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Trend analysis in L-fuzzy contexts with absent values

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Abstract

Sometimes we have to work with L-fuzzy context sequences where one or more values are missing. These sequences can represent, among other things, the evolution in time of an L-fuzzy context. The studies of tendencies that we have done so far used tools that are not valid when the L-fuzzy context has unknown values. In this work we address such situations and we propose new methods to tackle the problem. Besides, we use the study of tendencies to analyse relations between the objects and the attributes of L-fuzzy contexts and to replace the absent values taking into account the behaviour of the sequence.

Keywords: Trend analysis, L-fuzzy context, absent values, L-fuzzy concept, L-fuzzy context sequence.

1 Introduction

L-fuzzy Concept Analysis [8, 10, 20, 27, 34] is a mathematical tool for analysing data and formally representing conceptual knowledge. It extracts information from an L-fuzzy context by means of L-fuzzy concepts and can be seen as an extension of Formal Concept Analysis [24, 40]. This theory has been applied in image processing [7, 3].

There exist several relations between the same sets of objects and attributes. Such relations lead to the notion of L-fuzzy context sequence. In the particular case in which this sequence recovers the evolution in time of an L-fuzzy context, we try to predict future trends from the analysis of past behaviours. Besides, we have sometimes missing values in the sequence and, hence, the L-Fuzzy Concept Analysis for the treatment of incomplete information ([9, 8]) in the L-fuzzy context has been developed.

Furthermore, the study of the evolution of the relationship between objects and attributes by means of the search of trends is of special interest. Several works analysing the course of time in a Formal context can be found in the literature as, for instance, in [35, 41, 42].

In particular, in [41, 42] K.E. Wolff defines Temporal Concept Analysis where a Conceptual Time System is introduced such that the state and phase spaces are defined as concept lattices which represent the meaning of the states with respect to the chosen time description. Besides, hidden evolution trends are defined in [35, 39], using temporal matching in the case of Formal Concept Analysis.

The existence of Triadic contexts [29] gives us the possibility of using ternary relations to represent time. However, this approach is very demanding for our interest and for this reason it is only developed for formal contexts.

Trend analysis often refers to techniques for extracting an underlying pattern of behaviour in statistics. In this paper, we show a new and different method for L-fuzzy contexts with absent values that allows the detection of some regularities. This method establishes trends which can be used as a base for making decisions. In the paper, the behaviour of the observed data is described by the model and statements about tendencies are made. This is one of the main contributions of this work.

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Furthermore, these tendencies are used to analyse strong relations between the elements of the L-fuzzy contexts (objects and attributes) and, finally, to replace the absent values. Thus, we propose an algorithm for the replacement of absent values, which is necessary in order to study the L-fuzzy contexts by means of L-fuzzy concepts.

Additionally, we illustrate a possible application of both the theoretical developments and the proposed algorithm to replace absent values in a sport article selling company's data. The lattice L in this particular example is the set of closed intervals within $[0, 1]$, which have been proven to be appropriate to deal with uncertainty [37, 19, 18].

The paper is organized as follows: Section 2 provides a background about L-fuzzy Concept Analysis and L-fuzzy context sequences. Section 3 sets up a general study of Temporal Tendencies in the L-fuzzy context sequences. In the last part, we define Trend and Persistent Degrees and we study relations between objects and attributes by means of Trend and Persistent Matrices. In Section 4 we analyse L-fuzzy context sequences with absent values and propose an algorithm for replacing them. Then, we present an example of an application of the carried out trend analysis and absent-value replacement algorithm in Section 5. Finally, conclusions and future work are detailed.

2 Preliminaries

2.1 L-fuzzy concept analysis

The Formal Concept Analysis of Wille [24, 40] extracts information from a binary table that represents a formal context (X, Y, R) with X and Y finite sets of objects and attributes, respectively, and $R \subseteq X \times Y$. The hidden information consists of pairs (A, B), with $A \subseteq X$ and $B \subseteq Y$, called formal concepts. A formal concept can be interpreted as a group of objects A that shares the same set of attributes B.

In previous works [13, 14], we have defined L-fuzzy contexts (L, X, Y, R), where L is a complete lattice, X and Y are sets of objects and attributes, respectively, and $R \in L^{X\times Y}$ is an L-fuzzy relation between the objects and the attributes. This is an extension of Wille's Formal contexts when we want to study the relations between the objects and the attributes with values in a complete lattice L, instead of binary values.

We have defined the derivation operators 1 and 2 to work with these L-fuzzy contexts, given by means of the following expressions.

For all $A \in L^X$ and $B \in L^Y$, $A_1(y) = \inf_{x\in X}\{I(A(x), R(x, y))\}$ and $B_2(x) = \inf_{y\in Y}\{I(B(y), R(x, y))\}$, with I a L-fuzzy implication operator defined in the lattice $(L, \leq)$.

Although any L-fuzzy implication operator can be used to define the derivation operators, in this paper we work with residuated implications. Other authors have also used a residuated implication operator in their definitions of derivation operators [12, 32, 33].

The information stored in the context is visualized by means of L-fuzzy concepts, which are pairs $(M, M_1) \in L^X \times L^Y$ with $M \in fix(\varphi)$, where $fix(\varphi)$ is the set of fixed points of the operator φ, being defined from the derivation operators 1 and 2 as $\varphi(M) = (M_1)_2 = M_{12}$. These pairs, whose first and second components are said to be the L-fuzzy extension and intension respectively, represent a group of objects that share a group of attributes in a fuzzy way.

Using the usual order between L-fuzzy sets, that is, for all $M, N \in L^X$,

$$M \leq N \iff M(x) \leq N(x) \quad \text{for all } x \in X,$$

we define the set $\mathcal{L} = \{(M, M_1) \mid M \in fix(\varphi)\}$ with the order relation $\preceq$ given by the following expression

$$\text{for all } (M, M_1), (N, N_1) \in \mathcal{L},\ (M, M_1) \preceq (N, N_1) \text{ if } M \leq N.$$

Remark 1. *As is well known, it is easy to prove that $M \leq N$ is equivalent to $N_1 \leq M_1$.*

As φ is an order preserving operator, by the theorem of Tarski [38], the set $fix(\varphi)$ is a complete lattice and then $(\mathcal{L}, \preceq)$ is also a complete lattice that is said to be the L-fuzzy concept lattice associated with the L-fuzzy context (L, X, Y, R) [13, 14].

Besides, given $A \in L^X$ (or $B \in L^Y$), we can obtain the associated L-fuzzy concept applying the derivation operators twice. In the case of using a residuated implication, as we do in this work, the associated L-fuzzy concept is (A_{12}, A_1) (or (B_2, B_{21})).

We have also studied extensions of Formal Concept Analysis to the interval-valued case in [17], defining an interval-valued L-fuzzy context as $(\mathcal{J}[L], X, Y, R)$, with R an interval-valued L-Fuzzy relation between X and Y. For more insight into the topic, see [9, 22, 23].

Other important results about this theory can be found in [11, 15, 16, 25, 28, 31].

2.2 L-fuzzy context sequences

A first study of L-fuzzy context sequences when $L = [0,1]$ is tackled in [6]. We begin by recalling the main definition.

