d-Choquet integrals: Choquet integrals based on dissimilarities

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Abstract

The paper introduces a new class of functions from $[0,1]^n$ to $[0,n]$ called d-Choquet integrals. These functions are a generalization of the "standard" Choquet integral obtained by replacing the difference in the definition of the usual Choquet integral by a dissimilarity function. In particular, the class of all d-Choquet integrals encompasses the class of all "standard" Choquet integrals. We show that some d-Choquet integrals are aggregation/pre-aggregation/averaging/functions and some of them are not. The conditions under which this happens are stated and other properties of the d-Choquet integrals are studied.

Keywords: Choquet integral, d-Choquet integral, dissimilarity, pre-aggregation function, aggregation function, monotonicity, directional monotonicity

1. Introduction

The problem of finding an appropriate model of aggregation is crucial in many fields. One of the simplest tools to aggregate data is an additive aggregation function, i.e., a weighted arithmetic mean (a convex combination of the input values). However, in some cases, such functions are not appropriate to model even quite simple situations, which, on the other hand, can be treated with Choquet integrals [3, 11, 19, 20, 26, 27]. The origin of the Choquet integrals arises from Choquet beliefs, or capacities that generalize the notion of probability by relaxing additivity [10], and can be regarded as a generalization of additive aggregation functions replacing the requirement of additivity by that of comonotone additivity.

Some generalizations of the Choquet integral were proposed in the recent years. In [28] the product operator was replaced by a more general function and the authors studied the requirements that this function must satisfy so that the obtained generalization of the Choquet integral would be a pre-aggregation function [12, 29]. In the same pattern, using the distributivity of the product operator and then replacing its two instances by two different functions under some constraints, in [13, 30], generalizations of the Choquet

integral were obtained as either aggregation, pre-aggregation or ordered directionally increasing functions [5], depending on the properties satisfied by such two functions. Some Choquet-like integrals defined in terms of pseudo-addition and pseudo-multiplication are studied in [31]. A fuzzy t-conorm integral that is a generalization of Choquet integral (as well as of Sugeno integral) and is based on continuous t-conorms and continuous t-norms is introduced in [32]. A non-linear integral that need not be increasing is introduced in [36] and a concave integral generalizing the Choquet integral is introduced in [25]. A level dependent Choquet integral was also introduced in [21]. For a general review on the state-of-art on the generalizations of the Choquet Integral see [17].

In this paper, in order to generalize the Choquet integral, i.e. the n-ary function

$$C_\mu(x_1, \ldots, x_n) = \sum_{i=1}^{n} (x_{\sigma(i)} - x_{\sigma(i-1)}) \mu \left(A_{\sigma(i)} \right)$$

we replace the difference $x_{\sigma(i)} - x_{\sigma(i-1)}$ by $\delta(x_{\sigma(i)}, x_{\sigma(i-1)})$, where $\delta : [0,1]^2 \to [0,1]$ is a restricted dissimilarity function [6, 7], and refer to the obtained function as d-Choquet integral. This approach allows us to construct a wide class of new functions, d-Choquet integrals, which, unlike the "standard" Choquet integral, may be possibly outside of the scope of aggregation functions, since the monotonicity may be violated for some δ, and also the range of such functions can be wider than $[0,1]$.

The "standard" Choquet integral is known to be an averaging aggregation function which is comonotone additive, idempotent, self-dual for self-dual fuzzy measures, shift-invariant and positively homogeneous. According to the choice of a restricted dissimilarity function, the obtained d-Choquet integral possesses (or does not) some of the mentioned properties. In such a way we have a wide possibility to construct a function with desired properties.

The aim of the paper is to bring a theoretical study of the above described wide class of functions, that is the class of all d-Choquet integrals. Moreover, our work can be seen as the first step to the generalization of the Choquet integral to various settings where the difference cause problems (for example, intervals).

The structure of the paper is as follows. First, we present some preliminary concepts that help making the paper self-contained. In Section 3, we introduce the notion of d-Choquet integral, describe various construction methods and study its properties, such as comonotone additivity, idempotency, self-duality, shift-invariancy and homogeneity. We discuss the relation of the class of d-Choquet integrals with the classes of aggregation functions, averaging aggregation functions and pre-aggregation functions in Section 4. Finally, some important conclusions and future research are described in Section 5.

2. Preliminaries

In this section, we recall some basic notions and terminology that are necessary for our subsequent developments.

Definition 2.1. [6] A function $\delta : [0,1]^2 \to [0,1]$ is called a restricted dissimilarity function on $[0,1]$ if it satisfies, for all $x, y, z \in [0,1]$, the following conditions:

1. $\delta(x,y) = \delta(y,x)$;
2. $\delta(x,y) = 1$ if and only if $\{x,y\} = \{0,1\}$;
3. $\delta(x,y) = 0$ if and only if $x = y$;
4. if $x \leq y \leq z$, then $\delta(x,y) \leq \delta(x,z)$ and $\delta(y,z) \leq \delta(x,z)$.

Note that there is no the greatest neither the smallest restricted dissimilarity function. The minimal range of a restricted dissimilarity function consists of 3 values $\{0, a, 1\}$, where $a \in]0,1[$, attained for $\delta = \delta_a : [0,1]^2 \to [0,1]$ given by

$$\delta_a(x,y) = \begin{cases} 1 & \text{iff } \{x,y\} = \{0,1\} \\ 0 & \text{iff } x = y \\ a & \text{otherwise.} \end{cases}$$

Definition 2.2. [4] An n-ary aggregation function is a mapping $A : [0,1]^n \to [0,1]$ satisfying the following properties:

(A1) A is increasing[1] in each argument: for each $i \in \{1, \ldots, n\}$, if $x_i \leq y$, then

$$A(x_1, \ldots, x_n) \leq A(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n);$$

(A2) The boundary conditions: $A(0, \ldots, 0) = 0$ and $A(1, \ldots, 1) = 1$.

A conjunctive aggregation function is an aggregation function $A : [0,1]^n \to [0,1]$ such that $A \leq \min$.

A 0-positive aggregation function is an aggregation function $A : [0,1]^n \to [0,1]$ such that if $A(x_1, \ldots, x_n) = 0$ then there exists $i \in \{1, \ldots, n\}$ such that $x_i = 0$. On the other hand, if A is also conjunctive, then whenever there exists $i \in \{1, \ldots, n\}$ such that $x_i = 0$ one has that $A(x_1, \ldots, x_n) = 0$.

Definition 2.3. [22] An aggregation function $T : [0,1]^2 \to [0,1]$ is a t-norm if the following conditions hold, for all $x, y, z \in [0,1]$:

(T1) Commutativity: $T(x, y) = T(y, x)$;

(T2) Associativity: $T(x, T(y, z)) = T(T(x, y), z)$;

(T3) Neutral element: $T(x, 1) = x$.

An element $x \in\,]0,1]$ is said to be a non-trivial zero divisor of T if there exists $y \in\,]0,1]$ such that $T(x, y) = 0$. A t-norm is positive if and only if it has no non-trivial zero divisors, that is, if $T(x, y) = 0$ then either $x = 0$ or $y = 0$.

Example 2.4. Examples of t-norms are the Minimum, the Product, the Łukasiewicz t-norm, the Hamacher Product and the Drastic t-norm $T_M, T_P, T_{\text{Ł}}, T_{HP}, T_D : [0,1]^2 \to [0,1]$, defined, respectively, for each $x, y \in [0,1]$, by [22]:

$$
\begin{aligned}
T_M(x, y) &= \min\{x, y\}; \\
T_P(x, y) &= xy; \\
T_{\text{Ł}}(x, y) &= \max\{0, x + y - 1\}; \\
T_{HP}(x, y) &= \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise}; \end{cases} \\
T_D(x, y) &= \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise}. \end{cases}
\end{aligned}
$$

Observe that the T_M, T_P and T_{HP} are positive t-norms, and any t-norm T is conjunctive, since it always happens that $T \leq \min$.

Definition 2.5. [8] A function $O : [0,1]^2 \to [0,1]$ is said to be an overlap function if the following conditions hold, for all $x, y, z \in [0,1]$:

(O1) O is commutative;

(O2) $O(x, y) = 0$ if and only if $xy = 0$;

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[1]We consider that an increasing function may not be strictly increasing (and analogously for decreasing functions).

(O3) $O(x, y) = 1$ if and only if $xy = 1$;

(O4) O is increasing;

(O5) O is continuous.

Example 2.6. Examples of overlap functions are the Cuadras-Augé family of copulas $O_{B,\alpha} : [0, 1]^2 \to [0, 1]$, for $\alpha \in [0, 1]$, used in [8, Theorem 8] and [33], $O_{mM} : [0, 1]^2 \to [0, 1]$, presented in [14, Ex. 3.1.(i)], [15, Ex. 4] and [16, Ex. 3.1], the Eyraud-Farlie-Gumbel-Morgenstern (EFGM) 2-copulas $O_{\alpha} : [0, 1]^2 \to [0, 1]$, with $\alpha \in [-1, 1]$, found in [33], [1, Apendix A (A.2.1)] and [26], and the Geometric Mean $GM : [0, 1]^2 \to [0, 1]$, used in [18, Ex. 1], defined, respectively, for all $x, y \in [0, 1]$ by:

$$
\begin{aligned}
O_{B,\alpha}(x, y) &= (T_M(x, y))^{\alpha} \cdot (T_P(x, y))^{(1-\alpha)}, \text{ for } \alpha \in [0, 1]; \\
O_{mM}(x, y) &= \min\{x, y\} \max\{x^2, y^2\}; \\
O_{\alpha}(x, y) &= xy(1 + \alpha(1 - x)(1 - y)), \text{ for } \alpha \in [-1, 1]; \\
GM(x, y) &= \sqrt{xy}.
\end{aligned}
\tag{1}
$$

Definition 2.7. A function $N : [0, 1] \to [0, 1]$ is a negation function if it is a decreasing function such that $N(0) = 1$ and $N(1) = 0$. A negation N is called strong if $N(N(x)) = x$ for all $x \in [0, 1]$; it is called non-filling if $N(x) = 1$ if and only if $x = 0$; and it is called non-vanishing if $N(x) = 0$ if and only if $x = 1$.

Definition 2.8. [2] An implication function is a mapping $I : [0, 1]^2 \to [0, 1]$ satisfying the boundary conditions $I(0, 0) = I(0, 1) = I(1, 1) = 1$, $I(1, 0) = 0$ and the following properties:

- $(I1)$ $x \leq z$ implies $I(x, y) \geq I(z, y)$, for all $y \in [0, 1]$;

- $(I2)$ $y \leq z$ implies $I(x, y) \leq I(x, z)$, for all $x \in [0, 1]$.

We also recall some possible properties of implication functions that will be needed in the paper:

- (OP) $I(x, y) = 1$ if and only if $x \leq y$ (ordering property);

- (CS) $I(x, y) = I(N(y), N(x))$, where N is a strong negation (N-contrapositive symmetry);

- (NV) $I(x, y) = 0$ if and only if $x = 1$ and $y = 0$ (non-vanishing).

Note that for an arbitrary restricted dissimilarity function δ, the function $I_\delta : [0, 1]^2 \to [0, 1]$ given by

$$
I_\delta(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 1 - \delta(x, y) & \text{otherwise.} \end{cases}
$$

is an implication function satisfying (OP) and (NV).

Definition 2.9. An automorphism of $[0, 1]$ is a continuous, strictly increasing function $\varphi : [0, 1] \to [0, 1]$ such that $\varphi(0) = 0$ and $\varphi(1) = 1$. Moreover, the identity on $[0, 1]$ is denoted by Id.

It is well-known that a function $f : [0, 1]^n \to [0, 1]$ is additive if

$$
f(x_1 + y_1, \ldots, x_n + y_n) = f(x_1, \ldots, x_n) + f(y_1, \ldots, y_n)
\tag{2}
$$

for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in [0, 1]^n$ such that $(x_1 + y_1, \ldots, x_n + y_n) \in [0, 1]^n$. From now on, $[n]$ denotes the set $\{1, \ldots, n\}$.

Definition 2.10. Vectors $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in [0, 1]^n$ are comonotone if there exists a permutation $\sigma : [n] \to [n]$ such that $x_{\sigma(1)} \leq \ldots \leq x_{\sigma(n)}$ and $y_{\sigma(1)} \leq \ldots \leq y_{\sigma(n)}$.

Definition 2.11. A function $f : [0,1]^n \to [0,1]$ is called comonotone additive if Equality (2) holds for all comonotone vectors $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in [0,1]^n$ such that $(x_1 + y_1, \ldots, x_n + y_n) \in [0,1]^n$.

Definition 2.12. A function $\mu : 2^{[n]} \to [0,1]$ is called a fuzzy measure on $[n]$ if $\mu(\emptyset) = 0$, $\mu([n]) = 1$ and $\mu(A) \leq \mu(B)$ for all $A \subseteq B \subseteq [n]$.

Definition 2.13. [9] Let $\vec{r} = (r_1, \ldots, r_n)$ be a real n-dimensional vector such that $\vec{r} \neq \vec{0}$. A function $f : [0,1]^n \to [0,1]$ is $\vec{r}$-increasing if, for all $(x_1, \ldots, x_n) \in [0,1]^n$ and for all $c \in]0,1]$ such that $(x_1 + cr_1, \ldots, x_n + cr_n) \in [0,1]^n$, it holds

$$f(x_1 + cr_1, \ldots, x_n + cr_n) \geq f(x_1, \ldots, x_n).$$

Definition 2.14. [29] A function $f : [0,1]^n \to [0,1]$ is said to be an n-ary pre-aggregation function if $f(0, \ldots, 0) = 0$, $f(1, \ldots, 1) = 1$ and f is $\vec{r}$-increasing for some real n-dimensional vector $\vec{r} = (r_1, \ldots, r_n)$ such that $\vec{r} \neq \vec{0}$ and $r_i \geq 0$ for every $i = 1, \ldots, n$. In this case, we say that f is an $\vec{r}$-pre-aggregation function.

