf{a}} \colon [0,1]^n \to [0,1]$ given by

$$F_{\mathbf{a}}(\mathbf{x}) = x_{\sigma(1)}a_1 - x_{\sigma(2)}a_2 + \cdots + (-1)^{n-1}x_{\sigma(n)}a_n\,.$$

(2) If $\lambda \in [0,1]$, the O $[0, (1, -\lambda)]$-linear fusion function $F \colon [0,1]^2 \to [0,1]$ given by

$$F(\mathbf{x}) = \max(x_1, x_2) - \lambda \min(x_1, x_2)\,.$$

Observe that $F^c \colon [0,1]^2 \to [0,1]$ is the O $[1, (-1, \lambda)]$-linear fusion function given by

$$F^c(\mathbf{x}) = 1 - \max(x_1, x_2) + \lambda \min(x_1, x_2)\,.$$

(3) The O $[0, (1, -1)]$-linear fusion function $F \colon [0,1]^2 \to [0,1]$ given by

$$F(\mathbf{x}) = |x_1 - x_2|\,.$$

Then $F^c \colon [0,1]^2 \to [0,1]$ is the O $[1, (-1, 1)]$-linear fusion function given by

$$F^c(\mathbf{x}) = 1 - |x_1 - x_2|\,,$$

which is a restricted equivalence function (see [4]).

(4) The O $\left[\frac{1}{2}, (\frac{1}{2}, -1)\right]$-linear fusion function $F \colon [0,1]^2 \to [0,1]$ given by

$$F(\mathbf{x}) = \frac{1}{2}(1 + \max(x_1 - x_2) - 2\min(x_1 - x_2))\,.$$

(5) The O $\left[1, (-1, \frac{1}{2})\right]$-linear fusion function $F \colon [0,1]^2 \to [0,1]$ given by

$$F(\mathbf{x}) = 1 - \max(x_1, x_2) + \frac{1}{2}\min(x_1, x_2)\,.$$

(6) Let $\mathbf{w} \in [0,1]^n$ with $\sum_{i=1}^n w_i = 1$, the OWA operator $A\colon [0,1]^n \to [0,1]$ with respect to the weighting vector $\mathbf{w}$, given by

$$A(\mathbf{x}) = \mathbf{x}_\sigma \cdot \mathbf{w}$$

is the O $[0, \mathbf{w}]$-linear fusion function. Observe that if $F\colon [0,1]^n \to [0,1]$ is the weighted average corresponding to $\mathbf{w}$, then $A = \widehat{F}$ (see Example 4.7 (2) and Corollary 4.13).

Example 4.20. Let $p > 0$ and $\vec{r} = (t, \dots, t, s) \in \mathbb{R}^n$, where $s \leq t$. The function $F\colon [0,1]^n \to [0,1]$ given by

$$F(\mathbf{x}) = \frac{1}{n-1} \sum_{j=2}^{n} |x_1 - x_j|^p \,,$$

if $\mathbf{x} = (x_1, \dots, x_n) \in [0,1]^n$, is SOD $\vec{r}$-increasing.

Indeed, given $\mathbf{x} \in [0,1]^n$, $\sigma \in \mathcal{S}_n$ and $c \in \mathbb{R}^+$ such that $\mathbf{x}_\sigma \in [0,1]^n_{(\geq)}$, $\mathbf{x}_\sigma + c\vec{r} \in [0,1]^n$. Observe that

$$\mathbf{x} + c\vec{r}_{\sigma^{-1}} = (x_1 + ct, \dots, x_{i-1} + ct, x_i + cs, x_{i+1} + ct, \dots, x_n + ct)\,,$$

where $i = \sigma(n)$ and that $x_{\sigma(n)} = \min\{x_1, \dots, x_n\}$, so that $x_1 \geq x_i$. As $t \geq s$, we have

$$\begin{aligned} F(\mathbf{x} + c\vec{r}_{\sigma^{-1}}) &= \frac{1}{n-1}\left(\sum_{\substack{j=2\\ j\neq i}}^{n} |x_1 - x_j|^p + |(x_1 - x_i) + c(t-s)|^p\right) \\ &\geq \frac{1}{n-1}\sum_{j=2}^{n} |x_1 - x_j|^p = F(\mathbf{x})\,, \end{aligned}$$

as required. Note that, in the case where $\sigma(n) = 1$, the result follows readily since $x_1 \leq x_j$ and $s \leq t$. Thus,

$$|x_1 - x_j + cs - ct| \geq |x_1 - x_j|.$$

Corollary 4.21. *Let $0 < p \in \mathbb{R}$, $\vec{r} = (r_1, r_2) \in \mathbb{R}^2$ and consider the function $F\colon [0,1]^2 \to [0,1]$ given by*

$$F(x_1, x_2) = |x_1 - x_2|^p$$

if $(x_1, x_2) \in [0,1]^2$. Then

(1) $\mathcal{D}^{\uparrow}_{SOD}(F) = \mathcal{D}^{\uparrow}_{OD}(F) = \{\vec{r} \in \mathbb{R}^2 \mid r_1 \geq r_2\}$.
(2) $\mathcal{D}^{\uparrow}(F) = \mathcal{C}(F) = \mathcal{C}_{SOD}(F) = \mathcal{C}_{OD}(F) = \{\vec{r} \in \mathbb{R}^2 \mid r_1 = r_2\}$.

Proof. Consider $\varphi\colon [0,1] \to [0,1]$ given by $\varphi(x) = x^p$ if $x \in [0,1]$. Then $F = \varphi \circ F_1$, where $F_1(\mathbf{x}) = |x_1 - x_2|$, so it is consequence of Propositions 3.8 and 4.17. □

Proposition 4.22. *Let $0 < p \in \mathbb{R}$ and consider the function $F\colon [0,1]^2 \to [0,1]$ given by*

$$F(\mathbf{x}) = 1 - |x_1 - x_2|^p$$

if $\mathbf{x} = (x_1, x_2) \in [0,1]^2$. Set $\vec{r} = (r_1, r_2)$. Then

(1)

$$\begin{aligned} \mathcal{D}^{\uparrow}(F) = \mathcal{D}^{\uparrow}_{SOD}(F) = \mathcal{C}(F) = \mathcal{C}_{SOD}(F) = \mathcal{C}_{OD}(F) = \\ = \{\vec{r} \in \mathbb{R}^2 \mid r_1 = r_2\}. \end{aligned}$$

(2) $\mathcal{D}^{\uparrow}_{OD}(F) = \{\vec{r} \in \mathbb{R}^2 \mid r_1 \leq r_2\}$.

Proof. Let $M\colon [0,1]^2 \to [0,1]$ given by $M(\mathbf{x}) = |x_1 - x_2|^p$ if $\mathbf{x} = (x_1, x_2) \in [0,1]^2$. Then $F = M^c$, hence, by Proposition 3.5 and Corollary 4.21, it only rests to show the assertion on $\mathcal{D}^{\uparrow}_{\text{SOD}}(F)$. By Lemma 3.9, $\{\vec{r} \in \mathbb{R}^2 \mid r_1 = r_2\} = \mathcal{D}^{\uparrow}(F) \subseteq \mathcal{D}^{\uparrow}_{\text{SOD}}(F)$. Assume that F is SOD $\vec{r}$-increasing. Take $0 < x < 1$ and $\mathbf{x} = (x, x)$, so $\mathbf{x} \in [0,1]^2_{(\geq)}$; take $c > 0$ such that $\mathbf{x} + c\vec{r} \in [0,1]^2$. Then $F(\mathbf{x} + c\vec{r}) = 1 - |c(r_1 - r_2)|^p$. As F is SOD $\vec{r}$-increasing, it must be $F(\mathbf{x} + c\vec{r}) \geq F(\mathbf{x}) = 1$, hence $r_1 = r_2$. □

5. An operation between functions

We introduce here an operation between functions from $[0,1]^2$ to $[0,1]$ which generalizes, when applied to O linear fusion functions, for $n = 2$, the Choquet integral and the Łukasiewicz implication.

Definition 5.1. Let $F_i\colon [0,1]^2 \to [0,1]$, $i = 1, 2$, be two functions such that $F_1(x,x) = F_2(x,x)$ for all $x \in [0,1]$. Define $F_1 * F_2\colon [0,1]^2 \to [0,1]$ by

$$(F_1 * F_2)(\mathbf{x}) = \begin{cases} F_1(\mathbf{x}) \text{ if } \mathbf{x} \in [0,1]^2_{(\geq)}\,, \\ F_2(\mathbf{x}) \text{ if } \mathbf{x} \in [0,1]^2_{(\leq)}\,. \end{cases}$$

Proposition 5.2. *Let $F_i\colon [0,1]^2 \to [0,1]$, $i = 1, 2$, be two functions such that $F_1(x,x) = F_2(x,x)$ for all $x \in [0,1]$. Then, for $T \in \{\emptyset,\ SOD,\ OD\}$, the following hold.*

(1) $\mathcal{D}^{\uparrow}_T(F_1) \cap \mathcal{D}^{\uparrow}_T(F_2) \subseteq \mathcal{D}^{\uparrow}_T(F_1 * F_2)$.
(2) $\mathcal{C}_T(F_1) \cap \mathcal{C}_T(F_2) \subseteq \mathcal{C}_T(F_1 * F_2)$.

Proof. Set in this proof $F = F_1 * F_2$.

- (1) and (2) for T = OD.
  Let $\mathbf{x} \in [0,1]^2$, $\sigma \in \mathcal{S}_2$, $c > 0$ such that $\mathbf{x}_\sigma$, $\mathbf{x}_\sigma + c\vec{r} \in [0,1]^2_{(\geq)}$. These assertions follow immediately taking the following into account.
  (a) If $\sigma = \mathrm{id}$, $F(\mathbf{x} + c\vec{r}) = F_1(\mathbf{x} + c\vec{r})$, $F(\mathbf{x}) = F_1(\mathbf{x})$.
  (b) If $\sigma = (1\,2)$, $F(\mathbf{x} + c\vec{r}_\sigma) = F_2(\mathbf{x} + c\vec{r}_\sigma)$, $F(\mathbf{x}) = F_2(\mathbf{x})$.
- (1) and (2) for T = $\emptyset$.
  Let $\mathbf{x} \in [0,1]^2$ and $c > 0$ such that $\mathbf{x} + c\vec{r} \in [0,1]^2$.
  Assume that $\vec{r} \in \mathcal{D}^{\uparrow}(F_1) \cap \mathcal{D}^{\uparrow}(F_2)$. If $\mathbf{x}$, $\mathbf{x} + c\vec{r} \in [0,1]^2_{(\geq)}$, then $F(\mathbf{x}) = F_1(\mathbf{x}) \leq F_1(\mathbf{x} + c\vec{r}) = F(\mathbf{x} + c\vec{r})$ and $\vec{r} \in \mathcal{D}^{\uparrow}(F)$. Analogously if $\mathbf{x}$, $\mathbf{x} + c\vec{r} \in [0,1]^2_{(\leq)}$. Suppose $\mathbf{x} \in [0,1]^2_{(\geq)}$, $\mathbf{x} + c\vec{r} \in [0,1]^2_{(<)}$. Then $r_2 > r_1$. By Lemma 4.16 there exists $c' > 0$, $c > c'$, such that, with $\mathbf{z} = (z_1, z_2) = \mathbf{x} + (c - c')\vec{r}$, we have $z_1 = z_2$; as $\mathbf{z} + c'\vec{r} = \mathbf{x} + c\vec{r}$, one has, with $c'' = c - c'$,

  $$F(\mathbf{x}) = F_1(\mathbf{x}) \leq F_1(\mathbf{x} + c''\vec{r}) = F_1(\mathbf{z}) = F_2(\mathbf{z}) \leq F_2(\mathbf{z} + c'\vec{r}) = F(\mathbf{x} + c\vec{r})\,.$$

  Analogously for (2).
- (1) and (2) for T = SOD.
  Assume that $\vec{r} \in \mathcal{D}^{\uparrow}_{\text{SOD}}(F_1) \cap \mathcal{D}^{\uparrow}_{\text{SOD}}(F_2)$. Let $\mathbf{x} \in [0,1]^2$, $\sigma \in \mathcal{S}_2$ and $c > 0$ such that $\mathbf{x}_\sigma \in [0,1]^2_{(\geq)}$, $\mathbf{x}_\sigma + c\vec{r} \in [0,1]^2$.
  (a) Suppose that $\sigma = \mathrm{id}$, so $\mathbf{x} \in [0,1]^2_{(\geq)}$.
    – If $\mathbf{x} + c\vec{r} \in [0,1]^2_{(\geq)}$, then $F(\mathbf{x} + c\vec{r}) = F_1(\mathbf{x} + c\vec{r}) \geq F_1(\mathbf{x}) = F(\mathbf{x})$.
    – If $\mathbf{x} + c\vec{r} \in [0,1]^2_{(<)}$, then $r_1 < r_2$ and we may proceed as in the case T= $\emptyset$.
  (b) Suppose that $\sigma = (1\,2)$, so $\mathbf{x} \in [0,1]^2_{(\leq)}$.
    – If $\mathbf{x} + c\vec{r}_\sigma \in [0,1]^2_{(\leq)}$, $F(\mathbf{x} + c\vec{r}_\sigma) = F_2(\mathbf{x} + c\vec{r}_\sigma) \geq F_2(\mathbf{x}) = F(\mathbf{x})$.
    – Assume that $\mathbf{x} + c\vec{r}_\sigma \in [0,1]^2_{(>)}$. Apply Lemma 4.16 to $\vec{s} = \vec{r}_\sigma$ and argue analogous to the case T= $\emptyset$.
  Analogously for (2). □

Lemma 5.3. *Let $G = OL[\mu, \vec{a}]$ and $H = OL[\nu, \vec{b}]$, where $[\mu, \vec{a}]$, $[\nu, \vec{b}]$ belong to $\mathbb{R} \times \mathbb{R}^2$ and satisfy the conditions of Corollary 4.12, and set $\vec{a} = (a_1, a_2)$, $\vec{b} = (b_1, b_2)$. We have that $G(x, x) = H(x, x)$ for all $x \in [0, 1]$ if and only if $\mu = \nu$ and $a_1 + a_2 = b_1 + b_2$.*

Proof. ($\Rightarrow$) With $x = 0$ we have $\mu = \nu$. Take now for instance $x = 1$ and obtain $a_1 + a_2 = b_1 + b_2$.
($\Leftarrow$) Follows immediately. $\square$

Proposition 5.4. *Let $\vec{a} = (a_1, a_2)$, $\vec{b} = (b_1, b_2) \in \mathbb{R}^2$ be such that $s = a_1 + a_2 = b_1 + b_2$ and suppose that $[\mu, \vec{a}\,], [\mu, \vec{b}\,] \in \mathbb{R} \times \mathbb{R}^2$ satisfy the conditions of Proposition 3.7. Let $G = OL[\mu, \vec{a}\,]$ and $H = OL[\mu, \vec{b}\,]$ and consider $G * H$ (recall Lemma 4.3). Then the following statements hold.*

(1) $\mathcal{D}^{\uparrow}_{OD}(G * H) = \{\vec{r} \in \mathbb{R}^2 \,|\, \vec{r} \cdot \vec{a} \geq 0,\ \vec{r} \cdot \vec{b} \geq 0\}$.
(2) $\mathcal{C}_{OD}(G * H) = \{\vec{r} \in \mathbb{R}^2 \,|\, \vec{r} \cdot \vec{a} = \vec{r} \cdot \vec{b} = 0\}$.
(3)
$$\begin{aligned}\mathcal{D}^{\uparrow}_{SOD}(G * H) = \{&\vec{r} \in \mathbb{R}^2_{(\geq)} \,|\, \vec{r} \cdot \vec{a},\ \vec{r} \cdot \vec{b} \geq 0\} \cup \\ \cup\{&\vec{r} \in \mathbb{R}^2_{(\leq)} \,|\, \vec{r} \cdot \vec{a},\ \vec{r} \cdot \vec{a}^{\,d},\ \vec{r} \cdot \vec{b},\ \vec{r} \cdot \vec{b}^{\,d} \geq 0\}.\end{aligned}$$
(4)
$$\begin{aligned}\mathcal{C}_{SOD}(G * H) = \{&\vec{r} \in \mathbb{R}^2_{(\geq)} \,|\, \vec{r} \cdot \vec{a} = \vec{r} \cdot \vec{b} = 0\} \cup \\ \cup\{&\vec{r} \in \mathbb{R}^2_{(\leq)} \,|\, \vec{r} \cdot \vec{a} = \vec{r} \cdot \vec{a}^{\,d} = \vec{r} \cdot \vec{b} = \vec{r} \cdot \vec{b}^{\,d} = 0\}.\end{aligned}$$
(5) $\mathcal{D}^{\uparrow}(G * H) = \{\vec{r} \in \mathbb{R}^2 \,|\, \vec{r} \cdot \vec{a} \geq 0,\ \vec{r} \cdot \vec{b}^{\,d} \geq 0\}$.
(6) $\mathcal{C}(G * H) = \{\vec{r} \in \mathbb{R}^2 \,|\, \vec{r} \cdot \vec{a} = \vec{r} \cdot \vec{b}^{\,d} = 0\}$.

Proof. Set $F = G * H$,

(1) Let us assume that $\vec{r} \in \mathcal{D}^{\uparrow}_{\mathrm{OD}}(F)$. By Lemma 4.16 we can find $\mathbf{x} \in [0, 1]^2_{(\geq)}$ and $c > 0$ such that $\mathbf{x} + c\vec{r} \in [0, 1]^2_{(\geq)}$, where $\mu + (\mathbf{x} + c\vec{r}) \cdot \vec{a} = G(\mathbf{x} + c\vec{r}) = F(\mathbf{x} + c\vec{r}) \geq F(\mathbf{x}) = G(\mathbf{x}) = \mu + \mathbf{x} \cdot \vec{a}$, hence $\vec{r} \cdot \vec{a} \geq 0$ and $\vec{r} \in \mathcal{D}^{\uparrow}_{\mathrm{OD}}(G)$. Equally we can find $\mathbf{x} \in [0, 1]^2_{(\leq)}$ and $c > 0$ with $\mathbf{x} + c\vec{r}_\sigma \in [0, 1]^2_{(\leq)}$, from where $\vec{r} \in \mathcal{D}^{\uparrow}_{\mathrm{OD}}(H)$. Therefore, by Proposition 5.2 we have $\mathcal{D}^{\uparrow}_{\mathrm{OD}}(F) = \mathcal{D}^{\uparrow}_{\mathrm{OD}}(G) \cap \mathcal{D}^{\uparrow}_{\mathrm{OD}}(H)$.

(2) Proceed analogously as in (1).

(3) By Proposition 3.3, $\mathcal{D}^{\uparrow}_{\mathrm{SOD}}(F) \cap \mathbb{R}^2_{(\geq)} = \mathcal{D}^{\uparrow}_{\mathrm{OD}}(F) \cap \mathbb{R}^2_{(\geq)}$.

Let us assume that $\vec{r} \in \mathbb{R}^2_{(\leq)}$ satisfies $\vec{r} \cdot \vec{a},\ \vec{r} \cdot \vec{b},\ \vec{r} \cdot \vec{a}^{\,d},\ \vec{r} \cdot \vec{b}^{\,d} \geq 0$. Let us see that F is SOD $\vec{r}$-increasing. Let $\mathbf{x} \in [0, 1]^2$, $c > 0$ and $\sigma \in \mathcal{S}_2$ such that $\mathbf{x}_\sigma \in [0, 1]^2_{(\geq)}$, $\mathbf{x}_\sigma + c\vec{r} \in [0, 1]^2$.

Suppose first that $\sigma = \mathrm{id}$, then $\mathbf{x} \in [0, 1]^2_{(\geq)}$. We must show that $F(\mathbf{x} + c\vec{r}) \geq F(\mathbf{x})$.

(a) If $\mathbf{x} + c\vec{r} \in [0, 1]^2_{(\geq)}$, then $F(\mathbf{x} + c\vec{r}\,) - F(\mathbf{x}) = c\vec{r} \cdot \vec{a} \geq 0$.
(b) If $\mathbf{x} + c\vec{r} \in [0, 1]^2_{(<)}$, then by Lemma 5.3 there exists $c' \in (0, c)$ such that $\mathbf{x} + c'\vec{r} \in [0, 1]^2_{(=)}$. Therefore, if we put $\mathbf{y} = \mathbf{x} + c'\vec{r}$, then $\mathbf{y} + (c - c')\vec{r} = \mathbf{x} + c\vec{r} \in [0, 1]^2_{(\geq)}$. Thus, since $\mathbf{y} \in [0, 1]^2_{(=)}$,
$$\begin{aligned}F(\mathbf{x} + c\vec{r}\,) - F(\mathbf{x}) &= F(\mathbf{x} + c\vec{r}\,) - F(\mathbf{y}) + F(\mathbf{y}) - F(\mathbf{x}) \\ &= F\left(\mathbf{y} + (c - c')\vec{r}\right) - F(\mathbf{y}) + F\left(\mathbf{x} + c'\vec{r}\right) - F(\mathbf{x}) \\ &= (c - c')\,\vec{r}^{\,d} \cdot \vec{b} + c'\,\vec{r} \cdot \vec{a} \geq 0.\end{aligned}$$

Suppose now that $\sigma = (1\,2)$. We can assume that $\mathbf{x} \in [0, 1]^2_{(<)}$ and so $F(\mathbf{x}) = \mu + \mathbf{x}_\sigma \cdot \vec{b}$.

(c) If $\mathbf{x}_\sigma + c\vec{r} \in [0, 1]^2_{(\geq)}$, then $F(\mathbf{x} + c\vec{r}_\sigma) - F(\mathbf{x}) = c\vec{r} \cdot \vec{b} \geq 0$.
(d) If $\mathbf{x}_\sigma + c\vec{r} \in [0, 1]^2_{(<)}$, then by Lemma 5.3 there exists $c' \in (0, c)$ such that $\mathbf{x} + c'\vec{r}_\sigma \in [0, 1]^2_{(=)}$. Therefore, if we put $\mathbf{y} = \mathbf{x} + c'\vec{r}_\sigma$, then $\mathbf{y} + (c - c')\vec{r}_\sigma = \mathbf{x} + c\vec{r}_\sigma \in [0, 1]^2_{(\geq)}$. Thus, since $\mathbf{y} \in [0, 1]^2_{(=)}$,

$$\begin{aligned} F(\mathbf{x}+c\vec{r}_\sigma) - F(\mathbf{x}) &= F(\mathbf{x}+c\vec{r}_\sigma) - F(\mathbf{y}) + F(\mathbf{y}) - F(\mathbf{x}) \\ &= F\left(\mathbf{y}+(c-c')\vec{r}_\sigma\right) - F(\mathbf{y}) + F\left(\mathbf{x}+c'\vec{r}_\sigma\right) - F(\mathbf{x}) \\ &= (c-c')\vec{r}_\sigma \cdot \vec{a} + c'\vec{r}\cdot\vec{b} \geq 0. \end{aligned}$$

Therefore F is SOD $\vec{r}$-increasing.

Let us assume now that F is SOD $\vec{r}$-increasing, where $\vec{r} \in \mathbb{R}^2_{(\leq)}$. In particular, by Proposition 3.2, F is OD $\vec{r}$-increasing, hence $\vec{r}\cdot\vec{a}$, $\vec{r}\cdot\vec{b} \geq 0$ by 1. It rests to show that if $r_1 < r_2$ then $\vec{r}\cdot\vec{a}^{\,d}$, $\vec{r}\cdot\vec{b}^{\,d} \geq 0$. Indeed, from cases (b) and (d), the definition of SOD $\vec{r}$-increasingness for $\mathbf{y}$ ensures that $\vec{r}\cdot\vec{a}^{\,d}$, $\vec{r}\cdot\vec{b}^{\,d} \geq 0$.

(4) Proceed as in the preceding item with equalities instead of inequalities.

(5) Let $\vec{r} \in \mathcal{D}^{\uparrow}(F)$. By Lemma 5.3, there exist $\mathbf{x} \in [0,1]^2_{(\geq)}$ and $c > 0$ such that $\mathbf{x}+c\vec{r} \in [0,1]^2_{(\geq)}$. Thus, $0 \leq F(\mathbf{x}+c\vec{r}) - F(\mathbf{x}) = c\vec{r}\cdot\vec{a}$, so $\vec{r}\cdot\vec{a} \geq 0$. Similarly, by Lemma 5.3, there exist $\mathbf{x} \in [0,1]^2_{(\leq)}$ and $c > 0$ such that $\mathbf{x}+c\vec{r} \in [0,1]^2_{(\leq)}$. Therefore, $0 \leq F(\mathbf{x}+c\vec{r}) - F(\mathbf{x}) = c\vec{r}\cdot\vec{b}^{\,d}$, so $\vec{r}\cdot\vec{b}^{\,d} \geq 0$.

Conversely, let $\vec{r} \in \mathbb{R}^2$ be a vector such that $\vec{r}\cdot\vec{a} \geq 0$ and $\vec{r}\cdot\vec{b}^{\,d} \geq 0$. In order to prove that $\vec{r} \in \mathcal{D}^{\uparrow}(F)$, let $\mathbf{x} \in [0,1]^2$ and $c > 0$ such that $\mathbf{x}, \mathbf{x}+c\vec{r} \in [0,1]^2$. We have four cases.

- If $\mathbf{x}, \mathbf{x}+c\vec{r} \in [0,1]^2_{(\geq)}$, then $F(\mathbf{x}+c\vec{r}) - F(\mathbf{x}) = c\vec{r}\cdot\vec{a} \geq 0$.
- If $\mathbf{x}, \mathbf{x}+c\vec{r} \in [0,1]^2_{(\leq)}$, then $F(\mathbf{x}+c\vec{r}) - F(\mathbf{x}) = c\vec{r}\cdot\vec{b}^{\,d} \geq 0$.
- If $\mathbf{x} \in [0,1]^2_{(>)}$ and $\mathbf{x}+c\vec{r} \in [0,1]^2_{(<)}$, there is $c' \in (0,c)$ such that $\mathbf{x}+c'\vec{r} \in [0,1]^2_{(=)}$. If $\mathbf{y} = \mathbf{x}+c'\vec{r}$, then $\mathbf{y}+(c-c')\vec{r} = \mathbf{x}+c\vec{r}$. As $\mathbf{x}, \mathbf{y} = \mathbf{x}+c'\vec{r} \in [0,1]^2_{(\geq)}$ and $\mathbf{y}, \mathbf{y}+(c-c')\vec{r} = \mathbf{x}+c\vec{r} \in [0,1]^2_{(\leq)}$, then
$$\begin{aligned} F(\mathbf{x}+c\vec{r}) - F(\mathbf{x}) &= F(\mathbf{x}+c\vec{r}) - F(\mathbf{y}) + F(\mathbf{y}) - F(\mathbf{x}) \\ &= F\left(\mathbf{y}+(c-c')\vec{r}\right) - F(\mathbf{y}) + F\left(\mathbf{x}+c'\vec{r}\right) - F(\mathbf{x}) \\ &= (c-c')\vec{r}\cdot\vec{a} + c'\vec{r}\cdot\vec{b}^{\,d} \geq 0. \end{aligned}$$
- If $\mathbf{x} \in [0,1]^2_{(<)}$ and $\mathbf{x}+c\vec{r} \in [0,1]^2_{(>)}$, there is $c' \in (0,c)$ such that $\mathbf{x}+c'\vec{r} \in [0,1]^2_{(=)}$. If $\mathbf{y} = \mathbf{x}+c'\vec{r}$, then $\mathbf{y}+(c-c')\vec{r} = \mathbf{x}+c\vec{r}$. As $\mathbf{x}, \mathbf{y} = \mathbf{x}+c'\vec{r} \in [0,1]^2_{(\leq)}$ and $\mathbf{y}, \mathbf{y}+(c-c')\vec{r} = \mathbf{x}+c\vec{r} \in [0,1]^2_{(\geq)}$, then
$$\begin{aligned} F(\mathbf{x}+c\vec{r}) - F(\mathbf{x}) &= F(\mathbf{x}+c\vec{r}) - F(\mathbf{y}) + F(\mathbf{y}) - F(\mathbf{x}) \\ &= F\left(\mathbf{y}+(c-c')\vec{r}\right) - F(\mathbf{y}) + F\left(\mathbf{x}+c'\vec{r}\right) - F(\mathbf{x}) \\ &= (c-c')\vec{r}\cdot\vec{b}^{\,d} + c'\vec{r}\cdot\vec{a} \geq 0. \end{aligned}$$

In any case, $F(\mathbf{x}+c\vec{r}) \geq F(\mathbf{x})$ so $\vec{r} \in \mathcal{D}^{\uparrow}(F)$.

(6) Proceed as in the preceding item with equalities instead of inequalities. □

Example 5.5. Consider the Łukasiewicz implication $\mathrm{I}_L : [0,1]^2 \to [0,1]$ given by $\mathrm{I}_L(\mathbf{x}) = \min\{1, 1 - x_1 + x_2\}$ if $\mathbf{x} = (x_1, x_2) \in [0,1]^2$. Then we have

$$\mathrm{I}_L = \mathrm{OL}[1, (-1,1)] * \mathrm{OL}[1, (0,0)]$$

(by Corollary 4.12, $[1, (-1,1)]$ and $[1, (0,0)]$ define ordered linear fusion functions; as $1 = 1$ and $-1+1 = 0+0$, we may consider its $(*)$-product by Lemma 5.3). As an application of Proposition 5.4 we have:

(1) $\mathcal{D}^{\uparrow}(\mathrm{I}_L) = \mathcal{D}^{\uparrow}_{\mathrm{OD}}(\mathrm{I}_L) = \{\vec{r} \in \mathbb{R}^2 \mid r_1 \leq r_2\}$.

(2) $\mathcal{D}^{\uparrow}_{\mathrm{SOD}}(\mathrm{I}_L) = \mathcal{C}(\mathrm{I}_L) = \mathcal{C}_{\mathrm{SOD}}(\mathrm{I}_L) = \mathcal{C}_{\mathrm{OD}}(\mathrm{I}_L) = \{\vec{r} \in \mathbb{R}^2 \mid r_1 = r_2\}$.

Definition 5.6. Set $A = \{1, \ldots, n\}$ and, if $\alpha \in \mathcal{S}_n$, $A^\alpha = \left(A_1^\alpha, \ldots, A_n^\alpha\right)$, where

$$A_i^\alpha = \{\alpha(i), \alpha(i+1), \ldots, \alpha(n)\} = A \setminus \{\alpha(1), \ldots, \alpha(i-1)\}$$

Let now $\mathfrak{m} : 2^A \to [0,1]$ be a fuzzy measure (that is, $\mathfrak{m}$ satisfies $\mathfrak{m}(\emptyset) = 0$, $\mathfrak{m}(A) = 1$, and $\mathfrak{m}(X) \leq \mathfrak{m}(Y)$ if $X, Y \in 2^A$ and $X \subseteq Y$). If $(X_1, \ldots, X_n) \in (2^A)^n$, we put $\widetilde{\mathfrak{m}}(X_1, \ldots, X_n) = (\mathfrak{m}(X_1), \ldots, \mathfrak{m}(X_n))$

We set

$$C = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ & & \ddots & \ddots & \\ 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

for the $n \times n$ matrix obtained from the identity matrix I_n by putting -1 above the main diagonal and C^{T} for its transpose. Observe that, if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then $\mathbf{x}C \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{y}C^{\mathrm{T}}$.

The discrete Choquet integral of $\mathbf{x} \in [0,1]^n$ with respect to $\mathfrak{m}$ is the function

$$\mathrm{C}_{\mathfrak{m}} \colon [0,1]^n \to [0,1]$$

given by, if $\mathbf{x} \in [0,1]^n$ and $\alpha \in \mathcal{S}_n$ is such that $\mathbf{x}_\alpha \in [0,1]^n_{(\le)}$,

$$\mathrm{C}_{\mathfrak{m}}(\mathbf{x}) = \mathbf{x}_\alpha C \cdot \widetilde{\mathfrak{m}}(A^\alpha) = \mathbf{x}_\alpha \cdot \widetilde{\mathfrak{m}}(A^\alpha) C^{\mathrm{T}}.$$

Proposition 5.7. *Let* $\mathfrak{m} \colon 2^A \to [0,1]$ *be a fuzzy measure and consider the associated Choquet integral* $\mathrm{C}_{\mathfrak{m}} \colon [0,1]^n \to [0,1]$. *Then*

$$\mathcal{D}^{\uparrow}_{OD}(\mathrm{C}_{\mathfrak{m}}) = \{\vec{r} \in \mathbb{R}^n \mid \vec{r} \cdot \widetilde{\mathfrak{m}}(A^\alpha)C^{\mathrm{T}} \ge 0 \ \forall \alpha \in \mathcal{S}_n\},$$
$$\mathcal{C}_{OD}(\mathrm{C}_{\mathfrak{m}}) = \{\vec{r} \in \mathbb{R}^n \mid \vec{r} \cdot \widetilde{\mathfrak{m}}(A^\alpha)C^{\mathrm{T}} = 0 \ \forall \alpha \in \mathcal{S}_n\}.$$

Proof. Let $\mathbf{x} \in [0,1]^n$, $\alpha \in \mathcal{S}_n$ and $c \in \mathbb{R}^+$ with $\mathbf{x}_\alpha, \mathbf{x}_\alpha + c\vec{r} \in [0,1]^n_{(\le)}$. Then, with $\mathbf{y} = \mathbf{x} + c\vec{r}_{\alpha^{-1}}$, we have $\mathbf{y}_\alpha = \mathbf{x}_\alpha + c\vec{r} \in [0,1]^n_{(\le)}$ and so

$$\begin{aligned} \mathrm{C}_{\mathfrak{m}}(\mathbf{y}) &= \mathbf{y}_\alpha \cdot \widetilde{\mathfrak{m}}(A^\alpha)C^{\mathrm{T}} \\ &= \mathbf{x}_\alpha \cdot \widetilde{\mathfrak{m}}(A^\alpha)C^{\mathrm{T}} + c\vec{r} \cdot \widetilde{\mathfrak{m}}(A^\alpha)C^{\mathrm{T}} \\ &= \mathrm{C}_{\mathfrak{m}}(\mathbf{x}) + c(\vec{r} \cdot \widetilde{\mathfrak{m}}(A^\alpha)C^{\mathrm{T}}), \end{aligned}$$

whence the thesis. □

Corollary 5.8. *Let* $\mathfrak{m} \colon 2^A \to [0,1]$ *be a fuzzy measure. Then the associated Choquet integral* $\mathrm{C}_{\mathfrak{m}} \colon [0,1]^n \to [0,1]$ *is SOD* $\vec{r}$*-increasing for each* $\vec{r} \in \mathbb{R}^n$ *such that* $r_i \ge 0$, $1 \le i \le n$.