Definition 2.1. *An L-fuzzy context sequence is a sequence of tuples (L, X, Y, R_i), $i \in \{1, \dots, n\}$, $n \in \mathbb{N}$, with L a complete lattice, X and Y sets of objects and attributes respectively and $R_i \in L^{X \times Y}$, for all $i \in \{1, \dots, n\}$, a family of L-fuzzy relations between X and Y.*

In [2], a general study of these L-fuzzy context sequences using $n-$ary OWA operators of Lizasoain and Moreno ([30]) is developed. These operators are a generalization of the OWA operators of Yager.

Definition 2.2. [43] *A mapping $F : L^n \longrightarrow L$, where $L = [0,1]$ is called an OWA operator of dimension n if associated with F is a weighting $n-$tuple $W = (w_1, w_2, \dots, w_n)$ such that $w_i \in$[0,1] and $\sum_{1 \le i \le n} w_i = 1$, where $F(a_1, a_2, \dots, a_n) = w_1.b_1 + w_2.b_2 + \cdots + w_n.b_n$, with b_i the ith largest element in the collection $a_1, a_2, \dots, a_n$.*

However, Yager's OWA operators are not easy to be extended to any complete lattice L. The main difficulty is that Yager's construction is based on a previous arrangement of the real values to aggregate, which is not always possible in a partially ordered set. In order to overcome this problem Lizasoain and Moreno [30] built an ordered vector for each given vector in the lattice. This construction allows to define the n-ary OWA operator on any complete lattice which has Yager's OWA operator as a particular case.
Their contribution involves the construction, for each vector $(a_1, \dots, a_n) \in L^n$, of a totally ordered vector $(b_1, \dots, b_n)$ as shown in the following proposition.

Proposition 2.3. *Let $(L, \le_L)$ be a complete lattice. For any $(a_1, a_2, \dots, a_n) \in L^n$, consider the values*

- $b_1 = a_1 \vee \cdots \vee a_n \in L$
- $b_2 = [(a_1 \wedge a_2) \vee \cdots \vee (a_1 \wedge a_n)] \vee [(a_2 \wedge a_3) \vee \cdots \vee (a_2 \wedge a_n)] \vee \cdots \vee [a_{n-1} \wedge a_n] \in L$

  $\vdots$
- $b_k = \bigvee \{a_{j_1} \wedge \cdots \wedge a_{j_k} | \{j_1, \dots, j_k\} \subseteq \{1, \dots, n\}\} \in L$

  $\vdots$
- $b_n = a_1 \wedge \cdots \wedge a_n \in L$

Then $a_1 \wedge \cdots \wedge a_n = b_n \le_L b_{n-1} \le \cdots \le_L b_1 = a_1 \vee \cdots \vee a_n$.
Moreover, if the set $\{a_1, \dots, a_n\}$ is totally ordered, then the vector $(b_1, \dots, b_n)$ agrees with $(a_{\sigma(1)}, \dots, a_{\sigma(n)})$ for some permutation σ of $\{1, \dots, n\}$.

On the other hand, it is very easy to see that if $\{a_1, \dots, a_n\}$ is a chain, then b_k is the $k-$th order statistic.
This proposition allows us to generalize Yager's $n-$ary OWA operators from $[0,1]$ to any complete lattice. To do this, Lizasoain and Moreno give the following definition.

Definition 2.4. *Let $(L, \le_L, T, S)$ be a complete lattice endowed with a t-norm T and a t-conorm S. We will say that $(\alpha_1, \alpha_2, \dots, \alpha_n) \in L^n$ is a*

(i) weighting vector in $(L, \le_L, T, S)$ if $S(\alpha_1, \dots, \alpha_n) = 1_L$ and

(ii) distributive weighting vector in $(L, \le_L, T, S)$ if it also satisfies that $a = T(a, S(\alpha_1, \dots, \alpha_n)) = S(T(a, \alpha_1), \dots T(a, \alpha_n))$ for any $a \in L$.

Definition 2.5. *Let $(\alpha_1, \dots, \alpha_n) \in L^n$ be a distributive weighting vector in $(L, \le_L, T, S)$. For each $(a_1, \dots, a_n) \in L^n$, call $(b_1, \dots, b_n)$ the totally ordered vector constructed in Proposition 2.3. The function $F_\alpha : L^n \longrightarrow L$ given by*

$$F_\alpha(a_1, \dots, a_n) = S(T(\alpha_1, b_1), \dots, T(\alpha_n, b_n)),$$

$(a_1, \dots, a_n) \in L^n$, is called $n-$ary OWA operator.

Other important papers about OWA operators and measures are [36, 21, 26].
In order to summarize the information stored in the L-fuzzy context sequence, in [2] we defined aggregated relations using the totally ordered vectors defined by Lizasoain and Moreno.

Definition 2.6. *Let $(L, \leq_L, T, S)$ be a complete lattice endowed with a t-norm T and a t-conorm S. Let $(L, X, Y, R_i), i \in \{1, \ldots, n\}$, be an L-fuzzy context sequence, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ a distributive weighting vector and F_α the $n-$ary OWA operator associated with α. We can define an L-fuzzy relation R_{F_α} that aggregates the information of the different L-fuzzy contexts by means of this expression.*

For all $x \in X, y \in Y$,

$$R_{F_\alpha}(x, y) = F_\alpha(R_1(x, y), R_2(x, y), \ldots, R_n(x, y)) = S(T(\alpha_1, b_1(x, y)), T(\alpha_2, b_2(x, y)), \ldots, T(\alpha_n, b_n(x, y)),$$

with $(b_1(x, y), b_2(x, y), \ldots, b_n(x, y))$ the totally ordered vector constructed for $(R_1(x, y), R_2(x, y), \ldots, R_n(x, y))$.

The general study of L-fuzzy context sequences developed in the previous section uses different n-ary OWA operators to aggregate values. However, such a study may not allow one to make an analysis of their evolution in time.

To accomplish this analysis, in [4] we gave the following definition that tries to provide a value to estimate the relation between each object and each attribute at an instant h.

Definition 2.7. *In the complete lattice $(L, \leq_L, T, S)$ endowed with the t-norm T and the t-conorm S, let us consider the L-fuzzy context sequence (L, X, Y, R_i), $i \in \{1, \ldots, n\}$. Fixed $h \in \mathbb{N}, h \leq n$, let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a distributive weighting vector with $k = n - h + 1$ and F_α the $k-$ary OWA operator associated with α. We can define an L-fuzzy relation $R^h_{F_\alpha}$ that aggregates the information of the different L-fuzzy contexts by means of the following expression.*

$$R^h_{F_\alpha}(x, y) = F_\alpha(R_h(x, y), R_{h+1}(x, y), \ldots, R_n(x, y)) = S(T(\alpha_1, b^h_1(x, y)), T(\alpha_2, b^h_2(x, y)), \ldots, T(\alpha_k, b^h_k(x, y)), \quad \textit{for all } x \in X, y \in Y,$$

where $(b^h_1(x, y), b^h_2(x, y), \ldots, b^h_k(x, y))$ is the totally ordered vector for $(R_h(x, y),\ R_{h+1}(x, y), \ldots, R_n(x, y))$.

