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Asymptotic behaviour of the Urbanik semigroup

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Abstract

We revisit the product convolution semigroup of probability densities $e_c(t)$, $c > 0$ on the positive half-line with moments $(n!)^c$ and determine the asymptotic behaviour of e_c for large and small $t > 0$. This shows that $(n!)^c$ is indeterminate as Stieltjes moment sequence if and only if $c > 2$. When c is a natural number e_c is a Meijer-G function. From the results about e_c we obtain the asymptotic behaviour at $\pm\infty$ of the convolution roots of the Gumbel distribution.

1. Introduction

We consider a family of probability densities $e_c(t), c > 0$ on the half-line given by

$$e_c(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} t^{ix-1}\Gamma(1-ix)^c\,dx, \quad t > 0. \tag{1}$$

In this formula we use that $\Gamma(z)$ is a non-vanishing holomorphic function in the cut plane

$$\mathcal{A} = \mathbb{C} \setminus (-\infty, 0], \tag{2}$$

so we can define

$$\Gamma(z)^c = \exp(c\log\Gamma(z)), \quad z \in \mathcal{A}$$

using the holomorphic branch of $\log\Gamma$ which is 0 for $z = 1$.

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As far as we know it was proved first by Urbanik in [17, Section 4] that e_c is a probability density, and that the following product convolution equation holds

$$e_{c+d}(t) = \int_0^\infty e_c(t/x) e_d(x) \frac{dx}{x}, \quad c, d > 0. \tag{3}$$

Furthermore, it was noticed that

$$\int_0^\infty t^n e_c(t)\, dt = (n!)^c, \quad c > 0, n = 0, 1, \ldots. \tag{4}$$

Defining the probability measure τ_c on $(0, \infty)$ by

$$d\tau_c = e_c(t)\, dt = t e_c(t)\, dm(t), \quad c > 0, \tag{5}$$

where $dm(t) = (1/t)\, dt$ is the Haar measure on the locally compact abelian group $G = (0, \infty)$ under multiplication, we can write (3) as $\tau_c \diamond \tau_d = \tau_{c+d}$, where $\diamond$ denotes the (product) convolution of measures on the multiplicative group G. The family $(\tau_c)_{c>0}$ is a convolution semigroup in the sense of [7]. We propose to call this semigroup the Urbanik semigroup because of [17].

The continuous characters of the group G can be given as $t \to t^{ix}$, where $x \in \mathbb{R}$ is arbitrary, and in this way the dual group $\widehat{G}$ of G can be identified with the additive group of real numbers, and by the inversion theorem of Fourier analysis for LCA-groups, (1) is equivalent to

$$\widehat{\tau_c}(x) = \int_0^\infty t^{-ix}\, d\tau_c(t) = \exp(c \log(\Gamma(1 - ix)), \quad x \in \mathbb{R}. \tag{6}$$

To establish the existence of a product convolution semigroup (τ_c) satisfying (6) is therefore equivalent to proving that

$$\rho(x) := -\log \Gamma(1 - ix), \quad x \in \mathbb{R} \tag{7}$$

is a continuous negative definite function on $\mathbb{R}$ in the terminology of [7] or [14].

This was done in [17] by giving the Lévy-Khinchin representation of ρ, using Plana's formula, cf. [9, 8.341(3)] or [12, p. 187]:

$$\log \Gamma(z) = \int_0^\infty \left[\frac{e^{-zt} - e^{-t}}{1 - e^{-t}} + (z-1) e^{-t} \right] \frac{dt}{t}, \quad \mathrm{Re}\,(z) > 0. \tag{8}$$

In fact from (8) we get

$$-\log \Gamma(1 - ix) = \int_0^\infty \left[1 - e^{ixt} + \frac{itx}{1+t^2} \right] \frac{e^{-t}}{t(1 - e^{-t})}\, dt - iax, \tag{9}$$

where

$$a = \int_0^\infty \left[\frac{1}{(1+t^2)(1 - e^{-t})} - \frac{1}{t} \right] e^{-t}\, dt,$$

showing that $\rho(x) = -\log\Gamma(1-ix)$ is negative definite with the Lévy measure

$$d\mu = \frac{e^{-t}}{t(1-e^{-t})}\,dt$$

concentrated on $(0,\infty)$.