Proof. The i-th term of $\widetilde{\mathfrak{m}}(A^\alpha)C^{\mathrm{T}}$, with the convention $\mathfrak{m}(A^\alpha_{n+1}) = 0$, is $\mathfrak{m}(A^\alpha_i) - \mathfrak{m}(A^\alpha_{i+1}) \ge 0$ as $\mathfrak{m}$ is a fuzzy measure, whereby the thesis. □

Let us consider the Choquet integral for $n = 2$ in some detail. A fuzzy measure $\mathfrak{m} \colon \{1,2\}^2 \to L$ corresponds to

$$\begin{pmatrix} \emptyset & \{1\} & \{2\} & \{1,2\} \\ 0 & q_1 & q_2 & 1 \end{pmatrix},$$

where $0 \le q_1, q_2 \le 1$, so that $\mathfrak{m}$ is totally determined by the pair (q_1, q_2). We set $\mathfrak{m} \equiv (q_1, q_2)$. We have

$$\widetilde{\mathfrak{m}}(A^\alpha)C^{\mathrm{T}} = \begin{cases} (1, q_2)\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = (1 - q_2, q_2) \text{ if } \alpha = \mathrm{id}, \\ (1, q_1)\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = (1 - q_1, q_1) \text{ if } \alpha = (1\,2). \end{cases}$$

And the conditions of Proposition 5.7 $\vec{r} \cdot \widetilde{\mathfrak{m}}(A^\alpha)C^{\mathrm{T}} \ge 0 \ \forall \alpha \in S_2$ translate here to

$$\begin{cases} (r_1, r_2) \cdot (1 - q_1, q_1) \ge 0, \\ (r_1, r_2) \cdot (1 - q_2, q_2) \ge 0. \end{cases} \tag{1}$$

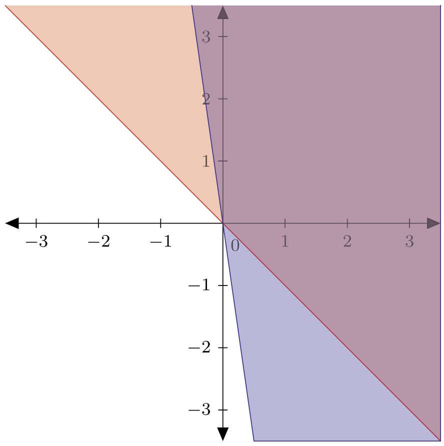

Fig. 6. Intersection of the semiplanes from the conditions in (1) for the case $\mathfrak{m} \equiv (1/2, 1/8)$.

This corresponds in the plane $\mathbb{R}^2$ to the intersection of the semiplanes rightwards and/or upwards the borders $(1-q_i)x_1 + q_i x_2 = 0$, $i = 1, 2$. For instance, the case $\mathfrak{m} \equiv (1/2, 1/8)$ is shown in Fig. 6.

Clearly, $\mathcal{C}_{\mathrm{OD}}(1/2, 1/2)$ corresponds to the line $x_1 + x_2 = 0$. If $q_1 \neq q_2$, then $\mathcal{C}_{\mathrm{OD}}(q_1, q_2) = \{(0, 0)\}$.

Let us set $\mathbf{q}_i = (1-q_i, q_i)$, $i = 1, 2$. So $\vec{r} \cdot \widetilde{\mathfrak{m}}(A^\alpha)C^{\mathrm{T}} \geq 0$ is $\vec{r} \cdot \mathbf{q}_i \geq 0$, $i = 1, 2$.

Observe that

$$\mathrm{C}_{(q_1,q_2)}\,\mathbf{x} = \begin{cases} \mathbf{x} \cdot \mathbf{q}_2 & \text{if } x_1 \leq x_2\,, \\ \mathbf{x} \cdot \mathbf{q}_1 & \text{if } x_1 \geq x_2\,. \end{cases}$$

Example 5.9. Let $\mu \in [0, 1)$, $\eta \in [0, 1-\mu]$, $q_1,\ q_2 \in [0, \eta]$ and set $\mathbf{q}_i = (1-q_i, q_i)$, $i = 1, 2$. For the corresponding Choquet integral we have

$$\mu + \eta\,\mathrm{C}_{(q_1,\ q_2)} = \mathrm{OL}[\mu, \eta\mathbf{q}_1] * \mathrm{OL}[\mu, \eta\mathbf{q}_2]$$

(by Corollary 4.12, each $[\mu, \eta(1-q_i, q_i)]$ defines an ordered linear fusion function, $i = 1, 2$; and as $\mu = \mu$ and $\eta - \eta q_1 + \eta q_1 = \eta - \eta q_2 + \eta q_2$, we may consider its $(*)$-product by Lemma 5.3). As a direct application of Proposition 5.4:

(1) $\mathcal{D}^{\uparrow}_{OD}\left(\mu + \eta\,\mathrm{C}_{(q_1,q_2)}\right) = \{\vec{r} \in \mathbb{R}^2 \mid \eta\vec{r} \cdot (1-q_i, q_i) \geq 0,\ i = 1, 2\}$.

(2) $\mathcal{C}_{OD}\left(\mu + \eta\,\mathrm{C}_{(q_1,q_2)}\right) = \{\vec{r} \in \mathbb{R}^2 \mid \eta\vec{r} \cdot (1-q_i, q_i) = 0,\ i = 1, 2\}$.

(3)

$$\begin{aligned} \mathcal{D}^{\uparrow}_{SOD}\left(\mu + \eta\,\mathrm{C}_{(q_1,q_2)}\right) = \{\vec{r} \in \mathbb{R}^2_{(\geq)} \mid \eta\vec{r} \cdot (1-q_i, q_i) \geq 0,\ i = 1, 2\,,\} \,\cup \\ \cup\{\vec{r} \in \mathbb{R}^2_{(\leq)} \mid \eta\vec{r} \cdot (1-q_i, q_i) \geq 0,\ \eta\vec{r} \cdot (q_i, 1-q_i) \geq 0,\ i = 1, 2\}. \end{aligned}$$

(4)

$$\begin{aligned} \mathcal{C}_{SOD}\left(\mu + \eta\,\mathrm{C}_{(q_1,q_2)}\right) = \{\vec{r} \in \mathbb{R}^2_{(\geq)} \mid \eta\vec{r} \cdot (1-q_i, q_i) = 0,\ i = 1, 2\} \,\cup \\ \cup\{\vec{r} \in \mathbb{R}^2_{(\leq)} \mid \eta\vec{r} \cdot (1-q_i, q_i) = \eta\vec{r} \cdot (q_i, 1-q_i) = 0,\ i = 1, 2\}. \end{aligned}$$

(5) $\mathcal{D}^{\uparrow}\left(\mu + \eta\,\mathrm{C}_{(q_1,q_2)}\right) = \{\vec{r} \in \mathbb{R}^2 \mid \eta\vec{r} \cdot (1-q_1, q_1) \geq 0,\ \eta\vec{r} \cdot (q_2, 1-q_2) \geq 0\}$.

(6) $\mathcal{C}\left(\mu + \eta\,\mathrm{C}_{(q_1,q_2)}\right) = \{\vec{r} \in \mathbb{R}^2 \mid \eta\vec{r} \cdot (1-q_1, q_1) = \eta\vec{r} \cdot (q_2, 1-q_2) = 0\}$.

For $\mu = 0$ and $\eta = 1$ we get the Choquet integral and let us observe that $\mathrm{OL}[0, \mathbf{q}_i]$ is an OWA operator, $i = 1, 2$.

Remark 5.10.

(1) Let $\mathbf{x} = (x_1, x_2) \in [0,1]^2$ and $\vec{r} = (r_1, r_2) \in \mathbb{R}^2$.
Let $M\colon [0,1]^2 \to [0,1]$ be given by $M(\mathbf{x}) = |x_1 - x_2|$. Then, M is OD $\vec{r}$-increasing if and only if $r_1 \geq r_2$ (see Corollary 4.21). We have that $I_M^0(\mathbf{x}) = \max\{0, x_1 - x_2\}$ defines an $\vec{r}$-increasing function such that $\widehat{I_M^0} = M$.
Consider now $M^c\colon [0,1]^2 \to [0,1]$, that is $M^c(\mathbf{x}) = 1 - |x_1 - x_2| = 1 - \max(x_1, x_2) + \min(x_1, x_2)$, an OD $\vec{r}$-increasing function if and only if $r_1 \leq r_2$ (see Proposition 4.22). Then $I_{M^c}^1 = I_L$ (the Łukasiewicz implication) is $\vec{r}$-increasing and $\widehat{I_L} = M^c$.
As $\widehat{I_L} = M^c$ we can deduce that in the hypothesis of Proposition 3.10 it is not true in general that $\widehat{F}$ is SOD $\vec{r}$-increasing: take $\vec{r} = (r_1, r_2)$ with $r_1 < r_2$; then I_L is $\vec{r}$-increasing by Example 5.5 but $\widehat{I_L}$ is not SOD $\vec{r}$-increasing by Proposition 4.22.

(2) Let $M\colon [0,1]^2 \to [0,1]$ as in (1). We have seen in Corollary 4.21 and Proposition 4.22 that $\mathcal{D}^{\uparrow}_{\mathrm{SOD}}(M) = \{\vec{r} \in \mathbb{R}^2 \mid r_1 \geq r_2\}$ and $\mathcal{D}^{\uparrow}_{\mathrm{SOD}}(M^c) = \{\vec{r} \in \mathbb{R}^2 \mid r_1 = r_2\}$, hence Proposition 3.5(2) is not true for $\mathrm{T} = \mathrm{SOD}$. And taking into account the proof of Proposition 3.5(2), neither is valid Proposition 3.4(3) for $\mathrm{T} = \mathrm{SOD}$.

6. General properties

In this section we present some results that characterize classical and weak monotonicity in terms of directional, SOD and OD monotonicity. Thus, they are a link between the different notions of monotonicity. Additionally, we extend a general property of OD monotone functions from [5] to the case of SOD monotone functions. We finish the section with two results on how OD and SOD monotone functions can be constructed by means of an aggregation of a series of functions with the same type of monotonicity.

Proposition 6.1. *Let $F\colon [0,1]^n \to [0,1]$ be a function and set $(\vec{e}_1, \ldots, \vec{e}_n)$ for the natural basis of $\mathbb{R}^n$. Then the following are equivalent.*

(1) F is increasing.
(2) F is $\vec{e}_i$-increasing $\forall i \in \{1, \ldots, n\}$.
(3) F is SOD $\vec{e}_i$-increasing $\forall i \in \{1, \ldots, n\}$.
(4) F is OD $\vec{e}_i$-increasing $\forall i \in \{1, \ldots, n\}$.

Proof. It is obvious that (1) $\iff$ (2), and (1) $\implies$ (3) as $\mathbf{x} \leq \mathbf{x} + c(\vec{e}_i)_{\sigma^{-1}}$ for all $c > 0$ and $\sigma \in \mathcal{S}_n$).

(3) $\implies$ (1) Let $\mathbf{x}, \mathbf{y} \in [0,1]^n$ such that $\mathbf{x} \leq \mathbf{y}$. We must show that $F(\mathbf{x}) \leq F(\mathbf{y})$ and so we may assume that $\mathbf{x} < \mathbf{y}$. Let $\sigma \in \mathcal{S}_n$ be such that $x_\sigma \in [0,1]^n_{(\geq)}$. Observe that $\mathbf{x} < \mathbf{y}$ is equivalent to $\mathbf{x}_\sigma < \mathbf{y}_\sigma$. Let $i = \max\{j \in \{1, \ldots, n\} \mid x_{\sigma(j)} < y_{\sigma(j)}\}$.

So we have the following scheme (it could be $i = n$):

$$\begin{array}{lccccccc}
(\mathbf{x}_\sigma): & x_{\sigma(1)} \geq & \ldots & \geq x_{\sigma(i-1)} \geq & x_{\sigma(i)} \geq & x_{\sigma(i+1)} \geq & \ldots & \geq x_{\sigma(n)} \\
 & \wedge\!\!| & \ldots & \wedge\!\!| & \wedge & \| & \ldots & \| \\
(\mathbf{y}_\sigma): & y_{\sigma(1)} & \ldots & y_{\sigma(i-1)} & y_{\sigma(i)} & y_{\sigma(i+1)} & \ldots & y_{\sigma(n)}
\end{array}$$

Let $c = y_{\sigma(i)} - x_{\sigma(i)}$. We have

$$\begin{aligned}
\mathbf{x}_\sigma + c\,\vec{e}_i &= (x_{\sigma(1)}, \ldots, x_{\sigma(i-1)}, x_{\sigma(i)} + c, x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) \\
&= (x_{\sigma(1)}, \ldots, x_{\sigma(i-1)}, y_{\sigma(i)}, y_{\sigma(i+1)}, \ldots, y_{\sigma(n)}) \in [0,1]^n .
\end{aligned}$$

Hence, as F is SOD $\vec{e}_i$-increasing, $F(\mathbf{x}) \leq F(\mathbf{z})$, where $\mathbf{z} = \mathbf{x} + c(\vec{e}_i)_{\sigma^{-1}}$. We have $\mathbf{z} = \mathbf{x} + c\vec{e}_j = (x_1, \ldots, x_j + c, \ldots, x_n) = (x_1, \ldots, y_j, \ldots, x_n)$, where $j = \sigma(i)$.

Set now, if $\mathbf{v} = (v_1, \ldots, v_n)$, $\mathbf{w} = (w_1, \ldots, w_n) \in [0,1]^n$ and $\mathbf{v} \leq \mathbf{w}$,

$$\mathrm{d}(\mathbf{v}, \mathbf{w}) = |\{k \mid v_k < w_k\}| .$$

We have that $\mathbf{z} \leq \mathbf{y}$ and $\mathrm{d}(\mathbf{z}, \mathbf{y}) = \mathrm{d}(\mathbf{x}, \mathbf{y}) - 1$, hence, if we reiterate the process, we conclude $F(\mathbf{x}) \leq F(\mathbf{z}) \leq \cdots \leq F(\mathbf{y})$, as required.

(3) $\iff$ (4) By the definitions it suffices to show (4) $\implies$ (3). Let $\mathbf{x} \in [0,1]^n$, $\sigma \in \mathcal{S}_n$, $c > 0$ and $i \in \{1, \ldots, n\}$ such that $\mathbf{x}_\sigma \in [0,1]^n_{(\geq)}$ and $\mathbf{x}_\sigma + c\vec{e}_i \in [0,1]^n$. Let us see that $F(\mathbf{x}) \leq F(\mathbf{x} + c(\vec{e}_i)_{\sigma^{-1}})$.

Proceed by induction. If $i = 1$, then $\mathbf{x}_\sigma \in [0,1]^n_{(\geq)}$ implies $\mathbf{x}_\sigma + c\vec{e}_1 \in [0,1]^n_{(\geq)}$ and the thesis follows as F is OD $\vec{e}_1$-increasing.

Assume that the result is true for $i - 1$, where $2 \leq i \leq n$.

If $\mathbf{x}_{\sigma(i-1)} \geq \mathbf{x}_{\sigma(i)} + c$, then $\mathbf{x}_\sigma + c\vec{e}_i \in [0,1]^n_{(\geq)}$ and the thesis follows as F is OD $\vec{e}_i$-increasing. Assume that $\mathbf{x}_{\sigma(i-1)} < \mathbf{x}_{\sigma(i)} + c$.

Set $t = x_{\sigma(i-1)} - x_{\sigma(i)}$. We have

$$\mathbf{x}_\sigma + t\vec{e}_i = (x_{\sigma(1)}, \ldots, \overset{i-1}{\smile}\!\!\!\!\!\! x_{\sigma(i-1)}, \overset{i}{\smile}\!\!\!\!\!\! x_{\sigma(i-1)}, x_{\sigma(i+1)}, \ldots) \in [0,1]^n_{(\geq)}\,,$$

hence $F(\mathbf{x}) \leq F(\mathbf{x} + t(\vec{e}_i)_{\sigma^{-1}})$ as F is OD $\vec{e}_i$-increasing.

Set $\mathbf{y} = \mathbf{x} + t(\vec{e}_i)_{\sigma^{-1}}$, so $F(\mathbf{x}) \leq F(\mathbf{y})$.

Consider the transposition $\tau = (i-1\, i)$. Observe that $\mathbf{y}_{(\sigma\tau)} = (\mathbf{y}_\sigma)_\tau = \mathbf{y}_\sigma$. Set $s = c - t$. As $x_{\sigma(i-1)} + c - t = x_{\sigma(i)} + c$ we have

$$\mathbf{y}_{(\sigma\tau)} + s\vec{e}_{i-1} = (x_{\sigma(1)}, \ldots, \overset{i-1}{\smile}\!\!\!\!\!\! x_{\sigma(i)} + c, \overset{i}{\smile}\!\!\!\!\!\! x_{\sigma(i-1)}, x_{\sigma(i+1)}, \ldots) \in [0,1]^n.$$

By the induction hypothesis, F is OD $\vec{e}_{i-1}$-increasing, hence

$$F(\mathbf{y}) \leq F(\mathbf{y} + s(\vec{e}_{i-1})_{(\sigma\tau)^{-1}})\,.$$

As

$$\begin{aligned}\mathbf{y} + s(\vec{e}_{i-1})_{(\sigma\tau)^{-1}} &= \mathbf{y} + s(\vec{e}_{i-1})_{(\tau\sigma^{-1})} = \mathbf{y} + s(\vec{e}_i)_{\sigma^{-1}} = \\ &= \mathbf{x} + (t+s)(\vec{e}_i)_{\sigma^{-1}} = \mathbf{x} + c(\vec{e}_i)_{\sigma^{-1}}\,,\end{aligned}$$

we have $F(\mathbf{x}) \leq F(\mathbf{y}) \leq F(\mathbf{x} + c(\vec{e}_i)_{\sigma^{-1}})$ and the proof is finished. □

Corollary 6.2. *Let $F\colon [0,1]^n \to [0,1]$ be a function. Then the following assertions are equivalent:*

(1) F is increasing.
(2) F is SOD $\vec{r}$-increasing for each $\vec{r} \in [0,\infty)^n$.
(3) F is OD $\vec{r}$-increasing for each $\vec{r} \in [0,\infty)^n$.

Recall that a function $F\colon [0,1]^n \to [0,1]$ is said to be *weakly increasing* if $\lambda \in \mathbb{R}$ and $(x_1, \ldots, x_n)$, $(x_1 + \lambda, \ldots, x_n + \lambda) \in [0,1]^n$ implies $F(x_1, \ldots, x_n) \leq F(x_1 + \lambda, \ldots, x_n + \lambda)$.

Proposition 6.3. *Let $F\colon [0,1]^n \to [0,1]$ and $\vec{1} = (1, \ldots, 1) \in \mathbb{R}^n$. The following assertions are equivalent.*

(1) F is weakly increasing.
(2) F is $\vec{1}$-increasing.
(3) F is SOD $\vec{1}$-increasing.
(4) F is OD $\vec{1}$-increasing.

Proof. The fact that (1) $\iff$ (2) is simply the definition of weakly increasingness and, by Proposition 3.3, (3) $\iff$ (4).

(2) $\implies$ (3) Let $\mathbf{x} \in [0,1]$, $\sigma \in \mathcal{S}_n$ and $c \in \mathbb{R}^+$ such that $\mathbf{x}_\sigma \in [0,1]^n_{(\geq)}$ and $\mathbf{x}_\sigma + c\vec{r} \in [0,1]^n$. Then $F(\mathbf{x} + c\vec{1}_{\sigma^{-1}}) = F(\mathbf{x} + c\vec{1}) \geq F(\mathbf{x})$ as F is $\vec{1}$-increasing.

(3) $\implies$ (2) Let $\mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n$ and $c \in \mathbb{R}^+$ such that $\mathbf{x} + c\vec{1} \in [0,1]^n$. Take $\sigma \in \mathcal{S}_n$ such that $\mathbf{x}_\sigma \in [0,1]^n_{(\geq)}$. We have that

$$\mathbf{x}_\sigma + c\vec{1} = \mathbf{x}_\sigma + c\vec{1}_\sigma = (\mathbf{x} + c\vec{1})_\sigma \in [0,1]^n$$

as $\mathbf{x} + c\vec{r} \in [0,1]^n$. So, by hypothesis,

$$F(\mathbf{x}) \leq F(\mathbf{x} + c\vec{1}_{\sigma^{-1}}) = F(\mathbf{x} + c\vec{1}),$$

and F is $\vec{1}$-increasing. □

The following theorem (see [5]) deals with OD increasingness along the linear combination of two directions.

Theorem 6.4 *([5]). Let $\vec{r}, \vec{s} \in \mathbb{R}^n$ and $a, b \in \mathbb{R}^+$. Let us assume that if $\mathbf{x} \in [0,1]^n$, $c \in \mathbb{R}^+$, $\sigma \in \mathcal{S}_n$, then*

$$\mathbf{x}_\sigma,\ \mathbf{x}_\sigma + c(a\vec{r} + b\vec{s}) \in [0,1]^n_{(\geq)} \implies \text{either } \mathbf{x}_\sigma + ca\vec{r} \in [0,1]^n_{(\geq)} \text{ or } \mathbf{x}_\sigma + cb\vec{s} \in [0,1]^n_{(\geq)}$$

Then if a function is both OD $\vec{r}$- and OD $\vec{s}$-increasing then it is also OD $(a\vec{r} + b\vec{s})$-increasing.

We present its extension to the case of SOD monotone functions. The proof is straightforward.

Theorem 6.5. *Let $\vec{r}, \vec{s} \in \mathbb{R}^n$ and $a, b \in \mathbb{R}^+$. Let us assume that if $\mathbf{x} \in [0,1]^n$, $c \in \mathbb{R}^+$, $\sigma \in \mathcal{S}_n$, then*

$$\mathbf{x}_\sigma \in [0,1]^n_{(\geq)},\ \mathbf{x}_\sigma + c(a\vec{r} + b\vec{s}) \in [0,1]^n \implies \text{either } \mathbf{x}_\sigma + ca\vec{r} \in [0,1]^n \text{ or } \mathbf{x}_\sigma + cb\vec{s} \in [0,1]^n$$

Then if a function is both SOD $\vec{r}$- and SOD $\vec{s}$-increasing then it is also SOD $(a\vec{r} + b\vec{s})$-increasing.

Finally, the next theorem shows a construction method of OD and SOD monotone functions, based on aggregating functions with the same type of monotonicity.

Theorem 6.6. *Let $\vec{r} \in \mathbb{R}^n$, let $A\colon [0,1]^n \to [0,1]$ be an aggregation function and $F_i\colon [0,1]^n \to [0,1]$, $1 \leq i \leq m$, functions. Define*

$$A(F_1, \ldots, F_m) : [0,1]^n \to [0,1]$$

by $A(F_1, \ldots, F_m)(\mathbf{x}) = A(F_1(\mathbf{x}), \ldots, F_m(\mathbf{x}))$. Then

(1) If F_i is OD $\vec{r}$-increasing $\forall i \in \{1, \ldots, m\}$, then $A(F_1, \ldots, F_m)$ is also OD $\vec{r}$-increasing.
(2) If F_i is SOD $\vec{r}$-increasing $\forall i \in \{1, \ldots, m\}$, then $A(F_1, \ldots, F_m)$ is also SOD $\vec{r}$-increasing.

Proof. It is straightforward. □

As a consequence, the sets of all SOD $\vec{r}$-increasing functions and of all OD $\vec{r}$-increasing functions for a given $\vec{r} \in \mathbb{R}^n$ are convex.

Corollary 6.7. *Let $\lambda_1, \ldots, \lambda_n \in [0,1]$ be such that $\lambda_1 + \cdots + \lambda_n = 1$. Let $T \in \{SOD, OD\}$ and $\vec{r} \in \mathbb{R}^n$. Then, if $F_i\colon [0,1]^n \to [0,1]$, $1 \leq i \leq n$, are $T\,\vec{r}$-increasing functions, then their convex combination $\lambda_1 F_1 + \cdots + \lambda_n F_n$ is also $T\,\vec{r}$-increasing.*

Proof. The result follows from Theorem 6.6, as $A(x_1, \ldots, x_n) = \lambda_1 x_1 + \cdots + \lambda_n x_n$ defines an aggregation function. □

7. Conclusions

We have defined the concept of strengthened ordered directional (SOD) monotonicity and studied some properties of the functions that are SOD monotone. We have also studied the relation between three notions of weaker forms of monotonicity, that of directional monotonicity, ordered directional monotonicity and strengthened ordered directional monotonicity. Additionally, we have introduced the family of linear fusion functions, the family of ordered linear fusion functions and an operation between functions that recover Choquet integrals and the Łukasiewicz implication.

Acknowledgements

This work is partially supported by the grants APVV-14-0013 and TIN2016-77356-P (AEI/FEDER, UE).

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3 Pointwise directional increasingness and geometric interpretation of directionally monotone functions

Associated publication:

- M. Sesma-Sara, L. De Miguel, A. F. Roldán López de Hierro, J. Lafuente, R. Mesiar, H. Bustince. Pointwise directional increasingness and geometric interpretation of directionally monotone functions, *Information Sciences*.
  - Status: Revised version submitted.
  - Impact factor (JCR 2017): 4.305.
  - Categories:
    - Computer Science, Information Systems. Ranking 12/148 (Q1).

Pointwise directional increasingness and geometric interpretation of directionally monotone functions

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Abstract

In the recent literature we find several generalizations of monotonicity, such as weak, directional, cone, ordered directional or strengthened directional monotonicity. The goal of this work is to study the geometric interpretation of these weaker forms of monotonicity, stressing their relations and singularities. In this attempt, we introduce the concept of pointwise directional increasingness (or directional increasingness at a point) to reveal the differences between these relaxations of monotonicity. The final part of the paper highlights some results and geometric aspects of these relaxations.

Keywords: Weak monotonicity, directional monotonicity, cone monotonicity, ordered directional monotonicity, strengthened ordered directional monotonicity

1. Introduction

Aggregation functions aim at finding a single number to represent a set of n values. This is a recurrent problem in practically every setting related with data processing. Consequently, the applicability of aggregation functions has proven to be extensive [1, 2, 3]. A function $A : [0, 1]^n \to [0, 1]$ is said to be

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Preprint submitted to Elsevier *February 22, 2019*

an aggregation function if it satisfies $A(0,\ldots,0)=0$, $A(1,\ldots,1)=1$ and A is increasing with respect to every argument.

Recent studies have shown that the monotonicity condition imposed to each argument of aggregation functions may be too restrictive (see [4]), as it leaves out of the aggregation framework non-monotone functions that otherwise are appropriate to fuse information [5, 6]. Hence, less restrictive monotonicity conditions have been proposed and studied. Weak [4], cone [5], directional [7], ordered directional [8] and strengthened ordered directional monotonicity [9] are some of the conditions which generate novel classes of functions, all of which are global properties that are studied for all the points in the domain $[0,1]^n$. These notions have been succesfully applied in various domains. For instance, pre-aggregation functions [10], which are not required to increase with respect to each argument but only to satisfy a directional monotonicity condition, have been shown to lead to improvements in the results of certain classification problems [11]. Additionally, ordered directionally monotone functions have been applied in image processing, particularly in the task of edge detection [8, 12].

In this work, we introduce the concept of pointwise directional monotonicity, a local property that studies the directions along which a function increases (or decreases) starting from an specific point $\mathbf{x}\in[0,1]^n$. The introduction of this notion enables to make explicit the differences between different relaxations of monotonicity presented in the literature [4, 7, 5, 8, 9], as we characterize each of the types of monotonicity in terms of pointwise directional monotonicity. In particular, this new definition permits to show relevant differences between the concepts of ordered directional monotonicity and strengthened ordered directional monotonicity and, thus, it allows to go further in the theoretical understandings developed in [9]. Finally, we provide a series of results that contribute to interpret the behavior of each notion of monotonicity geometrically. Additionally, we present some examples of the 2 and 3-dimensional cases as they can be graphically depicted and they are valid to illustrate the particularities of each relaxation of monotonicity.

The article is organized as follows. In Section 2 we recall the different notions of monotonicity defined in the literature. In Section 3, we introduce the concept of pointwise directional monotonicity to characterize each relaxation in terms of this new concept. In Section 4 we describe each relaxation of monotonicity geometrically and present examples of the 2 and 3-dimensional cases to illustrate our results. Finally, we state some concluding remarks and prospects for future investigations.

2. Different notions of monotonicity

We first introduce some theoretical notions in order to fix the notation for the subsequent sections. For any $n\in\mathbb{N}$ ($n\geq 2$), we consider elements $\mathbf{x}=(x_1,\ldots,x_n)\in[0,1]^n$. These elements are ordered according to the inherited partial order from $\mathbb{R}^n$, which is: for any $\mathbf{x}=(x_1,\ldots,x_n),\mathbf{y}=(y_1,\ldots,y_n)\in[0,1]^n$, $\mathbf{x}\leq\mathbf{y}$ if and only if $x_i\leq y_i$ for all $i=1,\ldots,n$.

Vectors in $\mathbb{R}^n$ are denoted as $\vec{r} \in \mathbb{R}^n$ and, specifically, $\vec{1}$ and $\vec{0}$ denote the vectors $(1,\ldots,1)$ and $(0,\ldots,0)$, respectively.

Let $\mathcal{S}_n$ denote the set of all permutations of the integers from 1 to n, i.e., $\mathcal{S}_n = \{\sigma : \{1,\ldots n\} \longrightarrow \{1,\ldots,n\} \mid \sigma$ is a bijective function$\}$. If $\mathbf{x} \in [0,1]^n$, we set $\mathbf{x}_\sigma = (x_{\sigma(1)},\ldots,x_{\sigma(n)})$ and, similarly, if $\vec{r} \in \mathbb{R}^n$, we set $\vec{r}_\sigma = (r_{\sigma(1)},\ldots,r_{\sigma(n)})$.

Aggregation functions are usually thought of as being increasing with respect to each argument. Let $n \in \mathbb{N}$ with $n \geq 2$, the n-ary function $A : [0,1]^n \longrightarrow [0,1]$ is an aggregation function if it satisfies that:

- $A(0,\ldots,0) = 0$, $A(1,\ldots,1) = 1$
- $A(x_1,\ldots,x_n) \leq A(y_1,\ldots,y_n)$, for every $(x_1,\ldots,x_n),(y_1,\ldots,y_n)$ such that $x_i \leq y_i$ for all $i = 1,\ldots,n$.

Note that aggregation functions are, in fact, non-decreasing rather than increasing. However, following the terminology in [7, 8, 9], in this work we say that a function is increasing when it is non-decreasing and strictly increasing when it is proper increasing.

Some operators, such as Lehmer means, Bajraktarevic means or the mode function are not monotone in this sense [13]. Hence, although valid for fusing information, they are not included in the framework of aggregation functions. This has been solved in the literature introducing some novel monotonicity conditions, such as weak monotonicity, directional monotonicity, cone monotonicity, ordered directional monotonicity and strengthened ordered directional monotonicity.

Definition 2.1 ([4]). A function $F\colon [0,1]^n \to [0,1]$ is said to be weakly increasing if the inequality

$$F(\mathbf{x} + c\vec{1}) \geq F(\mathbf{x})$$

holds for every $\mathbf{x} \in [0,1]^n$ and $c \in [0,1]$ such that $\mathbf{x} + c\vec{1} \in [0,1]^n$.

Definition 2.2 ([7]). Let $\vec{r} = (r_1,\ldots,r_n)$ be a real n-dimensional vector, $\vec{r} \neq \vec{0}$. A function $F\colon [0,1]^n \to [0,1]$ is said to be $\vec{r}$-increasing if for all points $\mathbf{x} \in [0,1]^n$ and for all $c > 0$ such that $\mathbf{x} + c\vec{r} \in [0,1]^n$, it holds that

$$F(\mathbf{x} + c\vec{r}) \geq F(\mathbf{x}) \ .$$

Definition 2.3 ([5]). Let $C \subset \mathbb{R}^n$ be a nonempty cone. A function $F\colon [0,1]^n \to [0,1]$ is said to be cone monotone with respect to C if F is directionally monotone in any direction $\vec{r} \in C$.

Definition 2.4 ([8]). Let $\vec{0} \neq \vec{r} \in \mathbb{R}^n$. A function $F\colon [0,1]^n \to [0,1]$ is said to be Ordered Directionally (OD) $\vec{r}$-increasing if for each $\mathbf{x} \in [0,1]^n$, and any permutation $\sigma \in \mathcal{S}_n$ with $x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)}$ and any $c > 0$ such that

$$1 \geq x_{\sigma(1)} + cr_1 \geq \cdots \geq x_{\sigma(n)} + cr_n \geq 0, \quad (1)$$

it holds that

$$F(\mathbf{x} + c\vec{r}_{\sigma^{-1}}) \geq F(\mathbf{x}),$$

where $\vec{r}_{\sigma^{-1}} = (r_{\sigma^{-1}(1)},\ldots,r_{\sigma^{-1}(n)})$.

Definition 2.5 ([9]). Let $\vec{0} \neq \vec{r} \in \mathbb{R}^n$. A function $F\colon [0,1]^n \to [0,1]$ is said to be Strengthened Ordered Directionally (SOD) $\vec{r}$-increasing if for each $\mathbf{x} \in [0,1]^n$, and any permutation $\sigma \in \mathcal{S}_n$ with $\mathbf{x}_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)}$ and any $c > 0$ such that $\mathbf{x}_\sigma + c\vec{r} \in [0,1]^n$, it holds that

$$F(\mathbf{x} + c\vec{r}_{\sigma^{-1}}) \geq F(\mathbf{x}),$$

where $\vec{r}_{\sigma^{-1}} = (r_{\sigma^{-1}(1)}, \ldots, r_{\sigma^{-1}(n)})$.

If a function F is SOD $\vec{r}$-increasing, then F is also OD $\vec{r}$-increasing. However, the converse does not hold in general. In the case that $\vec{r}$ satisfies $r_1 \geq \ldots \geq r_n$, then a function F is SOD $\vec{r}$-increasing if and only if F is OD $\vec{r}$-increasing.

For the sake of simplicity, we only refer to increasingness conditions, but decreasingness can be defined analogously.

In [9], one can find the following result on SOD monotonicity along the positive linear combination of two directions.

Theorem 2.6. *(Theorem 6.5 of [9]) Let $\vec{0} \neq \vec{r}, \vec{s} \in \mathbb{R}^n$ and $a, b \in \mathbb{R}^+$. Let us assume that for $\mathbf{x} \in [0,1]^n$, $c > 0$, $\sigma \in \mathcal{S}_n$, if $x_{\sigma(1)} \geq \ldots \geq x_{\sigma(n)}$ and $\mathbf{x}_\sigma + c(a\vec{r} + b\vec{s}) \in [0,1]^n$, then either $\mathbf{x}_\sigma + ca\vec{r} \in [0,1]^n$ or $\mathbf{x}_\sigma + cb\vec{s} \in [0,1]^n$. Thus, if a function is both SOD $\vec{r}$-and $\vec{s}$-increasing, then it is also SOD $(a\vec{r}+b\vec{s})$-increasing.*

Similar results as the preceding theorem can be found in [8] for OD monotone functions and in [7] for directionally monotone functions.