3 Temporal tendencies

In [4] we analysed temporal trends to identify the evolution in time of the L-fuzzy context sequence $(L, X, Y, R_i), i \in \{1, \ldots, n\}$, when L is a complete lattice. Our interest was in the study of the evolution of the relationship between the objects (or attributes) with respect to one or several attributes (or objects).

In the paper, we used residuated implication operators in the calculation of the L-fuzzy concepts associated with certain objects or attributes.

In [6] we performed a preliminary study in [0,1] and, in [4], we extended and deepened the results to any complete lattice L. In this case, we have to take into account that, except for a complete chain, the elements of the lattice L are not necessarily comparable. In both papers, the presence of all the values of the L-fuzzy context was needed.

We begin by showing in the next subsection the main results of the discussed works to address the problem of the absence of values.

3.1 Trend and persistent objects and attributes

The best way to study the evolution in time of an object or attribute is the study of their associated L-fuzzy concepts in the different L-fuzzy contexts of the sequence. This is the idea for the definitions and results of this subsection proved in [4].

Definition 3.1. *Consider $x_0 \in X, y_0 \in Y$. Let $(A_{i\{x_0\}}, B_{i\{x_0\}})$ and $(A_{i\{y_0\}}, B_{i\{y_0\}})$ be the L-fuzzy concepts associated with the crisp singletons $\{x_0\}$ and $\{y_0\}$ in the L-fuzzy context sequence (L, X, Y, R_i) with $i \leq n$.*

Then

(i) $Trend(x_0) = \{y \in Y \mid B_{i\{x_0\}}(y) \leq B_{i+1\{x_0\}}(y),$ for all $i < n\}$ is the attribute set whose membership degrees in the different intensions of the L-fuzzy concepts $(A_{i\{x_0\}}, B_{i\{x_0\}})$ are non decreasing.

(ii) $Trend(y_0) = \{x \in X \mid A_{i\{y_0\}}(x) \leq A_{i+1\{y_0\}}(x),$ for all $i < n\}$ is the object set whose membership degrees in the different extensions of the L-fuzzy concepts $(A_{i\{y_0\}}, B_{i\{y_0\}})$ are non decreasing.

We can say that they are the attributes that are more and more related to object x_0 and the objects more and more related to attribute y_0.

This is a very demanding definition but it allows to establish trends with a high degree of fulfillment.

Moreover, the following result is proved.

Proposition 3.2. *Consider $x \in X, y \in Y$. We can prove that $y \in Trend(x) \iff x \in Trend(y)$.*

We can extend this definition to the case of more than one object or attribute.

Definition 3.3. *For every $Z, T \neq \emptyset, Z \subseteq X$ and $T \subseteq Y$.*

(i) We define $Trend(Z)$ as $Trend(Z) = \{y \in Y \ \mid B_{i\{x\}}(y) \leq B_{i+1\{x\}}(y),$ for all $i < n,$ for all $x \in Z\}$.

(ii) And $Trend(T)$ as $Trend(T) = \{x \in X \ \mid A_{i\{y\}}(x) \leq A_{i+1\{y\}}(x),$ for all $i < n,$ for all $y \in T\}$.

In this case, the following can also be proven.

Proposition 3.4. *For every $Z, T \neq \emptyset, Z \subseteq X$ and $T \subseteq Y$.*

(i) If $Trend(Z) = T$, then $Z \subseteq Trend(T)$.

(ii) If $Trend(T) = Z$, then $T \subseteq Trend(Z)$.

As a particular case, we have the sets $Trend(X)$ and $Trend(Y)$ where the following result is provided.

Remark 2. $Trend(X) = Y \iff Trend(Y) = X$.

As the definition of $Trend$ is very demanding, in [4] we defined $Persistent$ objects and attributes in order to relax the exigence level.

Definition 3.5. *Given $x_0 \in X, y_0 \in Y$. Let $(A_{i\{x_0\}}, B_{i\{x_0\}})$ and $(A_{i\{y_0\}}, B_{i\{y_0\}})$ be the L-fuzzy concepts associated with the crisp singletons $\{x_0\}$ and $\{y_0\}$, in the L-fuzzy context sequence (L, X, Y, R_i) with $i \leq n$.*

(i) $Persistent(x_0) = \{y \in Y \ \mid B_{i\{x_0\}}(y) \geq B_{1\{x_0\}}(y),$ for all $i, 1 < i \leq n\}$ is the set of attributes whose membership degrees in the L-fuzzy intensions of the L-fuzzy concepts $(A_{i\{x_0\}}, B_{i\{x_0\}})$ are bigger than or equal to the values of the L-fuzzy concept $(A_{1\{x_0\}}, B_{1\{x_0\}})$.

(ii) $Persistent(y_0) = \{x \in X \ \mid A_{i\{y_0\}}(x) \geq A_{1\{y_0\}}(x),$ for all $i, 1 < i \leq n\}$ is the set of objects whose membership degrees in the L-fuzzy extensions of the L-fuzzy concepts $(A_{i\{y_0\}}, B_{i\{y_0\}})$ are bigger than or equal to the values of the L-fuzzy concept $(A_{1\{y_0\}}, B_{1\{y_0\}})$.

Fixed $j \leq n$, an alternative definition of Persitent(x_o) and Persistent(y_o) can be given as follows.

$$Persistent_j(x_0) = \{y \in Y \ \mid B_{i\{x_0\}}(y) \geq B_{j\{x_0\}}(y), \text{ for all } i, j < i \leq n\}.$$

$$Persistent_j(y_0) = \{x \in X \ \mid A_{i\{y_0\}}(x) \geq A_{j\{y_0\}}(x), \text{ for all } i, j < i \leq n\}.$$

With this definition, similar results to those of Proposition 3.2 and 3.4 hold.

Proposition 3.6. *Consider $x \in X, y \in Y$. The result is verified, $y \in Persistent(x) \iff x \in Persistent(y)$.*

We can extend this definition to the case of more than one object or attribute.

Definition 3.7. *For every $Z, T \neq \emptyset, Z \subseteq X$ and $T \subseteq Y$.*

(i) We define $Persistent(Z)$ as $Persistent(Z) = \{y \in Y \ \mid B_{i\{x\}}(y) \geq B_{1\{x\}}(y),$ for all $i < n,$ for all $x \in Z\}$.

(ii) And $Persistent(T)$ as $Persistent(T) = \{x \in X \ \mid A_{i\{y\}}(x) \geq A_{1\{y\}}(x),$ for all $i < n,$ for all $y \in T\}$.

Proposition 3.8. *For all $Z, T \neq \emptyset, Z \subseteq X$ and $T \subseteq Y$.*

(i) If $Persistent(Z) = T$, then $Z \subseteq Persistent(T)$.

(ii) If $Persistent(T) = Z$, then $T \subseteq Persistent(Z)$.

The Trend and Persistent definitions set up pairs of objects and attributes that can be used for a more complete analysis of the evolution of the L-fuzzy sequence $(L, X, Y, R_i), i \in \{1, \ldots, n\}$.