Another proof of the negative definiteness of ρ was given in [6] based on the Weierstrass product for Γ , where Log denotes the principal logarithm in the cut plane $\mathcal{A}$, cf. (2):

$$-\log\Gamma(z) = \gamma z + \operatorname{Log} z + \sum_{k=1}^{\infty}\left(\operatorname{Log}(1+z/k) - z/k\right), \quad z \in \mathcal{A}.$$

Clearly,

$$\rho_n(z) := \gamma z + \operatorname{Log} z + \sum_{k=1}^{n}\left(\operatorname{Log}(1+z/k) - z/k\right)$$

converges locally uniformly to $-\log\Gamma(z)$ for $z \in \mathcal{A}$, and since

$$\rho_n(1-ix) = \rho_n(1) - i\left(\gamma - \sum_{k=1}^{n}\frac{1}{k}\right)x + \sum_{k=1}^{n+1}\operatorname{Log}(1-ix/k)$$

is negative definite, because $\operatorname{Log}(1+iax)$ is so for $a \in \mathbb{R}$ and

$$\rho_n(1) = \gamma + \log(n+1) - \sum_{k=1}^{n}\frac{1}{k} > 0,$$

we conclude that the limit function $\rho(x) = -\log\Gamma(1-ix)$ is negative definite.

As noticed in [6, Lemma 2.1], (4) is a special case of

$$\int_0^\infty t^z e_c(t)\,dt = \Gamma(1+z)^c, \quad \operatorname{Re}(z) > -1, \tag{10}$$

and letting z tend to -1 along the real axis, we get

$$\int_0^\infty e_c(t)\frac{dt}{t} = \int_0^\infty e_c(1/t)\frac{dt}{t} = \infty, \quad c > 0. \tag{11}$$

It follows from (4) that $(n!)^c$ is a Stieltjes moment sequence for any $c > 0$, and while it is easy to see that it is S-determinate for $c \le 2$ in the sense, that there is only one measure on the half-line with these moments, namely τ_c, it is rather delicate to see that it is S-indeterminate for $c > 2$. This was proved in Theorem 2.5 in [6]. The proof was based on a relationship between τ_c and stable distributions, and it used heavily asymptotic results of Skorokhod from [15] and exposed in [19]. Further details are given at the end of this section.

The purpose of the present paper is to establish the asymptotic behaviour of the densities $e_c(t)$ for $t \to \infty$ and $t \to 0$. The behaviour for $t \to \infty$ will lead to a direct proof of the S-indeterminacy for $c > 2$.

We mention that the product convolution semigroup $(\tau_c)_{c>0}$ corresponds to the Bernstein function $f(s) = s$ in the following result from [6, Theorem 1.8].

Theorem 1.1. *Let f be a non-zero Bernstein function. The uniquely determined measure $\kappa = \kappa(f)$ with moments $s_n = f(1)\cdots f(n)$ is infinitely divisible with respect to the product convolution. The unique product convolution semigroup $(\kappa_c)_{c>0}$ with $\kappa_1 = \kappa$ has the moments*

$$\int_0^\infty x^n \, d\kappa_c(x) = (f(1)\cdots f(n))^c, \quad c > 0, n = 0, 1, \ldots. \tag{12}$$

It is an easy consequence of Carleman's criterion that the measures κ_c are S-determinate for $c \le 2$, cf. [6, Theorem 1.6].

In [6] we consider three Bernstein functions $f_\alpha, f_\beta, f_\gamma$ with corresponding product convolution semigroups $(\alpha_c)_{c>0}, (\beta_c)_{c>0}, (\gamma_c)_{c>0}$:

$$f_\alpha(s) = (1 + 1/s)^s, \quad f_\beta(s) = (1 + 1/s)^{-s-1}, \quad f_\gamma(s) = s(1 + 1/s)^{s+1}.$$

It is proved that the measures α_c, β_c have compact support, so they are clearly S-determinate for all $c > 0$, but γ_c is S-indeterminate for $c > 2$. Using that $\tau_c = \beta_c \diamond \gamma_c$, it is possible to infer that also τ_c is S-indeterminate, see [6] for details.