3. Pointwise directional monotonicity

The monotonicity relaxations in Section 2 are global properties which are demanded to the whole domain, i.e., the hypercube $[0,1]^n$. These concepts can be equivalently formulated from a local point of view, focusing on each point of the domain, in terms of pointwise directional increasingnes.

Definition 3.1. Let $F\colon [0,1]^n \to [0,1]$ be a function and let $\vec{0} \neq \vec{r} \in \mathbb{R}^n$. The function F is said to be $\vec{r}$-increasing at $\mathbf{x} \in [0,1]^n$ if for any $c > 0$ such that $\mathbf{x} + c\vec{r} \in [0,1]^n$, it holds that $F(\mathbf{x} + c\vec{r}) \geq F(\mathbf{x})$.

Remark 3.2. Although counterintuitive, the fact that a function is $\vec{r}$-increasing at a point $\mathbf{x}$ does not mean that it is $(-\vec{r})$-decreasing at $\mathbf{x}$. For instance, let $F : [0,1]^n \to [0,1]$ be given, for a fixed $\mathbf{x} \in [0,1]^n$, by:

$$F(\mathbf{y}) = \begin{cases} 0, & \text{if } \mathbf{y} = \mathbf{x} \\ 1, & \text{otherwise.} \end{cases}$$

One easily verifies that F is $\vec{r}$-increasing at $\mathbf{x}$ for any direction $\vec{r} \in [0,1]^n \backslash \{\vec{0}\}$ but it is not decreasing at $\mathbf{x}$ in any direction.

Remark 3.3. Weak and directional monotonicity, introduced in Section 2, can be directly expressed in terms of pointwise directional monotonicity:

(i) A function $F\colon [0,1]^n \to [0,1]$ is weakly increasing if and only if F is $\vec{1}$-increasing at $\mathbf{x}$, for all $\mathbf{x} \in [0,1]^n$.
(ii) Let $\vec{0} \neq \vec{r} \in \mathbb{R}^n$. A function $F\colon [0,1]^n \to [0,1]$ is $\vec{r}$-increasing if and only if F is $\vec{r}$-increasing at $\mathbf{x}$, for all $\mathbf{x} \in [0,1]^n$.

Consequently, a weakly increasing function can be seen as a function which satisfies pointwise directional increasingness along the fixed n-dimensional vector $\vec{1} = (1,\ldots,1)$, while a directional monotone function can be seen as a function which satisfies pointwise directional increasingness along a fixed n-dimensional real vector $\vec{r} = (r_1,\ldots,r_n)$. The case of cone monotonicity with respect to a cone $C \subset \mathbb{R}^n$ is equivalent to that of directional monotonicity considering all directions $\vec{r} \in C$.

For both weakly increasing and $\vec{r}$-directionally increasing functions, the direction is fixed and equal for all the points of the domain. On the contrary, for ordered and strengthened ordered directionally monotone ones, the direction varies from some regions to others. Specifically, if $\mathbf{x} \in [0,1]^n$ and $\sigma \in \mathcal{S}_n$ is a permutation such that $x_{\sigma(1)} \geq \ldots \geq x_{\sigma(n)}$, then the reordering $\vec{r}_{\sigma^{-1}}$ in Definition 2.5 denotes that r_1, the first component of the vector $\vec{r}$, affects the greatest component of the element $\mathbf{x}$, while r_2 affects the second greatest component, etc. Thus, the direction in which an OD (or SOD) function increases at a point $\mathbf{x}$ depends on the relative size of the components of $\mathbf{x}$. Let us formalize this idea.

Definition 3.4. For any $\sigma \in \mathcal{S}_n$, we define the subset $\Omega_\sigma \subset [0,1]^n$ as the set that contains all the elements of the hypercube which are decreasingly ordered through the permutation σ, i.e., the subset

$$\Omega_\sigma = \{\mathbf{x} \in [0,1]^n \mid x_{\sigma(1)} \geq x_{\sigma(2)} \geq \ldots \geq x_{\sigma(n)}\}.$$

Note that there are as many subsets as different permutations in $\mathcal{S}_n$ (recall that $|\mathcal{S}_n| = n!$). Moreover, any element $\mathbf{x} \in [0,1]^n$ belongs to some Ω_σ ($\sigma \in \mathcal{S}_n$), i.e., the collection of the subsets Ω_σ, where $\sigma \in \mathcal{S}_n$, is a cover of the hypercube $[0,1]^n$. It also holds that for any $\sigma_1, \sigma_2 \in [0,1]^n$, $\Omega_{\sigma_1} \cap \Omega_{\sigma_2} \neq \emptyset$. In fact, $\bigcap\limits_{\sigma \in \mathcal{S}_n} \Omega_\sigma = \Delta$, where Δ denotes the *diagonal* subset given by

$$\Delta = \{\mathbf{x} \in [0,1]^n \mid x_1 = x_2 = \ldots = x_n\}.$$

We denote the elements $(x,\ldots,x) \in \Delta$ by $\overline{x}$.

Proposition 3.5. *Let $\vec{0} \neq \vec{r} \in \mathbb{R}^n$. If $F\colon [0,1]^n \to [0,1]$ is $\vec{r}_{\sigma^{-1}}$-increasing at $\mathbf{x}$ for all $\mathbf{x} \in \Omega_\sigma$ and for all $\sigma \in \mathcal{S}_n$, then the function F is OD $\vec{r}$-increasing.*

Proof. Let $\vec{0} \neq \vec{r} \in \mathbb{R}^n$ and let $F\colon [0,1]^n \to [0,1]$ be $\vec{r}_{\sigma^{-1}}$-increasing at $\mathbf{x}$ for all $\mathbf{x} \in \Omega_\sigma$. Let $\mathbf{x} \in [0,1]^n$, $c > 0$ and $\sigma \in \mathcal{S}_n$ such that $x_{\sigma(1)} \geq \ldots \geq x_{\sigma(n)}$ and $1 \geq x_{\sigma(1)} + cr_1 \geq \ldots \geq x_{\sigma(n)} + cr_n \geq 0$. In this case, $\mathbf{x} \in \Omega_\sigma$ and therefore F is $\vec{r}_{\sigma^{-1}}$-increasing at $\mathbf{x}$. Hence, $F(\mathbf{x}+c\vec{r}_{\sigma^{-1}}) \geq F(\mathbf{x})$ and F is OD $\vec{r}$-increasing. $\square$

Note that in Proposition 3.5 we have only an if condition. Indeed, an example of a function that is OD $\vec{r}$-increasing but not $\vec{r}_{\sigma^{-1}}$-increasing at $\mathbf{x}$ for all $\mathbf{x} \in \Omega_\sigma$ can be found in later Example 3.11 for $\vec{r} = (r_1, r_2)$ such that $r_1 < r_2$. The converse only holds if SOD monotonicity is considered.

Theorem 3.6. *Let $\vec{0} \neq \vec{r} \in \mathbb{R}^n$. A function $F\colon [0,1]^n \to [0,1]$ is SOD $\vec{r}$-increasing if and only if it is $\vec{r}_{\sigma^{-1}}$-increasing at $\mathbf{x}$ for all $\mathbf{x} \in \Omega_\sigma$ and for all $\sigma \in \mathcal{S}_n$.*

Proof. Let $\mathbf{x} \in [0,1]^n$ and $\sigma \in \mathcal{S}_n$ such that $x_{\sigma(1)} \geq \ldots \geq x_{\sigma(n)}$. Thus, $\mathbf{x} \in \Omega_\sigma$ and, by Definitions 2.5 and 3.1, the condition for F to be SOD $\vec{r}$-increasing is equivalent to being $\vec{r}_{\sigma^{-1}}$-increasing at $\mathbf{x}$. Since $\mathbf{x}$ is arbitrarily chosen, this proves the statement. □

At this point, the difference between OD $\vec{r}$-increasingness and SOD $\vec{r}$-increasingness becomes decisive. Analytically, OD $\vec{r}$-increasingness imposes the restriction

$$1 \geq x_{\sigma(1)} + cr_1 \geq \cdots \geq x_{\sigma(n)} + cr_n \geq 0, \tag{2}$$

while SOD $\vec{r}$-increasingness does not. This condition of comonotonicity between $\mathbf{x}_\sigma$ and $\mathbf{x}_\sigma + c\vec{r}$, in the case of OD monotonicity, means that the increasingness condition is only imposed at the points $\mathbf{x}$ for which $\mathbf{x}$ and $\mathbf{x} + c\vec{r}_{\sigma^{-1}}$ belong to the same subset Ω_σ, for some $\sigma \in \mathcal{S}_n$. In other words, for each $\mathbf{x} \in [0,1]^n$, the increasingness condition is only imposed inside the region Ω_σ to which $\mathbf{x}$ belongs. In this case, we say that the function is $\vec{r}$-increasing at $\mathbf{x}$ within the region Ω_σ.[1]

Remark 3.7. Let F be an OD $\vec{r}$-increasing function for some $\vec{0} \neq \vec{r} \in \mathbb{R}^n$. We can distinguish a different behavior of the function F within Δ depending on the vector $\vec{r}$. Specifically, for any $\overline{x} \in \Delta$,

- if $\vec{r}$ satisfies that $r_1 \geq r_2 \geq \ldots \geq r_n$, then, $1 \geq x + cr_1 \geq \cdots \geq x + cr_n \geq 0$, and, consequently, the elements of Δ satisfy $F(\overline{x} + c\vec{r}_{\sigma^{-1}}) \geq F(\overline{x})$, for all $\sigma \in \mathcal{S}_n$; and
- if $r_i > r_j$ for some $i > j$, then none of the permutations satisfy Eq. (2) and the condition $F(\overline{x} + c\vec{r}_{\sigma^{-1}}) \geq F(\overline{x})$ is not imposed for any $\sigma \in \mathcal{S}_n$.

Remark 3.8. In the case $\vec{r} = \vec{1}$, for every $\sigma \in \mathcal{S}_n$ it holds that $\vec{r}_{\sigma^{-1}} = \vec{r}$. Therefore, the direction of increasingness is the same for all the points in the domain. Therefore, OD $\vec{1}$-monotonicity, SOD $\vec{1}$-monotonicity and weak monotonicity are equivalent conditions.

A characterization of OD $\vec{r}$-increasing functions in terms of pointwise directional monotonicity can also be stated from Proposition 3.5 and Remark 3.7.

[1] Pointwise directional increasingness of a function F within a region Ω coincides with pointwise directional increasingness of the function $F|_\Omega$, where $F|_\Omega$ denotes the restriction of F to the subset Ω.

Theorem 3.9. *Let $\vec{r} \neq \vec{0}$. A function $F\colon [0,1]^n \to [0,1]$ is OD $\vec{r}$-increasing if and only if F is $\vec{r}_{\sigma^{-1}}$-increasing at $\mathbf{x}$ within the region Ω_σ for all $\mathbf{x} \in \Omega_\sigma$ and for all $\sigma \in \mathcal{S}_n$.*

Unlike the characterization of SOD $\vec{r}$-increasingness as in Theorem 3.6, pointwise directional increasingness for the case of OD monotonicity (Theorem 3.9) is only considered within each corresponding region.

Given a function $F : [0,1]^n \to [0,1]$ and a point $\mathbf{x} \in [0,1]^n$, we can define the set of all directions $\vec{r} \in \mathbb{R}^n$ for which F is pointwise $\vec{r}$-increasing at $\mathbf{x}$:

$$D_F(\mathbf{x}) = \left\{\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\} \mid F \text{ is } \vec{r}\text{-increasing at } \mathbf{x}\right\}.$$

We can extend the definition of the set $D_F(\mathbf{x})$ to take into account all the points in a subset H of the domain of F, i.e., we define $D_F(H)$ as the set of all directions $\vec{r}$ for which a function F is $\vec{r}$-increasing at $\mathbf{x}$, for all $\mathbf{x} \in H$. Clearly, the next equality holds

$$D_F(H) = \bigcap_{\mathbf{x}\in H} D_F(\mathbf{x}).$$

Similarly, we can define the set D_F of directions $\vec{r}$ for which a function F is $\vec{r}$-increasing, and, in the same manner, we can define the sets OD_F and SOD_F:

$$D_F = \left\{\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\} \mid F \text{ is } \vec{r}\text{-increasing}\right\},$$
$$OD_F = \left\{\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\} \mid F \text{ is OD } \vec{r}\text{-increasing}\right\},$$
$$SOD_F = \left\{\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\} \mid F \text{ is SOD } \vec{r}\text{-increasing}\right\}.$$

As a consequence of Remark 3.3, we can characterize the sets D_F in terms of the directions of pointwise directional monotonicity:

$$D_F = \bigcap_{\mathbf{x}\in[0,1]^2} D_F(\mathbf{x}), \tag{3}$$

and if F is not $\vec{r}$-increasing for any $\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\}$, then $D_F = \emptyset$.

Likewise, we can characterize SOD_F as a consequence of Theorem 3.6,

$$SOD_F = \left\{\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\} \mid \vec{r}_{\sigma^{-1}} \in D_F(\Omega_\sigma)\ \forall \sigma \in \mathcal{S}_n\right\}, \tag{4}$$

and OD_F as a consequence of Theorem 3.9,

$$OD_F = \left\{\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\} \mid \vec{r}_{\sigma^{-1}} \in D_{F|_{\Omega_\sigma}}(\Omega_\sigma)\ \forall \sigma \in \mathcal{S}_n\right\}. \tag{5}$$

With respect to cone monotonicity [5], the concept deals with sets D_F which are cones, and clearly, for any cone C, a function F is cone monotone with respect to C if and only if $C \subset D_F$.

The following are two examples of functions that are not monotone with respect to each argument but satisfy the conditions of the discussed relaxations of monotonicity. Both are well-known functions that have been applied in different areas. The first example is the Lehmer mean [6, 14, 15] and the second one is a restricted equivalence function [16, 17, 18].

Example 3.10. Let $L : [0,1]^2 \to [0,1]$ be the binary Lehmer mean [13], given by

$$L(x,y) = \frac{x^2+y^2}{x+y},$$

with the convention $\frac{0}{0} = 0$. Then, for instance, it holds that

$$D_L\left(0, \frac{1}{2}\right) = \left\{\vec{r} \in \mathbb{R}^2 \setminus \{\vec{0}\} \mid r_2 \geq r_1\right\}.$$

Hence, L is not pointwise $(1,0)$-increasing at $(0, \frac{1}{2})$. Therefore, L is not an aggregation function. On the other hand, for every $\mathbf{x} \in [0,1]^2$ it holds that $(1,1) \in D_L(\mathbf{x})$. Therefore L is pointwise $(1,1)$-increasing at $\mathbf{x}$ for all $\mathbf{x} \in [0,1]^2$, i.e., L is weakly monotone.

Similarly, if we consider the weighted Lehmer mean [7] $L_\lambda : [0,1]^2 \to [0,1]$, given by

$$L_\lambda(x,y) = \frac{\lambda x^2 + (1-\lambda)y^2}{\lambda x + (1-\lambda)y},$$

for some $0 < \lambda < 1$ and again with the convention $\frac{0}{0} = 0$. In this case, the directions in which L_λ increases can be also characterized (see [7]):

$$D_{L_\lambda} = \bigcap_{\mathbf{x} \in [0,1]^2} D_{L_\lambda}(\mathbf{x}) = \left\{\vec{r} \in \mathbb{R}^2 \setminus \{\vec{0}\} \mid \vec{r} = c(1-\lambda, \lambda), \quad \text{for } c > 0\right\}.$$

Example 3.11. Let $F : [0,1]^2 \to [0,1]$ be given by

$$F(x,y) = 1 - |x-y|.$$

On the one hand, if $\vec{r} = (r_1, r_2) \in \mathbb{R}^2$ such that $r_1 = r_2 \neq 0$, it holds that $F(x,y) = F(x+cr_1, y+cr_2)$ for all $c > 0$ such that $(x+cr_1, y+cr_2) \in [0,1]^2$. Therefore, $\vec{r} \in D_F$. On the other hand, let $\vec{r} = (r_1, r_2) \neq \vec{0}$ such that $r_1 \neq r_2$. Thus, $F(0.5, 0.5) = 1$ and there exists $c > 0$ such that $F(0.5+cr_1, 0.5+cr_2) = 1 - |cr_1 - cr_2| < 1$. Hence, $\vec{r} \notin D_F(0.5, 0.5)$. Consequently, by Eq. (3),

$$D_F = \left\{\vec{r} \in \mathbb{R}^2 \setminus \{\vec{0}\} \mid r_1 = r_2\right\}.$$

In order to study SOD and OD monotonicity, it is clear that if $\vec{r} = (r_1, r_2) \in \mathbb{R}^2$ such that $r_1 = r_2 \neq 0$, then $\vec{r} \in SOD_F$ and $\vec{r} \in OD_F$.

Now, note that $|\mathcal{S}_2| = 2$ and, hence,

$$\Omega_1 = \{(x_1, x_2) \in [0,1]^2 | x_1 \geq x_2\} \tag{6}$$

and

$$\Omega_2 = \{(x_1, x_2) \in [0,1]^2 | x_1 \leq x_2\}. \tag{7}$$

Let $\vec{0} \neq \vec{r} \in \mathbb{R}^2$ such that $r_1 \neq r_2$, then we distinguish two cases:

- $r_1 > r_2$: Let $(0.75, 0.25) \in \Omega_1$. Note that in this case $\sigma = \sigma^{-1} = Id$. Then, $F(0.75, 0.25) = 0.5$ and

$$F(0.75 + cr_1, 0.25 + cr_2) = 1 - |0.5 + c(r_1 - r_2)| < 0.5,$$

for some $c > 0$ such that $(0.75 + cr_1, 0.25 + cr_2) \in [0,1]^2$. Hence, by Eq. (4), $\vec{r} \notin SOD_F$. Similarly, as $(0.75 + cr_1, 0.25 + cr_2) \in \Omega_1$, by Eq. (5), $\vec{r} \notin OD_F$.

- $r_1 < r_2$: Let $(0.75, 0.25) \in \Omega_1$. Then,

$$F(0.75 + cr_1, 0.25 + cr_2) = 1 - |0.5 + c(r_1 - r_2)|,$$

for some $c > 0$ such that $(0.75 + cr_1, 0.25 + cr_2) \in [0,1]^2$. If $(0.75 + cr_1, 0.25 + cr_2) \in \Omega_1$, then $0.5 + c(r_1 - r_2) \geq 0$ and therefore

$$F(0.75 + cr_1, 0.25 + cr_2) = 0.5 - c(r_1 - r_2) > 0.5.$$

It is straight to check that this also holds for any $(x, y) \in \Omega_1$. Consequently, $\vec{r} \in OD_F$. However, if $(0.75 + cr_1, 0.25 + cr_2) \in \Omega_2$, there exist $\vec{r}$ and $c > 0$ such that $F(0.75 + cr_1, 0.25 + cr_2) < 0.5$. Indeed, take $r_1 = -1$, $r_2 = 1$ and $c = 0.75$. Thus,

$$F(0.75 + cr_1, 0.25 + cr_2) = F(0, 1) = 0 < 0.5.$$

More generally, let $(0.5, 0.5) \in \Omega_1$. Then $F(0.5, 0.5) = 1$ and if $r_1 < r_2$, then there exists $c > 0$ such that $(0.5 + cr_1, 0.5 + cr_2) \in \Omega_2$ and it holds that $F(0.5 + cr_1, 0.5 + cr_2) = 1 - c(r_2 - r_1) < 1$. Therefore, by Eq. (4), $\vec{r} \notin SOD_F$.

Similarly, one can study the case of Ω_2 taking into account that $\vec{r}_{\sigma^{-1}} = (r_2, r_1)$. Finally, it holds that

$$SOD_F = \left\{ \vec{r} \in \mathbb{R}^2 \setminus \{\vec{0}\} \mid r_1 = r_2 \right\}, \text{ and}$$
$$OD_F = \left\{ \vec{r} \in \mathbb{R}^2 \setminus \{\vec{0}\} \mid r_1 \leq r_2 \right\}.$$

4. Geometric interpretation

In this section, we illustrate and develop a better understanding of the results in Section 3 and provide a geometric interpretation of the discussed relaxations of monotonicity. This section is organized so that we alternate general results from the n-dimensional case with their interpretation on the 2-dimensional case, as, on the one hand, it allows to make visualizations on the plane and, on the other hand, directions can be characterized in terms of their angle with respect to the non-negative horizontal axis.

Recall that it is possible to characterize each notion of monotonicity in terms of pointwise directional monotonicity (see Remark 3.3, Theorem 3.6 and Theorem 3.9). Hence, the study of pointwise directional monotonicity is useful to examine the effect of each relaxation of monotonicity geometrically, which is different for each point of the domain.

We focus on directional, OD and SOD monotonicity of functions $F : [0,1]^n \to [0,1]$, since weak and cone monotonicity follow readily from the case of directional monotonicity.

4.1. Points in which the monotonicity conditions are trivially satisfied

A particularity of each notion of monotonicity is the set of points that trivially satisfy the corresponding monotonicity conditions. For certain directions $\vec{r} \in \mathbb{R}^n$, there exist points $\mathbf{x} \in [0,1]^n$ for which the different monotonicity conditions are trivially satisfied. Indeed, let $F : [0,1]^n \to [0,1]$ and $\vec{r} \in \mathbb{R}^n$. In the case of directional monotonicity, let $\mathbf{x} \in [0,1]^n$ such that $\mathbf{x} + c\vec{r} \notin [0,1]^n$ for any $c > 0$. Then, F is trivially $\vec{r}$-increasing at $\mathbf{x}$. Similarly, let $\mathbf{x} \in [0,1]^n$ such that for all $\sigma \in \mathcal{S}_n$ for which $x_{\sigma(1)} \geq \ldots \geq x_{\sigma(n)}$, Eq. (2) does not hold for any $c > 0$. Then, F satisfies trivially the conditions of OD $\vec{r}$-increasingness at $\mathbf{x}$. Finally, in the case of SOD monotonicity, let $\mathbf{x} \in [0,1]^n$ such that for all $\sigma \in \mathcal{S}_n$ for which $x_{\sigma(1)} \geq \ldots \geq x_{\sigma(n)}$, it holds that $\mathbf{x}_\sigma + c\vec{r} \notin [0,1]^n$ for any $c > 0$. Then, F satisfies trivially the conditions of SOD $\vec{r}$-increasingness at $\mathbf{x}$. In Table 1 we include the relation of points that trivially satisfy the monotonicity conditions for directional, ordered directional and strengthened ordered directional monotonicity in the 2-dimensional case. Note that they depend on the angle α of the ray $\vec{r} = (r_1, r_2)$ with respect to the non-negative horizontal axis.

4.2. Directional monotonicity

Aside the points that satisfy the conditions trivially, the impact of directional monotonicity is similar on the rest of the points of the domain $[0,1]^n$ of a function F. For any $\mathbf{x} \in [0,1]^n$, F increases along a vector $\vec{r}$ at $\mathbf{x}$. In the particular case of $n = 2$, for any $(x_1, x_2) \in [0,1]^2$, a directionally monotone function increases along a fixed ray (r_1, r_2), whose angle with respect to the non-negative horizontal axis is $\alpha = \arctan(\frac{r_2}{r_1})$, when $r_1 \geq 0$, and $\alpha = \pi + \arctan(\frac{r_2}{r_1})$, when $r_1 \leq 0$, (see Figure 1). For a fixed $a \neq 0$, the convention $\arctan(\frac{a}{0}) = \text{sign}(a)\frac{\pi}{2}$ is considered.

4.3. OD and SOD monotonicity

With respect to OD and SOD monotone functions, the directions of pointwise increasingness vary depending on each of the $n!$ regions Ω_σ. For each point in the domain, the directions of pointwise increasingness are stated in Theorems 3.6 and 3.9.

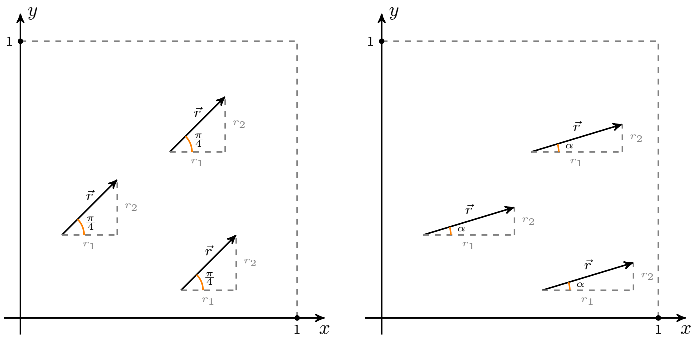
In the 2-dimensional case, there are only two possible permutations and, therefore, two different regions in which the directions of OD monotonicity vary. Let Ω_1 and Ω_2 be as in Eqs. (6) and (7), respectively. It holds that

$$\Delta = \Omega_1 \cap \Omega_2 = \{(x_1, x_2) \in [0,1]^2 \mid x_1 = x_2\}.$$

Theorems 3.6 and 3.9 can be simplified for the case of two variables.

Table 1: Directions (in terms of their angle α w.r.t. the non-negative horizontal axis) and points that trivially satisfy the monotonicity conditions for directional, ordered directional and strengthened ordered directional monotonicity.

<table>
<tr><th>Directions (α)</th><th>D monotonicity</th><th>OD monotonicity</th><th>SOD monotonicity</th></tr>
<tr><td>$\alpha = 0$</td><td>$x_1 = 1$</td><td rowspan="2">$x_1 = 1$ or $x_2 = 1$</td><td rowspan="3">$x_1 = 1$ or $x_2 = 1$</td></tr>
<tr><td>$0 < \alpha \leq \frac{\pi}{4}$</td><td rowspan="2">$x_1 = 1$ or $x_2 = 1$</td></tr>
<tr><td>$\frac{\pi}{4} < \alpha < \frac{\pi}{2}$</td><td>$x_1 = 1$ or $x_2 = 1$ or $x_1 = x_2$</td></tr>
<tr><td>$\alpha = \frac{\pi}{2}$</td><td>$x_2 = 1$</td><td rowspan="3">$x_1 = x_2$</td><td>$x_1 = x_2 = 1$</td></tr>
<tr><td>$\frac{\pi}{2} < \alpha < \pi$</td><td>$x_1 = 0$ or $x_2 = 1$</td><td>$x_1 = x_2 = 1$ or $x_1 = x_2 = 0$</td></tr>
<tr><td>$\alpha = \pi$</td><td>$x_1 = 0$</td><td>$x_1 = x_2 = 0$</td></tr>
<tr><td>$\pi < \alpha < \frac{5\pi}{4}$</td><td rowspan="2">$x_1 = 0$ or $x_2 = 0$</td><td>$x_1 = 0$ or $x_2 = 0$ or $x_1 = x_2$</td><td rowspan="3">$x_1 = 0$ or $x_2 = 0$</td></tr>
<tr><td>$\frac{5\pi}{4} \leq \alpha < \frac{3\pi}{2}$</td><td rowspan="2">$x_1 = 0$ or $x_2 = 0$</td></tr>
<tr><td>$\alpha = \frac{3\pi}{2}$</td><td>$x_2 = 0$</td></tr>
<tr><td>$\frac{3\pi}{2} < \alpha < 2\pi$</td><td>$x_1 = 1$ or $x_2 = 0$</td><td>$x_1 = 0$ or $x_1 = 1$
or $x_2 = 0$ or $x_2 = 1$</td><td>$x_1 = 0$ or $x_1 = 1$
or $x_2 = 0$ or $x_2 = 1$</td></tr>
</table>



a) Directions of increasingness for weak monotonicity

b) Directions of $\vec{r}$-increasingness for directional monotonicity

Figure 1: Examples of directions for weak monotonicity and directional monotonicity, A9 and b), respectively.

Corollary 4.1. *Let $\vec{0} \neq \vec{r} \in \mathbb{R}^2$. The function $F\colon [0,1]^2 \to [0,1]$ is SOD $\vec{r}$-increasing (respectively, OD $\vec{r}$-increasing) if and only if*

- *for any $(x_1, x_2) \in \Omega_1$, F is (r_1, r_2)-increasing at (x_1, x_2) (respectively, within Ω_1);*
- *for any $(x_1, x_2) \in \Omega_2$, F is (r_2, r_1)-increasing at (x_1, x_2) (respectively, within Ω_2).*

Moreover, the angles (with respect to the non-negative horizontal axis) along which a an OD and a SOD monotone function increases can be also stated.

Proposition 4.2. *Let Ω_1 and Ω_2 be the subsets defined in Eqs.* (6) *and* (7) *and let $\vec{0} \neq \vec{r} = (r_1, r_2) \in \mathbb{R}^2$. Let $F : [0,1]^2 \longrightarrow [0,1]$ be a SOD $\vec{r}$-increasing (respectively, OD $\vec{r}$-increasing) function. Then, it holds that*

- *for any $(x_1, x_2) \in \Omega_1$, the angle of the ray (w.r.t. the non-negative horizontal axis) in which the function increases at (x_1, x_2) (respectively, within Ω_1) is the following:*
  - *if $r_1 \geq 0$, then $\alpha = \arctan(\frac{r_2}{r_1})$;*
  - *if $r_1 < 0$, then $\alpha = \pi + \arctan(\frac{r_2}{r_1})$.*
- *for any $(x_1, x_2) \in \Omega_2$, the angle of the ray (w.r.t. the non-negative horizontal axis) in which the function increases at (x_1, x_2) (respectively, within Ω_2) is the following:*

– *if* $r_2 \geq 0$, *then* $\beta = \arctan(\frac{r_1}{r_2})$;
– *if* $r_2 < 0$, *then* $\beta = \pi + \arctan(\frac{r_1}{r_2})$.

Moreover, in every case it holds that $\beta + \alpha = \frac{\pi}{2}$.

Proof. Let $(x_1, x_2) \in \Omega_1$. The identity permutation σ satisfies $x_{\sigma(1)} \geq x_{\sigma(2)}$. Then, the direction of pointwise directional increasingness is $\vec{r} = (r_1, r_2)$ and hence, $\alpha = \arctan(\frac{r_2}{r_1})$ for vectors $\vec{r}$ such that $r_1 \geq 0$ and $\alpha = \pi + \arctan(\frac{r_2}{r_1})$ for vectors $\vec{r}$ such that $r_1 < 0$. Similarly, let $(x_1, x_2) \in \Omega_2$. The permutation σ, given by $\sigma(1) = 2$ and $\sigma(2) = 1$, satisfies $x_{\sigma(1)} \geq x_{\sigma(2)}$. Then, the direction of pointwise directional increasingness is $\vec{r} = (r_{\sigma^{-1}(1)}, r_{\sigma^{-1}(2)}) = (r_2, r_1)$ and hence $\beta = \arctan(\frac{r_1}{r_2})$ for vectors $\vec{r}$ such that $r_2 \geq 0$ and $\beta = \pi + \arctan(\frac{r_1}{r_2})$ for vectors $\vec{r}$ such that $r_2 < 0$.

The equality $\alpha + \beta = \frac{\pi}{2}$ follows from the fact that, for $a \neq 0$, it holds that $\arctan(\frac{1}{a}) = \mathrm{sign}(a)\frac{\pi}{2} - \arctan(a)$. Sometimes the actual sum is $\frac{5\pi}{2}$, which expresses the same angle as $\frac{\pi}{2}$. □

In Figure 2, four possible cases of different directions $\vec{r} = (r_1, r_2) \neq \vec{0}$ are depicted.

Back to the n-dimensional case, for an OD monotone function F, the geometric study of increasingness in each region Ω_σ reduces to study pointwise $\vec{r}_{\sigma^{-1}}$-increasingness of the functions $F|_{\Omega_\sigma}$. However, in the case of a SOD monotone function F, there exist points in the domain for which some interesting properties are satisfied.

This is the case of the points $\overline{x} \in \Delta \subset [0,1]^n$. Indeed, F must increase at $\overline{x}$ with respect to $\vec{r}$ and all its permutations.

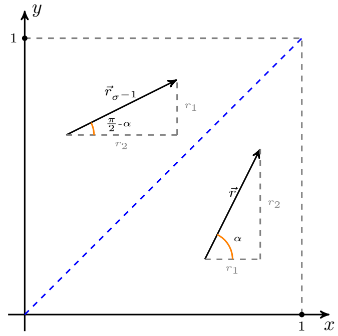
Proposition 4.3. *Let* $\vec{0} \neq \vec{r} \in \mathbb{R}^n$ *and let* $F : [0,1]^n \to [0,1]$ *be a SOD* $\vec{r}$*-increasing function. For any element* $\overline{x} \in \Delta$ *and any permutation* $\pi \in \mathcal{S}_n$, *it holds that* F *is* $\vec{r}_\pi$*-increasing at* $\overline{x}$.

Proof. Since $\overline{x} \in \Omega_\sigma$ for all $\sigma \in \mathcal{S}_n$, by Theorem 3.6, a SOD monotone function F must increase at $\overline{x}$ with respect to each of the $n!$ directions $\vec{r}_{\sigma^{-1}}$. Clearly, the result follows setting $\pi = \sigma^{-1}$. □

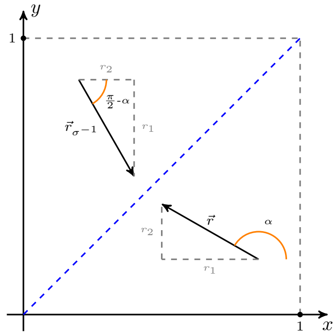
Example 4.4. For $n = 3$, there exist 6 different regions Ω_σ that are limited by the planes $z = x$, $z = y$ and $x = y$ (see Figure 3(a)). If a function $F : [0,1]^3 \to [0,1]$ is SOD $(0.7, 0.5, -0.2)$-increasing, then F is $(0.7, 0.5, -0.2)$, $(0.7, -0.2, 0.5)$, $(0.5, 0.7, -0.2)$, $(0.5, -0.2, 0.7)$, $(-0.2, 0.7, 0.5)$ and $(-0.2, 0.5, 0.7)$-increasing at $\overline{x}$, for all $\overline{x} \in \Delta$, as it is shown in Figure 3(b).

Note that, as a consequence of Remark 3.7, this is not the case for an OD monotone function.

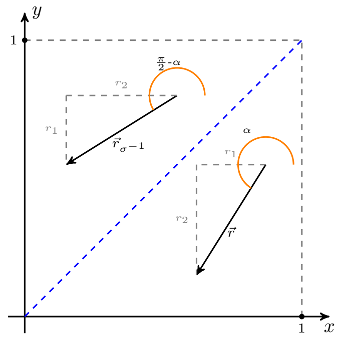
Additionally, if F is a SOD $\vec{r}$-increasing function, then, due to Theorem 2.6, F is increasing at $\overline{x}$ along any direction that is a positive linear combination of all the directions $\vec{r}_\pi$ for all $\pi \in \mathcal{S}_n$. In other words, for any $\overline{x} \in \Delta$, F is cone increasing at $\overline{x}$ with respect to the cone defined by the convex hull of the set $\{\vec{r}_\pi \in \mathbb{R}^n \mid \pi \in \mathcal{S}_n\}$.