Following this idea and Definition 3.1, the L-fuzzy context sequence tendencies studied can be completed with the construction of the binary Trend matrices.

Definition 3.9. *The Trend matrix $TM \subseteq X \times Y$ is defined as*

$$TM(x, y) = \begin{cases} 1 & \text{if } y \in Trend(x) (\text{equiv. } x \in Trend(y)) \\ 0 & \text{in other case.} \end{cases}$$

By Proposition 3.2, in order to obtain the Trend matrix, it is only necessary to compute $Trend(x)$, for all $x \in X$ or, similarly, $Trend(y)$, for all $y \in Y$.

We can consider now the formal context (X, Y, TM) and obtain its formal concepts to have a general view of the trends between the objects X and the attributes Y.

Definition 3.10. *Consider the formal context (X, Y, TM) with X set of objects, Y set of attributes and $TM \subseteq X \times Y$. The formal concepts of (X, Y, TM) are said to be the Trend formal concepts.*

It is also possible to do a different study using the definition of Persistent.

Definition 3.11. *The matrix $PM \subseteq X \times Y$ such that*

$$PM(x, y) = \begin{cases} 1 & \text{if } y \in Persistent(x) (x \in Persistent(y)) \\ 0 & \text{in other case.} \end{cases}$$

is said to be the Persistent Matrix.

We can now consider (X, Y, PM) and calculate their formal concepts to obtain information about the tendencies between the objects of X and the attributes of Y.

Definition 3.12. *Consider the formal context (X, Y, PM). The formal concepts of (X, Y, PM) are said to be the Persistent formal concepts.*

Moreover, as Persistent definition is less demanding than the Trend one, one can easily prove the following result.

Proposition 3.13. *If TM and PM are the Trend and Persistent matrices obtained from an L-fuzzy context sequence, then $TM \subseteq PM$ holds.*

If we denote by $\mathcal{L}(X, Y, TM)$ and $\mathcal{L}(X, Y, PM)$ the concept lattices of the formal contexts (X, Y, TM) and (X, Y, PM), respectively, then the following result is provided [24].

Proposition 3.14. *If $(A, B) \in \mathcal{L}(X, Y, TM)$ then there exists $(C, D) \in \mathcal{L}(X, Y, PM)$ such that $A \subseteq C$ and $B \subseteq D$.*

3.2 Trend and Persistent degrees

Tendency studies with absent values are not able to use the definitions given in the previous section. We need new tools for these situations. Thus, we propose the use of the Trend Degree and the Persistent Degree.

With the Trend Degree definition, we want to analyse the degree in which the values have increased in time. Moreover, we will use Persistent Degree when we want to study how the values have risen in relation to a starting point.

In both cases, we perform the study using the L-fuzzy concepts of the different L-fuzzy contexts.

Definition 3.15. *Consider $x_0 \in X, y_0 \in Y$. Let $(A_{i\{x_0\}}, B_{i\{x_0\}})$ and $(A_{i\{y_0\}}, B_{i\{y_0\}})$ be the L-fuzzy concepts associated with the crisp singletons $\{x_0\}$ and $\{y_0\}$ in the L-fuzzy context sequence (L, X, Y, R_i) with $i \leq n$.*

(i) Trend Degree (Td) of object x_0 for attribute y is the cardinality of the set of L-fuzzy concepts $(A_{i\{x_0\}}, B_{i\{x_0\}})$ whose intension membership degrees for attribute y are non decreasing.

$$Td(x_0)_y = Card(\{i < n \mid B_{i\{x_0\}}(y) \leq B_{i+1\{x_0\}}(y)\}).$$

(ii) Trend Degree (Td) of attribute y_0 for object x is the cardinality of the set of L-fuzzy concepts $(A_{i\{y_0\}}, B_{i\{y_0\}})$ whose extension membership degrees for object x are non decreasing.

$$Td(y_0)_x = Card(\{i < n \mid A_{i\{y_0\}}(x) \leq A_{i+1\{y_0\}}(x)\}).$$

Remark 3. *In the cases where is not possible to obtain the L-fuzzy concept $(A_{i+1\{x_0\}}, B_{i+1\{x_0\}})$, for any $i < n$, because there are missing elements in the relation, we take the minimum $j > i$ for which the L-fuzzy concept $(A_{j\{x_0\}}, B_{j\{x_0\}})$ exists.*

Analogously, we can compare the different L-fuzzy contexts with an initial one $(A_{k\{x_0\}}, B_{k\{x_0\}}), k \leq n$.

Definition 3.16. *Given $x_0 \in X, y_0 \in Y$. Let $(A_{i\{x_0\}}, B_{i\{x_0\}})$ and $(A_{i\{y_0\}}, B_{i\{y_0\}})$ be the L-fuzzy concepts associated with the crisp singletons $\{x_0\}$ and $\{y_0\}$, in the L-fuzzy context sequence (L, X, Y, R_i) with $i \leq n$. Consider $k \leq n$ the minimum value for which the L-fuzzy concept $(A_{k\{x_0\}}, B_{k\{x_0\}})$ exists.*

(i) Persistent Degree (Pd) of object x_0 for attribute y is the cardinality of the set of L-fuzzy concepts $(A_{i\{x_0\}}, B_{i\{x_0\}})$ whose intension membership degrees for attribute y are bigger than or equal to the values of the L-fuzzy concept $(A_{k\{x_0\}}, B_{k\{x_0\}})$.

$$Pd(x_0)_y = Card(\{i \mid k < i \leq n \, and \, B_{i\{x_0\}}(y) \geq B_{k\{x_0\}}(y)\}).$$

(ii) Persistent Degree (Pd) of attribute y_0 for object x is the cardinality of the set of L-fuzzy concepts $(A_{i\{y_0\}}, B_{i\{y_0\}})$ whose extension membership degrees for object x are bigger than or equal to the values of the L-fuzzy concept $(A_{k\{y_0\}}, B_{k\{y_0\}})$.

$$Pd(y_0)_x = Card(\{i \mid k < i \leq n \, and \, A_{i\{y_0\}}(x) \geq A_{k\{y_0\}}(x)\}).$$

With this definition, the following proposition is immediate.

Proposition 3.17. *(i) Consider $x \in X, y \in Y$. If $Td(x)_y$ exists then also does $Td(y)_x$ and $Td(x)_y = Td(y)_x$ holds.*

(ii) Analogously for $Pd(x)_y = Pd(y)_x$.

Using the previous definitions, at this point we show the definitions of Minimum and Average Trend Degree and Persistent Degree of an object or an attribute.

Definition 3.18. *Minimum Trend Degree (MinTd) and Average Trend Degree (AvTd) for $x_0 \in X$ are defined as*

(i) $MinTd(x_0) = \min_{y \in Y} Td(x_0)_y$.

(i) $AvTd(x_0) = \sum_{y \in Y} Td(x_0)_y / |Y|$.

The same can be defined for $y_0 \in Y$.

Definition 3.19. *Minimum Persistent Degree (MinPd) and Average Persistent Degree (AvPd) for $x_0 \in X$ are defined as*

(i) $MinPd(x_0) = \min_{y \in Y} Pd(x_0)_y$.