As noticed in [17], the measures $\tau_c,\ c \ge 1$ are also infinitely divisible for the additive structure, because $e_c(t)$ is completely monotonic. To see this, notice that the convolution equation (3) with $d = 1$ can be written

$$e_{c+1}(t) = \int_0^\infty e^{-tx} e_c(1/x) \, \frac{dx}{x}, \quad c > 0, \tag{13}$$

showing that $e_c(t)$ is completely monotonic for $c > 1$, and it tends to infinity for $t \to 0$ because of (11).

It is well-known that the exponential distribution τ_1 is infinitely divisible for the additive structure and with a completely monotonic density $e_1(t)$.

Urbanik also showed that τ_c is not infinitely divisible for the additive structure when $0 < c < 1$.

Formula (1) states roughly speaking that $te_c(t)$ is the Fourier transform of the Schwartz function $\Gamma(1 - ix)^c$ evaluated at $\log t$, thus showing that e_c is C^∞ on $(0, \infty)$. By Riemann-Lebesgue's Lemma we also see that $te_c(t)$ tends to zero for t tending to zero and to infinity. Much more will be obtained in the main results below.

2. Main results

Our main results are

Theorem 2.1. *For $c > 0$ we have*

$$e_c(t) = \frac{(2\pi)^{(c-1)/2}}{\sqrt{c}} \frac{\exp(-ct^{1/c})}{t^{(c-1)/(2c)}} \left[1 + \mathcal{O}\left(\frac{1}{t^{1/c}}\right)\right], \quad t \to \infty. \tag{14}$$

Remark 2.2. It is worth noticing that e_c can be expressed as a Meijer-G function when $c = 1, 2, \ldots$, namely as

$$e_c(t) = G^{c,0}_{0,c}\left(\begin{matrix} - \\ 0, \quad \ldots, \quad 0 \end{matrix} \mid t \right). \tag{15}$$

For an introduction to these functions see the recent paper [3]. Formula (15) follows e.g. by (31) below. The cases $c = 1, 2$ are particularly simple since

$$e_1(t) = e^{-t}, \quad e_2(t) = \int_0^\infty \exp(-x - t/x) \frac{dx}{x} = 2K_0(2\sqrt{t}).$$

In the last formula K_0 is a modified Bessel function, see [13, Chap. 10, Sec. 25].

Meijer-G functions have appeared recently in connection with random matrix problems, see [1],[8],[10].

Corollary 2.3. *The measure $\tau_c = e_c(t)\, dt$ is S-indeterminate for $c > 2$.*

Theorem 2.4. *For $c > 0$ we have*

$$e_c(t) = \frac{(\log(1/t))^{c-1}}{\Gamma(c)} + \mathcal{O}((\log(1/t))^{c-2}), \quad t \to 0. \tag{16}$$

Remark 2.5. Formula (16) shows that $e_c(t)$ tends to infinity as a power of $\log(1/t)$ when $c > 1$, but so slowly that multiplication with t forces the density to tend to zero. When $0 < c < 1$ the density $e_c(t)$ tends to zero.

In a short Section 4 we transfer our results to information about the Gumbel distribution.

3. Proofs

We will first give a proof of Theorem 2.1 in the case, where c is a natural number. Note that the asymptotic expression in (14) for $c = 1$ reduces to $e_1(t) = e^{-t}$. When $c = n + 1$, where n is a natural number, we know that $e_{n+1}(t)$ is the n'th product convolution power of e_1, hence

$$e_{n+1}(t) = \int_0^\infty \cdots \int_0^\infty e^{-\frac{t}{u_1 \cdots u_n}} e^{-u_1} \cdots e^{-u_n} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}.$$