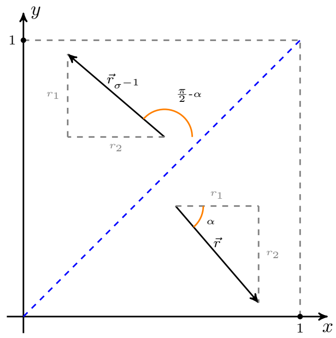
a) Vector $\vec{r}$ with $r_1 > 0$ and $r_2 > 0$.



b) Vector $\vec{r}$ with $r_1 < 0$ and $r_2 > 0$.



c) Vector $\vec{r}$ with $r_1 < 0$ and $r_2 < 0$.



d) Vector $\vec{r}$ with $r_1 > 0$ and $r_2 < 0$.

Figure 2: Examples of directions (r_1, r_2) for ordered and strengthened ordered directional monotonicity: case a) $r_1 > 0$ and $r_2 > 0$; b) $r_1 > 0$ and $r_2 < 0$; c) $r_1 < 0$ and $r_2 < 0$ and d) $r_1 > 0$ and $r_2 > 0$.

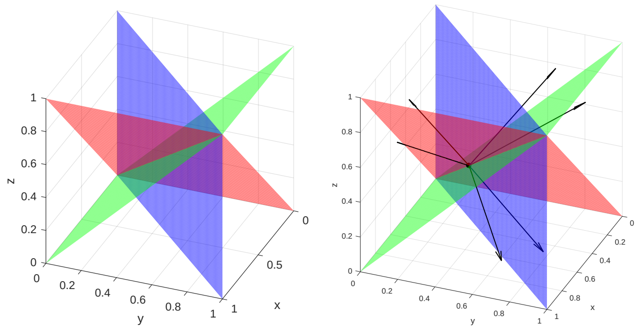


Figure 3: *(a):* Planes $z = x$ (red), $z = y$ (green) and $x = y$ (blue) that limit the 6 regions Ω_σ for all $\sigma \in \mathcal{S}_3$. *(b):* Vector $\vec{r} = (0.7, 0.5, -0.2)$ and its permutations from the point $\overline{x} = (0.3, 0.3, 0.3)$.

In the 2-dimensional case, if F is a SOD $\vec{r} = (r_1, r_2)$-increasing function, with $r_1 \neq -r_2$. Then, F is increasing at $\overline{x}$ along any direction in between (r_1, r_2) and (r_2, r_1) (see Figure 4(a)).

Depending on the vector $\vec{r} \in \mathbb{R}^n$, the number of distinct directions of pointwise increasingness at a point of Δ that are determined by the number of permutations is not necessarily as high as $n!$. Indeed, if $\vec{r}$ has repeated components, then different permutations $\pi \in \mathcal{S}_n$ yield identical vectors. In fact, let $\vec{r} \in \mathbb{R}^n$ such that there are $k < n$ distinct components in $\vec{r}$. Then, the number of distinct directions of pointwise increasingness at a point $\overline{x}$ that are determined by the number of permutations is $k!$.

On the other hand, as it can be derived from Figure 3, there exist points outside the diagonal that satisfy similar properties as those in Δ. We refer to the points in the intersection of two or more regions Ω_σ, which are precisely the points that have repeated components. In the 3-dimensional case, these points belong to the planes $z = x$, $z = y$ and $x = y$, and are the intersection of at most two different regions Ω_σ. The intersection of 3 or more regions results in Δ. In higher dimensions, the intersection of more regions can yield subsets different to Δ. However, since for a point $\mathbf{x}$ to be in the intersection of two regions Ω_σ, it implies that at least two of the components of $\mathbf{x}$ coincide, we reach the following lemma.

Lemma 4.5. *Let $k \leq n!$ and consider $\Omega_{\sigma_1}, \ldots, \Omega_{\sigma_k} \subset [0,1]^n$. If $\bigcap_{j=1}^{k} \Omega_{\sigma_j} \neq \Delta$, then $k < n$.*

With respect to the pointwise directional increasingness of the points in the intersection of various regions Ω_σ, we have the following result.

Proposition 4.6. *Let $\vec{0} \neq \vec{r} \in \mathbb{R}^n$, $F : [0,1]^n \to [0,1]$ be a SOD $\vec{r}$-increasing function and let $k \leq n$ and consider $\Omega_{\sigma_1}, \ldots, \Omega_{\sigma_k} \subset [0,1]^n$. For any element $\mathbf{x} \in \bigcap_{j=1}^{k} \Omega_{\sigma_j}$, it holds that F is $\vec{r}_{\sigma_j^{-1}}$-increasing at $\mathbf{x}$ for all $1 \leq j \leq k$.*

Remark 4.7. The case of $k = n$ in Proposition 4.6 recovers the situation of Proposition 4.3.

Of course, in the conditions of Proposition 4.6, F is cone increasing with respect to the convex hull of all the vectors $\vec{r}_{\sigma_j^{-1}}$.

Let us now study a particularity of SOD monotone function that derives from Proposition 4.6.

Theorem 4.8. *Let $\vec{0} \neq \vec{r} \in \mathbb{R}^n$, $F : [0,1]^n \to [0,1]$ be a SOD $\vec{r}$-increasing function, let $k \leq n$ and consider $\Omega_{\sigma_1}, \ldots, \Omega_{\sigma_k} \subset [0,1]^n$. If $\mathbf{x} \in \bigcap_{j=1}^{k} \Omega_{\sigma_j}$, then $F(\mathbf{x}) \leq F(\mathbf{y})$ for all $\mathbf{y} \in [0,1]^n$ such that $\mathbf{y} = \mathbf{x} + c\vec{r}_{\sigma_j^{-1}}$ for some $c > 0$ and for all $1 \leq j \leq k$. Moreover, if $\vec{r}$ is such that $\vec{r}_{\sigma_j^{-1}} = -\vec{r}_{\sigma_i^{-1}}$ for some $1 \leq i \neq j \leq k$ and $\mathbf{y} = \mathbf{x} + c\vec{r}_{\sigma_j^{-1}} \in \Omega_{\sigma_i}$, then $F(\mathbf{x}) = F(\mathbf{y})$.*

Proof. Let $F : [0,1]^n \to [0,1]$ be SOD $\vec{r}$-increasing. Let $\mathbf{x} \in \bigcap_{j=1}^{k} \Omega_{\sigma_j}$ and set $\mathbf{y} = \mathbf{x} + c\vec{r}_{\sigma_j^{-1}} \in [0,1]^n$ for $c > 0$. By Proposition 4.6, clearly it holds that $F(\mathbf{y}) \geq F(\mathbf{x})$.

Now, let $1 \leq i \leq k$ be such that $\vec{r}_{\sigma_j^{-1}} = -\vec{r}_{\sigma_i^{-1}}$ and $\mathbf{y} = \mathbf{x} + c\vec{r}_{\sigma_j^{-1}} \in \Omega_{\sigma_i}$. Since F is $\vec{r}$-increasing and $\mathbf{y} \in \Omega_{\sigma_i}$, by Theorem 3.6, F is $\vec{r}_{\sigma_i^{-1}}$ increasing at $\mathbf{y}$. Therefore,

$$F(\mathbf{x}) = F(\mathbf{y} - c\vec{r}_{\sigma_j^{-1}}) = F(\mathbf{y} + c\vec{r}_{\sigma_i^{-1}}) \geq F(\mathbf{y}).$$

Hence, $F(\mathbf{x}) = F(\mathbf{y})$. □

Theorem 4.8 can be readily visualized in the 2-dimensional case with the vectors $\vec{r} = (-r, r)$, whose angle is $\alpha = \frac{3\pi}{4}$, and $\vec{r} = (r, -r)$, whose angle is $\alpha = \frac{7\pi}{8}$ (or $-\frac{\pi}{4}$). The next two results are consequences of Theorem 4.8, nevertheless we include their proofs as they help grasping the situation.

Proposition 4.9. *Let $r > 0$ and F be a SOD $(r, -r)$-increasing function. For any $(x_1, x_2) \in [0,1]^2$, it holds that $F(x_m, x_m) \leq F(x_1, x_2)$, where $x_m = \dfrac{x_1 + x_2}{2}$.*

Proof. Let F be a SOD $(r, -r)$-increasing function, $(x_1, x_2) \in [0,1]^2$ and $x_m = \frac{x_1+x_2}{2}$. As a consequence of Proposition 4.3, F is $(r, -r)$ and $(-r, r)$-increasing at $\overline{x_m} = (x_m, x_m) \in \Delta$. We distinguish two cases depending on (x_1, x_2).

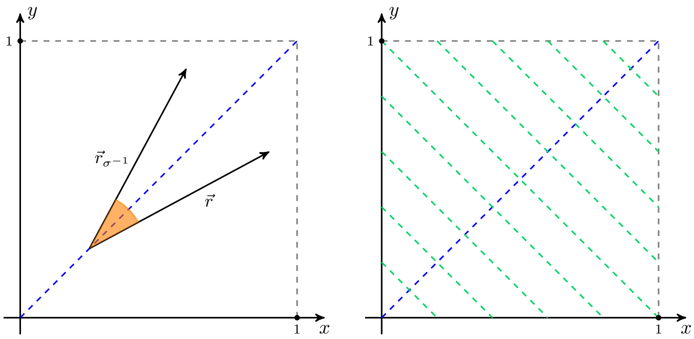


Figure 4: Graphical representation of the different behaviors of SOD $\vec{r}$-increasing functions. *(a):* directions in which a SOD (r_1, r_2)-increasing function increases at a point $\overline{x} \in \Delta$, with $r_1 \neq -r2$. *(b):* the parallel lines with direction $(-r, r)$.

- Case 1: Let $(x_1, x_2) \in \Omega_1$. If we set $c = \frac{x_1 - x_2}{2r} \geq 0$, it holds that $(x_m, x_m) + c(r, -r) = (x_1, x_2)$ and hence $F(x_m, x_m) \leq F(x_1, x_2)$.
- Case 2: Let $(x_1, x_2) \in \Omega_2$. If we set $c = \frac{x_2 - x_1}{2r} \geq 0$, it holds that $(x_m, x_m) + c(-r, r) = (x_1, x_2)$ and hence $F(x_m, x_m) \leq F(x_1, x_2)$.

$\square$

From Proposition 4.9, we conclude that if F is a SOD $(r, -r)$-increasing function, with $r > 0$, it holds that $F(\overline{x}) \leq F(\mathbf{y})$ for any $\mathbf{y} = (y_1, y_2)$ satisfying $y_1 + y_2 = 2x$, i.e., the value at the point of the diagonal is minimum with respect to the remaining points along the ray $(-r, r)$ (see green lines in Figure 4(b)).

Proposition 4.10. *Let $r > 0$ and F be a SOD $(-r, r)$-increasing function. For any $(x_1, x_2) \in [0, 1]^2$, it holds that $F(x_m, x_m) = F(x_1, x_2)$, where $x_m = \dfrac{x_1 + x_2}{2}$.*

Proof. Let F be a SOD $(-r, r)$-increasing function, $(x_1, x_2) \in [0, 1]^2$ and $x_m = \frac{x_1 + x_2}{2}$. By Proposition 4.3, F is $(-r, r)$ and $(r, -r)$-increasing at $\overline{x_m} = (x_m, x_m) \in \Delta$. We distinguish two cases depending on (x_1, x_2).

- Case 1: Let $(x_1, x_2) \in \Omega_1$. If we set $c = \frac{x_1 - x_2}{2r} \geq 0$, it holds that $(x_1, x_2) + c(-r, r) = (x_m, x_m)$. Since F is $(-r, r)$ increasing at (x_1, x_2), we find that $F(x_1, x_2) \leq F(x_m, x_m)$. Similarly, it holds that $(x_m, x_m) + c(r, -r) = (x_1, x_2)$. Since F is also $(r, -r)$ increasing at (x_m, x_m), we find that $F(x_m, x_m) \leq F(x_1, x_2)$. Therefore, $F(x_1, x_2) = F(x_m, x_m)$.
- Case 2: Let $(x_1, x_2) \in \Omega_2$. If we set $c = \frac{x_2 - x_1}{2r} \geq 0$, it holds that $(x_1, x_2) + c(r, -r) = (x_m, x_m)$. Since F is $(r, -r)$ increasing at (x_1, x_2)

(note that in Ω_2, $\sigma(1) = 2$ and $\sigma(2) = 1$), we find that $F(x_1, x_2) \leq F(x_m, x_m)$. Similarly, it holds that $(x_m, x_m)+c(-r, r) = (x_1, x_2)$. Since F is also $(-r, r)$ increasing at (x_m, x_m), we find that $F(x_m, x_m) \leq F(x_1, x_2)$. Therefore, $F(x_1, x_2) = F(x_m, x_m)$.

□

Proposition 4.10 yields that if F is a SOD $(-r, r)$-increasing function, with $r > 0$, then it holds that F is $(-r, r)$-constant at $\overline{x}$, for all $\overline{x} \in \Delta$, i.e., the function takes a constant value within the line in the direction $(-r, r)$ (see green lines in Figure 4(b)).

5. Conclusions

We have introduced the concept of pointwise directional monotonicity, a local property that has enabled to characterize various global relaxations of monotonicity defined in the literature. Specifically, we have developed theoretical knowledge about weak, directional, cone, ordered directional and strengthened ordered directional monotonicity. We have seen that SOD monotonicity is more restrictive than OD monotonicity and that the functions that satisfy the proposed relaxations have certain points in the domain that locally behave differently for each condition of monotonicity. We have described these behaviors from a geometric point of view.

As for future research, on the one hand, we intend to study the discussed notions of monotonicity for functions defined over a general lattice and analyze the effect of different order relations for each notion. On the other hand, as OD monotone functions have yielded good results to detect edges in images, we have the construction of an edge detector based on SOD monotone functions as a goal for future research.

Acknowledgements

This work has been supported by the Research Services of the Universidad Publica de Navarra, by the research project TIN2016-77356-P from MINECO, AEI/FEDER, UE, by the grant NPUII LQ1602 and with the support of the Ministry of Education, Science, Research and Sport of the Slovak Republic within the Research and Development Operational Programme for the project "University Science Park of STU Bratislava", ITMS 26240220084, co-funded by the European Regional Development Fund.

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4 Curve-based monotonicity: a generalization of directional monotonicity

Associated publication:

- A. F. Roldán López de Hierro, M. Sesma-Sara, J. Špirková, J. Lafuente, A. Pradera, R. Mesiar, H. Bustince. Curve-based monotonicity: a generalization of directional monotonicity, *International Journal of General Systems*.
  DOI: 10.1080/03081079.2019.1586684.
  - Status: Accepted (In press).
  - Impact factor (JCR 2017): 2.931.
  - Categories:
    * Computer Science, Theory & Methods. Ranking 15/103 (Q1).

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5 Weak and directional monotonicity of functions on Riesz spaces to fuse uncertain data

Associated publication:

- M. Sesma-Sara, R. Mesiar, H. Bustince. Weak and directional monotonicity of functions on Riesz spaces to fuse uncertain data, *Fuzzy Sets and Systems*.
  DOI: 10.1016/j.fss.2019.01.019.
  - Status: Accepted (In press).
  - Impact factor (JCR 2017): 2.675.
  - Categories:
    * Computer Science, Theory & Methods. Ranking 18/103 (Q1).
    * Mathematics, applied. Ranking 12/252 (Q1).
    * Statistics & Probability. Ranking 8/123 (Q1).

ARTICLE IN PRESS FSS:7585

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Weak and directional monotonicity of functions on Riesz spaces to fuse uncertain data

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Received 1 October 2018; received in revised form 28 January 2019; accepted 28 January 2019

Abstract

In the theory of aggregation, there is a trend towards the relaxation of the axiom of monotonicity and also towards the extension of the definition to other domains besides real numbers. In this work, we join both approaches by defining the concept of directional monotonicity for functions that take values in Riesz spaces. Additionally, we adapt this notion in order to work in certain convex sublattices of a Riesz space, which makes it possible to define the concept of directional monotonicity for functions whose purpose is to fuse uncertain data coming from type-2 fuzzy sets, fuzzy multisets, n-dimensional fuzzy sets, Atanassov intuitionistic fuzzy sets and interval-valued fuzzy sets, among others. Focusing on the latter, we characterize directional monotonicity of interval-valued representable functions in terms of standard directional monotonicity.

Keywords: Aggregation function; Directional monotonicity; Interval-valued function; Riesz space; Type-2 fuzzy set

1. Introduction

The theory of aggregation functions addresses the problem of obtaining a single number that is representative for a collection of values. This issue is prevalent in any process that involves working with real data. Before turning into an independent theory, there had been various works in the literature on the topic of aggregation [20,40,49]. The first monograph on aggregation functions was published [17] in 2001. Classically, an aggregation function A is a function defined on the unit hypercube with values in the unit interval that satisfies certain boundary conditions and is increasing with respect to every argument. This family of functions has been extensively used in different theoretical and applied fields [29,31,45].

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https://doi.org/10.1016/j.fss.2019.01.019
0165-0114/

Please cite this article in press as: M. Sesma-Sara et al., Weak and directional monotonicity of functions on Riesz spaces to fuse uncertain data, Fuzzy Sets Syst. (2019), https://doi.org/10.1016/j.fss.2019.01.019

Various works about the state-of-the-art of the theory of aggregation functions [30,41] declare that the following are two of the major trends in the aggregation theory:

1. To relax the monotonicity constraints in the definition of aggregation functions;
2. To extend the concept of aggregation functions to be capable of handling more scales apart from numbers.

The need of relaxing the monotonicity condition was originated due to the existence of functions that are valid to fuse information but do not qualify as aggregation functions because they violate the monotonicity condition. This is the case, for example of the Lehmer mean [11]. Consequently, and pursuing the creation of a framework of functions for fusing data, the notion of weak monotonicity was introduced [52]. This concept was then generalized by directional monotonicity [15], which studies the monotonicity of a function along a real vector, or a ray, in $\mathbb{R}^n$. Directional monotonicity was established as the new axiom replacing standard monotonicity and this lead to the introduction of a class of functions resembling aggregation functions but with relaxed monotonicity constrictions [39]. Subsequently, new notions of monotonicity have arised [7,14,48] and have been applied with success to problems of edge detection in computer vision [14,47] and fuzzy rule-based classification [37,38].

Regarding the second item, the theory of aggregation has been extended to work with posets [25,35], with graphs [51], with infinite sequences [42,46] and with intervals [16,22], among others. Furthermore, it is not unusual that there exists a degree of uncertainty around the data to aggregate (missing inputs, measurement errors, etc.) and therefore aggregation functions have been extended to work with values coming from structures that model uncertainty. This is the case of the different extensions of fuzzy sets, there are works in the literature describing how this type of uncertainty modeling techniques have been successfully applied to real problems, e.g., type-2 fuzzy sets [33,36], n-dimensional fuzzy sets [24], Atanassov intuitionistic fuzzy sets [21] and interval-valued fuzzy sets [10,12,16], among others.

In this work we combine both trends in the theory of aggregation, and based on the structure of Riesz spaces, we provide a framework to define directional monotonicity for functions that handle various types of uncertain data coming from different extensions of fuzzy sets. In particular, we define directional monotonicity for fusing type-2 fuzzy values, fuzzy multiset and n-dimensional fuzzy values, Atanassov intuitionistic fuzzy values and interval-valued fuzzy values. We also study the properties of this class of functions and, focusing on the interval-valued setting, we show the relation between directional monotonicity for interval-valued representable functions and standard directional monotonicity presented in [15]. This relation permits to construct examples belonging to this class of functions on the basis of two functions defined in $[0, 1]$ and with values in $[0, 1]$. Additionally, we study the particular case when the directions of increasingness are formed by interval values. We refer to this concept as interval directional monotonicity (IDM).

This work is organized as follows. In Section 2 we present some preliminary concepts and results in order to make the work self-contained. In Section 3 we recall the notion of a Riesz space and expound some of the specific instances of Riesz spaces that we use later in this work. In Section 4 we introduce the concept of directional monotonicity for functions that take values in a Riesz space, as well as some properties and how this notion can be modified in order to work in certain sublattices of a Riesz space. In Section 5 we make use of the mentioned sublattices to show how we can recover the concept of directional monotonicity in order to fuse data that comes from different extensions of fuzzy sets. In Section 6 we explicitly present the notion of directional monotonicity for interval-valued functions and give the relation between this notion for representable interval-valued functions and standard directional monotonicity. We also propose the concept of interval directional monotonicity, IDM, which results from restricting the directions of increasingness to vectors that are formed with intervals. We finalize this work with some conclusions in Section 7.

2. Preliminaries

2.1. Aggregation functions and directional monotonicity

We recall the definition of aggregation functions [5,18,32].

Definition 2.1. An aggregation function is a function $A: [0, 1]^n \to [0, 1]$ such that

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(i) $A(0,\ldots,0)=0$;
(ii) $A(1,\ldots,1)=1$;
(iii) A is increasing, i.e., if $(x_1,\ldots,x_n),(y_1,\ldots,y_n)\in[0,1]^n$ such that $x_i\leq y_i$ for all $1\leq i\leq n$, then $A(x_1,\ldots,x_n)\leq A(y_1,\ldots,y_n)$.

Seeking the relaxation of the monotonicity condition, in [52] the notion of weak monotonicity was introduced.

Definition 2.2 *([52]).* Let $F:[0,1]^n\to[0,1]$, we say that F is weakly increasing (weakly decreasing), if for all $c>0$ and $\mathbf{x}=(x_1,\ldots,x_n)\in[0,1]^n$ such that $0\leq x_i+c\leq 1$ for all $1\leq i\leq n$, it holds that $F(x_1+c,\ldots,x_n+c)\geq F(x_1,\ldots,x_n)$ $(F(x_1+c,\ldots,x_n+c)\leq F(x_1,\ldots,x_n))$.

Weak monotonicity can be seen as monotonicity along the ray $\vec{1}=(1,\ldots,1)$ and this concept is generalized by directional monotonicity in [15].

Definition 2.3 *([15]).* Let $F:[0,1]^n\to[0,1]$ and $\vec{r}\in\mathbb{R}^n$, we say that F is $\vec{r}$-increasing (decreasing), if for all $c>0$ and $\mathbf{x}\in[0,1]^n$ such that $\mathbf{x}+c\vec{r}\in[0,1]^n$, it holds that $F(\mathbf{x}+c\vec{r})\geq F(\mathbf{x})$ $(F(\mathbf{x}+c\vec{r})\leq F(\mathbf{x}))$.

The relaxation of monotonicity in the definition of aggregation functions by directional monotonicity has led to good results in fuzzy rule-based classification systems [38,39].

Let us now present two results about directionally monotone functions. The first deals with increasingness along the convex combination of two different directions.

Theorem 2.4 *([15]). Let $\vec{r},\vec{s}\in\mathbb{R}^n$ and $a,b\geq 0$ such that $a+b>0$ and let us set $\vec{u}=a\vec{r}+b\vec{s}$. Let $\mathbf{x}\in[0,1]^n$ and $c>0$ such that $\mathbf{x}+c\vec{u}\in[0,1]^n$ and either $\mathbf{x}+ca\vec{r}$ or $\mathbf{x}+cb\vec{s}\in[0,1]^n$. Then, if a function $F:[0,1]^n\to[0,1]$ is both $\vec{r}$- and $\vec{s}$-increasing, it is also $\vec{u}$-increasing.*

The second is a characterization of standard monotonicity in terms of directional monotonicity.

Theorem 2.5 *([15]). Let $n\in\mathbb{N}$, $F:[0,1]^n\to[0,1]$ and $\{\vec{e}_i\}_{i=1}^n$ be the canonical basis of $\mathbb{R}^n$. Then, the following are equivalent*

(i) F is increasing;
(ii) F is $\vec{e}_i$-increasing for all $1\leq i\leq n$.

2.2. Interval-valued aggregation functions

We call $L(\mathbb{R})$ the set of closed intervals of the real numbers, i.e., $L(\mathbb{R})=\{[x,y]\mid x,y\in\mathbb{R},x\leq y\}$. The restriction to the intervals in the unit interval is denoted by $L([0,1])$.

The set of closed intervals $L(\mathbb{R})=\{[x,y]\mid x,y\in\mathbb{R},x\leq y\}$ and the half-space $K(\mathbb{R})=\{(x,y)\in\mathbb{R}^2\mid x\leq y\}$ of $\mathbb{R}^2$ are isomorphic lattices with respect to the standard partial order of intervals $\leq_L$ defined by

$$[a,b]\leq_L[c,d] \text{ if and only if } a\leq c \text{ and } b\leq d.$$

The top and bottom elements of $(L([0,1]),\leq_L)$ are $1_L=[1,1]$ and $0_L=[0,0]$, respectively.

Thus, we can define the concept of an interval-valued (IV) aggregation function. We denote the product space as $L([0,1])^n=L([0,1])\times\ldots\times L([0,1])$ and the component-wise order in $L([0,1])^n$ by $\leq_{L^n}$.

Definition 2.6. Let $A:L([0,1])^n\to L([0,1])$. We say that A is an IV aggregation function if it satisfies the following conditions.

(i) $A(0_L,\ldots,0_L)=0_L$;
(ii) $A(1_L,\ldots,1_L)=1_L$;
(iii) A is increasing with respect to $\leq_L$.

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Interval-valued aggregation operators have been, and continue to be, extensively studied both from the theoretic and applied points of view [6,26,44].

2.3. Fuzzy sets and generalizations

We end the preliminaries section with this subsection about different generalizations of fuzzy sets. We present the definition of the concepts that are mentioned in this work and some remarks about their relation. The history and main properties of the different generalizations of fuzzy sets can be found in [13].

Let us start by recalling the concept of a fuzzy set [55].

Definition 2.7. Given a non-empty universe X, a fuzzy set A on X is a function $A : X \to [0,1]$. Given $x \in X$, the membership degree of x to the fuzzy set is $A(x)$.

We denote the set of all fuzzy sets over the universe X as $FS(X)$. Fuzzy sets are also known as type-1 fuzzy sets, due to the ideas presented in [56], where some extensions of fuzzy sets were presented, the so-called type-n fuzzy sets ($T_n FS$).

Definition 2.8. Given a non-empty universe X and $n > 1$, a type-n fuzzy set A on X is a function $A : X \to T_{n-1}FS([0,1])$.

In other words, a type-n fuzzy set is a fuzzy set in which the membership of the elements are described by a type-$(n-1)$ fuzzy set.

Among these extensions, type-2 fuzzy sets have been shown to hold a prominent position as they have been successfully applied in diverse fields [19,23]. The membership of an element $x \in X$ to a type-2 fuzzy set A is given by a type one fuzzy set on the universe $[0,1]$, therefore a type-2 fuzzy set can be identified with an operator $A : X \to [0,1]^{[0,1]}$, where $[0,1]^{[0,1]}$ is the set of functions whose domain and codomain is $[0,1]$.

An additional generalization of fuzzy sets that we discuss in this work is the so-called fuzzy multisets, which were introduced by Yager in [53].

Definition 2.9. Let $n \geq 1$ and $X \neq \emptyset$. A fuzzy multiset A on X is a function $A : X \to [0,1]^n$.

If in the preceding definition (Definition 2.9), if we refer to the membership of an element $x \in X$ by $A(x) = (A_1(x), \ldots, A_n(x))$ and it holds that $A_1(x) \leq \ldots \leq A_n(x)$ for all $x \in X$, then we say that A is a n-dimensional fuzzy set [4]. In the literature one can find an extensive list of works on fuzzy multisets and n-dimensional fuzzy sets and their application [43].

We address two more extensions of fuzzy sets in this work: Atanassov intuitionistic fuzzy sets (AIFS) [1] and interval-valued fuzzy sets (IVFS) [56]. To define the concept of an AIFS let us first set: $D([0,1]) = \{(x, y) \in [0,1]^2 \mid x + y \leq 1\}$.

Definition 2.10. Given a non-empty universe X, an Atanassov intuitionistic fuzzy set A on X is a function given by $A : X \to D([0,1])$. For $x \in X$, we have $A(x) = (\mu_A(x), \nu_A(x))$, where $\mu_A(x)$ denotes the membership of x to the AIFS and $\nu_A(X)$ its non-membership.

Finally, the concept of an interval-valued fuzzy set is defined as follows.

Definition 2.11. Given a non-empty universe X, an interval-valued fuzzy set A on X is a function given by $A : X \to L([0,1])$.

From a formal point of view, the last two concepts are equivalent, as there exists a one-to-one mapping between the set of all AIFSs on X, $AIFS(X)$, and the set of all IVFSs on X, $IVFS(X)$ (see [2,28]):

$$\begin{array}{rccc} \psi : & IVFS(X) & \to & AIFS(X) \\ & [\underline{A}, \overline{A}] & \mapsto & (\underline{A}, 1 - \overline{A}). \end{array}$$

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Moreover, IVFSs on X are n-dimensional fuzzy sets for $n = 2$. Hence, all the results for n-dimensional fuzzy values are valid for intervals in $L([0, 1])$.

3. Riesz spaces

Although we ultimately aim at interval-valued functions, the theoretical results in this work are developed in a more general framework of which the set of intervals is a relevant example. In particular, we deal with vector spaces and we consider vector spaces over $\mathbb{R}$ instead of general fields F.

A vector space V endowed with a partial order relation $\leq_V$ is said to be a partially ordered vector space if the order structure and the vector space structure are compatible, that is, if the following conditions hold for any $u, v \in V$:

(i) If $u \leq_V v$, then $u + w \leq_V v + w$ for every $w \in V$;
(ii) If $u \leq_V v$, then $\alpha u \leq_V \alpha v$ for every real $\alpha \geq 0$.

Condition (ii) can be equivalently formulated as follows: if $u \geq_V 0_V$, then $\alpha u \geq_V 0_V$ for every real $\alpha \geq 0$.

We denote the Cartesian product of such spaces as $V^n = V \times \ldots \times V$, which is a partially ordered vector space with respect to the product order $\leq_{V^n}$, which results from considering $\leq_V$ component-wise. Namely, if $\mathbf{v} = (v_1, \ldots, v_n), \mathbf{u} = (u_1, \ldots, u_n) \in V^n$, we say that $\mathbf{v} \leq_{V^n} \mathbf{u}$ if $v_i \leq_V u_i$ for $1 \leq i \leq n$.

Furthermore, if V with the order relation forms a lattice, then V is said to be a Riesz space (also known as vector-lattice). Note that, in this case, V^n forms a Riesz space as well.

All instances of ordered vector spaces that we mention in this work are in fact Riesz spaces. We provide a brief description of each in the next example.

Example 3.1. The following are various instances of Riesz spaces.

(1) The real line $\mathbb{R}$ with the standard linear order structure and operations is a Riesz space.
(2) The space $\mathbb{R}^n$ with $n \geq 2$ with the component-wise order is also a Riesz space.
(3) The space $\mathbb{R}^2$ with either the first or the second lexicographical order is a Riesz space.
(4) Let V be the space formed from all real functions defined on a non-empty set X with point-wise addition, scalar multiplication and order, respectively:
   - $(f + g)(x) = f(x) + g(x)$,
   - $(\alpha f)(x) = \alpha f(x)$,
   - $f \leq_V g$ if $f(x) \leq g(x)$,

   for all $x \in X$ and $\alpha \geq 0$. Thus, V is a Riesz space.
(5) The set $C(X)$ of continuous real functions on a topological space X with the point-wise order and linear operations is also a Riesz space.
(6) For every $0 < p \leq \infty$, the spaces L_p, spaces of functions whose absolute value to the p-th power is Lebesgue integrable for $0 < p < \infty$ and the set of all measurable bounded functions when $p = \infty$, are Riesz spaces with the almost everywhere (a.e.) point-wise order for functions. If $p \geq 1$, L_p is also a Banach space.
(7) For every $0 < p \leq \infty$, the sequence spaces ℓ_p are Riesz spaces with component-wise order. If $p \geq 1$, ℓ_p is also a Banach space.

For more insight about this example and about partially ordered vector spaces and Riesz spaces, see [54].

4. Weak and directional monotonicity in Riesz spaces

Let $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$, and let us denote by $\vec{0} \in V$ the identity element for addition. We can define monotonicity for functions whose inputs come from V^n and have values in V as follows.

Definition 4.1. Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$. We say that a function $F : V^n \to V$ is increasing (decreasing) if for all $\mathbf{x}, \mathbf{y} \in V^n$ such that $\mathbf{x} \leq_{V^n} \mathbf{y}$ it holds that $F(\mathbf{x}) \leq_V F(\mathbf{y})$ ($F(\mathbf{x}) \geq_V F(\mathbf{y})$).

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Moreover, we can define directional monotonicity for this class of functions, understanding that the directions are non-zero vectors from V^n.

Definition 4.2. Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$ and let $\mathbf{v} = (v_1, \ldots, v_n) \in V^n$ such that $v_i \neq \vec{0}$ for some $1 \leq i \leq n$. We say that a function $F : V^n \to V$ is $\mathbf{v}$-increasing ($\mathbf{v}$-decreasing) if for all $\mathbf{x} \in V^n$ and $c > 0$, it holds that $F(\mathbf{x} + c\mathbf{v}) \geq_V F(\mathbf{x})$ ($F(\mathbf{x} + c\mathbf{v}) \leq_V F(\mathbf{x})$). If F is both $\mathbf{v}$-increasing and $\mathbf{v}$-decreasing, we say that F is $\mathbf{v}$-constant.

Analogously, driven by the concept of weak monotonicity for real valued functions, we propose the concept of w-weak monotonicity, focusing on a fixed $\vec{0} \neq w \in V$.

Definition 4.3. Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$ and let $\vec{0} \neq w \in V$. We say that a function $F : V^n \to V$ is w-weakly increasing (decreasing) if for all $\mathbf{x} \in V^n$ and $c > 0$, it holds that $F(\mathbf{x} + c(w, \ldots, w)) \geq_V F(\mathbf{x})$ ($F(\mathbf{x} + c(w, \ldots, w)) \leq_V F(\mathbf{x})$).

Remark 4.4. Note that, in the conditions of Definition 4.3, if $u = cw$ for some real number $c > 0$, then u-weak increasingness (u-weak decreasingness) and w-weak increasingness (w-weak decreasingness) coincide. On the other hand, if $c < 0$, then u-weak increasingness (u-weak decreasingness) coincides with w-weak decreasingness (w-weak increasingness). Observe the generalization of the concept of weak monotonicity of real functions introduced in [52] (see also Definition 2.2). In fact, weak increasingness is just 1-weak increasingness.

4.1. Properties

In this section we discuss some properties of directionally monotone functions in this general setting. These properties serve as baseline and in the subsequent sections, where we focus on less general domains, we study which ones still hold true and which do not.

We start with a remark about the directions of increasingness for a function F when the ordered vector space $(V, \leq_V)$ we consider is in fact a normed space.