Finally, we recall some well-known properties of functions that will be used in the paper. A function $f : [0,1]^n \to [0,1]$ is called:

- idempotent if $f(x, \ldots, x) = x$, for all $x \in [0,1]$;

- averaging if $\min\{x_1, \ldots, x_n\} \leq f(x_1, \ldots, x_n) \leq \max\{x_1, \ldots, x_n\}$, for all $x_1, \ldots, x_n \in [0,1]$;

- shift-invariant if $f(x_1 + y, \ldots, x_n + y) = y + f(x_1, \ldots, x_n)$, for all $y, x_1, \ldots, x_n \in [0,1]$ such that $x_1 + y, \ldots, x_n + y \in [0,1]$;

- positively homogeneous if $f(rx_1, \ldots, rx_n) = rf(x_1, \ldots, x_n)$, for all $r, x_1, \ldots, x_n \in [0,1]$;

- self-dual if $f(x_1, \ldots, x_n) = 1 - f(1 - x_1, \ldots, 1 - x_n)$, for all $x_1, \ldots, x_n \in [0,1]$.

3. d-Choquet integral

In this section, we introduce the notion of d-Choquet integral and study its properties such as comonotone additivity, idempotency, self-duality, shift-invariancy and homogeneity. Moreover, construction of restricted dissimilarity function in terms of automorphisms and implications is considered and its influence on the obtained d-Choquet integral is discussed.

3.1. Definition of d-Choquet integral

The discrete Choquet integral on $[0,1]$ with respect to a fuzzy measure $\mu : 2^{[n]} \to [0,1]$ is defined as a mapping $C_\mu : [0,1]^n \to [0,1]$ such that

$$C_\mu(x_1, \ldots, x_n) = \sum_{i=1}^{n} (x_{\sigma(i)} - x_{\sigma(i-1)}) \mu\left(A_{\sigma(i)}\right) \tag{3}$$

where σ is a permutation on $[n]$ satisfying $x_{\sigma(1)} \leq \ldots \leq x_{\sigma(n)}$, with the convention $x_{\sigma(0)} = 0$ and $A_{\sigma(i)} = \{\sigma(i), \ldots, \sigma(n)\}$.

In order to generalize the Choquet integral, we replace the difference $x_{\sigma(i)} - x_{\sigma(i-1)}$ by $\delta(x_{\sigma(i)}, x_{\sigma(i-1)})$, where δ is a restricted dissimilarity function.

Definition 3.1. Let n be a positive integer and $\mu : 2^{[n]} \to [0,1]$ be a fuzzy measure on $[n]$. Let $\delta : [0,1]^2 \to [0,1]$ be a restricted dissimilarity function. An n-ary discrete d-Choquet integral on $[0,1]$ with respect to μ and δ is defined as a mapping $C_{\mu,\delta} : [0,1]^n \to [0,n]$ such that

$$C_{\mu,\delta}(x_1, \ldots, x_n) = \sum_{i=1}^{n} \delta(x_{\sigma(i)}, x_{\sigma(i-1)}) \mu\left(A_{\sigma(i)}\right) \tag{4}$$

where σ is a permutation on $[n]$ satisfying $x_{\sigma(1)} \leq \ldots \leq x_{\sigma(n)}$, with the convention $x_{\sigma(0)} = 0$ and $A_{\sigma(i)} = \{\sigma(i), \ldots, \sigma(n)\}$.

Remark 3.2. (i) The property $\delta(x, x) = 0$ for all $x \in [0, 1]$ ensures that if there exist several possible permutations such that $x_{\sigma(1)} \leq \ldots \leq x_{\sigma(n)}$, the result for any of them when applying (4) is the same. Moreover, since the ranges of δ and μ are subsets of $[0, 1]$, the range of $C_{\mu,\delta}$ is a subset of $[0, n]$. Hence, $C_{\mu,\delta}$ is well-defined.

(ii) Observe that, without loss of generality, it would be possible to replace in the above definition a restricted dissimilarity function δ by a pseudo-difference $\ominus : [0, 1]^2 \to [0, 1]$ characterized axiomatically as follows:

1. $x \ominus y = 0$ if and only if $x \leq y$;
2. $x \ominus y = 1$ if and only if $x = 1$ and $y = 0$;
3. for any $x \leq y \leq z$ it holds that $y \ominus x \leq z \ominus x$, and $z \ominus y \leq z \ominus x$.

Note that there is no δ, μ and $x_1, \ldots, x_n \in [0, 1]$ such that $C_{\mu,\delta}(x_1, \ldots, x_n) = n$. However, for any $x_1, \ldots, x_n \in [0, 1]$ such that $card(x_1, \ldots, x_n) = n$, and for any $a \in]0, 1[$, for the greatest fuzzy measure $\mu^\top$, see (6), it holds $C_{\mu^\top, \delta_a}(x_1, \ldots, x_n) = na$, and the range of $C_{\mu^\top, \delta_a}$ is contained in $[0, \max\{1, na\}]$. Consequently, for any $n > u = n - \epsilon \geq 1$, the value u is attained by $C_{\mu^\top, \delta_a}$ for $a = u/n$.

Then the related δ is given by $\delta(x, y) = \max\{x, y\} \ominus \min\{x, y\}$. For several appropriate $\ominus$ operations derived from nilpotent t-conorms one can refer to [37]. Observe that there are several alternative approaches to pseudo-differences, see, e.g. [[24, 32, 37] which are related to restricted dissimilarity functions only in particular cases.

Observe that, in general, the range of $C_{\mu,\delta}$ is a subset of $[0, n]$. Since, for some applications, it may be desired that the range of $C_{\mu,\delta}$ would be $[0, 1]$, we often impose the following condition:

(P1) $\delta(0, x_1) + \delta(x_1, x_2) + \ldots + \delta(x_{n-1}, x_n) \leq 1$ for all $x_1, \ldots, x_n \in [0, 1]$ where $x_1 \leq \ldots \leq x_n$.

Next proposition shows that the condition $(P1)$ assures that the range of $C_{\mu,\delta}$ is a subset of $[0, 1]$, i.e. we obtain $C_{\mu,\delta} : [0, 1]^n \to [0, 1]$.

Proposition 3.3. *Let $C_{\mu,\delta} : [0, 1]^n \to [0, n]$ be an n-ary discrete d-Choquet integral on $[0, 1]$ with respect to μ and δ given by Definition 3.1. If δ satisfies the condition $(P1)$, then*

$$C_{\mu,\delta}(x_1, \ldots, x_n) \in [0, 1]$$

for all $x_1, \ldots, x_n \in [0, 1]$ and for any measure μ.

Proof. Straightforwardly follows from (4). $\square$

Note that if δ satisfies (P1) and is continuous as a real function of two variables, then the range of $C_{\mu,\delta}$ is equal to $[0, 1]$.

Proposition 3.4. *If $\delta : [0, 1]^2 \to [0, 1]$ is a restricted dissimilarity function satisfying (P1) then each restricted dissimilarity function $\delta' : [0, 1]^2 \to [0, 1]$ such that $\delta' \leq \delta$ also satisfy (P1).*

Proof. Consider $0 = x_0 \leq x_1 \leq \ldots \leq x_n \leq 1$. Then, since $\delta' \leq \delta$ and δ satisfy (P1), we have that
$\sum_{i=1}^{n} \delta'(x_{i-1}, x_i) \leq \sum_{i=1}^{n} \delta(x_{i-1}, x_i) \leq 1$. $\square$

Now we give a sufficient condition under which a restricted dissimilarity function satisfies the condition $(P1)$.

Proposition 3.5. *Let $\delta : [0, 1]^2 \to [0, 1]$ be a restricted dissimilarity function and $f : [0, 1] \to [0, 1]$ be an increasing function such that $f(0) = 0$ and $f(1) = 1$. If $\delta(x, y) \leq |f(x) - f(y)|$ for all $x, y \in [0, 1]$, then the following statements hold:*

(i) *f is strictly increasing;*

(ii) *δ satisfies (P1);*

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(iii) $\delta(0, x) + \delta(x, y) \le f(y)$ *whenever* $x \le y$.

Proof. (i) Suppose that f is not strictly increasing. Then, there exists $x', y' \in [0, 1]$, with $x' \ne y'$ and $f(x') = f(y')$. Therefore, $\delta(x', y') \le |f(x') - f(y')| = 0$ leads to a contradiction with condition (3) of Def. 2.1.

(ii) Let n be a positive integer and $0 \le x_1 \le \ldots \le x_n \le 1$. Then

$$\delta(0, x_1) + \delta(x_1, x_2) + \ldots + \delta(x_{n-1}, x_n) \le -f(0) + f(x_1) - f(x_1) + f(x_2) - \ldots - f(x_{n-1}) + f(x_n) = f(x_n) - f(0) \le 1.$$

(iii) Take $x, y \in [0, 1]$ such that $x \le y$. Then it holds that $\delta(0, x) \le |f(0) - f(x)| = f(x)$ and $\delta(x, y) \le |f(x) - f(y)| = f(y) - f(x)$. Therefore, one has that $\delta(0, x) + \delta(x, y) \le f(y)$. $\square$

Example 3.6. Let μ be a fuzzy measure on $\{1, 2, 3\}$ defined by $\mu(\{1\}) = \mu(\{2\}) = \mu(\{3\}) = 0.3$, $\mu(\{1, 2\}) = 0.75$, $\mu(\{2, 3\}) = 0.55$ and $\mu(\{1, 3\}) = 0.6$.
(i) Then
$$C_\mu(0.2, 0.9, 0.6) = 0.2 \cdot 1 + 0.4 \cdot 0.55 + 0.3 \cdot 0.3 = 0.51.$$

It is easy to see that for $\delta(x, y) = |x - y|$ it holds $C_{\mu, \delta} = C_\mu$ for any possible inputs and any measure μ (as will be shown in Theorem 3.24).
(ii) However, if $\delta(x, y) = (x - y)^2$ we have

$$C_{\mu, \delta}(0.2, 0.9, 0.6) = 0.04 \cdot 1 + 0.16 \cdot 0.55 + 0.09 \cdot 0.3 = 0.155.$$

(iii) Finally, taking

$$\delta(x, y) = \begin{cases} 0, & \text{if } x = y; \\ \frac{|x-y|+1}{2}, & \text{otherwise,} \end{cases}$$

we obtain

$$C_{\mu, \delta}(0.2, 0.9, 0.6) = 0.6 \cdot 1 + 0.7 \cdot 0.55 + 0.65 \cdot 0.3 = 1.18,$$

where we can see that $C_{\mu, \delta}(0.2, 0.9, 0.6) > 1$. This may happen since for any increasing $f : [0, 1] \to [0, 1]$ with $f(0) = 0$ and $f(1) = 1$ there exist $x, y \in [0, 1]$ such that $\frac{|x-y|+1}{2} > |f(x) - f(y)|$, hence (P1) is not satisfied.

Remark 3.7. Clearly, for any fuzzy measure μ and restricted dissimilarity function δ, the 'boundary' conditions hold:
$$C_{\mu, \delta}(0, \ldots, 0) = 0 \qquad \text{and} \qquad C_{\mu, \delta}(1, \ldots, 1) = 1.$$

Note that, in fact, the second one should be called boundary condition only if the range of $C_{\mu, \delta}$ is a subset of $[0, 1]$, although the equality holds in any case, even if the range is not a subset of $[0, 1]$ - clearly, in this case, $C_{\mu, \delta}$ is not increasing (we study monotonicity of $C_{\mu, \delta}$ in Section 4).

Theorem 3.8. *Let* $\delta : [0, 1]^2 \to [0, 1]$ *be a restricted dissimilarity function. Consider* $f_\delta : [0, 1] \to [0, 1]$, *defined, for each* $x \in [0, 1]$, *by*

$$f_\delta(x) = \delta(x, 0)$$

and $\delta^* : [0, 1]^2 \to [0, 1]$, *defined, for each* $x, y \in [0, 1]$, *by*

$$\delta^*(x, y) = |f_\delta(x) - f_\delta(y)|.$$

Then δ^* *is a restricted dissimilarity function which satisfies (P1) if and only if* f_δ *is injective.*

Proof. ($\Rightarrow$) Take $x \neq y$ and suppose that $f_\delta(x) = f_\delta(y)$. Then one has that $\delta^*(x, y) = |f_\delta(x) - f_\delta(y)| = 0$, which is in contradiction with Condition 3. of Def. 2.1. Therefore, f_δ is injective.

($\Leftarrow$) Observe first that f_δ is increasing. In fact, if $x < y$ then by Condition 4. of Def. 2.1, one has that $\delta(0, x) \leq \delta(0, y)$ and, therefore, $f_\delta(x) = \delta(0, x) \leq \delta(0, y) = f_\delta(y)$. So, if $f_\delta(x) = 1$ then $f_\delta(1) \geq f_\delta(x) = 1$ and, therefore, once f_δ is injective, $f_\delta(x) = 1$ if and only if $x = 1$. Analogously, it is possible to prove that $f_\delta(x) = 0$ if and only if $x = 0$.

Clearly δ^* is symmetric and since f_δ is injective, then $\delta^*(x, y) = 0$ if and only if $\delta(x, 0) = \delta(y, 0)$ if and only if $f_\delta(x) = f_\delta(y)$ if and only if $x = y$. Analogously, $\delta^*(x, y) = 1$ if and only if either $f_\delta(x) = 1$ and $f_\delta(y) = 0$ or $f_\delta(x) = 0$ and $f_\delta(y) = 1$ if and only if either $x = 1$ and $y = 0$ or $x = 0$ and $y = 1$ if and only if $\{x, y\} = \{0, 1\}$.