For $t > 0$ fixed, the change of variables $u_j = t^{1/(n+1)} v_j, j = 1, \ldots, n$ leads to

$$e_{n+1}(t) = \int_0^\infty \cdots \int_0^\infty g(v_1, \ldots, v_n) e^{-t^{1/(n+1)} f(v_1,\ldots,v_n)} dv_1 \cdots dv_n, \tag{17}$$

with

$$g(v_1, \ldots, v_n) := \frac{1}{v_1 \cdots v_n}, \quad f(v_1, \ldots, v_n) := v_1 + \cdots + v_n + g(v_1, \ldots, v_n).$$

The phase function $f(v_1, \ldots, v_n)$ is convex in $C = \{v_1 > 0, \ldots, v_n > 0\}$ because the Hessian matrix of second derivatives is

$$Hf(v_1, \ldots, v_n) = g(v_1, \ldots, v_n) \begin{pmatrix} \frac{2}{v_1^2} & \frac{1}{v_1 v_2} & \cdots & \frac{1}{v_1 v_n} \\ \frac{1}{v_2 v_1} & \frac{2}{v_2^2} & \cdots & \frac{1}{v_2 v_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{v_n v_1} & \frac{1}{v_n v_2} & \cdots & \frac{2}{v_n^2} \end{pmatrix},$$

which is easily seen to be positive definite. The phase function therefore has a global minimum at the unique stationary point $\vec{v}_0$ such that $\vec{\nabla} f(\vec{v}_0) = \vec{0}$, that is, at $\vec{v}_0 = (1, \ldots, 1)$. At that point, the Hessian matrix of $f(\vec{v})$ is

$$A := Hf(1, \ldots, 1) = \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 2 \end{pmatrix},$$

with determinant $\det(A) = n + 1$.

By Laplace's asymptotic method for multiple dimensional Laplace transforms, cf. [18, Theorem 3, p. 495], we know that for $t \to \infty$,

$$e_{n+1}(t) = \left(\frac{2\pi}{t^{1/(n+1)}} \right)^{n/2} g(\vec{v}_0)(\det(A))^{-1/2} e^{-t^{1/(n+1)} f(\vec{v}_0)} \left[1 + \mathcal{O}\left(\frac{1}{t^{1/(n+1)}} \right) \right].$$

We have that $g(\vec{v}_0) = 1$ and $f(\vec{v}_0) = n + 1$, hence

$$e_{n+1}(t) = \frac{(2\pi)^{n/2}}{\sqrt{n+1}} \frac{e^{-(n+1)t^{1/(n+1)}}}{t^{n/(2(n+1))}} \left[1 + \mathcal{O}\left(\frac{1}{t^{1/(n+1)}} \right) \right], \tag{18}$$

which agrees with (14) for $c = n + 1$.

The proof of Theorem 2.1 for arbitrary $c > 0$ is more delicate. We first apply Cauchy's integral theorem to move the integration in (1) to an arbitrary horizontal line

$$L_a := \{z = x + ia \mid x \in \mathbb{R}\}, \quad a > 0. \tag{19}$$

Lemma 3.1. *With L_a as in* (19) *we have*

$$e_c(t) = \frac{1}{2\pi} \int_{L_a} t^{iz-1} \Gamma(1-iz)^c \, dz, \quad t > 0. \tag{20}$$

Proof: For $t, c > 0$ fixed, $f(z) = t^{iz-1}\Gamma(1-iz)^c$ is holomorphic in the simply connected domain $\mathbb{C} \setminus i(-\infty, -1]$, so the Lemma follows from Cauchy's integral theorem provided the integral

$$\int_0^a f(x+iy) \, dy$$

tends to 0 for $x \to \pm\infty$. We have

$$|f(x+iy)| = t^{-y-1} |\Gamma(1+y-ix)|^c$$

and since

$$|\Gamma(u+iv)| \sim \sqrt{2\pi} e^{-|v|\pi/2} |v|^{u-1/2}, \quad |v| \to \infty, \text{ uniformly for bounded real } u,$$

cf. [2, p.141, eq. 5.11.9],[9, 8.328(1)], the result follows. □

In the following we will use Lemma 3.1 with the line of integration $L = L_a$, where $a = t^{1/c} - 1$ for $t > 1$. Therefore, using the parametrization $z = x + i(t^{1/c} - 1)$ we get

$$e_c(t) = t^{-t^{1/c}} \frac{1}{2\pi} \int_{-\infty}^{\infty} t^{ix} \Gamma(t^{1/c} - ix)^c \, dx,$$

and after the change of variable $x = t^{1/c} u$

$$e_c(t) = t^{1/c - t^{1/c}} \frac{1}{2\pi} \int_{-\infty}^{\infty} t^{iut^{1/c}} \Gamma(t^{1/c}(1-iu))^c \, du. \tag{21}$$