Remark 4.5. Given $\mathbf{v} \in V^n$, for a function $F : V^n \to V$ it is equivalent to be $\mathbf{v}$-increasing and to be $k\mathbf{v}$-increasing for any positive constant k. Consequently, in the cases when the space V^n can be equipped with a norm $\|\cdot\|$, without loss of generality we can characterize each direction $\mathbf{v} \in V^n$ with the one that satisfies $\|\mathbf{v}\| = 1$.

Similarly, the next result shows that it is equivalent for a function F to increase along one direction and to decrease along the opposite one. Thus, without loss of generality, we can develop our results focusing on increasingness.

Proposition 4.6. *Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$ and let $\mathbf{v} \in V^n$ such that $v_i \neq \vec{0}$ for some $1 \leq i \leq n$. A function $F : V^n \to V$ is $\mathbf{v}$-increasing if and only if F is $(-\mathbf{v})$-decreasing.*

Proof. Let F be $\mathbf{v}$-increasing. Let $\mathbf{x} \in V^n$ and $c > 0$, then

$$F(\mathbf{x} + c(-\mathbf{v})) \leq_V F(\mathbf{x} + c(-\mathbf{v}) + c\mathbf{v}) = F(\mathbf{x}),$$

and therefore F is $(-\mathbf{v})$-decreasing. The converse statement follows similarly. □

Now, as in the real case with Theorem 2.4, we study whether a function $F : V^n \to V$ that is $\mathbf{v}$-increasing and $\mathbf{u}$-increasing is also increasing along a linear combination of $\mathbf{v}$ and $\mathbf{u}$.

Theorem 4.7. *Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$ and let $\mathbf{v}, \mathbf{u} \in V^n$ be such that $v_i \neq \vec{0}$ for some $1 \leq i \leq n$ and $u_j \neq \vec{0}$ for some $1 \leq j \leq n$. If a function $F : V^n \to V$ is $\mathbf{v}$-increasing and $\mathbf{u}$-increasing, then F is $(a\mathbf{v} + b\mathbf{u})$-increasing for any $a, b > 0$.*

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Proof. Let F be $\mathbf{v}$- and $\mathbf{u}$-increasing, $a, b > 0$ and $\mathbf{x} \in V^n$. Then, if $c > 0$,

$$F(\mathbf{x} + c(a\mathbf{v} + b\mathbf{u})) \geq_V F(\mathbf{x} + cb\mathbf{u}) \geq_V F(\mathbf{x}),$$

where the inequalities hold due to the $\mathbf{v}$- and $\mathbf{u}$-increasingness of F, respectively. □

The next two results deal with the directional increasingness of functions that are $\mathbf{v}$-increasing for some $\mathbf{v} \in V^n$.

Proposition 4.8. *Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$, $\mathbf{v} \in V^n$ such that $v_i \neq \vec{0}$ for some $1 \leq i \leq n$ and $F : V^n \to V$ be a $\mathbf{v}$-increasing function. Let $\varphi : V \to V$ be an increasing (decreasing) function. Then, the function $\varphi \circ F$ is $\mathbf{v}$-increasing (decreasing).*

Proof. Let F be $\mathbf{v}$-increasing and φ be increasing. Let $\mathbf{x} \in V^n$ and $c > 0$, then

$$(\varphi \circ F)(\mathbf{x} + c\mathbf{v}) = \varphi(F(\mathbf{x} + c\mathbf{v})) \geq_V \varphi(F(\mathbf{x})) = (\varphi \circ F)(\mathbf{x}).$$

The case in which φ is decreasing is analogous. □

Proposition 4.9. *Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$, $\mathbf{v} \in V^n$ such that $v_i \neq \vec{0}$ for some $1 \leq i \leq n$ and $F_1, \dots, F_k : V^n \to V$ be $\mathbf{v}$-increasing functions. Let $A : V^k \to V$ be an increasing (decreasing) function. Then, the function $A(F_1, \dots, F_k)$ is $\mathbf{v}$-increasing (decreasing).*

Proof. Let $F_1, \dots, F_k$ be $\mathbf{v}$-increasing functions and A be increasing. Let $\mathbf{x} \in V^n$ and $c > 0$, then

$$\begin{aligned} A(F_1, \dots, F_k)(\mathbf{x} + c\mathbf{v}) &= A(F_1(\mathbf{x} + c\mathbf{v}), \dots, F_k(\mathbf{x} + c\mathbf{v})) \\ &\geq_V A(F_1(\mathbf{x}), \dots, F_k(\mathbf{x})) = A(F_1, \dots, F_k)(\mathbf{x}). \end{aligned}$$

The case in which A is decreasing is analogous. □

Finally, as in the case of real functions (Theorem 2.5), we can characterize monotonicity in Riesz spaces in terms of directional monotonicity. To that end, let us define the set $V^+ = \left\{ v \in V \mid v \geq_V \vec{0} \right\}$.

Theorem 4.10. *Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$. A function $F : V^n \to V$ is increasing (decreasing) if and only if F is $\mathbf{v}$-increasing ($\mathbf{v}$-decreasing) for every $\mathbf{v} \in (V^+)^n$ such that $v_i \neq \vec{0}$ for some $1 \leq i \leq n$.*

Proof. Let F be increasing and let $\mathbf{x} \in V^n$. Now, given $c > 0$ and $\mathbf{0} \neq \mathbf{v} \in (V^+)^n$, it holds that $\mathbf{x} <_{V^n} \mathbf{x} + c\mathbf{v}$. Hence, since F is increasing, it also is $\mathbf{v}$-increasing.

Conversely, let F be $\mathbf{v}$-increasing for every $\mathbf{0} \neq \mathbf{v} \in (V^+)^n$. Let $\mathbf{x}, \mathbf{y} \in V^n$ such that $\mathbf{x} \leq_{V^n} \mathbf{y}$. Since the case $\mathbf{x} = \mathbf{y}$ is straight, we can assume $\mathbf{x} <_{V^n} \mathbf{y}$. Thus, we set $\mathbf{v} = \mathbf{y} - \mathbf{x}$. Clearly, $\mathbf{v} \in (V^+)^n$ and $\mathbf{v} \neq \mathbf{0}$. Therefore, since F is $\mathbf{v}$-increasing, it holds that $F(\mathbf{x}) \leq_V F(\mathbf{y})$. □

Theorem 4.7 and Theorem 4.10 derive the following result.

Corollary 4.11. *Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$. Let B be the set of vectors $\mathbf{v} \in (V^+)^n$ that span $(V^+)^n$. Then, a function $F : V^n \to V$ is increasing (decreasing) if and only if F is $\mathbf{v}$-increasing ($\mathbf{v}$-decreasing) for every $\mathbf{v} \in B$ such that $v_i \neq \vec{0}$ for some $1 \leq i \leq n$.*

4.2. Restriction of V to an interval sublattice

In this section we study directional monotononicity of functions as in Section 4 but whose domains and codomains are restricted. Let $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$ and let $r, s \in V$ such that $r \leq_V s$, then we set the following subset of V:

$$V_r^s = \{v \in V \mid r \leq_V v \leq_V s\}. \tag{1}$$

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V_r^s is a sublattice with top and bottom elements s and r, respectively. However, V_r^s is not a vector space and, hence, the definitions of directional and weak monotonicity for functions $F : (V_r^s)^n \to V_r^s$ need some adaptations.

Definition 4.12. Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$ and set V_r^s as in eq. (1). Let $\mathbf{v} = (v_1, \dots, v_n) \in V^n$ such that $v_i \neq \vec{0}$ for some $1 \leq i \leq n$. We say that a function $F : (V_r^s)^n \to V_r^s$ is $\mathbf{v}$-increasing ($\mathbf{v}$-decreasing) if for all $\mathbf{x} \in (V_r^s)^n$ and $c > 0$ such that $\mathbf{x} + c\mathbf{v} \in (V_r^s)^n$, it holds that $F(\mathbf{x} + c\mathbf{v}) \geq_V F(\mathbf{x})$ ($F(\mathbf{x} + c\mathbf{v}) \leq_V F(\mathbf{x})$). If F is both $\mathbf{v}$-increasing and $\mathbf{v}$-decreasing, we say that F is $\mathbf{v}$-constant.

Note that the direction $\mathbf{v}$ does not necessarily belong to the restricted set $(V_r^s)^n$, but to the original vector space V^n. We allow the directions to exit the restricted space and we require the condition of monotonicity to the points $\mathbf{x} \in (V_r^s)^n$ such that $\mathbf{x} + c\mathbf{v} \in (V_r^s)^n$. This resembles the original case ([15]), where directional monotonicity is studied for functions $F : [0, 1]^n \to [0, 1]$ and the directions of increasingness belong to the more general $\mathbb{R}^n$.

Definition 4.13. Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$, V_r^s as in eq. (1) and let $\vec{0} \neq w \in V$. We say that a function $F : (V_r^s)^n \to V_r^s$ is w-weakly increasing (decreasing) if for all $\mathbf{x} \in (V_r^s)^n$ and $c > 0$ such that $\mathbf{x} + c(w, \dots, w) \in (V_r^s)^n$, it holds that $F(\mathbf{x} + c(w, \dots, w)) \geq_V F(\mathbf{x})$ ($F(\mathbf{x} + c(w, \dots, w)) \leq_V F(\mathbf{x})$).

Real functions defined on a Cartesian product of a closed real interval with itself and taking values in the same interval are an example of the functions described in this section. In particular, aggregation functions can be seen as a particular case. Recall that an aggregation function is defined as $f : [0, 1]^n \to [0, 1]$ such that $f(0, \dots, 0) = 0$, $f(1, \dots, 1) = 1$ and f is increasing with respect to each component. Therefore, it suffices to set $V_r^s = [0, 1]$ as a subset of $V = \mathbb{R}$ and we recover the notion of directional monotonicity introduced in [15].

In relation to the properties that this class of functions satisfy, all the properties studied in Section 4.1 hold similarly for functions defined in V_r^s taking Definition 4.12 into account. The only result that needs an additional assumption is Theorem 4.7 and the new formulation is as follows.

Theorem 4.14. *Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$, let V_r^s be as in eq. (1), let $a, b > 0$ and $\mathbf{v}, \mathbf{u} \in V^n$ such that $v_i \neq \vec{0}$ for some $1 \leq i \leq n$ and $u_j \neq \vec{0}$ for some $1 \leq j \leq n$. Assume that for all $\mathbf{x} \in (V_r^s)^n$ and $c > 0$ such that $\mathbf{x} + c(a\mathbf{v} + b\mathbf{u}) \in (V_r^s)^n$, then $\mathbf{x} + ca\mathbf{v} \in (V_r^s)^n$ or $\mathbf{x} + cb\mathbf{u} \in (V_r^s)^n$. Thus, if a function $F : (V_r^s)^n \to V_r^s$ is $\mathbf{v}$-increasing and $\mathbf{u}$-increasing, then F is $(a\mathbf{v} + b\mathbf{u})$-increasing.*

4.3. Restriction of V to a convex cone

In this section we introduce the concepts of weak and directional monotonicity for functions that take values on a convex cone C of a Riesz space V. Note that the notions presented in this section are not a particular case of the developments in Section 4 since a convex cone C is not a vector space due to the non existence of an inverse for the addition in general.

Definition 4.15. Let V be a vector space over $\mathbb{R}$. We say that a subset $C \subset V$ is a cone if for every $x \in C$ and $a \geq 0$ it holds that $ax \in C$. A cone C is a convex cone if for all $a, b > 0$ and $x, y \in C$, it holds that $ax + by \in C$.

Let us point out that the set of closed real intervals $L(\mathbb{R})$ can be seen as a convex cone of the vector space $\mathbb{R}^2$ and that, indeed, there does not exist an inverse for the addition in general: $[2, 3] - [2, 4] = [0, -1] \notin L(\mathbb{R})$.

We now present the concepts of usual, directional and weak monotonicity for functions that take values on a convex cone C. These notions are also valid for interval-valued functions setting $C = L(\mathbb{R})$.

Definition 4.16. Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$ and let $C \subset V$ be a convex cone. We say that a function $F : C^n \to C$ is increasing (decreasing) if for all $\mathbf{x}, \mathbf{y} \in C^n$ such that $\mathbf{x} \leq_{V^n} \mathbf{y}$ it holds that $F(\mathbf{x}) \leq_V F(\mathbf{y})$ ($F(\mathbf{x}) \geq_V F(\mathbf{y})$).

Definition 4.17. Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$ and let $C \subset V$ be a convex cone and let $\mathbf{v} = (v_1, \dots, v_n) \in V^n$ such that $v_i \neq \vec{0}$ for some $1 \leq i \leq n$. We say that a function $F : C^n \to C$ is $\mathbf{v}$-increasing

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ARTICLE IN PRESS FSS:7585

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($\mathbf{v}$-decreasing) if for all $\mathbf{x} \in C^n$ and $c > 0$ such that $\mathbf{x}+c\mathbf{v} \in C^n$, it holds that $F(\mathbf{x}+c\mathbf{v}) \geq_V F(\mathbf{x})$ ($F(\mathbf{x}+c\mathbf{v}) \leq_V F(\mathbf{x})$). If F is both $\mathbf{v}$-increasing and $\mathbf{v}$-decreasing, we say that F is $\mathbf{v}$-constant.

Definition 4.18. Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$ and let $C \subset V$ be a convex cone and let $\vec{0} \neq v \in V$. We say that a function $F : C^n \to C$ is v-weakly increasing (decreasing) if for all $\mathbf{x} \in C^n$ and $c > 0$ such that $\mathbf{x} + c(v, \dots, v) \in C^n$, it holds that $F(\mathbf{x} + c(v, \dots, v)) \geq_V F(\mathbf{x})$ ($F(\mathbf{x} + c(v, \dots, v)) \leq_V F(\mathbf{x})$).

The restriction of V to a cone C has not great impact in the properties studied in Section 4.1, all the properties hold for functions $F : C^n \to C$ with minor adjustments. Note that although a convex cone C loses the vector space structure, it still is closed under convex combinations, and hence the adaptation of Theorem 4.7 for this framework is straightforward. This is relevant because the mentioned property is meaningful in the setting of directional monotonicity. In particular, for the interval-valued case with $C = L(\mathbb{R})$ and $V = \mathbb{R}^2$.

5. Directional monotonicity of functions to fuse data from different fuzzy settings

In this section we present some prominent particular cases of the theoretical developments of Section 4. We show that functions to fuse data from different fuzzy settings can be seen as either an interval sublattice or a convex cone of some of the Riesz spaces V presented in Example 3.1. Concretely, we study the cases of type-2 fuzzy sets, fuzzy multisets, n-dimensional fuzzy sets, interval-valued fuzzy sets and Atanassov intuitionistic fuzzy sets.

5.1. *Type-2 fuzzy values*

Let us set V as in Example 3.1 (4), the set of all real functions defined in a set X, and let $X = [0, 1]$. Thus, V with the point-wise order is a Riesz space.

Now, let $f_0, f_1 : [0, 1] \to \mathbb{R}$ be the functions given by $f_0(x) = 0$ and $f_1(x) = 1$ for all $x \in [0, 1]$, respectively. If we consider the interval sublattice $V_{f_0}^{f_1}$, we obtain that all the functions defined in $[0, 1]$ with values in $[0, 1]$ belong to $V_{f_0}^{f_1}$, and due to the definition of the point-wise order, no other function belongs to that subset. Therefore, we can see $V_{f_0}^{f_1}$ as $[0, 1]^{[0,1]}$ and, hence, from Section 4.2 we can retrieve a definition and properties of directional monotonicity for type-2 fuzzy valued functions.

5.2. *Fuzzy multiset values and n-dimensional fuzzy values*

Since $[0, 1]^n$ can be seen as an interval sublattice of the Riesz space $\mathbb{R}^n$, the developments in Section 4.2 are applicable to functions that are intended to fuse information coming from fuzzy multisets.

For the case of n-dimensional fuzzy sets, the set of n-dimensional fuzzy values is given by

$$L_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid 0 \leq x_1 \leq \dots \leq x_n \leq 1\}. \tag{2}$$

Clearly, L_n is the intersection of the interval sublattice $[0, 1]^n$ of the Riesz space $\mathbb{R}^n$ and the convex cone $\{(x_1, \dots, x_n) \mid x_1 \leq \dots \leq x_n\}$ of the same Riesz space $\mathbb{R}^n$. Therefore, we can take into account the adaptations in the definitions and properties of directional monotonicity in Sections 4.2 and 4.3, to define directional monotonicity for n-dimensional fuzzy valued functions.

5.3. *Interval-valued fuzzy values and Atanassov intuitionistic fuzzy values*

Since interval-valued fuzzy sets are a particular instance of n-dimensional fuzzy sets for $n = 2$, and since IVFSs and AIFSs are formally equivalent, the points made in the preceding subsection are also valid for interval-valued and Atanassov intuitionistic fuzzy sets.

However, in the following section of this work we study further the particular case of interval-valued functions, for intervals in $L([0, 1])$ (as is the case of interval-valued fuzzy values), giving explicit definitions, examples and properties.

Please cite this article in press as: M. Sesma-Sara et al., Weak and directional monotonicity of functions on Riesz spaces to fuse uncertain data, Fuzzy Sets Syst. (2019), https://doi.org/10.1016/j.fss.2019.01.019

Note that although we focus on functions defined and with values in $L([0,1])$, the developments in Section 4.3 generalize the concept of directional monotonicity for functions defined in $L(\mathbb{R})$ as well, since $L(\mathbb{R})$ can be seen as a convex cone of the vector space $\mathbb{R}^2$. Indeed, it is isomorphic to the set $K(\mathbb{R}) \subset \mathbb{R}^2$, which is a half-space as $K(\mathbb{R}) = \{(x,y) \in \mathbb{R}^2 \mid y - x \geq 0\}$, and therefore $K(\mathbb{R})$ is a convex cone of $\mathbb{R}^2$.

6. Weak and directional monotonicity on the interval-valued setting

6.1. Restriction to intervals in $L([0,1])$

In this subsection we present explicit definitions for interval-valued functions that are defined over $L([0,1])$, as they are recurrent in both theoretic and applied works in the literature [10,12,16]. This type of functions can be seen as the result of restricting the former space V to be the intersection of an interval sublattice and a convex cone, as in Sections 4.2 and 4.3, respectively. Note that these developments are equivalent for the case of any other closed interval, i.e., they are equivalent for $L([a,b])$.

We now present the explicit definitions for standard, directional and weak monotonicity for functions that take values on $L([0,1])$.

Definition 6.1. Let $n \in \mathbb{N}$. We say that a function $F : L([0,1])^n \to L([0,1])$ is increasing (decreasing) if for all $\mathbf{x}, \mathbf{y} \in L([0,1])^n$ such that $\mathbf{x} \leq_{L^n} \mathbf{y}$ it holds that $F(\mathbf{x}) \leq_L F(\mathbf{y})$ $(F(\mathbf{x}) \geq_L F(\mathbf{y}))$.

Definition 6.2. Let $n \in \mathbb{N}$ and let $\mathbf{v} = (a_1, b_1, \ldots, a_n, b_n) \in (\mathbb{R}^2)^n$ such that $(a_i, b_i) \neq \vec{0}$ for some $1 \leq i \leq n$. We say that a function $F : L([0,1])^n \to L([0,1])$ is $\mathbf{v}$-increasing ($\mathbf{v}$-decreasing) if for all $\mathbf{x} \in L([0,1])^n$ and $c > 0$ such that $\mathbf{x} + c\mathbf{v} \in L([0,1])^n$, it holds that $F(\mathbf{x} + c\mathbf{v}) \geq_L F(\mathbf{x})$ $(F(\mathbf{x} + c\mathbf{v}) \leq_L F(\mathbf{x}))$. If F is both $\mathbf{v}$-increasing and $\mathbf{v}$-decreasing, we say that F is $\mathbf{v}$-constant.

Definition 6.3. Let $n \in \mathbb{N}$ and let $\vec{0} \neq (a,b) \in \mathbb{R}^2$. We say that a function $F : L([0,1])^n \to L([0,1])$ is (a,b)-weakly increasing (decreasing) if for all $\mathbf{x} \in L([0,1])^n$ and $c > 0$ such that $\mathbf{x} + c(a,b,\ldots,a,b) \in (\mathbb{R}^2)^n$, it holds that $F(\mathbf{x} + c(a,b,\ldots,a,b)) \geq_L F(\mathbf{x})$ $(F(\mathbf{x} + c(a,b,\ldots,a,b)) \leq_L F(\mathbf{x}))$.

Example 6.4. The following are two examples of interval-valued functions and their directions of increasingness in terms of some parameters.

(1) Let $F : L([0,1])^2 \to L([0,1])$ be a function given by

$$F([\underline{x_1}, \overline{x_1}], [\underline{x_2}, \overline{x_2}]) = \left[\frac{\underline{x_1} + \underline{x_2}}{2}, \min\left(\frac{\underline{x_1} + \overline{x_2}}{2}, \frac{\overline{x_1} + \underline{x_2}}{2}\right)\right].$$

Then, given $a, b \in \mathbb{R}$, F is (a,b)-weakly increasing if and only if $a > 0$ and $a + b \geq 0$, or $a = 0$ and $b > 0$. Indeed, it follows the fact that, given $c > 0$,

$$F(([\underline{x_1}, \overline{x_1}], [\underline{x_2}, \overline{x_2}]) + c(a,b,a,b)) = F([\underline{x_1}, \overline{x_1}], [\underline{x_2}, \overline{x_2}]) + c\left[a, \frac{a+b}{2}\right].$$

(2) Let $\lambda \in]0,1[$ and let $L_\lambda : [0,1]^2 \to [0,1]$ be the weighted Lehmer mean [15], which is given (with the convention $\frac{0}{0} = 0$) by

$$L_\lambda(x,y) = \frac{\lambda x^2 + (1-\lambda)y^2}{\lambda x + (1-\lambda)y}.$$

Let $F : L([0,1])^2 \to L([0,1])$ be a function given by

$$F([\underline{x_1}, \overline{x_1}], [\underline{x_2}, \overline{x_2}]) = \left[\frac{1}{2} L_\lambda(\underline{x_1}, \underline{x_2}), L_\lambda(\overline{x_1}, \overline{x_2})\right],$$

(with the convention $\frac{0}{0} = 0$). F is a well-defined because $\frac{1}{2} L_\lambda(x,y) \leq L_\lambda(z,t)$ for all $x, y, z, t \in [0,1]$ such that $x \leq z$ and $y \leq t$.

Please cite this article in press as: M. Sesma-Sara et al., Weak and directional monotonicity of functions on Riesz spaces to fuse uncertain data, Fuzzy Sets Syst. (2019), https://doi.org/10.1016/j.fss.2019.01.019

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We show in the following subsection that the function F is $((1-\lambda, 1-\lambda), (\lambda, \lambda))$-increasing. In fact, it only increases along that particular direction (up to positive scalar multiplication).

The next result is an adaptation of Theorem 2.5 for the interval-valued case and succeeds to characterize regular monotonicity for interval-valued functions with respect to the partial order $\leq_L$. In it, we make use of the canonical basis of $(\mathbb{R}^2)^n$, i.e., the set of vectors $\{\mathbf{e}_i\}_{i=1}^{2n}$. In the case of $n=2$, the vectors of the canonical basis of $(\mathbb{R}^2)^2$ are the following:

$$\mathbf{e}_1 = ((1,0),(0,0))\,; \qquad \mathbf{e}_2 = ((0,1),(0,0))\,;$$
$$\mathbf{e}_3 = ((0,0),(1,0))\,; \qquad \mathbf{e}_4 = ((0,0),(0,1))\,.$$

Theorem 6.5. *Let $n \in \mathbb{N}$, let $\leq_L$ be the partial order on $L([0,1])$, let $F : L([0,1])^n \to L([0,1])$ and let $\{\mathbf{e}_i\}_{i=1}^{2n}$ be the canonical basis of $(\mathbb{R}^2)^n$. Then, F is increasing if and only if F is $\mathbf{e}_i$-increasing for all $1 \leq i \leq 2n$.*

Proof. Let F be increasing with respect to $\leq_L$ and let $\mathbf{x} = ([\underline{x_1}, \overline{x_1}], \dots, [\underline{x_n}, \overline{x_n}]) \in L([0,1])^n$. Now, given $c > 0$ such that $\mathbf{x} + c\mathbf{e}_i \in L([0,1])^n$, it is straight to check that $\mathbf{x} \leq_L \mathbf{x} + c\mathbf{e}_i$. Hence, the increasingness of F ensures $\mathbf{e}_i$-increasingness.

Conversely, let F be $\mathbf{e}_i$-increasing for all $1 \leq i \leq 2n$. Let $\mathbf{x}, \mathbf{y} \in L([0,1])^n$ such that $[\underline{x_i}, \overline{x_i}] \leq_L [\underline{y_i}, \overline{y_i}]$ for all $1 \leq i \leq n$. From the definition of $\leq_L$, it follows that $\underline{x_i} \leq \underline{y_i}$ and $\overline{x_i} \leq \overline{y_i}$ for all $1 \leq i \leq n$ and, hence, for each i there exist $a_i, b_i \geq 0$ with $a_i + b_i > 0$ such that $[\underline{y_i}, \overline{y_i}] = [\underline{x_i}, \overline{x_i}] + a_i(1,0) + b_i(0,1)$. Consequently,

$$\mathbf{y} = \mathbf{x} + \sum_{i=1}^{n} a_i \mathbf{e}_{2i-1} + \sum_{i=1}^{n} b_i \mathbf{e}_{2i},$$

and by the straight adaptation of Theorem 4.14 for $L([0,1])$, it holds that F is $\mathbf{v}$-increasing for $\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{e}_{2i-1} + \sum_{i=1}^{n} b_i \mathbf{e}_{2i}$. Hence, $F(\mathbf{x}) \leq_L F(\mathbf{y})$. □

6.2. Representable interval-valued functions

In this subsection we focus on the special class of IV functions $F : L([0,1])^n \to L([0,1])$ that verifies

$$F(\mathbf{x}) = F([\underline{x_1}, \overline{x_1}], \dots, [\underline{x_n}, \overline{x_n}]) = [f(\underline{x_1}, \dots, \underline{x_n}), g(\overline{x_1}, \dots, \overline{x_n})], \tag{3}$$

for some functions $f, g : [0,1]^n \to [0,1]$ such that $f(x_1, \dots, x_n) \leq g(y_1, \dots, y_n)$ whenever $x_i \leq y_i$ for all $1 \leq i \leq n$. This type of interval-valued functions are said to be representable [27].

Note that Example 6.4 brings an example of a non-representable function (see item (1)) and an example of a representable function (see item (2)).

Theorem 6.6. *Let $F : L([0,1])^n \to L([0,1])$ be an interval-valued function satisfying eq. (3) for some functions $f, g : [0,1]^n \to [0,1]$ such that $f(x_1, \dots, x_n) \leq g(y_1, \dots, y_n)$ if $x_i \leq y_i$ for $1 \leq i \leq n$. Let $\vec{a} = (a_1, \dots, a_n)$ and $\vec{b} = (b_1, \dots, b_n) \in \mathbb{R}^n$ be vectors such that $\vec{a}, \vec{b} \neq (0, \dots, 0)$. Then, F is $((a_1, b_1), \dots, (a_n, b_n))$-increasing if and only if f is $\vec{a}$-increasing and g is $\vec{b}$-increasing.*

Proof. Let F satisfy eq. (3) for $f, g : [0,1]^n \to [0,1]$ such that they satisfy $f(x_1, \dots, x_n) \leq g(y_1, \dots, y_n)$ if $x_i \leq y_i$ for $1 \leq i \leq n$ and assume that F is $((a_1, b_1), \dots, (a_n, b_n))$-increasing. Let us now show that f is $\vec{a}$-increasing. Let $(x_1, \dots, x_n) \in [0,1]^n$ and $c > 0$ such that $(x_1, \dots, x_n) + c\vec{a} \in [0,1]^n$. We can find $y_1, \dots, y_n \in [0,1]$ such that $x_i \leq y_i$ for all $1 \leq i \leq n$ and such that

$$([x_1, y_1], \dots, [x_n, y_n]) + c\,((a_1, b_1), \dots, (a_n, b_n)) \in L([0,1])^n.$$

Now, since F is $((a_1, b_1), \dots, (a_n, b_n))$-increasing, it follows that $f((x_1, \dots, x_n) + c\vec{a}) \geq f(x_1, \dots, x_n)$ and, hence, f is $\vec{a}$-increasing. Similarly, it can be shown that g is $\vec{b}$-increasing.

For the converse, let f and g be $\vec{a}$- and $\vec{b}$-increasing, respectively. Let $\mathbf{x} \in L([0,1])^n$ and $c > 0$ such that $\mathbf{x} + c\,((a_1, b_1), \dots, (a_n, b_n)) \in L([0,1])^n$. Then,

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$$\begin{aligned} F(\mathbf{x} + c((a_1, b_1), \ldots, (a_n, b_n))) &= F([\underline{x_1}, \overline{x_1}] + c(a_1, b_1), \ldots, [\underline{x_n}, \overline{x_n}] + c(a_n, b_n)) \\ &= [f(\underline{x_1} + ca_1, \ldots, \underline{x_n} + ca_n), g(\overline{x_1} + cb_1, \ldots, \overline{x_n} + cb_n)] \\ &= [f((\underline{x_1}, \ldots, \underline{x_n}) + c\vec{a}), g((\overline{x_1}, \ldots, \overline{x_n}) + c\vec{b})] \\ &\geq_L [f((\underline{x_1}, \ldots, \underline{x_n})), g((\overline{x_1}, \ldots, \overline{x_n}))] \\ &= F(\mathbf{x}), \end{aligned}$$

and, hence, F is $((a_1, b_1), \ldots, (a_n, b_n))$-increasing. □

As a direct consequence, we have the following corollary. Let us fix the notation $\mathcal{D}^{\uparrow}(F)$ to denote the set of vectors along which the function F is directionally increasing. Of course, $\mathcal{D}^{\uparrow}$ is a subset of the Riesz space where the directions of F are defined. In the case of an interval-valued function $F : L([0, 1])^n \to L([0, 1])$, it holds that $\mathcal{D}^{\uparrow}(F) \subset (\mathbb{R}^2)^n$.

Corollary 6.7. *Let $F : L([0, 1])^n \to L([0, 1])$ be an interval-valued function satisfying eq. (3) for some functions $f, g : [0, 1]^n \to [0, 1]$ such that $f(x_1, \ldots, x_n) \leq g(y_1, \ldots, y_n)$ if $x_i \leq y_i$ for $1 \leq i \leq n$. Then, it holds that*

$$\mathcal{D}^{\uparrow}(F) = \left\{((a_1, b_1), \ldots, (a_n, b_n)) \in (\mathbb{R}^2)^n \mid \vec{a} \in \mathcal{D}^{\uparrow}(f) \text{ and } \vec{b} \in \mathcal{D}^{\uparrow}(g)\right\}.$$

Consequently, one can define many instances of functions F that increase along some direction. Indeed, Theorem 6.6 shows that in the case of representable functions, the study of directions in which a function F is increasing (decreasing) is reduced to the study of directional monotonicity of component functions f and g. In particular, the function based on the weighted Lehmer mean in Example 6.4 item (2) increases only along the direction $((1 - \lambda, 1 - \lambda), (\lambda, \lambda))$ because the weighted Lehmer mean L_λ is only directionally increasing with respect to the vector $(1 - \lambda, \lambda)$, up to positive scalar multiplication.

In a similar manner, we can construct other instances of functions F and characterize the set of vectors along which it increases.

Example 6.8. Let $F : L([0, 1])^n \to L([0, 1])$ be a function given by

$$F([\underline{x_1}, \overline{x_1}], \ldots, [\underline{x_n}, \overline{x_n}]) = \left[\min(\underline{x_1}, \ldots, \underline{x_n}), \frac{1}{n}\sum_{i=1}^{n} \overline{x_i}\right].$$

Clearly, F is a well-defined representable interval-valued function and following the notation in eq. (3), the function f in this case is the minimum ($f = \min$) and the function g is the arithmetic mean ($g = AM$).

Now, these are the set of directions for which the minimum and the arithmetic mean are increasing:

$$\mathcal{D}^{\uparrow}(\min) = \{\vec{r} = (r_1, \ldots, r_n) \mid r_i \geq 0 \text{ for all } 1 \leq i \leq n\},$$

$$\mathcal{D}^{\uparrow}(AM) = \left\{\vec{r} = (r_1, \ldots, r_n) \mid \sum_{i=1}^{n} r_i \geq 0\right\}.$$

Therefore, by Corollary 6.7, the set of directions along which the function F is the following.

$$\mathcal{D}^{\uparrow}(F) = \left\{((a_1, b_1), \ldots, (a_n, b_n)) \in (\mathbb{R}^2)^n \mid a_i \geq 0 \text{ for all } 1 \leq i \leq n, \text{ and } \sum_{i=1}^{n} b_i \geq 0\right\}.$$

For example, one of the directions of increasingness for $n = 2$ is $((1, 1), (0, -1))$.

Interested readers can find numerous examples of directionally monotone functions in [7–9,15,52], which enable to construct directionally monotone representable IV functions.

A remarkable example of such an IV function is the interval-valued Choquet integral, as it has been proved to be useful in diverse applications [34].

Please cite this article in press as: M. Sesma-Sara et al., Weak and directional monotonicity of functions on Riesz spaces to fuse uncertain data, Fuzzy Sets Syst. (2019), https://doi.org/10.1016/j.fss.2019.01.019

ARTICLE IN PRESS FSS:7585

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Example 6.9. Let $X = \{1, \ldots, n\}$ and $m : 2^X \to [0, 1]$ be a fuzzy measure (see [50]). The discrete Choquet integral $C_m : [0, 1]^n \to [0, 1]$ is defined as:

$$C_m(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_{\sigma(i)} \left(m(\{\sigma(i), \ldots, \sigma(n)\}) - m(\{\sigma(i+1), \ldots, \sigma(n)\})\right),$$

where $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation such that $x_{\sigma(1)} \leq \ldots \leq x_{\sigma(n)}$ and, by convention, $\{x_{\sigma(n+1)}, x_{\sigma(n)}\} = \emptyset$.

The set of vectors for which C_m directionally increases was characterized in [15]:

$$\mathcal{D}^{\uparrow}(C_m) = \left\{ (r_1, \ldots, r_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} r_i (m(\{\sigma(i), \ldots, \sigma(n)\}) - m(\{\sigma(i+1), \ldots, \sigma(n)\})) \geq 0 \text{ for all } \sigma \in S_n \right\},$$

where S_n denotes the set of all permutations of n elements.