If $x \leq y \leq z$ then $f_\delta(x) \leq f_\delta(y) \leq f_\delta(z)$. Therefore $f_\delta(y) - f_\delta(x) \leq f_\delta(z) - f_\delta(x)$ and $f_\delta(z) - f_\delta(y) \leq f_\delta(z) - f_\delta(x)$. So, $\delta^*(x, y) \leq \delta^*(x, z)$ and $\delta^*(y, z) \leq \delta^*(x, z)$. Consequently, δ^* is a restricted dissimilarity function and once f_δ satisfies the conditions of Proposition 3.5, we have that δ^* satisfies (P1). $\square$

Proposition 3.9. *Let $\delta : [0, 1]^2 \to [0, 1]$ be a restricted dissimilarity function. If, for each $x, y \in [0, 1]$ such that $x \leq y$ it holds that $\delta(0, x) + \delta(x, y) \leq \delta(0, y)$, then δ satisfy (P1).*

Proof. For $n = 2$ it is straightforward from the premise that $\delta(0, x_1) + \delta(x_1, x_2) \leq \delta(0, x_2)$ whenever $x_1 \leq x_2$. Consider $k > 2$ and suppose that, for each $x_1, \ldots, x_k \in [0, 1]$ such that $x_1 \leq \ldots \leq x_k$, it holds that

$$\delta(0, x_1) + \delta(x_1, x_2) + \ldots + \delta(x_{k-1}, x_k) \leq \delta(0, x_k).$$

Then, for any $x_{k+1} \in [0, 1]$ such that $x_k \leq x_{k+1}$, we have that

$$\delta(0, x_1) + \delta(x_1, x_2) + \ldots + \delta(x_{k-1}, x_k) + \delta(x_k, x_{k+1}) \leq \delta(0, x_k) + \delta(x_k, x_{k+1}).$$

So, by the basic case, one has that $\delta(0, x_k) + \delta(x_k, x_{k+1}) \leq \delta(0, x_{k+1})$, and then

$$\delta(0, x_1) + \delta(x_1, x_2) + \ldots + \delta(x_k, x_{k+1}) \leq \delta(0, x_{k+1}).$$

Therefore, by induction, we have that for each $n \geq 2$,

$$\delta(0, x_1) + \delta(x_1, x_2) + \ldots + \delta(x_{n-1}, x_n) \leq \delta(0, x_n) \leq 1$$

which means that δ satisfies (P1). $\square$

Remark 3.10. Observe that if $\delta(x, 0) = x$ then $\delta^*(x, y) = |x - y|$.

3.2. d-Choquet integral based on restricted dissimilarity function constructed in terms of automorphisms

In [6], a construction method for restricted dissimilarity functions in terms of automorphisms was introduced.

Proposition 3.11. *[6] If φ_1, φ_2 are two automorphisms of $[0, 1]$, then the function $\delta : [0, 1]^2 \to [0, 1]$ defined by*

$$\delta(x, y) = \varphi_1^{-1}\Big(|\varphi_2(x) - \varphi_2(y)|\Big)$$

is a restricted dissimilarity function.

Note that if the restricted dissimilarity function δ is given in terms of automorphisms φ_1, φ_2 as in Proposition 3.11, we write $C_{\mu, \varphi_1, \varphi_2}$ instead of $C_{\mu, \delta}$.

Proposition 3.12. *Let n be a positive integer, $\delta : [0, 1]^2 \to [0, 1]$ be a restricted dissimilarity function given in terms of automorphisms φ_1, φ_2 as in Proposition 3.11. If $\varphi_1 \geq Id$ for all $x \in [0, 1]$, then δ satisfies (P1).*

Proof. It is enough to set $f = \varphi_2$ in Proposition 3.5 and observe that $\varphi_1 \geq Id$ implies $\varphi_1^{-1} \leq Id$. $\square$

Remark 3.13. (i) For instance the restricted dissimilarity functions $\delta(x,y) = (x-y)^2$, $\delta(x,y) = |\sqrt{x} - \sqrt{y}|$, $\delta(x,y) = |x^2 - y^2|$ and $\delta(x,y) = (\sqrt{x} - \sqrt{y})^2$ satisfy the condition $\delta(x,y) \leq |f(x) - f(y)|$ of Proposition 3.5 for $f = Id$, $f(x) = \sqrt{x}$, $f(x) = x^2$ and $f(x) = \sqrt{x}$, respectively (as well as the condition $\varphi_1 \geq Id$ of Proposition 3.12), hence they satisfy the $(P1)$. This means that the corresponding d-Choquet integrals have the ranges in $[0,1]$.

(ii) However, for the restricted dissimilarity function $\delta(x,y) = \sqrt{|x-y|}$, there does not exist a function f demanded in Proposition 3.5, hence it may happen that $(P1)$ is violated. For instance, $\delta(0, 0.1) + \delta(0.1, 1) = 1.2649 > 1$.

Given an automorphism φ and a restricted dissimilarity functions δ, the function $\delta^\varphi : [0,1]^2 \to [0,1]$ defined, for all $x, y \in [0,1]$, by $\delta^\varphi(x,y) = \varphi^{-1}(\delta(\varphi(x), \varphi(y)))$ is called of conjugate of δ.

Proposition 3.14. *Let φ be an automorphism and δ be a restricted dissimilarity function. Then the function δ^φ is also a restricted dissimilarity function. In addition, if δ satisfies (P1) and $\varphi \geq Id$ then δ^φ satisfies (P1).*

Proof. Trivially, δ^φ is commutative, and since φ and their inverse are bijective and increasing then $\delta^\varphi(x,y) = 0$ if and only if $\delta(\varphi(x), \varphi(y)) = 0$ if and only if $\varphi(x) = \varphi(y)$ if and only if $x = y$. Analogously, it holds that $\delta^\varphi(x,y) = 1$ if and only if $\delta(\varphi(x), \varphi(y)) = 1$ if and only if $\{\varphi(x), \varphi(y)\} = \{0,1\}$ if and only if $\{x, y\} = \{0,1\}$. Finally, if $x \leq y \leq z$ then one has that $\varphi(x) \leq \varphi(y) \leq \varphi(z)$ and therefore, it holds that $\delta(\varphi(x), \varphi(y)) \leq \delta(\varphi(x), \varphi(z))$ and $\delta(\varphi(y), \varphi(z)) \leq \delta(\varphi(x), \varphi(z))$. Hence, one has that $\delta^\varphi(x,y) \leq \delta^\varphi(x,z)$ and $\delta^\varphi(y,z) \leq \delta^\varphi(x,z)$. Therefore, δ^φ is also a restricted dissimilarity function. In addition, if δ satisfies (P1), $\varphi \geq Id$ and $0 = x_0 \leq x_1 \leq \ldots \leq x_n$ then one has that $0 = \varphi(x_0) \leq \varphi(x_1) \leq \ldots \leq \varphi(x_n)$ and since δ satisfies (P1) and $\varphi^{-1} \leq Id$, then

$$\sum_{i=1}^{n} \delta^\varphi(x_{i-1}, x_i) \leq \sum_{i=1}^{n} \delta(\varphi(x_{i-1}), \varphi(x_i)) \leq 1.$$

$\square$

Remark 3.15. Observe that if δ^φ satisfies (P1) it does not mean that δ satisfies (P1). For example, consider the restricted dissimilarity function given in Remark 3.13 (ii) and the automorphism $\varphi(x) = \sqrt{x}$. Then, by Proposition 3.14, $\delta^\varphi(x,y) = |\sqrt{x} - \sqrt{y}|$ is also a restricted equivalence function. However, δ^φ satisfies (P1) whereas δ does not satisfy (P1) as observed before in Remark 3.13 (ii). Indeed, if $x_1 \leq \ldots \leq x_n$ and $x_0 = 0$ then it holds that $\sum_{i=1}^{n} \delta^\varphi(x_{i-1}, x_i) = x_n$.

3.3. d-Choquet integral based on restricted dissimilarity function constructed in terms of strictly increasing functions f satisfying boundary conditions

In the following, we provide construction methods of restricted dissimilarity functions satisfying (P1) based on more general functions than automorphisms, namely, strictly increasing functions f satisfying boundary conditions.

Proposition 3.16. *Let $f : [0,1] \to [0,1]$ be a strictly increasing function such that $f(0) = 0$ and $f(1) = 1$. Then, $\delta_f : [0,1]^2 \to [0,1]$, defined, for all $x, y \in [0,1]$ by $\delta_f(x,y) = |f(x) - f(y)|$ is a restricted dissimilarity function satisfying (P1).*

Proof. Observe that (1) δ_f is trivially commutative. Moreover:
(2) Suppose that $\delta_f(x,y) = 1$. It follows that either $f(x) = 1$ and $f(y) = 0$, or $f(x) = 0$ and $f(y) = 1$, and, since f is strictly increasing, one has that either $x = 1$ and $y = 0$, or $x = 0$ and $y = 1$. Conversely, one has that $\delta_f(0,1) = \delta_f(1,0) = |f(1) - f(0)| = 1$.
(3) Suppose that $\delta_f(x,y) = 0$. It follows that $f(x) = f(y)$. Since f is strictly increasing, it follows that $x = y$. Conversely, $\delta_f(x,x) = |f(x) - f(x)| = 0$.
(4) Suppose that $x \leq y \leq z$. Then, it holds that $f(x) \leq f(y) \leq f(z)$. It follows that $f(y) - f(x) \leq f(z) - f(x)$ and $f(z) - f(y) \leq f(z) - f(x)$. Therefore one has that $\delta_f(x,y) \leq \delta_f(x,z)$ and $\delta_f(y,z) \leq \delta_f(x,z)$.

Finally, by Proposition 3.5, δ_f satisfies (P1). $\square$

Theorem 3.17. *Let* $f : [0,1] \to [0,1]$ *be a strictly increasing function such that* $f(0) = 0$ *and* $f(1) = 1$, $\delta : [0,1]^2 \to [0,1]$ *be a restricted dissimilarity function and* $M : [0,1]^2 \to [0,1]$ *a 0-positive conjunctive aggregation function. Then,* $\delta_{f,\delta,M} : [0,1]^2 \to [0,1]$, *defined, for all* $x, y \in [0,1]$ *by* $\delta_{f,\delta,M}(x,y) = M(|f(x) - f(y)|, \delta(x,y))$ *is a restricted dissimilarity function satisfying (P1).*

Proof. (1) One has that $\delta_{f,\delta,M}$ is trivially commutative. Moreover:
(2) Suppose that $\delta_{f,\delta,M}(x,y) = 1$, that is, $M(|f(x) - f(y)|, \delta(x,y)) = 1$. Since M is conjunctive, then

$$M(|f(x) - f(y)|, \delta(x,y)) = 1 \leq \min\{|f(x) - f(y)|, \delta(x,y)\}.$$

It follows that both $|f(x) - f(y)| = 1$ and $\delta(x,y) = 1$. Since δ is a restricted dissimilarity function, then $\{x, y\} = \{0, 1\}$. Conversely, one has that

$$\delta_{f,\delta,M}(0,1) = \delta_{f,\delta,M}(1,0) = M\{|f(1) - f(0)|, \delta(1,0)\} = M(1,1) = 1.$$

(3) Suppose that $\delta_{f,\delta,M}(x,y) = 0$, that is, $M(|f(x) - f(y)|, \delta(x,y)) = 0$. Since M is 0-positive, then it follows that either (i) $|f(x) - f(y)| = 0$ or (ii) $\delta(x,y) = 0$. In the case (i), one has that $f(x) = f(y)$, and, thus, $x = y$, since f is strictly increasing. Now, in the case (ii), it is immediate that $x = y$. Conversely, one has that:

$$\delta_{f,\delta,M}(x,x) = M(|f(x) - f(x)|, \delta(x,x)) = M(0,0) = 0.$$

(4) Suppose that $x \leq y \leq z$. Then, $\delta(x,y) \leq \delta(x,z)$, $\delta(y,z) \leq \delta(x,z)$ and $f(x) \leq f(y) \leq f(z)$. It follows that (i) $f(y) - f(x) \leq f(z) - f(x)$ and (ii) $f(z) - f(y) \leq f(z) - f(x)$. Considering (i), one has that:

$$\delta_{f,\delta,M}(x,y) = M(|f(x) - f(y)|, \delta(x,y)) \leq M(|f(z) - f(x)|, \delta(x,z)) = \delta_{f,\delta,M}(x,z).$$

Now, for (ii), it follows that:

$$\delta_{f,\delta,M}(y,z) = M(|f(y) - f(z)|, \delta(y,z)) \leq M(|f(x) - f(z)|, \delta(x,z)) = \delta_{f,\delta,M}(x,z).$$

Finally, since M is conjunctive, then

$$\delta_{f,\delta,M}(x,y) = M(|f(x) - f(y)|, \delta(x,y)) \leq \min\{|f(x) - f(y)|, \delta(x,y)\} \leq |f(x) - f(y)|.$$

Then, by Propositions 3.5, $\delta_{f,\delta,M}$ satisfies (P1). $\qquad\square$

Example 3.18. Some examples of restricted dissimilarity functions obtained by the construction method given by Theorem 3.17 are:

(i) Consider the discussion given in Example 2.4. Then, any positive t-norm T may play the role of M, as T_M, T_P and T_{HP}.

(ii) Observe that any overlap function is positive. Then, taking into account the overlap functions given in Example 2.6, the conjunctive overlap functions are O_{mM}, $O_{B,\alpha}$ (with $\alpha \in [0,1]$), O_{α} (with $\alpha \in [-1,1]$), and thus they can be used as the function M.

3.4. d-Choquet integral based on restricted dissimilarity function constructed in terms of implication function
Restricted equivalence function was constructed in terms of implication functions in [6].

Proposition 3.19. *[6] Let* $N : [0,1] \to [0,1]$ *be a strong negation. A function* $\delta : [0,1]^2 \to [0,1]$ *is a restricted equivalence function such that* $\delta(x,y) = \delta(N(x), N(y))$ *for all* $x, y \in [0,1]$ *if and only if there exists a function* $I : [0,1]^2 \to [0,1]$ *satisfying (I1), (OP), (CS) and (NV), such that*

$$\delta(x,y) = \max\{N(I(x,y)), N(I(y,x))\}.$$

We are going to construct restricted dissimilarity functions in a more simple way with weaker assumptions on the used negation.

Theorem 3.20. *Let $N : [0,1] \to [0,1]$ be a non-filling and non-vanishing negation and $I : [0,1]^2 \to [0,1]$ be a function satisfying $(I1)$, (OP), (CS) and (NV). Let $M : [0,1]^2 \to [0,1]$ be a symmetric aggregation function such that $M(x,y) = 1$ if and only if $1 \in \{x,y\}$, $M(x,y) = 0$ if and only if $x = y = 0$ and $M(u,v) \geq M(p,q)$ whenever $\max\{u,v\} \geq \max\{p,q\}$. Then a function $\delta : [0,1]^2 \to [0,1]$ defined, for all $x,y \in [0,1]$, by*

$$\delta(x,y) = M\left(N(I(x,y)), N(I(y,x))\right)$$

is a restricted equivalence dissimilarity function.

Proof.   1. The symmetry of δ follows from the symmetry of M.
   2. By the assumptions of the theorem, $\delta(x,y) = 1$ if and only if $I(x,y) = 0$ or $I(y,x) = 0$ which only holds if $\{x,y\} = \{0,1\}$.
   3. $\delta(x,y) = 0$ if and only if $I(x,y) = I(y,x) = 1$ which only holds if $x = y$.
   4. Let $x \leq y \leq z$. Then $I(z,x) \leq I(y,x) \leq I(x,y) \leq I(x,z)$, hence

$$\delta(x,y) = M\left(N(I(x,y)), N(I(y,x))\right) \leq M\left(N(I(x,z)), N(I(z,x))\right) = \delta(x,z).$$

Similarly for $\delta(y,z) \leq \delta(x,z)$.