(ii) $AvPd(x_0) = \sum_{y \in Y} Pd(x_0)_y / |Y|$.

The same can be defined for $y_0 \in Y$.

$MinTd$, $MinPd$, $AvTd$ and $AvPd$ definitions allow to establish preorder relations in the sets of objects X and attributes Y of the L-fuzzy context.

Definition 3.20. *Given $x_i, x_j \in X$ or $y_i, y_j \in Y$, we define the relations $\leq_{MinTd}, \leq_{MinPd}, \leq_{AvTd}, \leq_{AvPd}$ as follows.*

(i) $x_i \leq_{MinTd} x_j$ *if* $MinTd(x_i) \leq MinTd(x_j)$.

(i') $y_i \leq_{MinTd} y_j$ *if* $MinTd(y_i) \leq MinTd(y_j)$.

(ii) $x_i \leq_{AvTd} x_j$ *if* $AvTd(x_i) \leq AvTd(x_j)$.

(ii') $y_i \leq_{AvTd} y_j$ *if* $AvTd(y_i) \leq AvTd(y_j)$.

(iii) $x_i \leq_{MinPd} x_j$ *if* $MinPd(x_i) \leq MinPd(x_j)$.

(iii') $y_i \leq_{MinPd} y_j$ *if* $MinPd(y_i) \leq MinPd(y_j)$.

(iv) $x_i \leq_{AvPd} x_j$ *if* $AvPd(x_i) \leq AvPd(x_j)$.

(iv') $y_i \leq_{AvPd} y_j$ *if* $AvPd(y_i) \leq AvPd(y_j)$.

Proposition 3.21. *The four relations $MinTd$, $MinPd$, $AvTd$ and $AvPd$ are preorder relations.*

Proof. They are defined by means of the order between integer numbers. □

Remark 4. *Note that they are not order relations because antisymmetry is not verified.*

We can define tendencies between objects and attributes with these relations. We say that x_i is trendier than x_j, according to $MinTd$, if $MinTd(x_i) \geq MinTd(x_j)$. The same holds for the other relations.

The definitions of Trend and Persistent Degrees can be extended to sets with more than one element.

Definition 3.22. *For every $Z, T \neq \emptyset, Z \subseteq X$ and $T \subseteq Y$.*

(i) We define Minimum Trend Degree (MinTd) of Z as $MinTd(Z) = \min_{x \in Z} MinTd(x)$.

(ii) And Minimum Trend Degree (MinTd) of T as $MinTd(T) = \min_{x \in T} MinTd(x)$.

Definition 3.23. *For every $Z, T \neq \emptyset, Z \subseteq X$ and $T \subseteq Y$.*

(i) We define Average Trend Degree (AvTd) of Z as $AvTd(Z) = \sum_{x \in Z} AvTd(x)/|Z|$.

(ii) And Average Trend Degree (AvTd) of T as $AvTd(T) = \sum_{x \in T} AvTd(x)/|T|$.

Using the four relations $MinTd$, $MinPd$, $AvTd$ and $AvPd$ defined previously, we can order the set of objects X and the set of attributes Y.

These rankings are denoted by Minimum Trend Degree Ranking ($MinTdR$), Average Trend Degree Ranking ($AvTdR$), Minimum Persistent Degree Ranking ($MinPdR$) and Average Persistent Degree Ranking ($AvPdR$).

They can be represented by directed graphs and can be applied both to absent and to existing values.

3.3 Tendency matrices

The defined rankings establish priorities among the objects and the attributes, but it is not possible to establish tendencies in the relationship among the elements of both sets. In this sense, the study of tendency matrices will allow to perform a more complete study of the L-fuzzy context sequence. Furthermore, one of the goals of this paper goals is to use them in order to replace absent values.

These are the definitions.

Definition 3.24. *The Trend Degree Matrix (TdM) is defined as $TdM(x, y) = Td(x)_y$, for all $x \in X, y \in Y$.*

Definition 3.25. *The matrix defined as $PdM(x, y) = Pd(x)_y$, for all $x \in X, y \in Y$ is said to be the Persistent Degree Matrix.*

Recall that $Td(x)_y = Td(y)_x$ and $Pd(x)_y = Pd(y)_x$, for all $x \in X$, $y \in Y$, by Proposition 3.17.

In order to have more homogeneous information, we can now normalize the matrices dividing by $n - 1$.

Definition 3.26. *The matrices defined as*

$$NTdM(x, y) = TdM(x, y)/(n-1) \text{ and } NPdM(x, y) = PdM(x, y)/(n-1)$$

for all $x \in X$, $y \in Y$, are said to be the Normalized Trend Degree Matrix (NTdM) and the Normalized Persistent Degree Matrix (NPdM).

Next result establishes a relationship between $NTdM$ and $NPdM$ matrices and the TM and PM ones defined in Definition 3.9 and 3.11.

Proposition 3.27. *Let $x \in X$ and $y \in Y$. If $NTdM(x, y) = 1$ then $NPdM(x, y) = 1$. Moreover, when $NTdM(x, y) = 1$ the equalities $NTdM(x, y) = TM(x, y)$ and $NPdM(x, y) = PM(x, y)$ hold.*

Proof. Given $x \in X$ and $y \in Y$, if $NTdM(x, y) = 1$ then $TdM(x, y) = n - 1$. So, we have all the values for x and y in the L-fuzzy concepts of the sequence and also $PdM(x, y) = n - 1$. Finally, $NPdM(x, y) = 1$.

The last part of the proof is immediate. □

We can now consider the L-fuzzy context $(L, X, Y, NTdM)$, with $L = [0, 1]$ and obtain their L-fuzzy concepts to have a general overview of the tendencies between the objects X and the attributes Y.

Definition 3.28. *Consider the L-fuzzy context $(L, X, Y, NTdM)$, X, Y objects and attributes and $NTdM \in L^{X \times Y}$. The L-fuzzy concepts of $(L, X, Y, NTdM)$ are said to be the Normalized Trend Degree L-fuzzy concepts. We denote this set by $\mathcal{L}_{NTdM}$.*

The same can be done for $(L, X, Y, NPdM)$.

Definition 3.29. *Consider the L-fuzzy context $(L, X, Y, NPdM)$, with $L = [0, 1]$, X set of objects, Y set of attributes and $NPdM \in L^{X\times Y}$. The L-fuzzy concepts of $(L, X, Y, NPdM)$ are said to be the Normalized Persistent Degree L-fuzzy concepts. We denote this set by $\mathcal{L}_{NPdM}$.*

Remark 5. *$\mathcal{L}_{NTdM}$ and $\mathcal{L}_{NPdM}$ are complete lattices since the L-fuzzy concept set of any L-fuzzy context is a complete lattice* ([13, 14]).

We will see that some of these L-fuzzy concepts show the strongest relationship among objects and attributes.

First, we give the following definition for a general L-fuzzy context (L, X, Y, R) with $L = [0, 1]$, X object set and Y attribute set.