The definition of the discrete IV Choquet integral follows Auman's approach to define integrals for set-valued functions [3]. The discrete IV Choquet integral $\mathbf{C}_m : L([0, 1])^n \to L([0, 1])$ is given by

$$\mathbf{C}_m([\underline{x_1}, \overline{x_1}], \ldots, [\underline{x_n}, \overline{x_n}]) = \left[C_m(\underline{x_1}, \ldots, \underline{x_n}), C_m(\overline{x_1}, \ldots, \overline{x_n})\right].$$

Therefore, by Corollary 6.7, it holds that

$$\mathcal{D}^{\uparrow}(\mathbf{C}_m) = \Bigg\{ ((a_1, b_1), \ldots, (a_n, b_n)) \in (\mathbb{R}^2)^n \mid$$
$$\sum_{i=1}^{n} a_i (m(\{\sigma(i), \ldots, \sigma(n)\}) - m(\{\sigma(i+1), \ldots, \sigma(n)\})) \geq 0 \text{ and}$$
$$\sum_{i=1}^{n} b_i (m(\{\sigma(i), \ldots, \sigma(n)\}) - m(\{\sigma(i+1), \ldots, \sigma(n)\})) \geq 0, \text{ for all } \sigma \in S_n \Bigg\}.$$

6.3. Particular case: interval directions

In this section we study the particular case of directional monotonicity for functions $F : L([0, 1])^n \to L([0, 1])$ that increase along a direction formed by intervals, i.e., the cases in which such function F is $\mathbf{v}$-increasing for $\mathbf{v} = ([a_1, b_1], \ldots, [a_n, b_n]) \in L(\mathbb{R})^n$ (as opposed to $\mathbf{v} \in (\mathbb{R}^2)^n$). We refer to this notion as interval directional monotonicity (IDM).

Definition 6.10. Let $n \in \mathbb{N}$ and let $\mathbf{v} = ([a_1, b_1], \ldots, [a_n, b_n]) \in L(\mathbb{R})^n$ such that $[a_i, b_i] \neq 0_L$ for all $1 \leq i \leq n$. We say that a function $F : L([0, 1])^n \to L([0, 1])$ is IDM $\mathbf{v}$-increasing (IDM $\mathbf{v}$-decreasing) if for all $\mathbf{x} \in L([0, 1])^n$ and $c > 0$ such that $\mathbf{x} + c\mathbf{v} \in L([0, 1])^n$, it holds that $F(\mathbf{x} + c\mathbf{v}) \geq_L F(\mathbf{x})$ ($F(\mathbf{x} + c\mathbf{v}) \leq_L F(\mathbf{x})$). If F is both IDM $\mathbf{v}$-increasing and IDM $\mathbf{v}$-decreasing, we say that F is IDM $\mathbf{v}$-constant.

The restriction of the possible directions of increasingness from $(\mathbb{R}^2)^n$ to $L(\mathbb{R})^n$ has an impact in the properties studied in Section 4.1. However, all properties hold for functions $F : L([0, 1])^n \to L([0, 1])$ for which the vectors of increasingness belong in $L([0, 1])^n$, with the exception of Proposition 4.6, which deals with the inverse of addition. Note that there is no inverse of addition defined on $L([0, 1])^n$. The remaining properties of Section 4.1 hold similarly taking into account this new restriction.

Theorem 6.6 in Section 6.2 on representable interval-valued functions is also valid for IDM with some minor modifications. Theorem 6.6 is adapted as follows.

Theorem 6.11. *Let $F : L(\mathbb{R})^n \to L(\mathbb{R})$ be an interval-valued function satisfying eq. (3) for some functions $f, g : \mathbb{R}^n \to \mathbb{R}$ such that $f(x_1, \ldots, x_n) \leq g(y_1, \ldots, y_n)$ if $x_i \leq y_i$ for $1 \leq i \leq n$. Let $\vec{a} = (a_1, \ldots, a_n)$ and $\vec{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$ be vectors such that $\vec{a}, \vec{b} \neq (0, \ldots, 0)$ and $a_i \leq b_i$ for $1 \leq i \leq n$. Then, F is $([a_1, b_1], \ldots, [a_n, b_n])$-increasing if and only if f is $\vec{a}$-increasing and g is $\vec{b}$-increasing.*

Please cite this article in press as: M. Sesma-Sara et al., Weak and directional monotonicity of functions on Riesz spaces to fuse uncertain data, Fuzzy Sets Syst. (2019), https://doi.org/10.1016/j.fss.2019.01.019

With respect to finding a characterization of regular monotonicity in terms of IDM increasing functions, as in Theorem 6.5, note that in the conditions of Theorem 6.5 vectors $\mathbf{e}_{2i-1} \notin L(\mathbb{R})^n$ and hence it is not possible to find a characterization because the composition of vectors in $L(\mathbb{R})^n$ is not sufficient to reach every point $\mathbf{y} \in L([0,1])^n$ from a given point $\mathbf{x} \in L([0,1])^n$. Consequently, even if a function is IDM increasing for every possible direction, it need not be increasing. This fact is stated in the following remark.

Remark 6.12. A function $F : L([0,1])^n \to L([0,1])$ is not necessarily increasing even though F is IDM $\mathbf{v}$-increasing for all $\mathbf{v} \in L(\mathbb{R})^n$.

Indeed, let

$$\mathbf{x}_0 = ([0.2, 0.5], [0, 0], \dots, [0, 0]) \in L([0,1])^n,$$
$$\mathbf{y}_0 = ([0.4, 0.5], [0, 0], \dots, [0, 0]) \in L([0,1])^n.$$

Clearly, $[0.2, 0.5] \leq_L [0.4, 0.5]$. However, it does not necessarily hold that $F(\mathbf{x}_0) \leq_L F(\mathbf{y}_0)$ because there do not exist a constant $c > 0$ and a vector $\mathbf{v} \in L(R)^n$ such that $\mathbf{y}_0 = \mathbf{x}_0 + c\mathbf{v}$.

For example, let us define a function $F : L([0,1])^n \to L([0,1])$ in the following way

$$F([\underline{x_1}, \overline{x_1}], \dots, [\underline{x_n}, \overline{x_n}]) = [\overline{x_1} - \underline{x_1}, \overline{x_1} - \underline{x_1}].$$

It is straight to check that F is well-defined. Let us now consider an arbitrary direction $\mathbf{v} = ([a_1, b_1], \dots, [a_n, b_n]) \in L(\mathbb{R})^n$. Now, for $c > 0$ it holds that

$$\begin{aligned} F(([\underline{x_1}, \overline{x_1}], \dots, [\underline{x_n}, \overline{x_n}]) + c\mathbf{v}) &= [\overline{x_1} + b_1 - \underline{x_1} - a_1, \overline{x_1} + b_1 - \underline{x_1} - a_1] \\ &= [\overline{x_1} - \underline{x_1} + b_1 - a_1, \overline{x_1} - \underline{x_1} + b_1 - a_1] \\ &\geq_L F([\underline{x_1}, \overline{x_1}], \dots, [\underline{x_n}, \overline{x_n}]), \end{aligned}$$

because the fact that $\mathbf{v} \in L(\mathbb{R})^n$ implies that $b_1 - a_1 \geq 0$. Therefore, F is IDM $\mathbf{v}$-increasing for all $\mathbf{v} \in L(\mathbb{R})^n$.

However,

$$F(\mathbf{x}_0) = [0.3, 0.3] >_L [0.1, 0.1] = F(\mathbf{y}_0),$$

and hence, F is not increasing.

Nevertheless, although it is not possible to find a characterization of regular monotonicity, we are able to find a partial result.

Proposition 6.13. *Let $n \in \mathbb{N}$, let $\leq_L$ be the partial order. If F is increasing, then F is IDM $\mathbf{v}$-increasing for all $\mathbf{v} \in L([0,1])^n$.*

Proof. Let $\mathbf{x} = ([\underline{x_1}, \overline{x_1}], \dots, [\underline{x_n}, \overline{x_n}]) \in L([0,1])^n$. Now, given $c > 0$ such that $\mathbf{x} + c\mathbf{v} \in L([0,1])^n$, it is straight to check that $\mathbf{x} \leq_L \mathbf{x} + c\mathbf{v}$ because the fact that $\mathbf{v} \in L([0,1])^n$ implies that we are adding positive valued to every component. Hence, the increasingness of F ensures $\mathbf{v}$-increasingness. □

7. Conclusions

Based on the concept of Riesz spaces, we have proposed a framework to handle uncertain data originating from different extensions of fuzzy sets, such as type-2 fuzzy sets, fuzzy multisets, n-dimensional fuzzy sets, Atanassov intuitionistic fuzzy sets and interval-valued fuzzy sets. We have introduced the concept of directional monotonicity for functions that handle this sort of uncertainty, combining two of the tendencies in the research on aggregation theory, the relaxation of the monotonicity condition and the extension of the domain. Moreover, we have studied in depth this concept for the particular case of interval-valued functions and we have characterized it in terms of standard directional monotonicity for functions that take values in the unit interval. Thus, we have provided a tool to construct such functions.

As a goal for future research, we intend to study the class of interval-valued directionally monotone functions to see whether they produce as good results as standard directionally monotone functions in classification problems.

Please cite this article in press as: M. Sesma-Sara et al., Weak and directional monotonicity of functions on Riesz spaces to fuse uncertain data, Fuzzy Sets Syst. (2019), https://doi.org/10.1016/j.fss.2019.01.019

Acknowledgements

This work is supported by the research services of *Universidad Pública de Navarra*, by the project TIN2016-77356-P (AEI/FEDER, UE), by Slovak grant APVV-14-0013, and by Czech Project LQ1602 “IT4Innovations excellence in science”.

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Please cite this article in press as: M. Sesma-Sara et al., Weak and directional monotonicity of functions on Riesz spaces to fuse uncertain data, Fuzzy Sets Syst. (2019), https://doi.org/10.1016/j.fss.2019.01.019

6 New measures for comparing matrices and their application to image processing

Associated publication:

- M. Sesma-Sara, L. De Miguel, M. Pagola, A. Burusco, R. Mesiar, H. Bustince. New measures for comparing matrices and their application to image processing, *Applied Mathematical Modelling*, 61: 498-520, 2018.
  DOI: 10.1016/j.apm.2018.05.006.
  - Status: Published.
  - Impact factor (JCR 2017): 2.617.
  - Categories:
    * Engineering, multidisciplinary. Ranking 18/86 (Q1).
    * Mathematics, interdisciplinary applications. Ranking 15/103 (Q1).
    * Mechanics. Ranking 25/134 (Q1).

Applied Mathematical Modelling 61 (2018) 498–520

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

New measures for comparing matrices and their application to image processing

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A R T I C L E I N F O

Article history:
Received 29 May 2017
Revised 26 April 2018
Accepted 8 May 2018
Available online 16 May 2018

Keywords:
Matrix resemblance functions
Restricted equivalence functions
Inclusion grades
Fuzzy mathematical morphology
Defect detection
Motion detection

A B S T R A C T

In this work we present the class of matrix resemblance functions, i.e., functions that measure the difference between two matrices. We present two construction methods and study the properties that matrix resemblance functions satisfy, which suggest that this class of functions is an appropriate tool for comparing images. Hence, we present a comparison method for grayscale images whose result is a new image, which enables to locate the areas where both images are equally similar or dissimilar. Additionally, we propose some applications in which this comparison method can be used, such as defect detection in industrial manufacturing processes and video motion detection and object tracking.

1. Introduction

Measuring how similar or dissimilar two images are is a problem that is far from being closed. There exist many instances of similarity measures and indices [1–3], however there is no standard measure for comparing two images. Moreover, most techniques perform a pixel-wise comparison, which does not take into account the impact that the surrounding of a pixel has when deciding whether the images that are being compared are more or less similar. Another problem with the usual comparison methods is that the result is usually given by a number, which need not be representative in many cases.

Many industrial processes make use of image comparison techniques to guarantee certain quality standards. For instance, in the manufacturing process of printed circuit boards (PCB), all the products are compared with an image of an ideal PCB in order to detect any potential defect (see [4]). Another example of the usage of image comparison techniques is video motion detection. It is possible to detect objects that are moving in a video by adequately comparing its frames. Similarly, image comparison is also used for tamper detection as in [5]. All these instances of possible applications for

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https://doi.org/10.1016/j.apm.2018.05.006

image comparison techniques perform better when the information from the neighbourhoods of pixels is taken into account and they especially benefit from the result of the comparison being a new image.

In order to address these problems, in this work we present a method to compare images, but instead of carrying out a comparison pixel by pixel, we compare neighbourhoods of pixels, i.e., sets consisting of a central pixel and the ones that are adjacent to it. Thus, we include in the comparison the information that can be retrieved from the neighbourhood of each pixel, rather than considering just the pixel itself. Moreover, to avoid representing the difference with a number, the result of the method presented in this paper provides a new image for an outcome, which we call *comparison image*.

To develop a comparison method that preserves the aforementioned features we define a class of functions, the class of matrix resemblance functions, that are adequate to carry out a comparison between the neighbourhoods of two pixels, which ultimately are nothing other than matrices. Along with the definition, we present two different construction methods for this kind of functions. The first one is based on restricted equivalence functions [6] and the second one on inclusion grades [7,8]. Since our aim is to present a comparison method, we also study some properties that are usually demanded to comparison methods in order to be proper similarity measures (see [2,3,6]).

The structure of the paper is as follows: first, we present some preliminary concepts that help making the paper self-contained. In Section 3 the concept of matrix resemblance function is introduced, and in Section 4 two construction methods and some examples are presented. Section 5 exhibits the relation between matrix resemblance function and the erosion operator from fuzzy mathematical morphology. In Section 6, a summary of the properties that image comparison measures ought to satisfy is presented and the conditions under which matrix resemblance functions fulfil them are shown. In Section 7, we present a study of the particular cases where in the construction of matrix resemblance functions an aggregation function and a n-dimensional overlap function are used. In Section 8 an algorithm to compare images based on matrix resemblance functions is introduced and in Section 9 some illustrative examples of this image comparison method are presented. In Section 10 we include three fields in which our algorithm could be applied: tamper detection, defect detection in PCB manufacturing processes and a method to compare videos which can be applied to object motion detection and tracking.

2. Preliminaries

A fuzzy set $\mathcal{A}$ on a universe $X \neq \emptyset$ is a mapping $\mathcal{A}: X \to [0, 1]$. Given a point $x \in X$, $\mathcal{A}(x)$ refers to the membership degree of the point x in the fuzzy set $\mathcal{A}$. In the case of grayscale images, X would be the set of pixels and $\mathcal{A}(x)$ the intensity of the pixel x.

We consider the neighbourhood of a pixel to be a square set of pixels that are surrounding the central one, so they can be thought of as square matrices with values in the unit interval. We denote the set of this kind of $n \times n$ matrices by $\mathcal{M}_n([0, 1])$.

A fuzzy negation is a generalization of the negation in classical logic.

Definition 2.1. A function c: $[0, 1] \to [0, 1]$ is called a fuzzy negation if $c(0) = 1$, $c(1) = 0$ and c is decreasing. Additionally, c is said to be strict if it is continuous and strictly decreasing. A negation c is called strong negation if it is involutive, i.e., $c(c(x)) = x$ for all $x \in [0, 1]$.

Example 2.2. The function c_z: $[0, 1] \to [0, 1]$ given by $c_z(x) = 1 - x$ is a strong negation. It was given by Zadeh in [9].

Denoting $FS(X)$ the set of all fuzzy sets defined on the universe $X \neq \emptyset$, if $\mathcal{A} \in FS(X)$, we call c-complement of $\mathcal{A}$ to the fuzzy set given by the membership function $\mathcal{A}_c(x) = c(\mathcal{A}(x))$, where c is a fuzzy negation.

A restricted equivalence function [6], or a REF, is a function that enables a comparison between two numbers in the unit interval.

Definition 2.3. Let c be a strong negation. A function REF: $[0, 1]^2 \to [0, 1]$ is called a restricted equivalence function with respect to c if it satisfies the following conditions:

(REF1) $REF(x, y) = REF(y, x)$ for all $x, y \in [0, 1]$;
(REF2) $REF(x, y) = 1$ if and only if $x = y$;
(REF3) $REF(x, y) = 0$ if and only if $\{x, y\} = \{0, 1\}$;
(REF4) $REF(x, y) = REF(c(x), c(y))$ for all $x, y \in [0, 1]$;
(REF5) For all $x, y, z \in [0, 1]$, if $x \leq y \leq z$, then $REF(x, y) \geq REF(x, z)$ and $REF(y, z) \geq REF(x, z)$.

Example 2.4. $REF(x, y) = 1 - |x - y|$ is a restricted equivalence function with respect to the strong negation c_z.

An implication operator is a generalization of the implication in classical logic. It is defined as follows:

Definition 2.5. A function I: $[0, 1]^2 \to [0, 1]$ is called implication operator if it satisfies the following conditions:

(I1) If $x \leq z$, then $I(x, y) \geq I(z, y)$ for all $y \in [0, 1]$;
(I2) If $y \leq t$, then $I(x, y) \leq I(x, t)$ for all $x \in [0, 1]$;
(I3) $I(0, x) = 1$ for all $x \in [0, 1]$;
(I4) $I(x, 1) = 1$ for all $x \in [0, 1]$;
(I5) $I(1, 0) = 0$.

The following are some additional conditions that are frequently demanded to implication operators:

(I6) $I(1, x) = x$ for all $x \in [0, 1]$;
(I7) $I(x, I(y, z)) = I(y, (x, z))$;
(I8) $I(x, y) = 1$ if and only if $x \leq y$;
(I9) $I(x, 0) = c(x)$ is a strong negation;
(I10) $I(x, y) \geq y$;
(I11) $I(x, x) = 1$;
(I12) $I(x, y) = I(c(y), c(x))$ with c a strong negation;
(I13) I is a continuous function.

Example 2.6. The function $I_L(x, y) = \min(1, 1 - x + y)$ is called the Lukasiewicz implication and it satisfies all conditions (I1)–(I13) when considering c_z as strong negation. Conversely, a function that satisfies (I1)–(I13) for a strong negation c, is an isomorphic transformation of I_L, such that the related isomorphism φ generates the strong negation c as $c(x) = \varphi^{-1}(1 - \varphi(x))$ (see [6]).

A fuzzy inclusion grade, inclusion degree, or subsethood measure [8,10,11], is a function that indicates how included a fuzzy set is in another. There are three main axiomatizations for this concept; the one given by Kitainik [12] in 1987, the one by Sinha and Dougherty [7] in 1993, and the one by Young [13] in 1996. In 1999, Fan et al. [14] made some modifications to Young's axioms. We have chosen the axiomatization given by Sinha and Dougherty due to the fact that the second axiom allows to link this work to fuzzy mathematical morphology.

Given two fuzzy sets $\mathcal{A}, \mathcal{B} \in FS(X)$, we set the fuzzy sets $\mathcal{A} \vee \mathcal{B} \in FS(X)$ and $\mathcal{A} \wedge \mathcal{B} \in FS(X)$, given by $(\mathcal{A} \vee \mathcal{B})(x) = \max(\mathcal{A}(x), \mathcal{B}(x))$ and $(\mathcal{A} \wedge \mathcal{B})(x) = \min(\mathcal{A}(x), \mathcal{B}(x))$ for all $x \in X$, respectively.

Definition 2.7. A function $\sigma: FS(X) \times FS(X) \to [0, 1]$ is called an inclusion grade in the sense of Sinha and Dougherty if it satisfies the following axioms:

(IG1) $\sigma(\mathcal{A}, \mathcal{B}) = 1$ if and only if $\mathcal{A} \leq \mathcal{B}$ in Zadeh's sense[1];
(IG2) $\sigma(\mathcal{A}, \mathcal{B}) = 0$ if and only if there exists x_i such that $\mathcal{A}(x_i) = 1$ and $\mathcal{B}(x_i) = 0$;
(IG3) If $\mathcal{B} \leq \mathcal{C}$, then $\sigma(\mathcal{A}, \mathcal{B}) \leq \sigma(\mathcal{A}, \mathcal{C})$ for all $\mathcal{A} \in FS(X)$;
(IG4) If $\mathcal{B} \leq \mathcal{C}$, then $\sigma(\mathcal{C}, \mathcal{A}) \leq \sigma(\mathcal{B}, \mathcal{A})$ for all $\mathcal{A} \in FS(X)$;
(IG5) $\sigma(\mathcal{A}, \mathcal{B}) = \sigma(\pi(\mathcal{A}), \pi(\mathcal{B}))$, considering π a permutation of the elements of X and denoting $\pi(\mathcal{A})$ and $\pi(\mathcal{B})$ the sets in which the membership degrees are permuted by π, i.e., $\pi(\mathcal{A})(x) = \mathcal{A}(\pi(x))$;
(IG6) $\sigma(\mathcal{A}, \mathcal{B}) = \sigma(\mathcal{B}_c, \mathcal{A}_c)$, where c is a strong negation;
(IG7) $\sigma(\mathcal{B} \vee \mathcal{C}, \mathcal{A}) = \min(\sigma(\mathcal{B}, \mathcal{A}), \sigma(\mathcal{C}, \mathcal{A}))$ for all $\mathcal{A}, \mathcal{B}, \mathcal{C} \in FS(X)$;
(IG8) $\sigma(\mathcal{A}, \mathcal{B} \wedge \mathcal{C}) = \min(\sigma(\mathcal{A}, \mathcal{B}), \sigma(\mathcal{A}, \mathcal{C}))$ for all $\mathcal{A}, \mathcal{B}, \mathcal{C} \in FS(X)$;
(IG9) $\sigma(\mathcal{A}, \mathcal{B} \vee \mathcal{C}) \geq \max(\sigma(\mathcal{A}, \mathcal{B}), \sigma(\mathcal{A}, \mathcal{C}))$ for all $\mathcal{A}, \mathcal{B}, \mathcal{C} \in FS(X)$.

Subsequently, Burillo et al. [15] proved (IG3) and (IG9) to be equivalent.

Example 2.8. Let $X \neq \emptyset$ be finite. The function $\sigma: FS(X) \times FS(X) \to [0, 1]$ given by $\sigma(\mathcal{A}, \mathcal{B}) = \inf_x \{I_L(\mathcal{A}(x), \mathcal{B}(x))\}$ is an inclusion grade in the sense of Sinha and Dougherty.

Let us recall the definition of a n-ary aggregation function, i.e., an aggregation function with n arguments.

Definition 2.9. A n-ary aggregation function is a mapping f: $[0, 1]^n \to [0, 1]$ such that

(i) $f(0, \ldots, 0) = 0$,
(ii) $f(1, \ldots, 1) = 1$, and
(iii) f is increasing with respect to each component.

3. Matrix resemblance functions

3.1. Definition

Neighbourhoods of pixels can be represented as square matrices, so in order to compare them, matrix comparison techniques are needed. Hence the following definition.

Definition 3.1. A function $\Psi : \mathcal{M}_n([0, 1]) \times \mathcal{M}_n([0, 1]) \to [0, 1]$ is called a matrix resemblance function if it satisfies the following properties:

(MRF1) $\Psi(A, B) = 1$ if and only if $A = B$;

[1] A fuzzy set $\mathcal{A}$ in a universe X is included in another fuzzy set $\mathcal{B}$ of the same universe in Zadeh's sense if and only if $\mathcal{A}(x) \leq \mathcal{B}(x)$ for every $x \in X$.

(MRF2) $\Psi(A, B) = 0$ if and only if there exist i and j such that $\{a_{ij}, b_{ij}\} = \{0, 1\}$;
(MRF3) $\Psi(A, B) = \Psi(B, A)$ for all $A, B \in \mathcal{M}_n([0, 1])$.

Example 3.2. The function $\Psi(A, B) = \prod_{\substack{i=1\\j=1}}^{n} \left(1 - (a_{ij} - b_{ij})^2\right)$ is a matrix resemblance function.

The first and third conditions of the definition of matrix resemblance functions are readily justified as they are natural for a matrix comparison operator. The second property is based on the erosion operator of mathematical morphology. In Section 5 we present the relation between matrix resemblance functions and the erosion operator from fuzzy mathematical morphology.

It sometimes is useful for matrix resemblance functions to have some sort of monotonicity and hence we could add a fourth condition to Definition 3.1:

(MRF4) If $A \le B \le C$, then $\Psi(A, C) \le \Psi(A, B)$ and $\Psi(A, C) \le \Psi(B, C)$,

where $A \le B \le C$ means $a_{ij} \le b_{ij} \le c_{ij}$ for all $i, j \in \{1, \dots, n\}$.

In Section 6.2 we study the effect of this property to matrix resemblance functions. However, we have decided to leave this property out of the axiomatization since our intention is to generalise this concept to be able to compare other non-ordered structures.

Given an arbitrary matrix resemblance function, it is possible to construct another using two additional functions as in the next proposition.

Proposition 3.3. *Let ϕ, η: $[0, 1] \to [0, 1]$ be two functions such that $\phi(0) = \eta(0) = 0$, $\phi(1) = \eta(1) = 1$, $\eta(x) \in (0, 1)$ for all $x \in (0, 1)$, and ϕ is injective. If $\Psi : \mathcal{M}_n([0, 1])^2 \to [0, 1]$ is a matrix resemblance function, then the mapping $\Psi_{\phi,\eta} : \mathcal{M}_n([0, 1])^2 \to [0, 1]$ given by*

$$\Psi_{\phi,\eta}(A, B) = \eta(\Psi(\phi(A), \phi(B))),$$

where $\phi(A)_{ij} = \phi(a_{ij})$, is a matrix resemblance function.

Proof.

(MRF1) $\Psi_{\phi,\eta}(A, B) = \eta(\Psi(\phi(A), \phi(B))) = 1$ if and only if $\Psi(\phi(A), \phi(B)) = 1$ since $\eta(x) = 1$ only if $x = 1$. By the definition of Ψ, $\Psi(\phi(A), \phi(B)) = 1$ if and only if $\phi(A) = \phi(B)$, which holds if and only if $A = B$, since ϕ is injective.
(MRF2) Since $\eta(x) = 0$ only if $x = 0$, $\Psi_{\phi,\eta}(A, B) = 0$ if and only if $\Psi(\phi(A), \phi(B)) = 0$. This happens if and only if there exist i and j such that $\{\phi(a_{ij}), \phi(b_{ij})\} = \{0, 1\}$. Since ϕ is injective, $\{\phi(a_{ij}), \phi(b_{ij})\} = \{0, 1\}$ if and only if $\{a_{ij}, b_{ij}\} = \{0, 1\}$.
(MRF3) $\Psi_{\phi,\eta}(A, B) = \eta(\Psi(\phi(A), \phi(B))) = \eta(\Psi(\phi(B), \phi(A))) = \Psi_{\phi,\eta}(B, A)$. □

Corollary 3.4. *Let ϕ: $[0, 1] \to [0, 1]$ be an automorphism, i.e., a continuous strictly increasing function such that $\phi(0) = 0$ and $\phi(1) = 1$, and let Ψ be a matrix resemblance function, then the function $\Psi_\phi = \Psi_{\phi,\phi^{-1}}$ is a matrix resemblance function.*

Example 3.5. Consider the matrix resemblance function Ψ as in Example 3.2 and let ϕ be the automorphism given by $\phi(x) = x^2$. Thus, $\Psi(A, B) = \sqrt{\prod_{\substack{i=1\\j=1}}^{n} (1 - (a_{ij}^2 - b_{ij}^2)^2)}$ is a matrix resemblance function.

In the same vein, given a set of m matrix resemblance functions, it is possible to obtain another by aggregating them as in the next proposition.

Proposition 3.6. *Let $m \ge 2$ and F: $[0, 1]^m \to [0, 1]$ be an aggregation function with neither zero divisors, nor one divisors. If $\Psi_1, \dots, \Psi_m$ are matrix resemblance functions, then the mapping $\Psi = F(\Psi_1, \dots, \Psi_m) : \mathcal{M}_n([0, 1])^2 \to [0, 1]$ given by*

$$\Psi(A, B) = F(\Psi_1, \dots, \Psi_m)(A, B) = F(\Psi_1(A, B), \dots, \Psi_m(A, B)),$$

is a matrix resemblance function.

Proof.

(MRF1) $\Psi(A, B) = F(\Psi_1(A, B), \dots, \Psi_m(A, B)) = 1$ if and only if there exists $i \in \{1, \dots, m\}$ such that $\Psi_i(A, B) = 1$, since F has not one divisors. Without loss of generality, let us suppose that $\Psi_1(A, B) = 1$, which, by (MRF1), is equivalent to $A = B$.
(MRF2) $\Psi(A, B) = F(\Psi_1(A, B), \dots, \Psi_m(A, B)) = 0$ if and only if there exists $i \in \{1, \dots, m\}$ such that $\Psi_i(A, B) = 0$, due to the lack of zero divisors of F. In either case, by (MRF2), the former holds if and only if there exist $k, j \in \{1, \dots, n\}$ such that $\{a_{kj}, b_{kj}\} = \{0, 1\}$.
(MRF3) $\Psi(A, B) = F(\Psi_1(A, B), \dots, \Psi_m(A, B)) = F(\Psi_1(B, A), \dots, \Psi_m(B, A)) = \Psi(B, A)$.

□

Remark 3.7. In particular, averaging aggregation functions [16] do not have either one divisors or zero divisors. Therefore aggregating m matrix resemblance functions with an averaging aggregation function produces a matrix resemblance function. As a note, we recall that the monotonicity of aggregation functions implies that the averaging behaviour is equivalent to idempotency.

Example 3.8. The arithmetic mean of the two matrix resemblance functions from Examples 3.2 and 3.5 is a matrix resemblance function.

The next result provides information about the structure of the set $\mathcal{F}_n$ of matrix resemblance functions for a fixed n. Let us define an order $\leq$ over the set $\mathcal{F}_n$ by $\Psi_1 \leq \Psi_2$ if $\Psi_1(A, B) \leq \Psi_2(A, B)$ for all $A, B \in \mathcal{M}_n([0, 1])$. This is a partial order as it is induced from the order of [0,1].

Proposition 3.9. *Let $n \in \mathbb{N}$. $(\mathcal{F}_n, \leq)$ is a non-complete lattice with neither a maximal nor a minimal element.*

Proof. The set of matrix resemblance functions for a fixed n is a partially ordered set with $\leq$ and given $\Psi_1, \Psi_2 \in \mathcal{F}_n$ we can define the operations $\sqcup$ and $\sqcap$ by:

$$(\Psi_1 \sqcup \Psi_2)(A, B) = \max(\Psi_1, \Psi_2)(A, B) = \max(\Psi_1(A, B), \Psi_2(A, B)),$$
$$(\Psi_1 \sqcap \Psi_2)(A, B) = \min(\Psi_1, \Psi_2)(A, B) = \min(\Psi_1(A, B), \Psi_2(A, B)),$$

for all $A, B \in \mathcal{M}_n([0, 1])$. Now, $\Psi_1 \sqcup \Psi_2, \Psi_1 \sqcap \Psi_2 \in \mathcal{F}_n$. Indeed,

(MRF1)

$$\begin{aligned}(\Psi_1 \sqcup \Psi_2)(A, B) = 1 &\iff \max(\Psi_1(A, B), \Psi_2(A, B)) = 1 \\ &\iff \Psi_1(A, B) = 1 \text{ or } \Psi_2(A, B) = 1 \iff A = B.\end{aligned}$$

(MRF2)

$$\begin{aligned}(\Psi_1 \sqcup \Psi_2)(A, B) = 0 &\iff \max(\Psi_1(A, B), \Psi_2(A, B)) = 0 \\ &\iff \Psi_1(A, B) = 0 \text{ and } \Psi_2(A, B) = 0 \\ &\iff \text{there exist } i \text{ and } j \text{ such that } \{a_{ij}, b_{ij}\} = \{0, 1\}.\end{aligned}$$

(MRF3)

$$\begin{aligned}(\Psi_1 \sqcup \Psi_2)(A, B) &= \max(\Psi_1(A, B), \Psi_2(A, B)) \\ &= \max(\Psi_1(B, A), \Psi_2(B, A)) = (\Psi_1 \sqcup \Psi_2)(B, A).\end{aligned}$$

The case of $\Psi_1 \sqcap \Psi_2$ is analogous.
Furthermore, the supremum and the infimum of all matrix resemblance functions are, respectively,

$$\Psi_{sup}(A, B) = \begin{cases} 0 & \text{if } \exists\, i, j \text{ s.t. } \{a_{ij}, b_{ij}\} = \{0, 1\}, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$\Psi_{inf}(A, B) = \begin{cases} 1 & \text{if } A = B, \\ 0 & \text{otherwise,} \end{cases}$$

and since neither is a matrix resemblance function, $(\mathcal{F}_n, \leq)$ is a non-complete lattice. □

4. Constructions

Once the definition of matrix resemblance functions is set, we need an algebraic expression to work with them and study their properties. In this paper we provide two construction methods, the first one being based on restricted equivalence functions and the second one on inclusion grades.

Both methods make use of a function F: $[0, 1]^N \to [0, 1]$ that satisfies a set of specific properties. We enlist these properties and refer to them as (F1), (F2) and (F3).

(F1) $F(x_1, \ldots, x_N) = 1$ if and only if $x_i = 1$ for every $1 \leq i \leq N$,
(F2) $F(x_1, \ldots, x_N) = 0$ if and only if there exists $1 \leq j \leq N$ such that $x_j = 0$,
(F3) $F\big((x_i)_{i=1}^N\big) = F\big((x_{\pi(i)})_{i=1}^N\big)$ for every permutation π of $\{1, \ldots, N\}$.

The first two are the properties proposed in [17] for aggregation operators.

Example 4.1. The minimum, the product or the geometric mean are well-known examples of such a function F.

Remark 4.2. Observe that although all functions in the previous example are in fact aggregation functions, we do not require F to be monotone and hence it need not be an aggregation function. The case in which F is an aggregation function and other particular instances of F are further studied in Section 7.

Example 4.3. Let F: $[0, 1]^2 \to [0, 1]$ be such that

$$F(x, y) = \begin{cases} 1, & \text{if } x = y = 1; \\ 0, & \text{if } xy = 0; \\ 0.2, & \text{if } x = y = 0.9; \\ 0.5 & \text{otherwise.} \end{cases}$$

Thus, F verifies the conditions (F1)–(F3), but is not an aggregation function since it is not increasing.

4.1. First construction method

The next theorem constitutes the first construction method for matrix resemblance functions that we present in this work.

Theorem 4.4. *Let β be a function satisfying (REF1)–(REF3) and $H : [0, 1]^{n^2} \to [0, 1]$ a function satisfying (F1) and (F2), then the mapping $\Psi : \mathcal{M}_n([0, 1])^2 \to [0, 1]$ given by*

$$\Psi(A, B) = \underset{\substack{i=1\\j=1}}{\overset{n}{H}} (\beta(a_{ij}, b_{ij})), \tag{1}$$

where $\underset{\substack{i=1\\j=1}}{\overset{n}{H}} (\beta(a_{ij}, b_{ij})) = H(\beta(a_{11}, b_{11}), \ldots, \beta(a_{nn}, b_{nn}))$, *is a matrix resemblance function.*

Proof.