$\square$

Remark 3.21. Note that M considered in Theorem 3.20 can be easily shown to have form $M(x,y) = M(\max\{x,y\}, \max\{x,y\})$, that is $M(x,y) = d_M(\max\{x,y\})$ where $d_M : [0,1] \to [0,1]$ is the diagonal section of M. Clearly, d_M is increasing, $d_M(x) = 0$ if and only if $x = 0$, and $d_M(x) = 1$ if and only if $x = 1$.

Remark 3.22. Observe that if M' is a bivariate aggregation function such that $M'(x,x) \in \{0,1\}$ implies that $x \in \{0,1\}$, then $M(x,y) = M'(\max\{x,y\}, \max\{x,y\})$ satisfies the conditions of the Theorem 3.20. Therefore, each positive t-norm and each overlap function generate such aggregation function.

From now on, the d-Choquet integral with respect to a restricted dissimilarity function δ given as in Theorem 3.20 will be denoted by $C_{\mu,N,I,M}$, where μ is a fuzzy measure.

Example 3.23. Recall that the d-Choquet integral $C_{\mu,\delta}$ has range in $[0,1]$ if δ satisfies the condition $(P1)$. Hence, in terms of N, I and M the following hold:

(i) It is easy to check (the justification is made in Corollary 3.27) that $C_{\mu,N,I,M}$ has range in $[0,1]$ for $I(x,y) = \min\{1, 1 - x + y\}$, $N(x) = 1 - x$ and $M = \max$. Note that in this case we obtain the "standard" Choquet integral.

(ii) $C_{\mu,N,I,M}$ has range in $[0,1]$ for the same implication as in item (i) with any non-filling and non-vanishing negation $N(x) \leq 1 - x$, for instance $N(x) = 1 - \sqrt{x}$; and any symmetric aggregation function $M(x,y) \leq \max\{x,y\}$ satisfying the assumptions of Theorem 3.20, for instance

$$M(x,y) = \begin{cases} 1, & \text{if } \max\{x,y\} = 1; \\ \alpha \max\{x,y\}, & \text{otherwise}, \end{cases}$$

where $\alpha \in]0,1]$. Or, in more general, $M(x,y) = g(\max\{x,y\})$ where $g : [0,1] \to [0,1]$ is an increasing function such that $g(x) = 0$ if and only if $x = 0$, $g(x) = 1$ if and only if $x = 1$ and $g(x) \leq x$ for all $x \in [0,1]$.

(iii) $C_{\mu,N,I,M}$ has range in $[0,1]$ for the same N and M as in item (ii) and

$$I(x,y) = \begin{cases} 1, & \text{if } x < 1; \\ y, & \text{if } x = 1. \end{cases}$$

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3.5. Relation between d-Choquet integrals and the "standard" Choquet integral

Clearly, $C_{\mu,\delta}$ for the restricted dissimilarity function $\delta(x,y) = |x - y|$ is equal to the "standard" Choquet integral.

Theorem 3.24. *Let n be a positive integer, $\mu : 2^{[n]} \to [0,1]$ be a fuzzy measure on $[n]$, $\delta : [0,1]^2 \to [0,1]$ be the function $\delta(x,y) = |x - y|$, $C_{\mu,\delta} : [0,1]^n \to [0,1]$ be an n-ary discrete d-Choquet integral on $[0,1]$ with respect to μ and δ given by Definition 3.1 and $C_\mu : [0,1]^n \to [0,1]$ be an n-ary discrete Choquet integral on $[0,1]$ with respect to μ given by Equation (3). Then*

$$C_{\mu,\delta}(x_1, \ldots, x_n) = C_\mu(x_1, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in [0,1]$.

Proof. Straightforwardly follows from Equations (3) and (4). $\qquad\qquad\qquad\qquad\qquad$ $\square$

Corollary 3.25. *Let n be a positive integer, $\mu : 2^{[n]} \to [0,1]$ be a fuzzy measure on $[n]$ and $\delta : [0,1]^2 \to [0,1]$ be a restricted equivalence function such that $f_\delta = Id$. Then*

$$C_{\mu,\delta^*}(x_1, \ldots, x_n) = C_{\mu,\delta_{f_\delta}}(x_1, \ldots, x_n) = C_\mu(x_1, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in [0,1]$.

Corollary 3.26. *Let n be a positive integer, $\mu : 2^{[n]} \to [0,1]$ be a fuzzy measure on $[n]$. Then*

$$C_{\mu,Id,Id}(x_1, \ldots, x_n) = C_\mu(x_1, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in [0,1]$.

We can also recover the "standard" Choquet integral for appropriate choice of N, I and M when restricted dissimilarity function δ is given in terms of implications.

Corollary 3.27. *Let n be a positive integer, μ be a fuzzy measure on $[n]$, $N : [0,1] \to [0,1]$ be defined by $N(x) = 1 - x$ for all $x \in [0,1]$, $M : [0,1]^2 \to [0,1]$ be the maximum and $I : [0,1]^2 \to [0,1]$ be defined by $I(x,y) = \min\{1, 1 - x + y\}$ for all $x, y \in [0,1]$. Then*

$$C_{\mu,N,I,M}(x_1, \ldots, x_n) = C_\mu(x_1, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in [0,1]$.

Proof. The proof follows from Theorem 3.24 and the observation:

$$\delta(x,y) = \max\{N(I(x,y)), N(I(y,x))\} = N\left(I\left(\max\{x,y\}, \min\{x,y\}\right)\right) = \max\{x,y\} - \min\{x,y\} = |x - y|.$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$$

3.6. Properties of d-Choquet integrals

It is well-known that each Choquet integral is comonotone additive, however, a d-Choquet integral is comonotone additive only if δ is comonotone additive.

Theorem 3.28. *Let $n \geq 2$ be an integer. An n-ary d-Choquet integral $C_{\mu,\delta}$ given by Definition 3.1 is comonotone additive for any fuzzy measure μ on $[n]$ if and only if the restricted dissimilarity function δ is comonotone additive.*

Proof. ⇐ Let δ be comonotone additive. Then for any comonotone vectors $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in [0, 1]^n$ such that $(x_1 + y_1, \ldots, x_n + y_n) \in [0, 1]^n$ we have

$$C_{\mu,\delta}(x_1 + y_1, \ldots, x_n + y_n) = \sum_{i=1}^{n} \delta(x_{\sigma(i)} + y_{\sigma(i)}, x_{\sigma(i-1)} + y_{\sigma(i-1)})\mu\left(A_{\sigma(i)}\right)$$

$$= \sum_{i=1}^{n} \delta(x_{\sigma(i)}, x_{\sigma(i-1)})\mu\left(A_{\sigma(i)}\right) + \sum_{i=1}^{n} \delta(y_{\sigma(i)}, y_{\sigma(i-1)})\mu\left(A_{\sigma(i)}\right) = C_{\mu,\delta}(x_1, \ldots, x_n) + C_{\mu,\delta}(y_1, \ldots, y_n).$$

⇒ Since $C_{\mu,\delta}$ is comonotone additive for each fuzzy measure μ then, in particular, it is comonotone additive for the fuzzy measures $\mu_\perp, \mu_\top : 2^{[n]} \to [0, 1]$ defined, respectively, by:

$$\mu_\perp(A) \;=\; \begin{cases} 1 & \text{if } A = [n]; \\ 0 & \text{if } A \neq [n] \end{cases} \tag{5}$$

$$\mu_\top(A) \;=\; \begin{cases} 1 & \text{if } A \neq \emptyset; \\ 0 & \text{if } A = \emptyset. \end{cases} \tag{6}$$

Let $(x_1, x_2), (y_1, y_2) \in [0, 1]^2$ be comonotone ordered pairs such that $(x_1 + y_1, x_2 + y_2) \in [0, 1]^2$. Then, one has that $(x_1, \ldots, x_1, x_2), (y_1, \ldots, y_1, y_2) \in [0, 1]^n$ are comonotone vectors such that $(x_1 + y_1, \ldots, x_1 + y_1, x_2 + y_2) \in [0, 1]^n$. It follows that

$$\begin{aligned} \delta(x_{\sigma(1)}, 0) + \delta(y_{\sigma(1)}, 0) \;&=\; C_{\mu_\perp,\delta}(x_1, \ldots, x_1, x_2) + C_{\mu_\perp,\delta}(y_1, \ldots, y_1, y_2) \\ &=\; C_{\mu_\perp,\delta}(x_1 + y_1, \ldots, x_1 + y_1, x_2 + y_2) \text{ (since } C_{\mu_\perp,\delta} \text{ is comonotone additive)} \\ &=\; \delta(x_{\sigma(1)} + y_{\sigma(1)}, 0) \end{aligned}$$

On the other hand, one has that

$$\begin{aligned} \delta(x_1, x_2) + \delta(y_1, y_2) \;&=\; C_{\mu_\top,\delta}(x_1, \ldots, x_1, x_2) + C_{\mu_\perp,\delta}(y_1, \ldots, y_1, y_2) - (\delta(x_{\sigma(1)}, 0) + \delta(y_{\sigma(1)}, 0)) \\ &=\; C_{\mu_\top,\delta}(x_1 + y_1, \ldots, x_1 + y_1, x_2 + y_2) - (\delta(x_{\sigma(1)}, 0) + \delta(y_{\sigma(1)}, 0)) \\ &\quad (\text{since } C_{\mu_\top,\delta} \text{ is comonotone additive}) \\ &=\; C_{\mu_\top,\delta}(x_1 + y_1, \ldots, x_1 + y_1, x_2 + y_2) - \delta(x_{\sigma(1)} + y_{\sigma(1)}, 0)) \\ &=\; \delta(x_1 + y_1, x_2 + y_2) \end{aligned}$$

Therefore, δ is comonotone additive. □

Corollary 3.29. *Let $n \geq 2$ be an integer. An n-ary d-Choquet integral $C_{\mu,\delta}$ given by Definition 3.1 is comonotone additive, for any fuzzy measure μ on $[n]$ if and only if $\delta(x, y) = |x - y|$ for all $x, y \in [0, 1]$.*

Proof. ⇒ Observe that binary monotone function is additive if and only if it is a polynomial of degree 1, see [23]. Hence, on the domain $S_1 = \{[x, y] \mid x \leq y\}$, δ is additive if and only if $\delta(x, y) = ax + by + c$ and from the conditions $\delta(0, 0) = \delta(1, 1) = 0$, $\delta(0, 1) = 1$ it follows that $\delta(x, y) = y - x$ on S_1. From the symmetry of δ it follows that $\delta(x, y) = x - y$ on the domain $S_2 = \{[x, y] \mid y \leq x\}$, hence $\delta(x, y) = |x - y|$ on $[0, 1]^2$.
⇐ The proof in this direction is obvious. □

Remark 3.30. (i) It is worth pointing out that according to Corollary 3.29 and Theorem 3.24 the only comonotone additive d-Choquet integral with respect to an arbitrary considered fuzzy measure μ on $[n]$ given by Definition 3.1 is the "standard" Choquet integral, i.e. the d-Choquet integral w.r.t. the restricted dissimilarity function $\delta(x, y) = |x - y|$. However, we point out that there may exist a restricted dissimilarity function δ' different from $\delta(x, y) = |x - y|$, such that, for some specific fuzzy measure μ', it holds that $C_{\mu',\delta'}$

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is comonotone additive. For example, this happens when one considers the fuzzy measure $\mu_\perp$ defined in Equation (5) and the restricted dissimilarity function $\delta_\sqrt{}(x,y) = (\sqrt{x} - \sqrt{y})^2$. In fact, for the fuzzy measure $\mu_\perp$, the corresponding $C_{\mu_\perp,\delta}$ is comonotone additive if and only if $\delta(0, x+y) = \delta(0,x) + \delta(0,y)$, for all $x, y \in [0,1]$ such that $x + y \in [0,1]$. In particular, if $\delta(x,0) = x$, for all $x \in [0,1]$, then $C_{\mu_\perp,\delta}$ is minimum, i.e., it is a "standard" Choquet integral with respect to $\mu_\perp$.

(ii) For any δ and μ, the d-Choquet integral gives back the measure, i.e., for any subset E of $[n]$, $C_{\mu,\delta}(1_E) = \mu(E)$, where 1_E denotes the n-tuple which takes the value 1 at position i if $i \in E$ and 0 otherwise.

(iii) If $\delta(x,y) = |g(x) - g(y)|$ for some $g : [0,1] \to [0,1]$ strictly increasing, such that $g(x) = 0$ and $g(1) = 1$, then $C_{\mu,\delta}(x_1, \ldots, x_n) = C_\mu(g(x_1), \ldots, g(x_n))$

Another property of the Choquet integral is idempotency. In the following theorem we show that $C_{\mu,\delta}$ is idempotent only if δ has neutral element 0.

Theorem 3.31. *Let n be a positive integer. An n-ary d-Choquet integral $C_{\mu,\delta}$ given by Definition 3.1 is idempotent for any fuzzy measure μ on $[n]$ if and only if the restricted dissimilarity function δ satisfies $\delta(0,x) = x$ for all $x \in [0,1]$.*

Proof. Observe that

$$C_{\mu,\delta}(x,\ldots,x) = \delta(0,x)\mu(\{1,\ldots,n\}) = \delta(0,x)$$

and the proof is obvious. □

Corollary 3.32. *Let δ be a restricted dissimilarity function, n be a positive integer and μ be a fuzzy measure on $[n]$. The following statements are equivalent:*

1. *$C_{\mu,\delta}$ is idempotent;*
2. *$f_\delta = Id$; and*
3. *$C_{\mu,\delta^*} = C_\mu$.*

Proof. $(1 \Rightarrow 2)$ For each $x \in [0,1]$, by Theorem 3.31, it holds that $f_\delta(x) = \delta(x,0) = x = Id(x)$.