Definition 3.30. *Let (L, X, Y, R) be an L-fuzzy context with $L = [0, 1]$. Consider $x_0 \in X, y_0 \in Y$ and $0 < \alpha \leq 1$. We say that the pair (x_0, y_0) is an $\alpha-$level depending pair in (L, X, Y, R) if $(A_{\{x_0\}}, B_{\{x_0\}})$, the L-fuzzy concept associated with the crisp singleton $\{x_0\}$, is such that $B_{\{x_0\}}(y_0) \geq \alpha$. Analogously, we can define an $\alpha-$level depending pair associated with an attribute $y_0 \in Y$, obtaining the L-fuzzy concept $(A_{\{y_0\}}, B_{\{y_0\}})$ and choosing an object x_0 such that $A_{\{y_0\}}(x_0) \geq \alpha$.*

Remark 6. *Notice that there is not necessarily an $\alpha-$level depending pair for every $x_0 \in X$ neither for every $y_0 \in Y$. Moreover, an object x_0 or an attribute y_0 can be part of more than one $\alpha-$level depending pair.*

This definition can be applied to the Normalized Trend and Persistent Degree L-fuzzy contexts.

We can obtain all the $\alpha-$level depending pairs associated with the objects or the attributes.

Proposition 3.31. *Let (L, X, Y, R) be an L-fuzzy context and let $0 < \alpha \leq 1$. The sets of $\alpha-$level depending pairs associated with the objects and the attributes are coincident.*

Proof. Let $x_0 \in X$ and $y_0 \in Y$. Let $(A_{\{x_0\}}, B_{\{x_0\}})$ and $(A_{\{y_0\}}, B_{\{y_0\}})$ be the L-fuzzy concepts associated with x_0 and y_0. $B_{\{x_0\}}(y_0) \geq \alpha$ if and only if $A_{\{y_0\}}(x_0) \geq \alpha$ since $A_{\{y_0\}}(x_0) = B_{\{x_0\}}(y_0) = R(x_0, y_0)$. $\square$

The most interesting case is when $\alpha = 1$.

Definition 3.32. *The pair (x_0, y_0) verifying $B_{\{x_0\}}(y_0) = A_{\{y_0\}}(x_0) = 1$ is said to be a strong depending pair. We denote the strong depending pairs of the L-fuzzy context (L, X, Y, R) by S_R.*

When we work with Normalized Trend Degree L-fuzzy contexts, these strong depending pairs play an important role. If $(x_0, y_0) \in S_{NTdM}$, then the relation between object x_0 and attribute y_0 increases in time. In other case (when (x_0, y_0) is not a strong depending pair), we do not know what happens. Maybe this relationship decreases, fluctuates or we have not enough information to establish a clear diagnosis. In order to distinguish these possibilities, we could take into account the number of absent values in the different contexts of the sequence. We define the Absent Matrix (AM) to quantify this number.

$$AM(x, y) = Card(\{i \leq n \mid R_i(x, y) \text{ does not exist}\})$$

for all $x \in X$, $y \in Y$.

This matrix will be of special interest in the last part of the paper.

4 Replacement of absent values in L-fuzzy context sequences

In the first part of the paper we have studied tendencies in L-fuzzy context sequences with absent values. We can use some of the results for the replacement of missing values.

In the past, we analysed L-fuzzy contexts with absent values ([5, 9]). In that case, we worked with an only L-fuzzy context and we had not other information to use. However, in the case of an L-fuzzy context sequence, if we have an absent value $R_i(x_0, y_0)$ of the L-fuzzy context (L, X, Y, R_i), we think that the corresponding values in the other L-fuzzy contexts $R_k(x_0, y_0), k \neq i$, play an important role in the replacement of $R_i(x_0, y_0)$.

4.1 L-fuzzy context sequences with absent values

In previous papers, we took as departure point an interval-valued L-fuzzy context ([17]) in which there were unknown values ([5, 9]) and we replaced the absent values using different kind of attribute implications. In [9], we defined a coherent method to estimate the absent values of an interval-valued L-Fuzzy context based on the rest of the available information in the interval-valued L-fuzzy context. This method allows to apply the L-fuzzy concept theory to obtain information from the completed L-fuzzy context when $L = [0, 1]$.

In order to do this, we used the association rules defined by [1] to set up the implications between attributes. Let us consider B and C two attribute sets such that $\text{supp}(B) = \sum_{x\in X} \mathbf{B}_2(x) > 0$, being $\mathbf{B}(x) = 1$ if $x \in B$ and $\mathbf{B}(x) = 0$ in other case, and 2 the derivation operator.

An association rule is an implication $B \Rightarrow C$, that is verified with a certain support and degree of confidence.

Then, we defined the support and confidence of an implication between attributes as follows [9].

Definition 4.1. *Consider $L = [0, 1]$. Let (L, X, Y, R) be an L-Fuzzy context and $B, C \subseteq Y$ two sets of attributes. The* support *of the implication $B \Rightarrow C$ is given by*

$$\text{supp}(B \Rightarrow C) = \text{supp}(B \cup C) = \frac{\displaystyle\sum_{x\in X} (\mathbf{B} \cup \mathbf{C})_2(x)}{|X|}$$

and represents the percentage of objects that share the attributes of B and C.

The confidence *of the implication is*

$$\text{conf}(B \Rightarrow C) = \frac{\text{supp}(B \cup C)}{\text{supp}(B)} = \frac{\displaystyle\sum_{x\in X} (\mathbf{B} \cup \mathbf{C})_2(x)}{\displaystyle\sum_{x\in X} \mathbf{B}_2(x)}$$

and represents the percentage of objects that verify the implication, that is, the percentage of objects that having the attributes of B to a certain degree also have those of C to the same degree.

This definition can be extended to the interval-valued case [9].

Definition 4.2. *Consider $L = [0, 1]$. Let $(\mathcal{J}[L], X, Y, R)$ be an interval-valued L-Fuzzy context. Given the attribute sets $B, C \subseteq Y$, we define the* support *of the implication $B \Rightarrow C$, $\text{supp}(B \Rightarrow C)$, as the interval*

$$\left[\frac{\displaystyle\sum_{x\in X} l_{(\mathbf{B}\cup\mathbf{C})_2}(x)}{|X|}, \frac{\displaystyle\sum_{x\in X} u_{(\mathbf{B}\cup\mathbf{C})_2}(x)}{|X|}\right]$$

where the membership function of the derived set $\mathbf{B}_2$ is $\mathbf{B}_2(x) = [l_{\mathbf{B}_2}(x), u_{\mathbf{B}_2}(x)]$ and the confidence *of the implication, $\text{conf}(B \Rightarrow C)$, is given by the interval*

$$\left[\frac{\displaystyle\sum_{x\in X} l_{(\mathbf{B}\cup\mathbf{C})_2}(x)}{\displaystyle\sum_{x\in X} u_{\mathbf{B}_2}(x)}, \frac{\displaystyle\sum_{x\in X} u_{(\mathbf{B}\cup\mathbf{C})_2}(x)}{\displaystyle\sum_{x\in X} l_{\mathbf{B}_2}(x)} \wedge 1\right].$$

We used implications between attributes to estimate the value of the absent data. The confidence of an implication between attributes gives us the percentage of objects that, containing the attributes of the antecedent to a certain degree, also have those of the consequent to the same degree. Different types of implications between attributes can be used in the process, with negation of attributes and associated with labels.