(MRF1)

$$\Psi(A, B) = \underset{\substack{i=1\\j=1}}{\overset{n}{H}} (\beta(a_{ij}, b_{ij})) = 1 \underset{(F1)}{\iff} \beta(a_{ij}, b_{ij}) = 1 \text{ for all } i, j \underset{(REF2)}{\iff} a_{ij} = b_{ij} \text{ for all } i, j.$$

(MRF2)

$$\Psi(A, B) = \underset{\substack{i=1\\j=1}}{\overset{n}{H}} (\beta(a_{ij}, b_{ij})) = 0 \underset{(F2)}{\iff} \exists i, j \text{ s.t. } \beta(a_{ij}, b_{ij}) = 0 \underset{(REF3)}{\iff} \exists i, j \text{ s.t. } \{a_{ij}, b_{ij}\} = \{0, 1\}.$$

(MRF3)

$$\Psi(A, B) = \underset{\substack{i=1\\j=1}}{\overset{n}{H}} (\beta(a_{ij}, b_{ij})) \underset{(REF1)}{=} \underset{\substack{i=1\\j=1}}{\overset{n}{H}} (\beta(b_{ij}, a_{ij})) = \Psi(B, A).$$

□

Remark 4.5. If Ψ is a matrix resemblance function constructed by the pair (β, H) with the previous method, then the matrix resemblance function $\Psi_{\phi,\eta}$ obtained from the application of Proposition 3.3 is generated by ($\beta\phi$, $\eta \circ H$), where $\beta\phi(x, y) = \beta(\phi(x), \phi(y))$.

Example 4.6. An example of a matrix resemblance function as in (1) can be found in Example 3.2 considering H the product and $\beta(x, y) = 1 - (x - y)^2$, i.e.,

$$\Psi(A, B) = \prod_{i,j=1}^{n} \left(1 - (a_{ij} - b_{ij})^2\right). \tag{2}$$

If we applied this matrix resemblance function to the matrices

$$A = \begin{pmatrix} 0.1 & 0.9 & 0.7 \\ 0.1 & 0.7 & 0.1 \\ 0.8 & 0.2 & 0.2 \end{pmatrix}, \quad B = \begin{pmatrix} 0.6 & 0.7 & 0.3 \\ 0.3 & 0.6 & 0.7 \\ 0.6 & 0.7 & 0.9 \end{pmatrix},$$

we would obtain $\Psi(A, B) = 0.1351$; and understanding each value as the gray level of a pixel, we can represent the result as in Fig. 1.

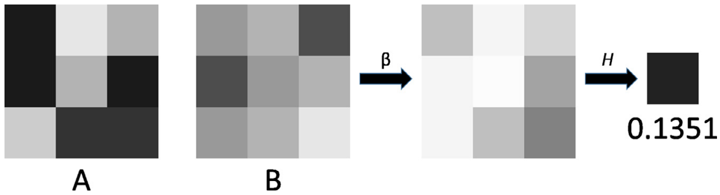

Fig. 1. Representation of the application of Ψ (as in (2)).

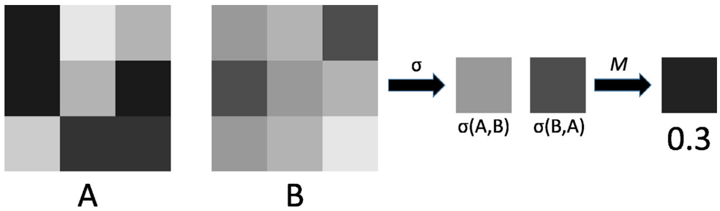

Fig. 2. Representation of the application of Ψ (as in (4)).

4.2. Second construction method

The following result provides the second construction method for matrix resemblance functions:

Theorem 4.7. *Let $\sigma : \mathcal{M}_n([0,1])^2 \to [0,1]$ be a function that satisfies (IG1) and (IG2) and let M: $[0,1]^2 \to [0,1]$ be a function satisfying (F1)–(F3), then the mapping $\Psi : \mathcal{M}_n([0,1])^2 \to [0,1]$ given by*

$$\Psi(A,B) = M(\sigma(A,B), \sigma(B,A)), \tag{3}$$

is a matrix resemblance function.

Proof.

(MRF1)

$$\Psi(A,B) = M(\sigma(A,B), \sigma(B,A)) = 1 \underset{(F1)}{\iff} \sigma(A,B) = \sigma(B,A) = 1$$
$$\underset{(IG1)}{\iff} A \leq B \text{ and } B \leq A \iff A = B.$$

(MRF2)

$$\Psi(A,B) = M(\sigma(A,B), \sigma(B,A)) = 0 \underset{(F2)}{\iff} \sigma(A,B) = 0 \text{ or } \sigma(B,A) = 0$$
$$\iff \exists i,j \text{ such that } a_{ij} = 1 \text{ and } b_{ij} = 0 \text{ or } a_{ij} = 0 \text{ and } b_{ij} = 1$$
$$\iff \exists i,j \text{ s.t. } \{a_{ij}, b_{ij}\} = \{0,1\}.$$

(MRF3)

$$\Psi(A,B) = M(\sigma(A,B), \sigma(B,A)) \underset{(F3)}{=} M(\sigma(B,A), \sigma(A,B)) = \Psi(B,A).$$

□

Example 4.8. Considering $\sigma(A,B) = \inf_{i,j}\{I_L(a_{ij}, b_{ij})\}$ and M the minimum, the function

$$\Psi(A,B) = \min\left(\inf_{i,j}\{I_L(a_{ij}, b_{ij})\}, \inf_{i,j}\{I_L(b_{ij}, a_{ij})\}\right) \tag{4}$$

is a matrix resemblance function as in Theorem 4.7. Moreover, applying this Ψ to the matrices from Example 4.6, we obtain 0.3 (see Fig. 2).

Remark 4.9. The matrix resemblance function in Example 4.8 can be built using either construction methods. Indeed,

$$\Psi(A,B) = \min\left(\inf_{i,j}\{I_L(a_{ij}, b_{ij})\}, \inf_{i,j}\{I_L(b_{ij}, a_{ij})\}\right)$$

$$= \min\left(\inf_{i,j}\{\min(1, 1 - a_{ij} + b_{ij})\}, \inf_{i,j}\{\min(1, 1 - b_{ij} + a_{ij})\}\right)$$
$$= \inf_{i,j}\{\min(1, 1 - a_{ij} + b_{ij}), \min(1, 1 - b_{ij} + a_{ij})\}$$
$$= \inf_{i,j}\{\min(1, 1 - a_{ij} + b_{ij}, 1 - b_{ij} + a_{ij})\}$$
$$= \inf_{i,j}\{1 - |a_{ij} - b_{ij}|\}$$
$$= \min_{i,j}\left(1 - |a_{ij} - b_{ij}|\right),$$

which is a function constructed by the first method using H the minimum and $\beta(x, y) = 1 - |x - y|$, the restricted equivalence function from Example 2.4.

4.3. Relation between both constructions

In this section we study the cases where both constructions are equivalent, i.e., whether it is possible to reach the expression given by one of the constructions from the other.

Let us start recalling two theorems that characterize the restricted equivalence functions and the inclusion grades in the sense of Sinha and Dougherty. The first can be found in [6] (Theorem 7).

Theorem 4.10. *A function REF*: $[0, 1]^2 \to [0, 1]$ *is a restricted equivalence function if and only if there exists a function I*: $[0, 1]^2 \to [0, 1]$ *satisfying (I1), (I8), (I12) and for all x,* $y \in [0, 1]$:

$$I(x, y) = 0 \text{ if and only if } x = 1 \text{ and } y = 0, \tag{5}$$

such that

$$REF(x, y) = \min(I(x, y), I(y, x)). \tag{6}$$

The second theorem, the characterization of inclusion grades, corresponds to Theorem 5.4 of [18].

Theorem 4.11. *Let X be a finite universe. A mapping* $\sigma : FS(X) \times FS(X) \to [0, 1]$ *satisfies all Sinha–Dougherty axioms if and only if there exists a fuzzy implicator I satisfying (I8), (I12) and (5) for all x,* $y \in [0, 1]$, *such that for all* $\mathcal{A}$ *and* $\mathcal{B}$ *in* $FS(X)$:

$$\sigma(\mathcal{A}, \mathcal{B}) = \inf_{x \in X} I(\mathcal{A}(x), \mathcal{B}(x)). \tag{7}$$

Note that one of the conditions in Theorem 4.11 is that X is finite, and since we are working with finite dimensional matrices, we are assuming that to be the case. Hence, in the preceding theorem the infimum is actually the minimum.

Proposition 4.12. *Let I*: $[0, 1]^2 \to [0, 1]$ *be a function. The requested conditions to the function I in Theorems 4.10 and 4.11 are equivalent.*

Proof. Let I: $[0, 1]^2 \to [0, 1]$ be a function that satisfies all the requirements in Theorem 4.10, i.e., (I1), (I8), (I12), and (5). Now, since (I1) is clear, it suffices to show that I satisfies conditions (I2)–(I5).

(I2) Straightforward from (I1) and (I12).
(I4) Straightforward from (I8).
(I3) Straightforward from (I4) and (I12).
(I5) Straightforward from (5).

Some of the preceding dependencies can be found in [19].

The converse is immediate since every implication I satisfies (I1), and the remaining conditions coincide. □

As a consequence of the aforementioned result, one can see that both constructions are similar in the case of β being a restricted equivalence function and σ an inclusion grade. However, a study on the functions H and M is still required to show when they are actually equivalent. Recall that H is a function with n^2 inputs that satisfies (F1) and (F2), and M is a function with 2 inputs that satisfies the three conditions (F1)–(F3).

By Theorem 4.10, we know that given Ψ constructed by means of a restricted equivalence function, there exists a function I such that:

$$\Psi(A, B) = \underset{\substack{i=1 \\ j=1}}{\overset{n}{H}} (\min(I(a_{ij}, b_{ij}), I(b_{ij}, a_{ij}))) \tag{8}$$

Similarly, a matrix resemblance function can also be constructed using inclusion grades, in which case by Theorem 4.11 there exists a function I such that:

$$\Psi(A, B) = M\left(\min_{i,j}(I(a_{ij}, b_{ij})), \min_{i,j}(I(b_{ij}, a_{ij}))\right). \tag{9}$$

Now, the result in Proposition 4.12 indicates that in the case functions H and M verify that for two sequences $(x_i)_{i=1}^N, (y_i)_{i=1}^N \subset [0,1]^N$ the following equation holds

$$M(\min((x_i)_{i=1}^N), \min((y_i)_{i=1}^N)) = H((\min(x_i, y_i))_{i=1}^N), \tag{10}$$

then both constructions will be equivalent. In the next theorem we show an instance of such functions H and M.

Theorem 4.13. *Let H be the function minimum for n^2 arguments and M the function minimum for two arguments. Then the first construction with a restricted equivalence function, as in (8), and the second with an inclusion grade , as in (9), are equivalent.*

Proof. Since such functions H and M satisfy (10), given a matrix resemblance function as in (8) we can obtain the same as in (9). The converse is analogous. □

But there are other examples of H and M that do not satisfy (10) and hence they produce different matrix resemblance functions for each construction:

Example 4.14. Let H be the geometric mean for N arguments, M the geometric mean for 2 arguments, $(x_i)_{i=1}^N = (0.4)_{i=1}^N$ and $(y_i)_{i=1}^N = (0.7)_{i=1}^N$. Thus,

$$H((\min(x_i, y_i))_{i=1}^N) = H((0.4)_{i=1}^N) = 0.4$$
$$M(\min((x_i)_{i=1}^N), \min((y_i)_{i=1}^N)) = M(0.4, 0.7) = \sqrt{0.28} \neq 0.4$$

Theorem 4.13 ensures that if H and M are the minimum (with the corresponding arity) and a matrix resemblance function is constructed in terms of a restricted equivalence function, then it can also be constructed in terms of an inclusion grade, and viceversa.

The following two results expose that the minimum is the only idempotent function that satisfies (F1)–(F3) and (10). Recall that a function F: $[0,1]^N \to [0,1]$ is said to be idempotent if $F(x, \ldots, x) = x$ for all $x \in [0,1]$.

Theorem 4.15. *Let H: $[0,1]^N \to [0,1]$ be a function satisfying (F1) and (F2), and let M: $[0,1]^2 \to [0,1]$ be a function that satisfies (F1)–(F3). Let d: $[0,1] \to [0,1]$ be the diagonal section of H, i.e., $d(x) = H(x, \ldots, x)$. If H and M satisfy (10), then $M(x, y) = d(\min(x, y))$.*

Proof. Let H be a function that satisfies (F1), (F2) and M a function that satisfies (F1)–(F3), such that (10) is satisfied. Let x_0, $y_0 \in [0,1]$ such that $x_0 \le y_0$. Then,

$$H((\min(x_0, y_0))_{i=1}^N) = H((x_0)_{i=1}^N) = d(x_0),$$
$$M(\min((x_0)_{i=1}^N), \min((y_0)_{i=1}^N)) = M(x_0, y_0),$$

and, since H and M satisfy (10), $M(x_0, y_0) = d(x_0)$.

Thus, since x_0 and y_0 are arbitrarily taken, for every x, $y \in [0,1]$ such that $x \le y$, it holds that $M(x, y) = d(x)$. Hence $M(x, y) = d(\min(x, y))$. □

Theorem 4.16. *Let H: $[0,1]^N \to [0,1]$ be an idempotent function satisfying (F1) and (F2), and let M: $[0,1]^2 \to [0,1]$ be a function that satisfies (F1)–(F3). Then H and M satisfy (10) if and only if H and M are the N-ary and 2-ary minimum, respectively.*

Proof. Let H be an idempotent function that satisfies (F1), (F2) and M a function that satisfies (F1)–(F3), such that (10) is satisfied.

By Theorem 4.15, it holds that $M(x, y) = d(\min(x, y))$, where d is the diagonal section of H. Since H is idempotent, it holds that $d(x) = H(x, \ldots, x) = x$. Hence M coincides with the 2-ary minimum.

Now, by (10), for any two sequences $(x_i)_{i=1}^N, (y_i)_{i=1}^N \subset [0,1]^N$, the value of $H((\min(x_i, y_i))_{i=1}^N)$ must coincide with the minimum of all the inputs $\{x_1, \ldots, x_N, y_1, \ldots, y_N\}$, i.e.,

$$H((\min(x_i, y_i))_{i=1}^N) = \min(x_1, \ldots, x_N, y_1, \ldots, y_N). \tag{11}$$

Assume that H is not the N-ary minimum, then there exists a sequence $(\lambda_i)_{i=1}^N \subset [0,1]^N$ such that $H(\lambda_1, \ldots, \lambda_N) \neq \min(\lambda_1, \ldots, \lambda_N)$. Thus, if we take $(\gamma_i)_{i=1}^N \subset [0,1]^N$ such that $\min(\lambda_i, \gamma_i) = \lambda_i$ for $1 \le i \le N$ we reach that

$$H((\min(\lambda_i, \gamma_i))_{i=1}^N) \neq \min(\lambda_1, \ldots, \lambda_N) = \min(\lambda_1, \ldots, \lambda_N, \gamma_1, \ldots, \gamma_N),$$

which contradicts (11).

The converse implication is immediate. □

5. Matrix resemblance functions and fuzzy mathematical morphology

Mathematical morphology is a theory for processing images based on their form and structure. It was originally developed for binary images [20,21], but later on it was generalized as fuzzy mathematical morphology for grayscale images [22–25].

In this section we study the relation between the class of matrix resemblance functions and the theory of fuzzy mathematical morphology. This theory has four main operations to transform an image via an structuring element: dilation, erosion, opening and closing. The mentioned relation comes from the second construction method for matrix resemblance functions, as, under certain assumptions, inclusion grades in the sense of Sinha and Dougherty are related to the erosion operator from fuzzy mathematical morphology.

Generally, the erosion operator ε is defined as an operator that commutes with the infimum, i.e.,

$$\varepsilon(\inf Y) = \inf_{y\in Y} \varepsilon(y).$$

In particular, given X a finite universe and $\mathcal{A}, \mathcal{B} \in FS(X)$, an expression of erosion operator with respect to an structuring element $\mathcal{B}$, $\varepsilon(\mathcal{A}, \mathcal{B}) \in FS(X)$, is given by $\varepsilon(\mathcal{A}, \mathcal{B})(z) = \inf_{x\in X}\{I_L(\mathcal{B}(x-z), \mathcal{A}(x))\}$, as in [23,24]. In this case, the fuzzy erosion operator coincides with an inclusion grade in the sense of Sinha and Dougherty, applied to a set translated by z, i.e., $\varepsilon(\mathcal{A}, \mathcal{B})(z) = \sigma(\mathcal{B}_z, \mathcal{A})$.

This can be translated to the framework of matrix resemblance functions. Let $k \in \mathbb{N}$ and let $A, B \in \mathcal{M}_{2k+1}([0, 1])$, if we consider the following indexation for the elements of a matrix:

$$\begin{pmatrix} (-k,-k) & \dots & (-k,0) & \dots & (-k,k) \\ \vdots & & \vdots & & \vdots \\ (0,-k) & \dots & \mathbf{(0,0)} & \dots & (0,k) \\ \vdots & & \vdots & & \vdots \\ (k,-k) & \dots & (k,0) & \dots & (k,k) \end{pmatrix},$$

(0,0) refers to the central element of the matrix (or central pixel of the neighbourhood) and, thus, we get

$$\Psi(A, B) = M(\sigma(A, B), \sigma(B, A)) = M(\varepsilon(B, A)((0,0)), \varepsilon(A, B)((0,0))),$$

which relates the fuzzy erosion operator with matrix resemblance functions.

6. Some properties of matrix resemblance functions

There exist some properties that are expected for comparison measures to satisfy (see [2,3,6]). Bustince et al. [3] proposed a set of properties that should be met by any global comparison measure for images. Some of these properties are straightforward from Definition 3.1. Namely, comparison measures are normally asked for *symmetry*, i.e., the difference between two images ought not to depend on the order in which they are compared. Matrix resemblance functions satisfy this condition due to (MRF3). Another property is that a comparison measure should yield that the images are equal if and only if they are exactly equal pixel-wise, which happens for matrix resemblance functions because of (MRF1). This last condition is stronger than the property *reflexivity* in [3]. Additionally, it is often required that the comparison measure between a binary image (in black and white) and its complement is 0, and (MRF2) ensures that.

This subsection goes over some of these usually demanded features and studies in which cases matrix resemblance functions fulfil them.

6.1. Invariance under permutation

As it is mentioned in [7], a permutation of the inputs can be used for modelling certain domain transformations; such as shifts, rotations and reflections. Thus, since a function that measures the similarity between two images ought to provide the same results when comparing images which have been transformed by any of the aforementioned operators, we study the conditions under which a matrix resemblance function Ψ is invariant under permutation.

We say that a matrix resemblance function Ψ is invariant under permutation if $\Psi(A, B) = \Psi(\pi(A), \pi(B))$, for all $A, B \in \mathcal{M}_n([0, 1])$ and all permutations π of the set of indices.

Invariance under such permutation would mean that, as far as it concerns to the result of the comparison, it is the same to compare two images or their transformations; either their shifts, their rotations, or their reflections.

In the case of matrix resemblance functions constructed as in the first method, we reach the following result:

Lemma 6.1. *Let $n \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_n \in [0, 1]$. There exists a function $\beta\colon [0, 1]^2 \to [0, 1]$ satisfying (REF1)–(REF3) such that for all $1 \le i \le n$, there exist $x, y \in [0, 1]^2$ such that $\beta(x, y) = \lambda_i$.*

Proof. The cases of $\lambda_i = 0$ and $\lambda_i = 1$ for some i are trivial, since $\beta(0, 1) = 0$ and $\beta(x_0, x_0) = 1$ for any $x_0 \in [0, 1]$.

Let us suppose that for all $1 \le i \le n$, $0 \neq \lambda_i \neq 1$. Thus, we can define a function β as follows:

$$\beta(x, y) = \begin{cases} 0, & \text{if} \quad \{x, y\} = \{0, 1\}, \\ 1, & \text{if} \quad x = y, \\ \lambda_1, & \text{if} \quad \{x, y\} = \{\frac{1}{n}, 1\}, \\ \lambda_2, & \text{if} \quad \{x, y\} = \{\frac{2}{n}, 1\}, \\ \vdots & \vdots \quad \vdots \\ \lambda_{n-1}, & \text{if} \quad \{x, y\} = \{\frac{n-1}{n}, 1\}, \\ \lambda_n, & \quad\quad \text{otherwise.} \end{cases}$$

□

Theorem 6.2. *Let*

$$\Psi(A, B) = \mathop{H}_{\substack{i=1\\j=1}}^{n} (\beta(a_{ij}, b_{ij}))$$

be a matrix resemblance function as in (1). Then Ψ is invariant under permutation if and only if H satisfies (F3).

Proof. Let β be a function as in (1) and let us assume Ψ is invariant under permutation, i.e., $\Psi(A, B) = \Psi(\pi(A), \pi(B))$, for all $A, B \in \mathcal{M}_n([0, 1])$ and all permutations π.

Now, let us suppose there exists a permutation τ of $\{1, \ldots, n^2\}$ such that

$$H(x_1, \ldots, x_{n^2}) \neq H(x_{\tau(1)}, \ldots, x_{\tau(n^2)}).$$

By Lemma 6.1 every x_i is an image of some function β. Thus, we can choose matrices A and B to be such that

$$\begin{aligned} \beta(a_{11}, b_{11}) &= x_1, \\ \beta(a_{12}, b_{12}) &= x_2, \\ &\vdots \\ \beta(a_{nn}, b_{nn}) &= x_{n^2}. \end{aligned}$$

Thus, we reach the following:

$$\begin{aligned} \Psi(A, B) &= \mathop{H}_{\substack{i=1\\j=1}}^{n} (\beta(a_{ij}, b_{ij})) \\ &= \mathop{H}_{i=1}^{n^2} (x_1, \ldots, x_{n^2}) \\ &\neq \mathop{H}_{i=1}^{n^2} (x_{\tau(1)}, \ldots, x_{\tau(n^2)}) \\ &= \Psi(\tau(A), \tau(B)), \end{aligned}$$

which contradicts the fact that Ψ is invariant under permutation. Hence H satisfies (F3).

The converse implication is straightforward.

□

In the case of constructing Ψ with the second construction method, we obtain the next result.

Proposition 6.3. *Let Ψ be a matrix resemblance function*

$$\Psi(A, B) = M(\sigma(A, B), \sigma(B, A)),$$

as in (3). If σ satisfies (IG5), then Ψ is invariant under permutation.

Proof. Let Ψ be as in (3) and σ satisfying (IG5). Then

$$\Psi(A, B) = M(\sigma(A, B), \sigma(B, A)) \underset{(IG5)}{=} M(\sigma(\pi(A), \pi(B)), \sigma(\pi(B), \pi(A))) = \Psi(\pi(A), \pi(B)),$$

for all $A, B \in \mathcal{M}_n([0, 1])$ and all permutations π. □

Nevertheless, the converse does not hold. For instance, let us consider the function M defined as:

$$M(x, y) = \begin{cases} 1, & \text{if } x = y = 1; \\ 0, & \text{if } xy = 0; \\ 0.5 & \text{otherwise,} \end{cases}$$

and let σ be:

$$\sigma(A,B)=\begin{cases}0, & \text{if } \exists 1\le i,j\le n \text{ s.t. } a_{ij}=0 \text{ and } b_{ij}=1;\\ 1, & \text{if } A\le B \text{ in Zadeh's sense;}\\ b_{11}-a_{11} & \text{otherwise.}\end{cases}$$

Then, σ does not satisfy (IG5) and yet Ψ is invariant under permutation. Indeed, let $A, B \in \mathcal{M}_n([0,1])$, we have three different cases:

- The case where $\Psi(A,B)=1$, which implies that $A=B$ and therefore $\pi(A)=\pi(B)$ for all π. Hence $\Psi(A,B)=\Psi(\pi(A),\pi(B))$.
- The case where $\Psi(A,B)=0$. We can assume $\sigma(A,B)=0$, then for all permutations π it holds that $\sigma(\pi(A),\pi(B))=0$ and hence

$$\Psi(A,B)=M(\sigma(A,B),\sigma(B,A))=0=M(\sigma(\pi(A),\pi(B)),\sigma(\pi(B),\pi(A)))=\Psi(\pi(A),\pi(B)).$$

- The case where $0<\Psi(A,B)<1$. In this situation, it holds that $0<\sigma(A,B)\sigma(B,A)<1$, which means that $0<\sigma(\pi(A),\pi(B))\sigma(\pi(B),\pi(A))<1$ and hence

$$\Psi(A,B)=M(\sigma(A,B),\sigma(B,A))=0.5=M(\sigma(\pi(A),\pi(B)),\sigma(\pi(B),\pi(A)))=\Psi(\pi(A),\pi(B)).$$

Clearly, Ψ is invariant under any permutation but σ does not satisfy (IG5).

6.2. Monotonicity

It is natural to ask that a comparison measure's result decreases when comparing an image with another that is darker and darker (or clearer). Similarly, the result should be higher when we compare an image with another that is more akin to itself. In the case of matrix resemblance functions, that monotonicity property is represented by (MRF4). The property *Reaction to lightening and darkening* from [3] is a consequence of this property. We present here some conditions to ensure monotonicity for each construction.

Proposition 6.4. *Consider*

$$\Psi(A,B)=\underset{\substack{i=1\\j=1}}{\overset{n}{H}}(\beta(a_{ij},b_{ij}))$$

as in (1) with H increasing. If β satisfies (REF5), then Ψ satisfies (MRF4). Moreover, if H is strictly increasing in $(0,1]^{n^2}$, then the converse holds.

Proof. Let H be increasing and suppose that β satisfies (REF5), i.e., for all $x, y, z \in [0,1]$, if $x\le y\le z$, then $\beta(x,y)\ge\beta(x,z)$ and $\beta(y,z)\ge\beta(x,z)$.

Consider $A, B, C \in \mathcal{M}_n([0,1])$ such that $A\le B\le C$. Since $a_{ij}\le b_{ij}\le c_{ij}$ for all i, j, then $\beta(a_{ij}, c_{ij})\le\beta(a_{ij}, b_{ij})$ and since H is increasing,

$$\Psi(A,C)=\underset{\substack{i=1\\j=1}}{\overset{n}{H}}(\beta(a_{ij},c_{ij}))\le\underset{\substack{i=1\\j=1}}{\overset{n}{H}}(\beta(a_{ij},b_{ij}))=\Psi(A,B).$$

Similarly, it holds that $\beta(a_{ij}, c_{ij})\le\beta(b_{ij}, c_{ij})$ and thus,

$$\Psi(A,C)=\underset{\substack{i=1\\j=1}}{\overset{n}{H}}(\beta(a_{ij},c_{ij}))\le\underset{\substack{i=1\\j=1}}{\overset{n}{H}}(\beta(b_{ij},c_{ij}))=\Psi(B,C).$$

Now, for H strictly increasing and Ψ satisfying (MRF4), suppose that there exist $x\le y\le z$ such that $\beta(x,y)<\beta(x,z)$ or $\beta(y,z)<\beta(x,z)$.

Thus, consider the constant matrix A with x in all its entries, B with y in all its entries and C with z in all its entries. Clearly $A\le B\le C$, but

$$\Psi(A,C)=\underset{\substack{i=1\\j=1}}{\overset{n}{H}}(\beta(x,z))>\underset{\substack{i=1\\j=1}}{\overset{n}{H}}(\beta(x,y))=\Psi(A,B),$$

or

$$\Psi(A,C)=\underset{\substack{i=1\\j=1}}{\overset{n}{H}}(\beta(x,z))>\underset{\substack{i=1\\j=1}}{\overset{n}{H}}(\beta(y,z))=\Psi(B,C),$$

which contradicts (MRF4). Hence β satisfies (REF5). □

In particular, Ψ satisfies the monotonicity condition when we consider H to be an aggregation function that fulfills (F1), (F2) and we take β a restricted equivalence function.

The following result regards to the second construction method.

Theorem 6.5. *Let Ψ be such that*

$$\Psi(A,B) = M(\sigma(A,B), \sigma(B,A)),$$

as in (3), with M increasing. If σ satisfies, for all $A, B, C \in \mathcal{M}_n([0,1])$ such that $A \leq B \leq C$, the following conditions:

(a) $\sigma(C, A) \leq \sigma(C, B)$, and
(b) $\sigma(C, A) \leq \sigma(B, A)$;

then Ψ satisfies (MRF4). Besides, if $M(1,x) = x$, the converse also holds.

Proof. Consider $A, B, C \in \mathcal{M}_n([0,1])$ such that $A \leq B \leq C$. Thus, since M is increasing,

$$\Psi(A,C) = M(\sigma(A,C), \sigma(C,A)) \underset{(IG1)}{=} M(1, \sigma(C,A)) \underset{(b)}{\leq} M(1, \sigma(B,A)) \underset{(IG1)}{=} M(\sigma(A,B), \sigma(B,A)) = \Psi(A,B),$$

and

$$\Psi(A,C) = M(\sigma(A,C), \sigma(C,A)) \underset{(IG1)}{=} M(1, \sigma(C,A)) \underset{(a)}{\leq} M(1, \sigma(C,B)) \underset{(IG1)}{=} M(\sigma(B,C), \sigma(C,B)) = \Psi(B,C).$$

Now, suppose that Ψ satisfies (MRF4) and $M(1,x) = x$. Let $A, B, C \in \mathcal{M}_n([0,1])$ such that $A \leq B \leq C$. Then

$$\begin{aligned}\sigma(C,A) &= M(1, \sigma(C,A)) \underset{(IG1)}{=} M(\sigma(A,C), \sigma(C,A)) = \Psi(A,C) \\ &\underset{(MRF4)}{\leq} \Psi(B,C) = M(\sigma(B,C), \sigma(C,B)) \underset{(IG1)}{=} M(1, \sigma(C,B)) = \sigma(C,B),\end{aligned}$$

and

$$\begin{aligned}\sigma(C,A) &= M(1, \sigma(C,A)) \underset{(IG1)}{=} M(\sigma(A,C), \sigma(C,A)) = \Psi(A,C) \\ &\underset{(MRF4)}{\leq} \Psi(A,B) = M(\sigma(A,B), \sigma(B,A)) \underset{(IG1)}{=} M(1, \sigma(B,A)) = \sigma(B,A).\end{aligned}$$

□

Remark 6.6. If in the construction of Ψ as in Theorem 4.7 we consider M to be a t-norm, we can assure that Ψ satisfies (MRF4) if and only if σ satisfies (a) and (b).

Corollary 6.7. *Let Ψ be as in (3) with M increasing. If σ satisfies (IG3) and (IG4), then Ψ satisfies (MRF4).*

Proof. If σ satisfies (IG3) and (IG4), then it satisfies (a) and (b) and we are under the conditions of Theorem 6.5. □

6.3. Comparing the complements: $\Psi(A,B) = \Psi(A_c, B_c)$

Some comparison measures between two images are required to produce the same result when applied to their c-complements (see Property 4 in [3]). In the case of matrix resemblance functions, this translates into studying under what conditions we get the following equality:

$$\Psi(A,B) = \Psi(A_c, B_c),$$

where c is a strong negation and A_c and B_c denote the matrices $\left(c(a_{ij})\right)_{i,j=1}^n$ and $(c(b_{ij}))_{i,j=1}^n$ respectively.

Proposition 6.8. *Let Ψ be a matrix resemblance function as in (1) and β satisfy the additional property (REF4). Then it holds that $\Psi(A,B) = \Psi(A_c, B_c)$.*

Proof.

$$\Psi(A_c, B_c) = \underset{\substack{i=1\\j=1}}{\overset{n}{H}} (\beta(c(a_{ij}), c(b_{ij}))) \underset{(REF4)}{=} \underset{\substack{i=1\\j=1}}{\overset{n}{H}} (\beta(a_{ij}, b_{ij})) = \Psi(A,B).$$

□

The converse of the preceding result does not hold in general. Consider, for instance,

$$\beta(x,y)=\begin{cases}1, & \text{if } x=y;\\ 0, & \text{if } \{x,y\}=\{0,1\};\\ 0.4, & \text{if } \{x,y\}=\{0.2,0.3\};\\ 0.6 & \text{otherwise,}\end{cases} \tag{12}$$

which satisfies (REF1)–(REF3), and

$$H(x_1,\dots,x_N)=\begin{cases}1, & \text{if } x_i=1 \text{ for all } 1\le i\le N;\\ 0, & \text{if } \exists i \text{ s.t. } x_i=0;\\ 0.5 & \text{otherwise,}\end{cases} \tag{13}$$

which satisfies (F1) and (F2). Therefore, by Theorem 4.4, $\Psi(A,B)=\underset{\substack{i=1\\ j=1}}{\overset{n}{H}}(\beta(a_{ij},b_{ij}))$ is a matrix resemblance function.

Moreover, let us show that Ψ satisfies $\Psi(A,B)=\Psi(A_c,B_c)$ for all $A,B\in\mathcal{M}_n([0,1])$. Indeed, let $A,B\in\mathcal{M}_n([0,1])$. Firstly, it holds that

$$\begin{aligned}\Psi(A,B)=0 &\iff \text{there exist } i,j \text{ such that } \{a_{ij},b_{ij}\}=\{0,1\}\\ &\iff \text{there exist } i,j \text{ such that } \{c(a_{ij}),c(b_{ij})\}=\{0,1\}\\ &\iff \Psi(A_c,B_c)=0.\end{aligned} \tag{14}$$

Secondly, it holds that

$$\Psi(A,B)=1 \iff A=B \iff A_c=B_c \iff \Psi(A_c,B_c)=1. \tag{15}$$

Now, due to the definition of function H, there are only three different cases:

- $\Psi(A,B)=0$: By (14), it holds that $\Psi(A,B)=\Psi(A_c,B_c)=0$ for all $A,B\in\mathcal{M}_n([0,1])$.
- $\Psi(A,B)=1$: By (15), it holds that $\Psi(A,B)=\Psi(A_c,B_c)=1$ for all $A,B\in\mathcal{M}_n([0,1])$.
- $\Psi(A,B)=0.5$: Since (14) and (15) cover all the situations in which $\Psi(A_c,B_c)=0$ and $\Psi(A_c,B_c)=1$, the only possibility is $\Psi(A_c,B_c)=0.5$.

Consider now the usual negation $c_z(x)=1-x$ and the constant matrices $A=(0.2)_{i,j=1}^n$ and $B=(0.3)_{i,j=1}^n$. Thus, $A_{c_z}=(0.8)_{i,j=1}^n$ and $B_{c_z}=(0.7)_{i,j=1}^n$.

In this manner,

$$\beta(a_{ij},b_{ij})=\beta(0.2,0.3)=0.4\neq 0.6=\beta(0.8,0.7)=\beta(c_z(a_{ij}),c_z(b_{ij})).$$

In the case of constructing Ψ as in Theorem 4.7, we obtain the following result.