$(2 \Rightarrow 3)$ For each $x_1, \ldots, x_n \in [0,1]$, one has that

$$
\begin{aligned}
C_{\mu,\delta^*}(x_1, \ldots, x_n) &= \sum_{i=1}^{n} \delta^*(x_{\sigma(i)}, x_{\sigma(i-1)})\mu(A_{\sigma(i)}) \\
&= \sum_{i=1}^{n} |f_\delta(x_{\sigma(i)}) - f_\delta(x_{\sigma(i-1)})|\mu(A_{\sigma(i)}) \\
&= \sum_{i=1}^{n} (x_{\sigma(i)} - x_{\sigma(i-1)})\mu(A_{\sigma(i)}) \\
&= C_\mu(x_1, \ldots, x_n).
\end{aligned}
$$

$(3 \Rightarrow 1)$ For each $x \in [0,1]$, it holds that

$$C_{\mu,\delta}(x, \ldots, x) = \delta(x,0) = f_\delta(x) = C_{\mu,\delta^*}(x, \ldots, x) = C_\mu(x, \ldots, x) = x.$$

□

Corollary 3.33. *Let n be a positive integer and $\varphi_1, \varphi_2 : [0,1] \to [0,1]$ be automorphisms of $[0,1]$. An n-ary d-Choquet integral $C_{\mu,\varphi_1,\varphi_2}$ is idempotent for any fuzzy measure μ on $[n]$ if and only if $\varphi_1 = \varphi_2$.*

Proof. The proof follows from the observation:

$$\delta(0,x) = \varphi_1^{-1}\Big(\varphi_2(x) - \varphi_2(0)\Big) = \varphi_1^{-1}\Big(\varphi_2(x)\Big).$$

$\square$

Corollary 3.34. *Let n be a positive integer, $N : [0,1] \to [0,1]$ a strong negation and $I : [0,1]^2 \to [0,1]$ an implication function. An n-ary d-Choquet integral $C_{\mu,N,I,\max}$ given as in Theorem 3.20 is idempotent for any fuzzy measure μ on $[n]$ if and only if $I(x,0) = N(x)$ for all $x \in [0,1]$.*

Proof. The proof follows from the observation:

$$\delta(0,x) = \max\Big\{N\left(I(0,x)\right), N\left(I(x,0)\right)\Big\} = N\left(I(x,0)\right).$$

$\square$

Now observe that a d-Choquet integral $C_{\mu,\delta}$ is self-dual whenever the fuzzy measure μ is self-dual, i.e. $\mu\left([n] \setminus A\right) = 1 - \mu(A)$ for all $A \subseteq [n]$, and the restricted dissimilarity function δ satisfies certain conditions.

Theorem 3.35. *Let n be a positive integer. An n-ary d-Choquet integral $C_{\mu,\delta}$ given by Definition 3.1 is self-dual if the fuzzy measure μ on $[n]$ is self-dual and the restricted dissimilarity function δ satisfies $\delta(1-x,1-y) = \delta(x,y)$ for all $x,y \in [0,1]$ and $\delta(0,x_1)+\delta(x_1,x_2)+\ldots+\delta(x_n,1) = 1$ for all $x_1,\ldots,x_n \in [0,1]$ such that $x_1 \le \ldots \le x_n$.*

Proof. The proof for $n=1$ is obvious. Now let $n \ge 2$ and let σ be a permutation on $[n]$ with $x_{\sigma(1)} \le \ldots \le x_{\sigma(n)}$, with the convention $x_{\sigma(0)} = 0$, $x_{\sigma(n+1)} = 1$, $A_{\sigma(i)} := \{\sigma(i),\ldots,\sigma(n)\}$ and $A_{\sigma(n+1)} = \emptyset$. Consider $y_i = 1-x_i$, $\sigma'(i) = \sigma(n+1-i)$ and $B_{\sigma'(i)} := \{\sigma'(i),\ldots,\sigma'(n)\}$ for each $i \in [n]$. Observe that, $y_{\sigma'(i)} = 1-x_{\sigma(n+1-i)}$ and therefore $y_{\sigma'(1)} \le \ldots \le y_{\sigma'(n)}$. So, one has that $B_{\sigma'(i)} = \{\sigma(1),\ldots,\sigma(n+1-i)\} = [n] \setminus A_{\sigma(n+2-i)}$. In addition, consider that, by convention, $y_{\sigma'(0)} = 0$ or, equivalently, that $x_{\sigma(n+1)} = 1$. Then we have

$$
\begin{aligned}
C_{\mu,\delta}(1-x_1,\ldots,1-x_n) &= C_{\mu,\delta}(y_1,\ldots,y_n)\\
&= \sum_{i=1}^{n} \delta(y_{\sigma'(i)}, y_{\sigma'(i-1)})\mu\left(B_{\sigma'(i)}\right)\\
&= \sum_{i=1}^{n} \delta(1 - x_{\sigma(n+1-i)}, 1 - x_{\sigma(n+2-i)})\mu\left([n] \setminus A_{\sigma(n+2-i)}\right)\\
&= \delta\left(1 - x_{\sigma(n)}, 1 - x_{\sigma(n+1)}\right)\mu\left([n] \setminus A_{\sigma(n+1)}\right) + \delta\left(1 - x_{\sigma(n-1)}, 1 - x_{\sigma(n)}\right)\mu\left([n] \setminus A_{\sigma(n)}\right) + \ldots\\
&\quad + \delta\left(1 - x_{\sigma(2)}, 1 - x_{\sigma(3)}\right)\mu\left([n] \setminus A_{\sigma(3)}\right) + \delta\left(1 - x_{\sigma(1)}, 1 - x_{\sigma(2)}\right)\mu\left([n] \setminus A_{\sigma(2)}\right)\\
&= \left(\delta\left(x_{\sigma(n)}, x_{\sigma(n+1)}\right)\right)(1 - \mu(\emptyset)) + \delta\left(x_{\sigma(n-1)}, x_{\sigma(n)}\right)\left(1 - \mu\left(A_{\sigma(n)}\right)\right) + \ldots\\
&\quad + \delta\left(x_{\sigma(2)}, x_{\sigma(3)}\right)\left(1 - \mu\left(A_{\sigma(3)}\right)\right) + \delta\left(x_{\sigma(1)}, x_{\sigma(2)}\right)\left(1 - \mu\left(A_{\sigma(2)}\right)\right)\\
&= \left(\delta\left(x_{\sigma(1)}, x_{\sigma(2)}\right) + \ldots + \delta\left(x_{\sigma(n)}, x_{\sigma(n+1)}\right)\right) - \left(\delta\left(x_{\sigma(n-1)}, x_{\sigma(n)}\right)\mu\left(A_{\sigma(n)}\right) + \ldots\right.\\
&\quad \left. + \delta\left(x_{\sigma(1)}, x_{\sigma(2)}\right)\mu\left(A_{\sigma(2)}\right)\right)\\
&= \left(\delta\left(x_{\sigma(0)}, x_{\sigma(1)}\right) + \delta\left(x_{\sigma(1)}, x_{\sigma(2)}\right) + \ldots + \delta\left(x_{\sigma(n)}, x_{\sigma(n+1)}\right)\right) - \left(\delta\left(x_{\sigma(n-1)}, x_{\sigma(n)}\right)\mu\left(A_{\sigma(n)}\right) + \ldots\right.\\
&\quad \left. + \delta\left(x_{\sigma(1)}, x_{\sigma(2)}\right)\mu\left(A_{\sigma(2)}\right) + \delta\left(x_{\sigma(0)}, x_{\sigma(1)}\right)\mu\left(A_{\sigma(1)}\right)\right)\\
&= 1 - C_{\mu,\delta}(x_1,\ldots,x_n).
\end{aligned}
$$

$\square$

Corollary 3.36. *Let n be a positive integer and δ be a restricted dissimilarity function such that f_δ is injective and self-dual. Then C_{μ,δ^*} is self-dual if the fuzzy measure μ on $[n]$ is self-dual.*

Proof. Since f_δ is injective then, by Theorem 3.8, δ^* is a restricted dissimilarity function. Consider $x, y \in [0, 1]$. Then, since f_δ is self-dual, $\delta^*(x, y) = |f_\delta(x) - f_\delta(y)| = |f_\delta(1 - x) - f_\delta(1 - y)|\delta^*(1 - x, 1 - y)$ and

$$\delta^*(0, x_1) + \delta^*(x_1, x_2) + \ldots + \delta^*(x_n, 1) = \delta(x_1, 0) + \delta(x_2, 0) - \delta(x_1, 0) + \ldots + \delta(1, 0) - \delta(x_n, 0) = \delta(1, 0) = 1,$$

for all $x_1, \ldots, x_n \in [0, 1]$ such that $x_1 \leq \ldots \leq x_n$. Therefore, from Theorem 3.35, C_{μ, δ^*} is self-dual. $\square$

Corollary 3.37. *Let n be a positive integer and $\varphi_1, \varphi_2 : [0, 1] \to [0, 1]$ be automorphisms of $[0, 1]$. An n-ary d-Choquet integral $C_{\mu, \varphi_1, \varphi_2}$ is self-dual if the fuzzy measure μ on $[n]$ satisfies $\mu([n] \setminus A) = 1 - \mu(A)$ for all $A \subseteq [n]$, $\varphi_1 = Id$ and*

$$\varphi_2(x) = \begin{cases} \varphi(x), & \text{if } x \in [0, 1/2]; \\ 1 - \varphi(1 - x), & \text{if } x \in]1/2, 1], \end{cases}$$

where φ is an automorphism of $[0, 1/2]$.

Proof. It is easy to check that $\varphi_1 = Id$ implies $\delta(0, x_1) + \delta(x_1, x_2) + \ldots + \delta(x_n, 1) = 1$ for all $0 \leq x_1 \leq \ldots \leq x_n \leq 1$. Moreover, since

- $\varphi_2(1/2) = 1/2$;
- if $x \in [0, 1/2[$, then $\varphi_2(x) = \varphi(x)$ and $\varphi_2(1 - x) = 1 - \varphi(x)$;
- if $x \in]1/2, 1]$, then $\varphi_2(x) = 1 - \varphi(1 - x)$ and $\varphi_2(1 - x) = \varphi(1 - x)$,

we have $\varphi_2(1 - x) = 1 - \varphi_2(x)$ for all $x \in [0, 1]$. Then

$$\delta(1 - x, 1 - y) = \varphi_1^{-1}\Big(|\varphi_2(1 - x) - \varphi_2(1 - y)|\Big) = |1 - \varphi_2(x) - 1 + \varphi_2(y)| = \delta(x, y).$$

$\square$

Remark 3.38. Note that from the geometrical point of view, the point $(1/2, 1/2)$ is the center of symmetry of the function φ_2 in Corollary 3.37. Moreover, φ_2 is a generator of the standard negation, that is

$$\varphi_2^{-1}\Big(1 - \varphi_2(x)\Big) = 1 - x.$$

Lemma 3.39. *Let $n \geq 2$ be an integer, μ be a fuzzy measure on $[n]$ and δ be a restricted dissimilarity function. If $C_{\mu, \delta}$ given by Definition 3.1 is shift-invariant then $\delta(x + y, 0) = y + \delta(x, 0)$, for each $x, y \in [0, 1]$ such that $x + y \in [0, 1]$.*

Proof. Consider $x, y \in [0, 1]$ such that $x + y \in [0, 1]$. Then one has that

$$C_{\mu, \delta}(x + y, \ldots, x + y) = \delta(x + y, 0)\mu\left(A_{\sigma(1)}\right) + \sum_{i=2}^n \delta(x + y, x + y)\mu\left(A_{\sigma(i)}\right) = \delta(x + y, 0)$$

and, similarly, $C_{\mu, \delta}(x, \ldots, x) = \delta(x, 0)$. Then, since $C_{\mu, \delta}$ is an n-ary shift-invariant d-Choquet integral, it follows that:

$$\delta(x + y, 0) = C_{\mu, \delta}(x + y, \ldots, x + y) = y + C_{\mu, \delta}(x, \ldots, x) = y + \delta(x, 0).$$

$\square$

In the following theorem, the shift-invariancy of the d-Choquet integral is studied and demanded properties of the restricted dissimilarity measure δ are stated.

Theorem 3.40. *Consider the fuzzy measure $\mu_\perp$ defined by Equation (5). Let $n \geq 2$ be an integer, μ be a fuzzy measure on $[n]$ such that $\mu \neq \mu_\perp$ and δ be a restricted dissimilarity function. The n-ary d-Choquet integral $C_{\mu, \delta}$ given by Definition 3.1 is shift-invariant if and only if*

1. $\delta(x_1 + y, x_2 + y) = \delta(x_1, x_2)$, *for all* $x_1, x_2, y \in [0, 1]$ *such that* $x_1 + y, x_2 + y \in [0, 1]$;
2. $\delta(x, 0) = x$, *for all* $x \in [0, 1]$.

Proof. First observe that any permutation σ on $[n]$ which order the vector $(x_1, \ldots, x_n)$ also order the vector $(x_1 + y, \ldots, x_n + y)$.

$\Leftarrow$ Let δ satisfy the two properties and μ be a fuzzy measure on $[n]$ such that $\mu \neq \mu_\perp$. Then it follows that

$$
\begin{aligned}
C_{\mu,\delta}(x_1 + y, \ldots, x_n + y) &= \delta(x_{\sigma(1)} + y, 0)\mu\left(A_{\sigma(1)}\right) + \sum_{i=2}^{n} \delta(x_{\sigma(i)} + y, x_{\sigma(i-1)} + y)\mu\left(A_{\sigma(i)}\right) \\
&= \left(y + \delta(x_{\sigma(1)}, 0)\right)\mu\left(A_{\sigma(1)}\right) + \sum_{i=2}^{n} \delta(x_{\sigma(i)}, x_{\sigma(i-1)})\mu\left(A_{\sigma(i)}\right) \\
&= y + \delta(x_{\sigma(1)}, 0) + \sum_{i=2}^{n} \delta(x_{\sigma(i)}, x_{\sigma(i-1)})\mu\left(A_{\sigma(i)}\right) \\
&= y + \sum_{i=1}^{n} \delta(x_{\sigma(i)}, x_{\sigma(i-1)})\mu\left(A_{\sigma(i)}\right) \\
&= y + C_{\mu,\delta}(x_1, \ldots, x_n),
\end{aligned}
$$

and, thus, $C_{\mu,\delta}$ is shift-invariant.