We estimate each absent value analysing the behaviour of the attribute that we are considering with respect to the other objects.

This method only uses the values of the L-fuzzy context to replace the absent ones. We will see in next section how to deal with the problem using the studies of tendencies in the L-fuzzy context sequence.

4.2 Replacement of absent values process using Tendency matrices

We want to take advantage of the previous tendency study to replace these absent values. To do this, we will look at the values of the rest of L-fuzzy contexts of the sequence.

Let $i \in \{1,\dots,n\}, x_0 \in X$ and $y_0 \in Y$ be such that $R_i(x_0,y_0)$ is an absent value of the L-fuzzy context (L,X,Y,R_i). As we want to do a complete study of the L-fuzzy context (L,X,Y,R_i) by means of its L-fuzzy concepts, we need to replace this missing value.

If (x_0,y_0) is an α-level depending pair in the L-fuzzy context $(L,X,Y,NTdM)$ with α <1, close to 1, then the values of the different relations are increasing and there are few absent values among them. We will replace the absent value by an aggregation of the previous and the following values in the sequence.

Otherwise, if $AM(x_0,y_0)$ is sufficiently small (there is not growth) then we will take an aggregation of all the non-absent values $R_k(x_0,y_0), k \in \{1,\dots,n\}$. This replacement is the same for all the absent values in position (x_0,y_0).

Finally, if (x_0,y_0) is not an α-level depending pair and $AM(x_0,y_0)$ is sufficiently large, we have not enough information to replace the absent value using the sequence of L-fuzzy contexts. In this case, we can replace the absent value using only the values of the L-fuzzy context (L,X,Y,R_i).

To carry out this goal, using the Normalized Trend Degree (NTdM) and the Absent Matrices (AM), we define an algorithm (see Algorithm 1) that we call *Replacement of Absent Values Process (RAVP).*

Algorithm 1 Replacement of Absent Values Process (RAVP)

Input: An L-fuzzy context (L,X,Y,R_i), for a fixed $i \in \{1,\dots,n\}$, such that $R_i(x_0,y_0)$ is an absent value for some $x_0 \in X$ and $y_0 \in Y$.

Output: A value for $R_i(x_0,y_0)$.

if (x_0,y_0) is an α-level depending pair **then**

  $R_i(x_0,y_0) = Agr(R_l(x_0,y_0), R_k(x_0,y_0))$, with $l < i < k$ verifying that l is the greatest value less than i and k the smallest value greater than i for which $R_l(x_0,y_0)$ and $R_k(x_0,y_0)$ exist.

else

  if $AM(x_0,y_0) < n/2$ **then**

    Aggregate the non absent values:

$$R_i(x_0,y_0) = Agr(R_1(x_0,y_0),\dots R_n(x_0,y_0)).$$

  else

    Apply implications between attributes ([9]).

  end if

end if

5 Illustrative example of an application

Consider the L-fuzzy context sequence $(L,X,Y,R_i), i = 1,\dots,5$, that represents the sales of sports articles $X = \{x_1,x_2,x_3\}$ in some establishments $Y = \{y_1,y_2,y_3\}$ during 5 months. Every interval-valued observation of the relations $R_i \in \mathcal{J}([0,1])^{X\times Y}$, represents the variation of the percentage of the daily product sales in each establishment along a month. Note that if we consider $(\mathcal{J}([0,1]),\leq)$ with the usual order ($[a_1,c_1] \leq [a_2,c_2] \iff a_1 \leq a_2$ and $c_1 \leq c_2$), this is a complete but not totally ordered lattice.

R_1	y_1	y_2	y_3
x_1	$[0.7, 0.8]$	$[1, 1]$	$[0.8, 1]$
x_2	$[0, 0]$	$[0.1, 0.4]$	$[0.1, 0.3]$
x_3	$[0, 0]$	$[0.1, 0.3]$	$[0, 0.4]$

R_2	y_1	y_2	y_3
x_1	$[1, 1]$	$[0.8, 1]$	$[1, 1]$
x_2	$[0.8, 0.9]$	$[-, -]$	$[0.1, 0.3]$
x_3	$[0, 0]$	$[0, 0.2]$	$[0.2, 0.4]$

R_3	y_1	y_2	y_3
x_1	$[1, 1]$	$[1, 1]$	$[1, 1]$
x_2	$[0.6, 0.8]$	$[0.5, 0.5]$	$[0.7, 0.8]$
x_3	$[0, 0]$	$[0.1, 0.2]$	$[0.2, 0.4]$

R_4	y_1	y_2	y_3
x_1	$[0.5, 0.5]$	$[0.4, 0.6]$	$[0.6, 0.8]$
x_2	$[0.1, 0.3]$	$[-, -]$	$[0.3, 0.5]$
x_3	$[-, -]$	$[0.8, 0.9]$	$[0.8, 1]$

R_5	y_1	y_2	y_3
x_1	$[0.1, 0.4]$	$[0, 0.2]$	$[0, 0.2]$
x_2	$[0, 0]$	$[0.6, 0.8]$	$[0, 0.2]$
x_3	$[0.8, 1]$	$[1, 1]$	$[0.9, 1]$

We can obtain $MinTd$ and $AvTd$ from every object and attribute.
$MinTd(x_1) = 1$, $MinTd(x_2) = 1$, $MinTd(x_3) = 3$
$MinTd(y_1) = 1$, $MinTd(y_2) = 1$, $MinTd(y_3) = 2$
$AvTd(x_1) = 1.7$, $AvTd(x_2) = 1.7$, $AvTd(x_3) = 3.3$
$AvTd(y_1) = 2$, $AvTd(y_2) = 2$, $AvTd(y_3) = 2.7$
Then, although values are different, the Minimum Trend Degree and the Average Trend Degree rankings are coincident in this case (see Figure 1).

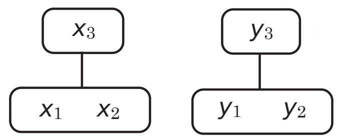

Figure 1: Minimum and Average Trend Degree Rankings.

A similar process can be done to calculate $MinPd$ and $AvPd$ from every object and attribute.
$MinPd(x_1) = 1$, $MinPd(x_2) = 2$, $MinPd(x_3) = 2$
$MinPd(y_1) = 2$, $MinPd(y_2) = 1$, $MinPd(y_3) = 2$
$AvPd(x_1) = 1.7$, $AvPd(x_2) = 3$, $AvPd(x_3) = 3$
$AvPd(y_1) = 3$, $AvPd(y_2) = 1.7$, $AvPd(y_3) = 3$
Also in this case both rankings are coincident (See Figure 2).

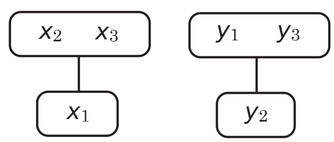

Figure 2: Minimum and Average Persistent Degree Rankings.

In this case, the resulting matrices TdM and PdM are the following ones.