Proposition 6.9. *Let Ψ be a matrix resemblance function as in (3) and σ satisfy (IG6). Then it holds that $\Psi(A,B)=\Psi(A_c,B_c)$.*

Proof.

$$\Psi(A_c,B_c)=M(\sigma(A_c,B_c),\sigma(B_c,A_c)) \underset{(IG6)}{=} M(\sigma(B,A),\sigma(A,B))=\Psi(B,A)=\Psi(A,B).$$

□

The converse of Proposition 6.9 does not hold. Indeed, we can consider M to be the function in (13) for $N=2$ and

$$\sigma(A,B)=\inf_{i,j}\{\min(1,1-a_{ij}^2+b_{ij}^2)\}, \tag{16}$$

which satisfies (IG1) and (IG2). By Theorem 4.7, $\Psi(A,B)=M(\sigma(A,B),\sigma(B,A))$ is a matrix resemblance function.

Note that, as in the preceding counterexample, Ψ satisfies $\Psi(A,B)=\Psi(A_c,B_c)$ for all $A,B\in\mathcal{M}_n([0,1])$ since the aforementioned three possible cases coincide.

Now, if we consider the matrices

$$A=\begin{pmatrix}0.3 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix} \quad\text{and}\quad B=\begin{pmatrix}0.2 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix},$$

then

$$\sigma(A,B)=0.95\neq 0.85=\sigma(B_c,A_c).$$

6.4. Shift invariance

This property, that we have called shift invariance, is related to constant enlightening and darkening of an image. It is sometimes required that a comparison measure gives the same result whenever we compare two images and the same two images constantly enlightened or darkened in the same amount, i.e., adding the same positive or negative amount to each pixel without exceeding the allowed range. This is equivalent to examining when

$$\Psi(A+\lambda J_n, B+\lambda J_n) = \Psi(A,B)$$

holds, where J_n is the $n\times n$ constant matrix with 1 in every entry and provided $a_{ij}+\lambda, b_{ij}+\lambda \in [0,1]$ for $1\le i, j\le n$, i.e., for all $\lambda \in [0, 1-\max(a_{ij}, b_{ij})]$.

In the case of the first construction we attain the next result after a simple computation.

Proposition 6.10. *Let Ψ be a matrix resemblance function as in (1) and β satisfy $\beta(x+\lambda, y+\lambda) = \beta(x,y)$ for all $\lambda \in [0, 1-\max(a_{ij}, b_{ij})]$. Then, $\Psi(A+\lambda J_n, B+\lambda J_n) = \Psi(A,B)$ for all $A, B \in \mathcal{M}_n([0,1])$ and for all $\lambda \in [0, 1-\max(a_{ij}, b_{ij})]$.*

Nevertheless, the converse does not hold. For example, consider β as in (12) and H as in (13). Thus, Ψ is a matrix resemblance function and let us show that $\Psi(A+\lambda J_n, B+\lambda J_n) = \Psi(A,B)$ for all $A, B \in \mathcal{M}_n([0,1])$ and for all $\lambda \in [0, 1-\max(a_{ij}, b_{ij})]$. Let $A, B\in \mathcal{M}_n([0,1])$. Firstly, it holds that

$$\Psi(A,B) = 0 \iff \Psi(A+\lambda J_n, B+\lambda J_n) = 0 \quad \text{for all } \lambda \in [0, 1-\max(a_{ij}, b_{ij})] \tag{17}$$

Indeed, if $\Psi(A,B)=0$, then there exist i, j such that $\{a_{ij}, b_{ij}\} = \{0,1\}$. Therefore $\max(a_{ij}, b_{ij}) = 1$ and hence $\lambda = 0$, which implies $\Psi(A+\lambda J_n, B+\lambda J_n) = 0$.

On the other hand, if $\Psi(A+\lambda J_n, B+\lambda J_n) = 0$, then there exist i, j such that $\{a_{ij}+\lambda, b_{ij}+\lambda\} = \{0,1\}$. Therefore $\lambda = 0$. This means that $\Psi(A,B)=0$ if and only if $\Psi(A+\lambda J_n, B+\lambda J_n) = 0$.

Secondly, for all $\lambda \in [0, 1-\max(a_{ij}, b_{ij})]$, it holds that

$$\Psi(A,B) = 1 \iff A = B \iff A+\lambda J_n = B+\lambda J_n \iff \Psi(A+\lambda J_n, B+\lambda J_n) = 1. \tag{18}$$

Now, due to the definition of function H, there are only three different cases:

- $\Psi(A,B) = 0$: By (17), $\Psi(A+\lambda J_n, B+\lambda J_n) = \Psi(A,B) = 0$ for all $\lambda \in [0, 1-\max(a_{ij}, b_{ij})]$.
- $\Psi(A,B) = 1$: By (18), $\Psi(A+\lambda J_n, B+\lambda J_n) = \Psi(A,B) = 1$ for all $\lambda \in [0, 1-\max(a_{ij}, b_{ij})]$.
- $\Psi(A,B) = 0.5$: Since (17) and (18) cover all the situations in which $\Psi(A+\lambda J_n, B+\lambda J_n) = 0$ and $\Psi(A+\lambda J_n, B+\lambda J_n) = 1$, it holds that $\Psi(A_c, B_c) = 0.5$ for all $\lambda \in [0, 1-\max(a_{ij}, b_{ij})]$.

Now, if we consider the matrices $A = (0.2)_{i,j=1}^n$ and $B = (0.3)_{i,j=1}^n$ and $\lambda = 0.1$, we get

$$\beta(a_{ij}, b_{ij}) = \beta(0.2, 0.3) = 0.4 \neq 0.6 = \beta(0.3, 0.4) = \beta(a_{ij}+\lambda, b_{ij}+\lambda).$$

Similarly, if we construct Ψ in the manner of Theorem 4.7, we obtain the following.

Proposition 6.11. *Let Ψ be a matrix resemblance function as in (3). If $\sigma(A+\lambda J_n, B+\lambda J_n) = \sigma(A,B)$ for all $\lambda \in [0, 1-\max(a_{ij}, b_{ij})]$, then $\Psi(A,B) = \Psi(A+\lambda J_n, B+\lambda J_n)$ for all $\lambda \in [0, 1-\max(a_{ij}, b_{ij})]$.*

Once again, the converse implication does not hold. If we consider M as the function in (13) for $N=2$, which also satisfies (F3), σ as in (16), it holds that Ψ is a matrix resemblance function that is shift invariant, as in the preceding counterexample.

Now, consider $\lambda = 0.2$, and the constant matrices $A = (0.6)_{i,j=1}^n$ and $B = (0.3)_{i,j=1}^n$. Thus,

$$\sigma(A,B) = \inf_{i,j}\{\min(1, 1-0.6^2+0.3^2)\} = 1-0.6^2+0.3^2 = 0.73$$

$$\sigma(A+\lambda J_n, B+\lambda J_n) = \inf_{i,j}\{\min(1, 1-(0.6+0.2)^2+(0.3+0.2)^2)\} = 1-0.8^2+0.5^2 = 0.61.$$

6.5. Homogeneity and migrativity

Homogeneity and migrativity are two properties related to how a perturbation in both or one of the images, respectively, affects on the result. A function is said to be homogeneous of order $k>0$ if when each argument is multiplied by a factor $\lambda>0$, then the result is multiplied by λ^k. In the context of image comparison, multiplying an image by a factor equates to enlightening or darkening the image proportionally and the fact that matrix resemblance functions were homogeneous would mean that

$$\Psi(\lambda A, \lambda A) = \lambda^k \Psi(A,A) = \lambda^k,$$

but by (MRF1), $\Psi(\lambda A, \lambda A) = 1$, hence matrix resemblance functions are not homogeneous operators.

As for migrativity, the property of α-migrativity for a class of binary functions was introduced in [26] and it was studied for aggregation operators in [27]. Given $\alpha\in(0,1)$, a function F: $[0,1]^2 \to [0,1]$ is said to be α-migrative if $F(\alpha x, y) = F(x, \alpha y)$ for all $\alpha \ge 0$ such that $(\alpha x, \alpha y) \in [0,1]^2$. In the case of matrix resemblance functions, a MRF Ψ is said to be α-migrative if $\Psi(\alpha A, B) = \Psi(A, \alpha B)$ for some $\alpha \ge 0$ such that $\alpha A, \alpha B \in \mathcal{M}_n([0,1])$. However, matrix resemblance functions do not satisfy this property due to (MRF2). Indeed, consider A and B such that there exist i, j such that $a_{ij} = 0$ and $b_{ij} = 1$. Thus, $\Psi(\alpha A, B) = 0$ and $\Psi(A, \alpha B)$ need not be equal to 0. Therefore matrix resemblance functions are not migrative operators.

6.6. Additivity

Some similarity measures fulfill a property known as additivity (see [28]). In general, a comparison measure between two fuzzy sets $m : FS(X)^2 \to \mathbb{R}$ is said to be additive if there exists a function $h : [0, 1]^2 \to \mathbb{R}$ so that m can be decomposed in the following way:

$$m(A, B) = \sum_{x \in X} h(A(x), B(x)).$$

In that case h is said to be the additive generator of m.

However, matrix resemblance functions are not additive. Indeed, if they were, there would exist a function h such that

$$\Psi(A, B) = \sum_{i,j=1}^{n} h(a_{ij}, b_{ij}). \tag{19}$$

Now, consider A to be the 3×3 constant matrix with the value 0.8 in all its entries and set $B = A$.

Thus, $1 = \Psi(A, B) = \sum_{i,j=1}^{3} h(0.8, 0.8)$ and hence $h(0.8, 0.8) = \frac{1}{9}$. But if we modify the values a_{11} and b_{11} to be 1 and 0, respectively, then $0 = \Psi(A, B) = \sum_{i,j=1}^{n} h(a_{ij}, b_{ij})$ and therefore $h(0.8, 0.8) = 0$, a contradiction.

7. Special cases of functions *H* and *M*

In this section we study some special cases of the functions H in the first construction and M in the second. Recall that H is a function with n^2 inputs and M is a function with 2 inputs. Additionally, for the first construction H is not required to satisfy condition (F3), although, as seen in Section 6.1, it is convenient for a proper comparison measure as it ensures invariance under permutation.

7.1. H and M aggregation functions

The first two conditions of n-ary aggregation functions (see Definition 2.9) are trivially satisfied by any function that verifies (F1) and (F2), since the latter are more restrictive.

The third condition of aggregation functions is the one about monotonicity. Aggregation functions are increasing with respect to each component and functions H and M do not need to be. Nevertheless, in Section 6.2 we show that if H and M satisfy an additional monotonicity condition, i.e., H and M are increasing, we are under the conditions to apply Proposition 6.4 and Theorem 6.5, respectively. Therefore, in the cases H and M are aggregation functions, by the mentioned results, we know in which cases we have monotone matrix resemblance functions, which is an important property for an image comparison measure.

Moreover, in both constructions these functions are intended to aggregate the values resulting of applying the function β, in the first case, and the function σ, in the second. Hence, it seems natural to use an aggregation function for that purpose. However, increasingness is not strictly required, as in some situations it might be better to use some functions H and M that are not monotone.

7.2. H and M n-dimensional overlap functions

Overlap functions are a particular instance of aggregation functions that are symmetric and continuous. They were first introduced as bivariate functions in [29], and were later generalized to the n-dimensional setting in [30].

Since overlap functions verify (F1)–(F3) conditions, as well as a monotonicity and a continuity condition, they belong to an adequate family of functions that can be used as H and M in both constructions of matrix resemblance functions; as a n^2-ary function in the first case and a bivariate one in the second. Let us start with the definition of n-dimensional overlap functions.

Definition 7.1. A function G: $[0, 1]^n \to [0, 1]$ is said to be a n-dimensional overlap function if it satisfies (F1)–(F3) and:

(i) G is increasing with respect to each component,
(ii) G is a continuous function.

Example 7.2. The minimum, the product and the geometric mean are examples of n-dimensional overlap functions.

Overlap functions satisfy an increasingness condition, which is important for the image comparison property studied in Section 6.2. Moreover, overlap functions are continuous and hence, their use in the construction of matrix resemblance functions can lead to obtaining continuous matrix resemblance functions. Continuity can be considered a desirable property for comparison measures as it ensures a certain degree of robustness, i.e., comparing two images and the same images having been slightly altered produce similar results due to the continuity of the comparison operator.

Let us further study the cases in which a matrix resemblance function Ψ is continuous. In the case of the first construction with H an overlap function, if β is continuous then so is Ψ, as it can be seen of a composition of continuous functions.

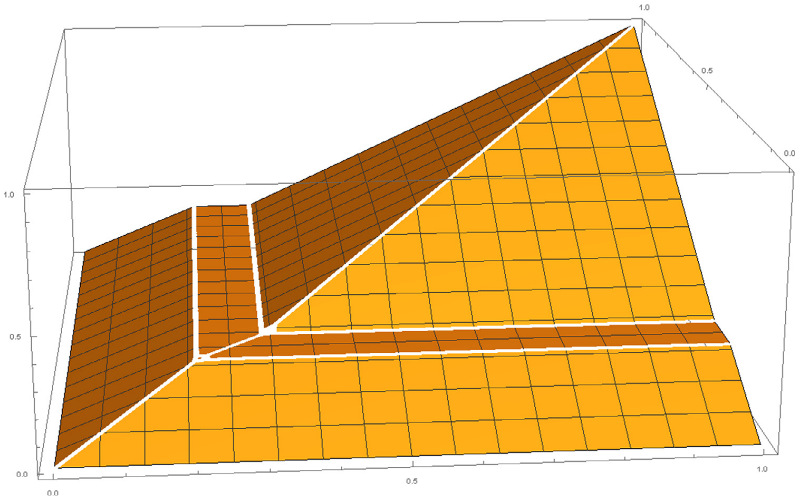

Fig. 3. Graphical representation of the function H in (20) for $N = 2$.

However, the converse does not hold. Indeed, let the function H: $[0, 1]^N \to [0, 1]$ be defined as:

$$H(x_1, \ldots, x_N) = \max\left(\min(x_1, \ldots, x_N, 0.2), \frac{0.56 \min(x_1, \ldots, x_N) - 0.07}{0.49}\right), \tag{20}$$

which is an overlap function, and let β: $[0, 1]^2 \to [0, 1]$ be:

$$\beta(x, y) = \begin{cases} 0.2, & \text{if } \{x, y\} = \{0.15, 0.9\}, \\ 1 - |x - y|, & \text{otherwise.} \end{cases} \tag{21}$$

Thus, one easily verifies that β is not continuous and yet the matrix resemblance function constructed as $\Psi(A, B) = \underset{\substack{i=1\\j=1}}{\overset{n}{H}} (\beta(a_{ij}, b_{ij}))$ is continuous. This is due to the fact that β has the discontinuity in the area where H is constant (see Fig. 3 for a graphical representation of H in the two dimensional case).

However, if H is an idempotent overlap function, we can characterize the continuity of Ψ in terms of the continuity of β:

Theorem 7.3. *Let Ψ be a matrix resemblance function as in (1) with H an idempotent overlap function. Then Ψ is continuous if and only if β is continuous.*

Proof. Let H be an idempotent n^2-dimensional overlap function and $\Psi(A, B) = \underset{\substack{i=1\\j=1}}{\overset{n}{H}} (\beta(a_{ij}, b_{ij}))$ a continuous matrix resemblance function and let us assume that β is not continuous.

Recall that a real function f of N arguments is continuous if for any sequence $(x_{ij})_{i=1}^N$ such that $\lim_{j\to\infty} x_{ij} = y_i$, then $\lim_{j\to\infty} f(x_{1j}, \ldots, x_{Nj}) = f(y_1, \ldots, y_N)$.

Thus, since β is not continuous there exists a sequence (x_k, y_k) such that $\lim_{k\to\infty}(x_k, y_k) = (x, y)$ for some $x, y \in [0, 1]$, but $\lim_{k\to\infty}\beta(x_k, y_k) \neq \beta(x, y)$.

Now, consider A_k and B_k the constant matrices with x_k and y_k in all their entries, respectively. Thus, $\lim_{k\to\infty}(A_k, B_k) = (A, B)$, where A is the constant matrix with x in all its entries and B is the constant y matrix.

Since Ψ is continuous, it holds that $\lim_{k\to\infty} \Psi(A_k, B_k) = \Psi(A, B)$. But since H is idempotent, it holds that

$$\Psi(A_k, B_k) = \underset{\substack{i=1\\j=1}}{\overset{n}{H}} (\beta(x_k, y_k)) = \beta(x_k, y_k), \text{ and}$$

$$\Psi(A, B) = \underset{\substack{i=1\\j=1}}{\overset{n}{H}} (\beta(x, y)) = \beta(x, y).$$

Which contradicts the fact that $\lim_{k\to\infty}\beta(x_k, y_k) \neq \beta(x, y)$. Therefore β is continuous.

The converse implication is immediate, since Ψ is the composition of continuous functions. $\square$

8. Image comparison algorithm

In this section we present a method to measure the difference, or similarity, between two grayscale images. The following algorithm is underpinned by the concept of matrix resemblance functions as a method to compare neighbourhoods of pixels. The reason for a neighbourhood-based comparison as an alternative of proceeding pixel-wise is that we wish to take into account the visual impact that an alteration has in its proximity (see [2]).

One of the main contributions of this paper is that instead of obtaining a number for a result, we get another image, that we will call comparison image and will allow us to set similarity regions, which means that not only will we get a global idea of how similar the images are, but also we will be able to extract areas in the images where they are equally similar or equally different.

The final step of the algorithm is to cluster the comparison image in a variable number of similarity regions, depending on our purpose. For that we use a variation of the k-means algorithm [31,32]. The k-means algorithm is a well-known clustering algorithm that divides a set of data into k groups, being k fixed beforehand. The first step is to set k centroids, one for each group, and then classify each datum in the class of the closest centroid. The next step is to recalculate the centroids as the arithmetic mean of the data that belong to each group, then it proceeds to redistribute all the data according to the closeness to the new centroids. The process continues until the groups remain unchanged for two consecutive iterations. When using this algorithm for image segmentation, it is usual that the closeness of the pixels to each centroid is computed based on the intensity of each pixel, but the variation of the algorithm that we use also takes into account their spatial distribution.

Our comparison algorithm takes two images of the same size and returns two other images, the comparison image and the clustered image by the k-means algorithm. Then, for each pixel of the first input image and the corresponding one in the second, it considers their neighbourhood and compares them using a matrix resemblance function, then sets the number resulting from that local comparison to the pixel from the comparison image in the position that is being considered. Once the loop is finished, the user must decide the number of clusters for the algorithm to apply the clustering technique to the comparison image and get the one divided in similarity regions.

Once we have the comparison image C and the clustered image SC, we are able to visually inspect which areas are more similar and which more different. Since matrix resemblance functions give results closer to 1 when matrices are similar, the regions that are more akin will appear clearer in the comparison image and the more different regions will be darker.

Furthermore, in the clustered image SC, for each cluster we can compute the arithmetic mean of the values from C that are in that cluster and get a number that expresses a similarity measure in each region. In this way the image is divided in zones and we provide a local similarity measure.

9. Illustrative examples

In this section we present some examples that illustrate the algorithm proposed in Section 8.

These examples show the advantages of considering a new image rather than a number as the result of a comparison measure. In this way it is possible to extract location information, such as where both images are more different and how different they are in that area. Besides, a number can lead us to confusion when using it as a measure to compare two images; if we obtain a number that is close to 1 we might think that both images are nearly identical, nevertheless, as we will see in the following examples, this is not always the case. The choice of the parameter k in the final part of the algorithm is made according to the nature of each example to better illustrate the different regions of similarity.

For the examples we have considered the matrix resemblance function of Example 4.8, which can be obtained using either construction method given in this paper, and the images that we use can be found in http://decsai.ugr.es/cvg/index2.php.

9.1. Example 1

In the first example, Fig. 4, we compare an image with another which is the result of enlightening and darkening the division in three areas of the first (see Fig. 4.*a* and 4.*b*). The result of applying Algorithm 1 are the images in Fig. 4.*c* (the comparison image) and in Fig. 4.*d* (the result of applying the k-means clustering algorithm to the comparison image).

In the comparison image we can see the regions where the images are more similar (the lighter regions) and the ones where the images are more different (the darker ones). In this case, the fact that the bottom part of Fig. 4.*c* is the brightest denotes that it is the region where Fig. 4.*a* and 4.*b* are the most similar. Similarly, the fact that the top left part of Fig. 4.*c* is the darkest denotes that Fig. 4.*a* and 4.*b* are the least similar on the top left part. Furthermore, if we computed the arithmetic mean of the pixels of the comparison image we would get a global image comparison measure, i.e., the result of the comparison would be a number. However, that number in the case of Fig. 4 would be 0.7737 and this number does not provide too much information. Using the whole comparison image as a result, we are able to distinguish the zones where both pictures are more similar and computing the arithmetic mean to each of the three regions obtained by the k-means algorithm we obtain a local measure for each region of similarity (see Fig. 5).

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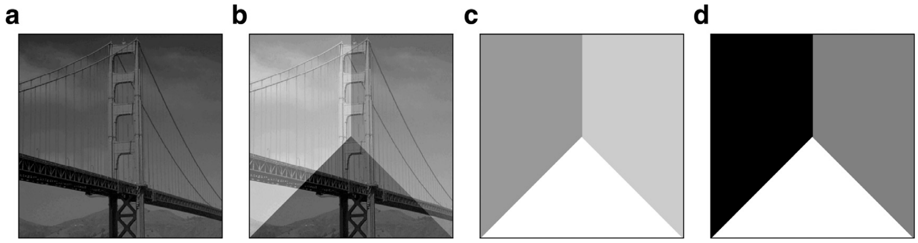

Fig. 4. *a:* Original picture, *b:* Locally enlightened and darkened picture, *c:* Comparison image and *d:* Result of the k-means clustering algorithm with $k = 3$.

Algorithm 1 Image comparison measure algorithm.

Require: Two images (of the same size) to compare: A and B
Ensure: C the comparison image and SC the image clustered in similarity regions

```
for each pixel in A do
    Consider its neighbourhood in A and the corresponding neighbourhood in B
    Compare both neighbourhoods using a matrix resemblance function
    Define in C a pixel in that position and whose value is the result of the comparison
end for
Show the comparison image C
Ask for the number k of clusters needed
Perform the spatial k-means clustering algorithm with k clusters and save it in SC
```

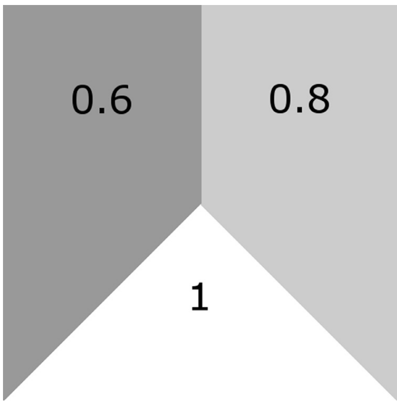

Fig. 5. Comparison measure of each similarity region.

9.2. Example 2

Another possibility is to compare an image with another in which noise has been added. For the next example we use the image in Fig. 6.*a* and we compare it to Fig. 6.*b*, a new image in which we have added different intensities of Gaussian noise to different areas.

In this case, it is difficult to accurately tell, in plain sight, the regions where each intensity of Gaussian noise has been applied. However, the comparison image that results from the Algorithm shows that images Fig. 6.*a* and 6.*b* are more similar in the centre.

The mean pixel intensity of the comparison image is 0.604 and if we compute as before the arithmetic mean of each similarity region given by the k-means segmentation, we reach the results that appear in Fig. 7.

The difference between the average of the values in each region indicates that the k-means algorithm segments the comparison image to obtain well defined similarity regions, areas where the images we compare are equally similar or dissimilar.

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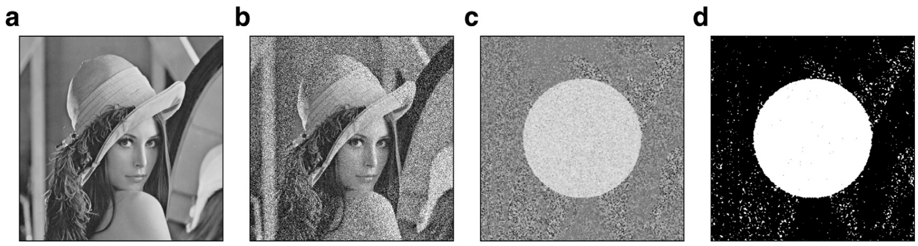

Fig. 6. *a:* Original picture, *b:* Picture with 2 different Gaussian noises ($\sigma = 0.1$ and $\sigma = 0.01$), *c:* Comparison image and *d:* Result of the k-means clustering algorithm with $k = 2$.

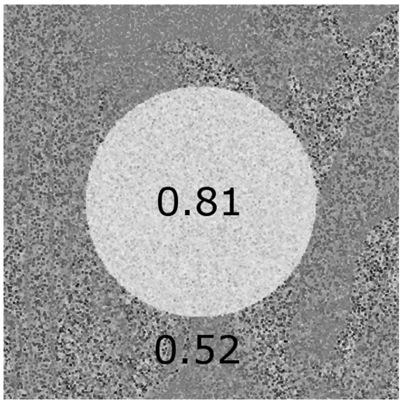

Fig. 7. Comparison measure of each similarity region.

10. Three possible applications

In this section we expose some possible applications for the previously presented MRF-based algorithm. Besides the application as an image comparison measure as such, this method has also potential applications in such fields as pattern matching, vision information retrieval, tamper and damaged areas detection for image reconstruction algorithms, defect detection in industrial processes, video motion detection and object tracking, etc. In this section we present some examples for the last three.

10.1. Tamper and damaged areas detection

One possible application of the proposed comparison method is tamper detection [5,33]. In Fig. 8 we show an example, similar to the ones given in [33], of a tampered image with image synthesis attacks. As it appears in Fig. 8.*d*, the algorithm successfully locates the tampered areas.

Additionally, this algorithm could be used for image reconstruction techniques as it successfully locates the damaged areas. The next example (Fig. 9) is similar to the one that can be found in [34], it consists of an image and a damaged version of it.

10.2. Defect detection in industrial processes

The aim of this section is to present some examples of the performance of our method for defect detection in PCBs, showing the applicability of our algorithm in this field.

Visual inspection systems play a crucial role in manufacturing processes, as they benefit in the goal of having a 100% rate of defect-free products. In this section we focus on the case of defect detection in the assembly of printed circuit boards (PCB). PCBs are a basic component in any electronic device and therefore it is important that they do not have any defects to ensure the proper performance of the device in question. Defects in PCBs are sorted in two types, functionals and cosmetic [35]. There are several proposals of automatic optical inspection systems to detect either kind of defects in the manufacturing production of PCBs [36,37].

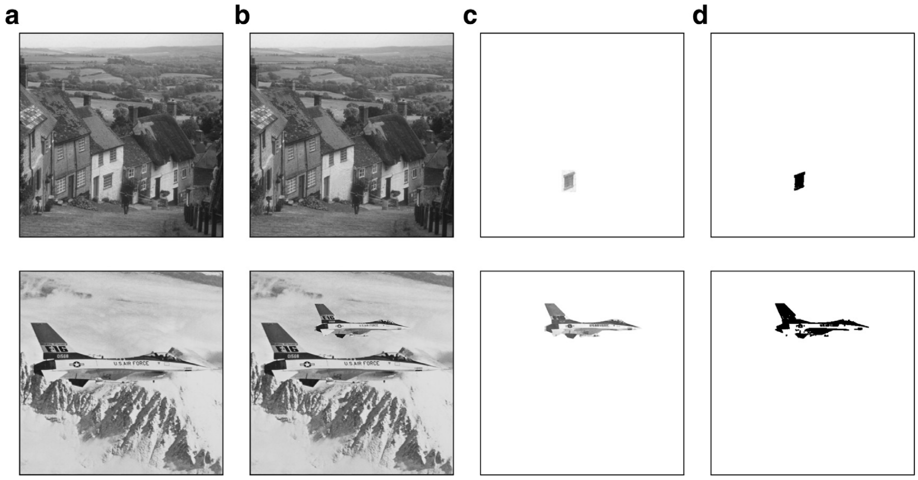

Fig. 8. *a:* Original images, *b:* Tampered images, *c:* Comparison images and *d:* Results of the k-means clustering algorithm with $k = 2$.

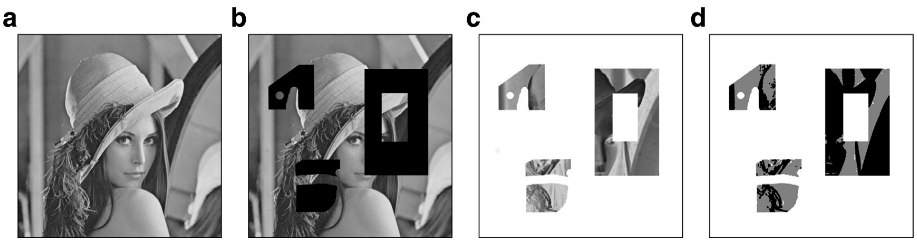

Fig. 9. *a:* Original image. *b:* Damaged image. *c:* Comparison images. *d:* Results of the k-means clustering algorithm with $k = 3$.

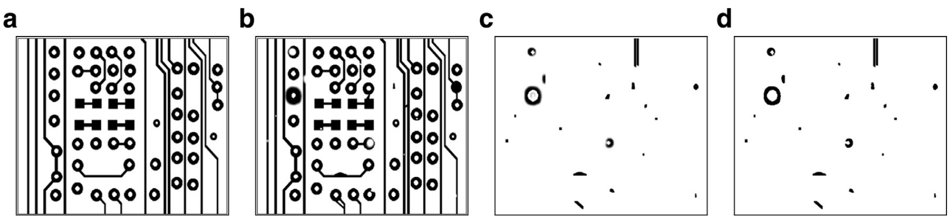

Fig. 10. *a:* Image of good PCB patterns, *b:* Image of defected PCB patterns, *c:* Comparison image and *d:* Results of the k-means clustering algorithm with $k = 2$.

The use of our algorithm based on matrix resemblance functions as a PCB inspection algorithm would be categorised as a referential approach [4], as it would be a model-based technique.

In Fig. 10 one can see an example, as the one in [37], of defect detection and location using Algorithm 1 to compare a well assembled PCB image with a defective one.

The defects that appear in Fig. 10.*b* are detected and located by our algorithm (Fig. 10.*d*).

10.3. Video motion detection and object tracking

The image comparison method presented in this work could also be used for videos, specifically for motion detection or object tracking, i.e., locating an object that is moving in a video.

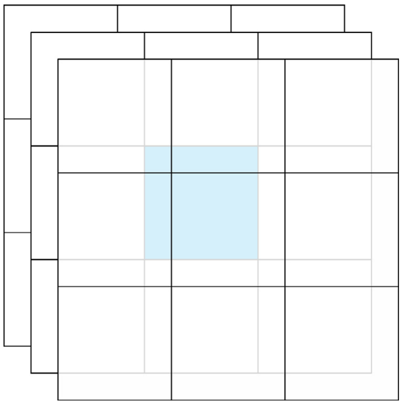

Fig. 11. $3 \times 3 \times 3$ neighbourhood of a pixel in a video. The blue square represents the pixel. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

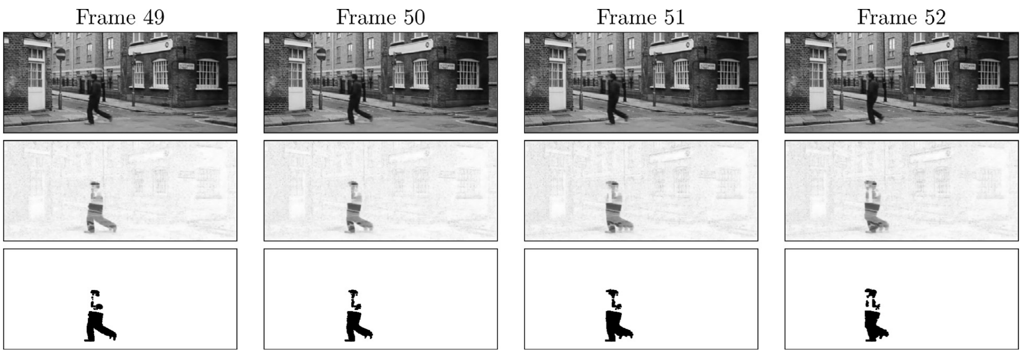

Fig. 12. *First row:* Grayscale conversion of the original frames from the clip. *Second row:* Results of comparing the clips. *Third row:* Results of the k-means algorithm for $k = 2$.

In this subsection, we show how the comparison algorithm with matrix resemblance functions could be applied to object tracking in a video. The idea is to use the algorithm to compare two videos instead of two images; a video in which an object moves and another in which either the object does not appear or it is still.

The first step to carry a comparison between two videos, that have exactly the same duration, is to extract their frames. Thus, we have a series of images and the amount of pixels to manage is the result of adding all the pixels from each frame. In this case, we consider that the neighbourhood of a pixel (i, j, k), i.e., the pixel in the row i, column j of the frame k, is formed by the adjacent pixels in the same frame k, the corresponding ones in the previous frame $k-1$ and those from the following frame $k+1$. So, a neighbourhood of a pixel in a video can be seen as a 3-D matrix and two neighbourhoods can be compared using matrix resemblance functions as before. In Fig. 11 we show a representation of an instance of a neighbourhood of a pixel in a video. Note that, since the definition of matrix resemblance function can be straightforwardly generalized to the case of 3 dimensional matrices, the same algorithm can be used.

Let us now present an example of the usage of matrix resemblance functions to detect a person who is crossing a street. We use a video from a human motion database[2] that is described in [38]. In order to reproduce a video in which the street is empty, we build a new one consisting in a copy of a frame in which the street is empty from the original video.

The video of the following example is a conversion to grayscale of a movie-clip from the 2002 film *About a boy* directed by Paul and Chris Weitz. In Fig. 12 a glimpse of the results are shown. It is apparent that a extraction of the object in motion, the pedestrian in this case, is achieved and that it is reasonably determined by the k-means algorithm.

[2] http://serre-lab.clps.brown.edu/resource/hmdb-a-large-human-motion-database/

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11. Conclusions

In this paper a method for comparing images is presented which not only considers the information provided by each pixel, but also the impact that the surrounding of each pixel has in the comparison. For that purpose, the concept of matrix resemblance function is introduced and two construction methods are presented. Furthermore, since the result of the comparison is a new image, we are able to identify areas in which both images are equally similar and equally dissimilar. Due to this fact, the comparison method presented in this paper is versatile when it comes to possible applications. We have seen that the method could yield good results when applied to tamper detection, location of defect detection in manufacturing processes and video motion detection and object tracking.

Acknowledgments

This work is partially supported by the research services of *Universidad Publica de Navarra* and by the project TIN2016-77356-P (AEI/FEDER, UE). R. Mesiar is supported by Slovak grant APVV-14-0013, and by Czech Project LQ1602 “IT4Innovations excellence in science”.

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