$\Rightarrow$ From Lemma 3.39, since $C_{\mu,\delta}$ is shift-invariant, then, for each $x \in [0, 1]$, one has that $\delta(0 + x, 0) = x + \delta(0, 0) = x$. Now, consider $x_1, x_2, y \in [0, 1]$ such that $x_1 + y, x_2 + y \in [0, 1]$. Without loss of generality, since δ is commutative, we can assume that $x_1 \leq x_2$. Then, one has that

$$
\begin{aligned}
&C_{\mu,\delta}(x_2 + y, \ldots, x_2 + y, x_1 + y, x_2 + y, \ldots, x_2 + y) \\
&= \delta(x_1 + y, 0)\mu\left(A_{\sigma(1)}\right) + \delta(x_1 + y, x_2 + y)\mu\left(A_{\sigma(2)}\right) + \sum_{i=3}^{n} \delta(x_2 + y, x_2 + y)\mu\left(A_{\sigma(i)}\right) \\
&= \delta(x_1 + y, 0) + \delta(x_1 + y, x_2 + y)\mu\left(A_{\sigma(2)}\right) \\
&= y + \delta(x_1, 0) + \delta(x_1 + y, x_2 + y)\mu\left(A_{\sigma(2)}\right) \text{ by Lemma 3.39}
\end{aligned}
$$

and, similarly,

$$
C_{\mu,\delta}(x_2, \ldots, x_2, x_1, x_2 \ldots, x_2) = \delta(x_1, 0) + \delta(x_1, x_2)\mu\left(A_{\sigma(2)}\right). \tag{7}
$$

Since $\mu \neq \mu_\perp$, then there exists $i_0 \in [n]$ such that $\mu([n] - \{i_0\}) > 0$. So, consider $z_i = x_2$ if $i \neq i_0$ and $z_{i_0} = x_1$. Then any permutation σ on $[n]$ ordering $(z_1, \ldots, z_n)$ (and therefore also $(z_1 + y, \ldots, z_n + y)$) is such that $\sigma(1) = i_0$, which implies that $A_{\sigma(2)} = [n] - \{i_0\}$, that is $\mu(A_{\sigma(2)}) > 0$. Then, since $C_{\mu,\delta}$ is shift-invariant, it follows that:

$$
\begin{aligned}
\delta(x_1 + y, x_2 + y) &= \delta(z_{\sigma(1)} + y, z_{\sigma(2)} + y) \\
&= \frac{C_{\mu,\delta}(z_1 + y, \ldots, z_n + y) - \delta(z_{\sigma(1)} + y, 0)}{\mu\left(A_{\sigma(2)}\right)} \\
&= \frac{y + C_{\mu,\delta}(z_1, \ldots, z_2) - y - \delta(z_{\sigma(1)}, 0)}{\mu\left(A_{\sigma(2)}\right)} \text{ by Lemma 3.39} \\
&= \frac{\delta(z_{\sigma(1)}, 0) + \delta(z_{\sigma(1)}, z_{\sigma(2)})\mu\left(A_{\sigma(2)}\right) - \delta(z_{\sigma(1)}, 0)}{\mu\left(A_{\sigma(2)}\right)} \text{ by Eq. (7)} \\
&= \delta(z_{\sigma(1)}, z_{\sigma(2)}) \\
&= \delta(x_1, x_2).
\end{aligned}
$$

$\square$

Corollary 3.41. *Let $n \geq 2$ be an integer, μ be a fuzzy measure on $[n]$ and δ be a restricted equivalence function. The n-ary d-Choquet integral $C_{\mu,\delta}$ given by Definition 3.1 is shift-invariant if and only if $C_{\mu,\delta} = C_\mu$.*

Proof. It follows that:

$\Leftarrow$ Straightforward, since each Choquet integral is shift-invariant for any fuzzy measure μ.

$\Rightarrow$ Let δ be a restricted equivalence function and μ be a fuzzy measure on $[n]$ such that $C_{\mu,\delta}$ is an n-ary shift-invariant d-Choquet integral. If $\mu = \mu_\perp$ then, by Lemma 3.39, for each $x_1, \ldots, x_n \in [0,1]$, it holds that $\delta(x_{\sigma(1)}, 0) = x_{\sigma(1)} + \delta(0,0) = x_{\sigma(1)}$. Thus, we have that

$$C_{\mu_\perp,\delta}(x_1, \ldots, x_n) = \delta(x_{\sigma(1)}, 0) = x_{\sigma(1)} = C_{\mu_\perp}(x_1, \ldots, x_n).$$

Therefore, $C_{\mu,\delta} = C_\mu$. Now, if $\mu \neq \mu_\perp$ then, by Theorem 3.40, for each $x_1, x_2, y \in [0,1]$ such that $x_1 + y, x_2 + y \in [0,1]$, we have that $\delta(x_1, 0) = x_1$ and $\delta(x_1 + y, x_2 + y) = \delta(x_1, x_2)$. So, taking $x_2 = 0$, we obtain $\delta(x_1 + y, y) = \delta(x_1, 0) = x_1$, for all $x_1, y \in [0,1]$ with $x_1 + y \leq 1$. Hence, one has that $\delta(x, y) = x - y$, for all $x, y \in [0,1]$ with $x \geq y$. Similarly, it can be shown that $\delta(x, y) = y - x$, for all $x, y \in [0,1]$ with $x \leq y$, and, thus $\delta(x, y) = |x - y|$, for all $x, y \in [0,1]$. Therefore, $C_{\mu,\delta} = C_\mu$. $\square$

Remark 3.42. (i) According to Corollary 3.41, the situation with shift-invariancy is the same as with the comonotone additivity. The only shift-invariant d-Choquet integral $C_{\mu,\delta}$ is the "standard" Choquet integral, i.e. the d-Choquet integral w.r.t. the restricted dissimilarity function $\delta(x, y) = |x - y|$.

(ii) With respect to automorphisms, the only shift-invariant d-Choquet integral $C_{\mu,\varphi_1,\varphi_2}$ is $C_{\mu,Id,Id}$. This can be shown by the following: for $x_1 \geq x_2$ the condition $\delta(x_1 + y, x_2 + y) = \delta(x_1, x_2)$ is equivalent with

$$\varphi_1^{-1}\Big(\varphi_2(x_1 + y) - \varphi_2(x_2 + y)\Big) = \varphi_1^{-1}\Big(\varphi_2(x_1) - \varphi_2(x_2)\Big) \tag{8}$$

which is equivalent with $\varphi_2(x_1 + y) - \varphi_2(x_2 + y) = \varphi_2(x_1) - \varphi_2(x_2)$. For $x_2 = 0$ we obtain $\varphi_2(x_1 + y) = \varphi_2(x_1) + \varphi_2(y)$, thus Equation (8) holds if and only if φ_2 is additive, i.e., $\varphi_2 = Id$. The case $x_1 \leq x_2$ is similar. Now observe that $\delta(x + y, 0) = y + \delta(x, 0)$ is equivalent with

$$\varphi_1^{-1}\Big(\varphi_2(x + y)\Big) = y + \varphi_1^{-1}\Big(\varphi_2(x)\Big). \tag{9}$$

Setting $x = 0$ we obtain $\varphi_1^{-1}\Big(\varphi_2(y)\Big) = y$, hence, Equation (9) holds if and only if $\varphi_1 = \varphi_2$. Note that the assertions also straightforwardly follows from Corollary 3.41.

The homogeneity of a d-Choquet integral is strongly related to the homogeneity of the relevant restricted dissimilarity function.

Theorem 3.43. *Let n be a positive integer. An n-ary d-Choquet integral $C_{\mu,\delta}$ given by Definition 3.1 is positively homogeneous for any fuzzy measure μ on $[n]$ if and only if the restricted dissimilarity function δ is positively homogeneous.*

Proof. The proof is straightforward. $\square$

Corollary 3.44. *Let n be a positive integer and δ be a restricted dissimilarity function. If δ is positively homogeneous then $C_{\mu,\delta^*} = C_\mu$, and, therefore, it is positively homogeneous for any fuzzy measure μ on $[n]$.*

Proof. Immediate, since whenever δ is positively homogeneous then $f_\delta = Id$. $\square$

Lemma 3.45. *Let φ be an automorphism of $[0,1]$. Then the following assertions are equivalent:*

(i) $\varphi(cx) = \varphi(c)\varphi(x)$ *for all* $c, x \in [0,1]$;

(ii) $\varphi^{-1}(\varphi(cx) - \varphi(cy)) = c\varphi^{-1}(\varphi(x) - \varphi(y))$ *for all* $c, x, y \in [0,1]$ *where* $x \geq y$.

Proof. (i) $\Rightarrow$ (ii) Since (i) is Cauchy equation and its unique solutions are $\varphi(x) = x^a$ where $a \in]0, \infty[$, we have:

$$\varphi^{-1}\left(\varphi(cx) - \varphi(cy)\right) = ((cx)^a - (cy)^a)^{\frac{1}{a}} = c\left(x^a - y^a\right)^{\frac{1}{a}} c\varphi^{-1}\left(\varphi(x) - \varphi(y)\right).$$

(ii) $\Rightarrow$ (i) Let (ii) holds and let us consider a fixed c. Then

$$\varphi\left(\frac{1}{c}\varphi^{-1}\left(\varphi(cx) - \varphi(cy)\right)\right) = \varphi(x) - \varphi(y),$$

setting $u = cx$ and $v = cy$ we obtain

$$\varphi\left(\frac{1}{c}\varphi^{-1}\left(\varphi(u) - \varphi(v)\right)\right) = \varphi\left(\frac{1}{c}u\right) - \varphi\left(\frac{1}{c}v\right)$$

and setting $t = \varphi(u)$, $r = \varphi(v)$ we have

$$\varphi\left(\frac{1}{c}\varphi^{-1}(t - r)\right) = \varphi\left(\frac{1}{c}\varphi^{-1}(t)\right) - \varphi\left(\frac{1}{c}\varphi^{-1}(r)\right).$$

Now let as denote $g_c(z) = \varphi\left(\frac{1}{c}\varphi^{-1}(z)\right)$. Hence,

$$g_c(t - r) = g_c(t) - g_c(r)$$

and denoting $a = t - r$ we obtain

$$g_c(a) + g_c(r) = g_c(a + r).$$

Since the last formula is Cauchy equation, we have $g_c(a) = p_c a$ for all $a \in [0, 1]$ and from $g_c(\varphi(cx)) = \varphi(x)$ it follows $\varphi(x) = p_c\varphi(cx)$. For $a = 1$ we receive $p_c = g_c(1) = \varphi\left(\frac{1}{c}\right)$, thus $\varphi(x) = \varphi\left(\frac{1}{c}\right)\varphi(cx)$. Finally, setting $r = \frac{1}{c}$ and $s = cx$, it follows $\varphi(rs) = \varphi(r)\varphi(s)$ and the proof is complete. $\qquad\square$

Corollary 3.46. *Let n be a positive integer. An n-ary d-Choquet integral $C_{\mu,\varphi_1,\varphi_2}$ is positively homogeneous for any fuzzy measure μ on $[n]$ if and only if $\varphi_1 = \varphi_2$ and $\varphi_2(\lambda x) = \varphi_2(\lambda)\varphi_2(x)$ for all $\lambda, x \in [0, 1]$.*

Proof. The homogeneity of δ, i.e.,

$$\delta(\lambda x, \lambda y) = \lambda\delta(x, y)$$

is, for $x \geq y$, equivalent to

$$\varphi_1^{-1}\left(\varphi_2(\lambda x) - \varphi_2(\lambda y)\right) = \lambda\varphi_1^{-1}\left(\varphi_2(x) - \varphi_2(y)\right)$$

Setting $x = 1$ and $y = 0$ we obtain

$$\varphi_1^{-1}\left(\varphi_2(\lambda)\right) = \lambda\varphi_1^{-1}\left(1\right) = \lambda,$$

for all $\lambda \in [0, 1]$, hence $\varphi_1 = \varphi_2$. Moreover, by Lemma 3.45, it follows that

$$\varphi_2^{-1}\left(\varphi_2(\lambda x) - \varphi_2(\lambda y)\right) = \lambda\varphi_2^{-1}\left(\varphi_2(x) - \varphi_2(y)\right)$$

is equivalent to $\varphi_2(\lambda x) = \varphi_2(\lambda)\varphi_2(x)$ for all $\lambda, x \in [0, 1]$.

The proof for $x \leq y$ is similar. $\qquad\square$

Remark 3.47. (i) Note that $\varphi_2(\lambda x) = \varphi_2(\lambda)\varphi_2(x)$ is a Cauchy equation with the only solutions $\varphi_2(x) = x^p$ for some $p > 0$. Then $\delta_p(x, y) = |x^p - y^p|^{\frac{1}{p}}$. Observe that, in particular, δ_p satisfies (P1) if and only if $p \leq 1$.

(ii) From Corollary 3.46 and Corollary 3.33 it follows that a d-Choquet integral $C_{\mu,\varphi_1,\varphi_2}$ is idempotent whenever it is positively homogeneous.

(iii) The only d-Choquet integral $C_{\mu,\delta}$ (or $C_{\mu,\varphi_1,\varphi_2}$) which is idempotent, comonotone additive, self-dual, shift-invariant and positively homogeneous is the one for $\delta(x, y) = |x - y|$ (or $C_{\mu,Id,Id}$) where the fuzzy measure μ satisfies $\mu([n] \setminus A) = 1 - \mu(A)$ for all $A \subseteq [n]$, that is, "standard" Choquet integral w.r.t. a fuzzy measure μ being self-dual.

Example 3.48. (i) Let us consider the restricted dissimilarity function $\delta(x,y) = \left(\sqrt{x} - \sqrt{y}\right)^2$. Since δ can be obtained in terms of automorphisms $\varphi_1(x) = \varphi_2(x) = \sqrt{x}$, it is easy to check that the d-Choquet integral $C_{\mu,\sqrt{x},\sqrt{x}}$ is idempotent and positively homogeneous for any fuzzy measure μ.

(ii) Now let $\mu(A) = \frac{|A|}{n}$ for all $A \subseteq [n]$, $\varphi_1 = Id$ and

$$\varphi_2(x) = \begin{cases} 2x^2, & \text{if } x \in [0, 1/2]; \\ -2x^2 + 4x - 1, & \text{if } x \in]1/2, 1]. \end{cases}$$

Then C_{μ,Id,φ_2} is self-dual. For instance,

$$C_{\mu,Id,\varphi_2}(0.9, 0.2, 0.4) = 0.46$$

and

$$C_{\mu,Id,\varphi_2}(0.1, 0.8, 0.6) = 0.54.$$

(iii) Let us consider the smallest fuzzy measure $\mu_\perp$ given by $\mu_\perp([n]) = 1$ and $\mu_\perp(A) = 0$ for any $A \subset [n]$. Then

$$C_{\mu_\perp,\delta}(x_1, \ldots, x_n) = \delta(\min\{x_1, \ldots, x_n\}, 0)$$

for any restricted dissimilarity function δ.