TdM	y_1	y_2	y_3
x_1	2	1	2
x_2	1	2	2
x_3	3	3	4

PdM	y_1	y_2	y_3
x_1	2	1	2
x_2	4	2	3
x_3	3	2	4

and the normalized ones

$NTdM$	y_1	y_2	y_3
x_1	0.5	0.25	0.5
x_2	0.25	0.5	0.5
x_3	0.75	0.75	1

$NPdM$	y_1	y_2	y_3
x_1	0.5	0.25	0.5
x_2	1	0.5	0.75
x_3	0.75	0.5	1

Remark 7. *$NTdM$ and $NPdM$ are not necessarily comparable with the usual order ($M \leq N$ if and only if $M(x,y) \leq N(x,y), \forall x \in X, \forall y \in Y$). For instance, $NTdM(x_2, y_1) < NPdM(x_2, y_1)$ but $NTdM(x_3, y_2) > NPdM(x_3, y_2)$ in this example.*

Moreover, we can come back to Proposition 3.27.

Remark 8. *If $NPdM(x,y) = 1$ then $NTdM(x,y) = 1$ is not necessarily verified. For example, $NPdM(x_2, y_1)$=1 but $NTdM(x_2, y_1) = 0.25 \neq 1$*

We define now the L-fuzzy concepts associated with the crisp singletons that represent the objects and the attributes.
For every object $x \in X$, we can define the L-fuzzy set $\mathbf{x} \in L^X$ such that $\mathbf{x}(x) = 1$ and $\mathbf{x}(z) = 0$, for any $z \in X$, $z \neq x$.
Then the pair $\mathcal{C}_\mathbf{x} = (A_{\{x\}}, B_{\{x\}}) = (\mathbf{x}_{12}, \mathbf{x}_1)$ is said to be the L-fuzzy concept derived from object x. An implication operator is used in the definition of operators 1 and 2 for obtaining the pair (see Section 2.1).
The same can be defined for an attribute $y \in Y$.
For the L-fuzzy context $(L, X, Y, NTdM)$, and using the Luckasiewicz implication operator, we obtain the following L-fuzzy concepts.

$x_1 : (\{x_1/1, x_2/0.75, x_3/1\}, \{y_1/0.5, y_2/0.25, y_3/0.5\})$
$x_2 : (\{x_1/0.75, x_2/1, x_3/1\}, \{y_1/0.25, y_2/0.5, y_3/0.5\})$
$x_3 : (\{x_1/0.5, x_2/0.5, x_3/1\}, \{y_1/0.75, y_2/0.75, y_3/1\})$
$y_1 : (\{x_1/0.5, x_2/0.25, x_3/0.75\}, \{y_1/1, y_2/0.75, y_3/1\})$
$y_2 : (\{x_1/0.25, x_2/0.5, x_3/0.75\}, \{y_1/0.75, y_2/1, y_3/1\})$
$y_3 : (\{x_1/0.5, x_2/0.5, x_3/1\}, \{y_1/0.75, y_2/0.75, y_3/1\})$

And for the L-fuzzy context $(L, X, Y, NPdM)$ we have

$x_1 : (\{x_1/1, x_2/1, x_3/1\}, \{y_1/0.5, y_2/0.25, y_3/0.5\})$
$x_2 : (\{x_1/0.5, x_2/1, x_3/0.75\}, \{y_1/1, y_2/0.5, y_3/0.75\})$
$x_3 : (\{x_1/0.5, x_2/0.75, x_3/1\}, \{y_1/0.75, y_2/0.5, y_3/1\})$
$y_1 : (\{x_1/0.5, x_2/1, x_3/0.75\}, \{y_1/1, y_2/0.5, y_3/0.75\})$
$y_2 : (\{x_1/0.25, x_2/0.5, x_3/0.5\}, \{y_1/1, y_2/1, y_3/1\})$
$y_3 : (\{x_1/0.5, x_2/0.75, x_3/1\}, \{y_1/0.75, y_2/0.5, y_3/1\})$

In this case, we have an only strong depending pair $S_{NTdM} = \{(x_3, y_3)\}$ for $(L, X, Y, NTdM)$ but $S_{NPdM} = \{(x_3, y_3), (x_2, y_1)\}$ for $(L, X, Y, NPdM)$.

Thus, if we decrease the level of demand to $\alpha = 0.75$ then we have $[S_{NTdM} = \{(x_3, y_1), (x_3, y_2), (x_3, y_3)\}$, for the L-fuzzy context $(L, X, Y, NTdM)$. The result is $S_{NPdM} = \{(x_2, y_1), (x_2, y_3), (x_3, y_1), (x_3, y_3)\}$ for the L-fuzzy context $(L, X, Y, NPdM)$.

These sets represent the strongest relations among objects and attributes.

Finally, recall that we have three absent values, $R_2(x_2, y_2), R_4(x_2, y_2)$ and $R_4(x_3, y_1)$.

Consider $\alpha = 0.7$. Following the $RAVP$, as $NTdM(x_3, y_1) = 0.75 > 0.7$ then (x_3, y_1) is an 0.7-level depending pair in the L-fuzzy context $(L, X, Y, NTdM)$. Thereby we apply Step 1 to replace $R_4(x_3, y_1)$ and we aggregate $R_3(x_3, y_1) = [0, 0]$ and $R_5(x_3, y_1) = [0.8, 1]$.

Now, we can use different aggregation operators. For instance, if we take the average of the lower and upper bounds we obtain the result $[0.4, 0.5]$ and if we take the minimum of the lower bounds and the maximum of the upper ones $[0, 1]$.

For $R_2(x_2, y_2)$ and $R_4(x_2, y_2)$, we apply Step 4 since (x_2, y_2) is not an 0.7-level depending pair and $AM(x_2, y_2) = 2 < n/2$ being $n = 5$ the number of contexts of the sequence.

Then, we have to aggregate $R_1(x_2, y_2) = [0.1, 0.4]$, $R_3(x_2, y_2) = [0.5, 0.5]$ and $R_5(x_2, y_2) = [0.6, 0.8]$ to replace the values $R_2(x_2, y_2)$ and $R_4(x_2, y_2)$.

For instance, if we take the average of lower and upper bounds we obtain the result $[0.4, 0.57]$ for replacing both values and if we take the minimum of the lower bounds and the maximum of the upper ones, $[0.1, 0.8]$.

Conclusions and future work

The method described in [9] and that proposed in this paper are complementary because in the first case the absent values are replaced looking at the values of the context (studying dependencies between attributes), whereas in the second one looking at the data (for the same object and attribute) of the sequence of contexts that represent the evolution in time. We believe that this new algorithm is more suitable since we have all the information that provides the sequence.

Besides, if we look at the computational complexity, the study of the implications between attributes (Section 4.1) has higher cost than the study of tendencies proposed in this paper since the latter only requires the concepts associated with the objects and attributes.

As a future work, we have to define what we understand as a sufficiently high level for α in every case. It is also important to decide if the value $n/2$ is the most appropriate to define a sufficiently large or small value for when $AM(x_0, y_0)$. $AM(x_0, y_0) = 1$ is the best situation (the only absent value is the one that we want to replace).

Finally, it is also important to study the differences between the results of $RAVP$ using different aggregation operators.

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