If we take the biggest fuzzy measure $\mu^\top$ given by $\mu^\top(\emptyset) = 0$ and $\mu^\top(A) = 1$ for any non-empty subset A of $[n]$, we have that

$$C_{\mu^\top,\delta}(x_1, \ldots, x_n) = \sum \delta(x_{\sigma(i)}, x_{\sigma(i-1)})$$

for any restricted dissimilarity function δ.

(iv) For $n = 3$, let μ_{med} be the fuzzy measure defined by $\mu_{med}(E) = 0$ if the cardinal of E is smaller than or equal to 1, and $\mu_{med}(E) = 1$ otherwise. Then

$$C_{\mu_{med},\delta}(x_1, x_2, x_3) = \delta(\min\{x_1, x_2, x_3\}, 0) + \delta(med(x_1, x_2, x_3), \min\{x_1, x_2, x_3\})$$

for any restricted dissimilarity function δ, where med denotes the median of the considered inputs.

4. Averageness and monotonicity of d-Choquet integrals

In this section we show that unlike "standard" Choquet integrals, the d-Choquet integrals are neither averaging nor monotone in general. We also deal with directional monotonicity and study conditions under which the d-Choquet integrals are pre-aggregation functions.

4.1. Averageness

Theorem 4.1. *Let n be a positive integer, μ be a fuzzy measure on $[n]$, δ be a restricted dissimilarity function and $C_{\mu,\delta}$ be the n-ary d-Choquet integral with respect to μ and δ. Then the following assertions are equivalent:*

(i) There exist $x_1, \ldots, x_n \in [0, 1]$ such that $C_{\mu,\delta}(x_1, \ldots, x_n) < \min\{x_1, \ldots, x_n\}$.
(ii) There exists $x \in [0, 1]$ such that $\delta(x, 0) < x$.

Proof. (ii) $\Rightarrow$ (i) Let $x \in [0, 1]$ be such that $\delta(x, 0) < x$. Then $C_{\mu,\delta}(x, \ldots, x) = \delta(x, 0) < x$.

(i) $\Rightarrow$ (ii) Let $x_1, \ldots, x_n \in [0, 1]$ be such that $C_{\mu,\delta}(x_1, \ldots, x_n) < \min\{x_1, \ldots, x_n\}$. The proof follows from the observations that

$$C_{\mu,\delta}(x_1, \ldots, x_n) = \delta(x_{\sigma(1)}, 0) + \sum_{i=2}^{n} \delta(x_{\sigma(i)}, x_{\sigma(i-1)})\mu\left(A_{\sigma(i)}\right) \geq \delta(x_{\sigma(1)}, 0)$$

and

$$C_{\mu,\delta}(x_1, \ldots, x_n) < x_{\sigma(1)}.$$

$\square$

Corollary 4.2. *Let n be a positive integer, μ be a fuzzy measure on $[n]$ and let $\delta : [0,1]^2 \to [0,1]$ be a restricted dissimilarity function given in terms of automorphisms φ_1, φ_2 as in Proposition 3.11. Let $C_{\mu,\varphi_1,\varphi_2}$ be the n-ary d-Choquet integral with respect to μ and δ. Then the following assertions are equivalent:*

(i) There exist $x_1, \ldots, x_n \in [0,1]$ such that $C_{\mu,\varphi_1,\varphi_2}(x_1, \ldots, x_n) < \min\{x_1, \ldots, x_n\}$.
(ii) There exists $x \in [0,1]$ such that $\varphi_1(x) > \varphi_2(x)$.

Proof. The proof follows from Theorem 4.4, the equality $\delta(x,0) = \varphi_1^{-1}(\varphi_2(x))$ and the fact that $\varphi_1^{-1}(\varphi_2(x)) < x$ if and only if $\varphi_1(x) > \varphi_2(x)$. $\qquad\square$

Theorem 4.3. *Let n be a positive integer, μ be a fuzzy measure on $[n]$, δ be a restricted dissimilarity function and $C_{\mu,\delta}$ be the n-ary d-Choquet integral with respect to μ and δ. Then the property:*

(i) There exist $x_1, \ldots, x_n \in [0,1]$ such that $C_{\mu,\delta}(x_1, \ldots, x_n) > \max\{x_1, \ldots, x_n\}$.

holds whenever the following property holds:

(ii) There exists $x \in [0,1]$ such that $\delta(x,0) > x$.

Moreover, if $C_{\mu,\delta}(x_1, \ldots, x_n) \leq C_{\mu,\delta}(y_1, \ldots, y_n)$ whenever $x_1 \leq y_1, \ldots, x_n \leq y_n$, then (i) and (ii) are equivalent.

Proof. (ii) $\Rightarrow$ (i) Let $x \in [0,1]$ be such that $\delta(x,0) > x$. Then $C_{\mu,\delta}(x, \ldots, x) = \delta(x,0) > x$.
(i) $\Rightarrow$ (ii) Let $x_1, \ldots, x_n \in [0,1]$ be such that $C_{\mu,\delta}(x_1, \ldots, x_n) > \max\{x_1, \ldots, x_n\}$. The proof follows from the observations that

$$x_{\sigma(n)} < C_{\mu,\delta}(x_1, \ldots, x_n) \leq C_{\mu,\delta}(x_{\sigma(n)}, \ldots, x_{\sigma(n)}) = \delta(x_{\sigma(n)}, 0).$$

$\qquad\square$

Corollary 4.4. *Let n be a positive integer, μ be a fuzzy measure on $[n]$ and let $\delta : [0,1]^2 \to [0,1]$ be a restricted dissimilarity function given in terms of automorphisms φ_1, φ_2 as in Proposition 3.11. Let $C_{\mu,\varphi_1,\varphi_2}$ be the n-ary d-Choquet integral with respect to μ and δ. Then the property:*

(i) There exist $x_1, \ldots, x_n \in [0,1]$ such that $C_{\mu,\varphi_1,\varphi_2}(x_1, \ldots, x_n) > \max\{x_1, \ldots, x_n\}$.

holds whenever the following property holds:

(ii) There exists $x \in [0,1]$ such that $\varphi_1(x) < \varphi_2(x)$.

Moreover, if $C_{\mu,\delta}(x_1, \ldots, x_n) \leq C_{\mu,\delta}(y_1, \ldots, y_n)$ whenever $x_1 \leq y_1, \ldots, x_n \leq y_n$, then (i) and (ii) are equivalent.

Proof. The proof follows from Theorem 4.1, the equality $\delta(x,0) = \varphi_1^{-1}(\varphi_2(x))$ and the fact that $\varphi_1^{-1}(\varphi_2(x)) > x$ if and only if $\varphi_1(x) < \varphi_2(x)$. $\qquad\square$

Having these results in hands we can state the sufficient and necessary condition under which a d-Choquet integral is averaging.

Theorem 4.5. *Let n be a positive integer, μ be a fuzzy measure on $[n]$, δ be a restricted dissimilarity function and $C_{\mu,\delta}$ be the n-ary d-Choquet integral with respect to μ and δ such that $C_{\mu,\delta}(x_1, \ldots, x_n) \leq C_{\mu,\delta}(y_1, \ldots, y_n)$ whenever $x_1 \leq y_1, \ldots, x_n \leq y_n$. Then the following assertions are equivalent:*

(i) $\min\{x_1, \ldots, x_n\} \leq C_{\mu,\delta}(x_1, \ldots, x_n) \leq \max\{x_1, \ldots, x_n\}$ for all $x_1, \ldots, x_n \in [0,1]$.
(ii) $\delta(x,0) = x$ for all $x \in [0,1]$.

Proof. Immediately follows from Theorem 4.1 and Theorem 4.3. $\qquad\square$

Corollary 4.6. *Let n be a positive integer, μ be a fuzzy measure on $[n]$ and let $\delta : [0,1]^2 \to [0,1]$ be a restricted dissimilarity function. Then $\min \leq C_{\mu,\delta} \leq \max$ if and only if $C_{\mu,\delta^*} = C_\mu$.*

Proof. Straightforward from Theorem 4.5 and the fact that $\delta(x,0) = x$ for all $x \in [0,1]$ if and only if $\delta^*(x,y) = |x-y|$. $\square$

Theorem 4.7. *Let n be a positive integer, μ be a fuzzy measure on $[n]$ and let $\delta : [0,1]^2 \to [0,1]$ be a restricted dissimilarity function given in terms of automorphisms φ_1, φ_2 as in Proposition 3.11. Let $C_{\mu,\varphi_1,\varphi_2}$ be the n-ary d-Choquet integral with respect to μ and δ such that $C_{\mu,\varphi_1,\varphi_2}(x_1,\ldots,x_n) \leq C_{\mu,\varphi_1,\varphi_2}(y_1,\ldots,y_n)$ whenever $x_1 \leq y_1, \ldots, x_n \leq y_n$. Then the following assertions are equivalent:*

(i) $\min\{x_1,\ldots,x_n\} \leq C_{\mu,\varphi_1,\varphi_2}(x_1,\ldots,x_n) \leq \max\{x_1,\ldots,x_n\}$ *for all* $x_1,\ldots,x_n \in [0,1]$.
(ii) $\varphi_1(x) = \varphi_2(x)$ *for all* $x \in [0,1]$.

Proof. Immediately follows from Corollary 4.2 and Corollary 4.4. $\square$

Note that in this case $C_{\mu,\varphi_1,\varphi_2}$ can be represented as a φ_1-transform of the standard Choquet integral C_μ.

4.2. Monotonicity

Now we give the sufficient and necessary condition under which a d-Choquet integral is increasing. Note that, since, by Remark 3.7, the boundary conditions are satisfied, any increasing d-Choquet integral is an aggregation function.

Theorem 4.8. *Let n be a positive integer, δ be a restricted dissimilarity function and $C_{\mu,\delta}$ be an n-ary d-Choquet integral with respect to μ and δ. Then the following assertions are equivalent:*

(i) For any fuzzy measure μ on $[n]$, $C_{\mu,\delta}(x_1,\ldots,x_n) \leq C_{\mu,\delta}(y_1,\ldots,y_n)$ whenever $x_1 \leq \ldots \leq x_n$, $y_1 \leq \ldots \leq y_n$, $x_1 \leq y_1,\ldots,x_n \leq y_n$.
(ii) $\delta(0,x_1) + \delta(x_1,x_2) + \ldots + \delta(x_{m-1},x_m) \leq \delta(0,y_1) + \delta(y_1,y_2) + \ldots + \delta(y_{m-1},y_m)$ for all $m \in [n]$ and $x_1,\ldots,x_m,y_1,\ldots,y_m \in [0,1]$ where $x_1 \leq \ldots \leq x_m$, $y_1 \leq \ldots \leq y_m$, $x_1 \leq y_1,\ldots,\leq x_m \leq y_m$.

Proof. (ii) $\Rightarrow$ (i) Let $x_1,\ldots,x_n,y_1,\ldots,y_n \in [0,1]$ be such that $x_1 \leq \ldots \leq x_n$, $y_1 \leq \ldots \leq y_n$, $x_1 \leq y_1,\ldots,x_n \leq y_n$. For simplicity let us denote $\mu(\{k,k+1,\ldots,n\})$ by μ_k for all $k \in [n]$. From (ii) it follows $(\mu_1 - \mu_2)\delta(0,x_1) \leq (\mu_1 - \mu_2)\delta(0,y_1)$ and $(\mu_2 - \mu_3)(\delta(0,x_1) + \delta(x_1,x_2)) \leq (\mu_2 - \mu_3)(\delta(0,y_1) + \delta(y_1,y_2))$, hence

$$(\mu_1 - \mu_3)\delta(0,x_1) + (\mu_2 - \mu_3)\delta(x_1,x_2) \leq (\mu_1 - \mu_3)\delta(0,y_1) + (\mu_2 - \mu_3)\delta(y_1,y_2).$$

Since $(\mu_3 - \mu_4)(\delta(0,x_1) + \delta(x_1,x_2) + \delta(x_2,x_3)) \leq (\mu_3 - \mu_4)(\delta(0,y_1) + \delta(y_1,y_2) + \delta(y_2,y_3))$, we have

$$(\mu_1-\mu_4)\delta(0,x_1)+(\mu_2-\mu_4)\delta(x_1,x_2)+(\mu_3-\mu_4)\delta(x_2,x_3) \leq (\mu_1-\mu_4)\delta(0,y_1)+(\mu_2-\mu_4)\delta(y_1,y_2)+(\mu_3-\mu_4)\delta(y_2,y_3).$$

Repeating a similar procedure we obtain

$$(\mu_1 - \mu_n)\delta(0,x_1) + (\mu_2 - \mu_n)\delta(x_1,x_2) + \ldots + (\mu_{n-1} - \mu_n)\delta(x_{n-2},x_{n-1})$$
$$\leq \quad (\mu_1 - \mu_n)\delta(0,y_1) + (\mu_2 - \mu_n)\delta(y_1,y_2) + \ldots + (\mu_{n-1} - \mu_n)\delta(y_{n-2},y_{n-1})$$

and, finally, from $\mu_n(\delta(0,x_1) + \delta(x_1,x_2) + \ldots + \delta(x_{n-1},x_n)) \leq \mu_n(\delta(0,y_1) + \delta(y_1,y_2) + \ldots + \delta(y_{n-1},y_n))$, it follows that

$$\mu_1\delta(0,x_1) + \mu_2\delta(x_1,x_2) + \ldots + \mu_n\delta(x_{n-1},x_n) \leq \mu_1\delta(0,y_1) + \mu_2\delta(y_1,y_2) + \ldots + \mu_n\delta(y_{n-1},y_n), \quad (10)$$

i.e., $C_{\mu,\delta}(x_1,\ldots,x_n) \leq C_{\mu,\delta}(y_1,\ldots,y_n)$.

(i) $\Rightarrow$ (ii) Now let $x_1,\ldots,x_m,y_1,\ldots,y_m \in [0,1]$ be such that $x_1 \leq \ldots \leq x_m$, $y_1 \leq \ldots \leq y_m$, $x_1 \leq y_1,\ldots,\leq x_m \leq y_m$. Then $C_{\mu,\delta}(x_1,\ldots,x_n) \leq C_{\mu,\delta}(y_1,\ldots,y_n)$, i.e., Equation (10) holds, and taking the fuzzy measure μ^m, for any $m \in [n]$, given by

$$\mu^m(A) = \begin{cases} 1, & \text{if } |A| \geq n - m + 1; \\ 0, & \text{otherwise}, \end{cases}$$

we have that $\mu^m([n]_k) = 1$ for each $k \leq m$ and $\mu^m([n]_k) = 0$ for each $k > m$, with $[n]_k = \{k, \ldots, n\}$. So, it holds that

$$
\begin{aligned}
C_{\mu^m,\delta}(x_1, \ldots, x_n) &= \delta(0, x_1)\mu^m([n]_1) + \delta(x_1, x_2)\mu^m([n]_2) + \ldots + \delta(x_{n-1}, x_n)\mu^m([n]_n) \\
&= \delta(0, x_1) + \delta(x_1, x_2) + \ldots + \delta(x_{m-1}, x_m)
\end{aligned}
$$

and, analogously, we have that $C_{\mu^m,\delta}(y_1, \ldots, y_n) = \delta(0, y_1) + \delta(y_1, y_2) + \ldots + \delta(y_{m-1}, y_m)$. Therefore, it holds that $\delta(0, x_1) + \delta(x_1, x_2) + \ldots + \delta(x_{m-1}, x_m) \leq \delta(0, y_1) + \delta(y_1, y_2) + \ldots + \delta(y_{m-1}, y_m)$, for all $m \in [n]$. □

Corollary 4.9. *Let n be a positive integer, δ be a restricted dissimilarity function and $C_{\mu,\delta}$ be an n-ary d-Choquet integral with respect to μ and δ. If for all $m \in [n]$ there exists an increasing function $f_m : [0,1] \to [0,1]$ such that $\delta(0, x_1) + \delta(x_1, x_2) + \ldots + \delta(x_{m-1}, x_m) = f_m(x_m)$ for all $x_1, \ldots, x_m \in [0,1]$ where $x_1 \leq \ldots \leq x_m$, then for any fuzzy measure μ on $[n]$, $C_{\mu,\delta}(x_1, \ldots, x_n) \leq C_{\mu,\delta}(y_1, \ldots, y_n)$ whenever $x_1 \leq y_1, \ldots, x_n \leq y_n$.*

Proof. The proof directly follows from Theorem 4.8. □

Corollary 4.10. *Let n be a positive integer, $\delta : [0,1]^2 \to [0,1]$ be a restricted dissimilarity function given in terms of automorphisms φ_1, φ_2 as in Proposition 3.11. Let $C_{\mu,\varphi_1,\varphi_2}$ be an n-ary d-Choquet integral with respect to μ and δ. If $\varphi_1 = Id$, then for any fuzzy measure μ on $[n]$, $C_{\mu,\delta}(x_1, \ldots, x_n) \leq C_{\mu,\delta}(y_1, \ldots, y_n)$ whenever $x_1 \leq y_1, \ldots, x_n \leq y_n$.*

Proof. The proof directly follows from Corollary 4.9 taking $f = \varphi_2$. □

Remark 4.11. It is worth pointing out that taking $\varphi_1 = Id$, as in Corollary 4.10, we obtain:

$$
C_{\mu,Id,\varphi_2}(x_1, \ldots, x_n) = \sum_{i=1}^n \left(\varphi_2\left(x_{\sigma(i)}\right) - \varphi_2\left(x_{\sigma(i-1)}\right) \right) \mu\left(A_{\sigma(i)}\right) = C_\mu(\varphi_2(x_1), \ldots, \varphi_2(x_n)),
$$

that is, C_{μ,Id,φ_2} is fully determined by a "standard" Choquet integral C_μ, which also means that C_{μ,Id,φ_2} is an aggregation function (note that also $(P1)$ is satisfied according to Proposition 3.12).

Theorem 4.12. *Let n be a positive integer and consider the fuzzy measure $\mu_\perp$ defined in Equation (5). Then, for any restricted dissimilarity function δ, the n-ary d-Choquet integral $C_{\mu_\perp,\delta}$ satisfies $C_{\mu_\perp,\delta}(x_1, \ldots, x_n) \leq C_{\mu_\perp,\delta}(y_1, \ldots, y_n)$ whenever $x_1 \leq y_1, \ldots, x_n \leq y_n$.*

Proof. The proof immediately follows from the monotonicity of δ. □

Example 4.13. We give examples of d-Choquet integrals that are not averaging or/and increasing, i.e. these functions are not averaging aggregation functions (hence, they are not "standard" Choquet integrals) and the ones that are not increasing (hence they are even not aggregation functions). Let $C_{\mu,\delta}$ be an n-ary d-Choquet integral given by Definition 3.1 with respect to the fuzzy measure μ and the restricted dissimilarity function:

- $\delta(x, y) = (x - y)^2$, then $C_{\mu,\delta}$ is neither averaging (it is not above minimum) nor increasing for some μ;

- $\delta(x, y) = \sqrt{|x - y|}$, then $C_{\mu,\delta}$ is neither averaging (it is not under maximum) nor increasing for some μ, note that in this case, in contrast to all the other cases of this example, the range of $C_{\mu,\delta}$ is not a subset of $[0, 1]$;

- $\delta(x, y) = |\sqrt{x} - \sqrt{y}|$, then $C_{\mu,\delta}$ is not averaging (it is not under maximum) for some μ, but it is increasing for any μ;

- $\delta(x, y) = |x^2 - y^2|$, then $C_{\mu,\delta}$ is not averaging for some μ (it is not above minimum), but it is increasing for any μ;

- $\delta(x, y) = (\sqrt{x} - \sqrt{y})^2$, then $C_{\mu,\delta}$ is averaging for any μ, but it is not increasing for some μ;

- δ given as in Example 3.48 (ii), then $C_{\mu,\delta}$ is not averaging for some μ (it is neither above minimum nor under maximum), but it is increasing for any μ. Note that in this case δ can be expressed:

$$
\delta(x,y) = \begin{cases}
2|x^2 - y^2|, & \text{if } x, y \in \left[0, \frac{1}{2}\right]; \\
2|2x - x^2 - 2y + y^2|, & \text{if } x, y \in \left]\frac{1}{2}, 1\right]; \\
-2y^2 + 4y - 1 - 2x^2, & \text{if } x \in \left[0, \frac{1}{2}\right], \ y \in \left]\frac{1}{2}, 1\right]; \\
-2x^2 + 4x - 1 - 2y^2, & \text{if } x \in \left]\frac{1}{2}, 1\right], \ y \in \left[0, \frac{1}{2}\right].
\end{cases}
$$

4.3. Directional monotonicity

From the results of the previous subsection it is clear that, in general, an n-ary d-Choquet integral is not $\vec{r}$-increasing for a vector $\vec{r} = (r_1, \ldots, r_n)$ such that there exists $k \in \{1, \ldots, n\}$ such that $r_i \neq 0$ if and only if $i = k$. Now we study the situation for the vector $\vec{r} = (1, \ldots, 1)$.

Theorem 4.14. *Let n be a positive integer and $C_{\mu,\delta} : [0,1]^n \to [0, n]$ be an n-ary d-Choquet integral with respect to a fuzzy measure μ and a restricted dissimilarity function δ. Then*

(i) $C_{\mu,\delta}$ is $\vec{1}$-increasing (i.e., weakly increasing in the sense of [38])for any fuzzy measure μ whenever

$$\delta(x + c, y + c) \geq \delta(x, y)$$

for all $x, y, c \in [0, 1]$ such that $x + c, y + c \in [0, 1]$;

(ii) $C_{\mu,\delta}$ is $\vec{1}$-increasing for any fuzzy measure μ whenever for all $m \in [n]$ there exists an increasing function $f_m : [0, 1] \to [0, 1]$ such that

$$\delta(0, x_1) + \delta(x_1, x_2) + \ldots + \delta(x_{m-1}, x_m) = f_m(x_m)$$

for all $x_1, \ldots, x_m \in [0, 1]$ where $x_1 \leq \ldots \leq x_m$.

Proof. (i)

$$
\begin{aligned}
& C_{\mu,\delta}(x_1 + c, \ldots, x_n + c) \\
= \ & \delta\left(x_{\sigma(1)} + c, 0\right) + \sum_{i=2}^{n} \delta\left(x_{\sigma(i)} + c, x_{\sigma(i-1)} + c\right) \mu\left(A_{\sigma(i)}\right) \\
\geq \ & \delta\left(x_{\sigma(1)}, 0\right) + \sum_{i=2}^{n} \delta\left(x_{\sigma(i)}, x_{\sigma(i-1)}\right) \mu\left(A_{\sigma(i)}\right) \\
= \ & C_{\mu,\delta}(x_1, \ldots, x_n).
\end{aligned}
$$

(ii) Directly follows from Corollary 4.9. $\qquad\square$

Corollary 4.15. *Let n be a positive integer, $\delta : [0,1]^2 \to [0,1]$ be a restricted dissimilarity function given in terms of automorphisms φ_1, φ_2 as in Proposition 3.11. Let $C_{\mu,\varphi_1,\varphi_2} : [0,1]^n \to [0, n]$ be an n-ary d-Choquet integral with respect to μ and δ. Then $C_{\mu,\delta}$ is $\vec{1}$-increasing for any fuzzy measure μ whenever at least one of the following conditions is satisfied:*

(i) φ_2 is convex;

(ii) $\varphi_1 = Id$.

Proof. (i) Let $x, y, c \in [0, 1]$ be such that $x + c, y + c \in [0, 1]$ and $x > y$ (the proof for $x < y$ is similar). Then, since φ_2 is convex, we have $\varphi_2(x + c) - \varphi_2(x) \geq \varphi_2(y + c) - \varphi_2(y)$, hence

$$\delta(x + c, y + c) = \varphi_1^{-1}\left(\varphi_2(x + c) - \varphi_2(y + c)\right) \geq \varphi_1^{-1}\left(\varphi_2(x) - \varphi_2(y)\right) = \delta(x, y)$$

and finally, by Theorem 4.14 (i) it follows that $C_{\mu,\delta}$ is $\vec{1}$-increasing for any fuzzy measure μ.

(ii) Directly follows from Corollary 4.10. □

Now we give the conditions under which an n-ary d-Choquet integral is $\vec{1}$-pre-aggregation function.

Corollary 4.16. *Let n be a positive integer, $\delta : [0, 1]^2 \to [0, 1]$ be a restricted dissimilarity function given in terms of automorphisms φ_1, φ_2 as in Proposition 3.11. Then the n-ary d-Choquet integral with respect to μ and δ is a $\vec{1}$-pre-aggregation function for any fuzzy measure μ on $[n]$ whenever at least one of the following conditions is satisfied:*

(i) $\varphi_1 = Id$;

(ii) $\varphi_1 > Id$ and φ_2 is convex.

Proof. Immediately follows from Corollary 4.15 and Proposition 3.12. □

Example 4.17. We give examples of d-Choquet integrals that are (or are not) aggregation functions/$\vec{1}$-pre-aggregation functions. Let $C_{\mu,\delta}$ be an n-ary d-Choquet integral given by Definition 3.1 with respect to the fuzzy measure μ and the restricted dissimilarity function:

- $\delta(x, y) = (x - y)^2$, then $C_{\mu,\delta}$ is a $\vec{1}$-pre-aggregation function for any μ, however it is not an aggregation function for some μ (since it is not increasing);

- $\delta(x, y) = \sqrt{|x - y|}$, then $C_{\mu,\delta}$ is $\vec{1}$-increasing for any μ, however it is neither a $\vec{1}$-pre-aggregation function nor an aggregation function for some μ (since its range is not a subset of $[0, 1]$) and it is not increasing for some μ;

- $\delta(x, y) = |\sqrt{x} - \sqrt{y}|$, then $C_{\mu,\delta}$ is an aggregation function for any μ (hence also a $\vec{1}$-pre-aggregation function);

- $\delta(x, y) = |x^2 - y^2|$, then $C_{\mu,\delta}$ is an aggregation function for any μ (hence also a $\vec{1}$-pre-aggregation function);

- $\delta(x, y) = \sqrt{|\sqrt{x} - \sqrt{y}|}$, then $C_{\mu,\delta}$ is not $\vec{1}$-increasing for some μ (hence it is not increasing as well) and its range is not a subset of $[0, 1]$ for some μ.

The fact that the last one, namely $C_{\mu,\delta}$, is not $\vec{1}$-increasing for some μ follows from the following counterexample: let $n = 6$, $c = 0.1$ and $\mu(A) = 1$ for all nonempty $A \subseteq [n]$, then

$$C_{\mu,\delta}(0.1, 0.2, 0.3, 0.4, 0.5, 0.6) \doteq 0.562341 \geq 0.555946 \doteq C_{\mu,\delta}(0.2, 0.3, 0.4, 0.5, 0.6, 0.7).$$

Moreover, for $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$ we have:

$$\sqrt{\sqrt{x_1} - \sqrt{0}} + \sqrt{\sqrt{x_2} - \sqrt{x_1}} + \sqrt{\sqrt{x_3} - \sqrt{x_2}} \doteq 1.24129 > 1,$$

hence the condition $(P1)$ is not satisfied, so the range of $C_{\mu,\delta}$ is not a subset of $[0, 1]$ for some fuzzy measures μ.

5. Conclusions

We have proposed a generalization of the Choquet integral in terms of replacing the standard difference by a restricted dissimilarity function. This approach results in a class of functions, d-Choquet integrals, that encompasses the class of all "standard" Choquet integrals, but is much wider and, based on the choice of a restricted dissimilarity function, the d-Choquet integral satisfies or does not satisfy the characteristic properties of the "standard" Choquet integrals such as increasingness, pre-increasingness, range in $[0, 1]$, comonotone additivity, idempotency, self-duality, shift-invariancy and positive homogeneity.

Note that this class of functions can be useful in all those applications where fuzzy integrals, and specially Choquet integrals, have shown themselves valuable, ranging from image processing to decision making or classification. In particular, these new functions can be used to replace the standard Choquet integral where comparison between inputs using the usual difference is not possible either because it is difficult to define it properly, as in the case of intervals.

In this sense, in our future work we intend to make a research of possibilities to apply our results in image processing, multi-criteria decision making and classification problems, and, specially, to use them to extend to the interval-valued setting the notion of Choquet integral in such a way that classical fuzzy algorithms which make use of the Choquet integral can be appropriately defined.

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