Stirling's formula for Γ with Binet's remainder term is, see [9, 8.341(1)] or [12, p. 176],

$$\Gamma(z) = \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z+\mu(z)}, \quad \mathrm{Re}\,(z) > 0, \tag{22}$$

where

$$\mu(z) = \int_0^{\infty} \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) \frac{e^{-zt}}{t} \, dt, \quad \mathrm{Re}\,(z) > 0. \tag{23}$$

Notice that $\mu(z)$ is the Laplace transform of a positive function, so we have the estimates for $z = r + is, r > 0$

$$|\mu(z)| \le \mu(r) \le \frac{1}{12r}, \tag{24}$$

where the last inequality is a classical version of Stirling's formula, thus showing that the estimate is uniform in $s \in \mathbb{R}$.

Inserting this in (21), we get after some simplification

$$e_c(t) = (2\pi)^{c/2-1} t^{1/c-1/2} e^{-ct^{1/c}} \int_{-\infty}^{\infty} e^{ct^{1/c} f(u)} g_c(u) M(u,t)\, du, \tag{25}$$

where

$$f(u) := iu + (1-iu)\,\mathrm{Log}(1-iu), \quad g_c(u) := (1-iu)^{-c/2} \tag{26}$$

and

$$M(u,t) := \exp[c\mu(t^{1/c}(1-iu))]. \tag{27}$$

From (24) we get $M(u,t) = 1 + \mathcal{O}(t^{-1/c})$ for $t \to \infty$, uniformly in u. We shall therefore consider the behaviour of

$$\int_{-\infty}^{\infty} e^{ct^{1/c} f(u)} g_c(u)\, du. \tag{28}$$

From here we need to apply the saddle point method to obtain the approximation of (28) for large positive t. For convenience, we use Theorem 1 in [11]. We have that the only saddle point of the phase function $f(u)$ is $u = 0$ and $f(0) = f'(0) = 0$, $f''(0) = -1$, $f'''(0) \neq 0$; also $g_c(0) = 1$. Then, the parameters used in that theorem are $m = 2$, $p = 3$, $\phi = \pi$, $N = 0$, $M = 1$ and the large variable used in the theorem is $x \equiv ct^{1/c}$. We have that the steepest descendent path used in the theorem is $\Gamma = \Gamma_0 \bigcup \Gamma_1 = (-\infty, 0) \bigcup (0, \infty)$, that is, it is just the original integration path in the above integral, and therefore does not need any deformation. From [11, Theorem 1] with the notation used there, we read that the integral (28) has an expansion of the form

$$e^{xf(0)}[c_0\Psi_0(x) + c_1\Psi_1(x) + c_2\Psi_2(x) + \cdots],$$

with $\Psi_n(x) = \mathcal{O}(x^{-(n+1)/2})$ and c_n is independent of x. Because the factors c_{2n+1} vanish we find

$$c_0\Psi_0(x) + c_1\Psi_1(x) + c_2\Psi_2(x) + \cdots = c_0\Psi_0(x)[1 + \mathcal{O}(x^{-1})]$$

with $c_0 = 1$ and

$$\Psi_0(x) = a_0(x)\Gamma\left(\frac{1}{2}\right)\left|\frac{2}{xf''(0)}\right|^{1/2}$$

with

$$a_0(x) = e^{-xf(0)} A_0(x) B_0, \quad A_0(x) = e^{xf(0)}, \quad B_0 = g_c(0),$$

hence $a_0(x) = B_0 = 1$. Using all these data we finally obtain

$$\int_{-\infty}^{\infty} e^{ct^{1/c} f(u)} g_c(u) du = \frac{\sqrt{2\pi}}{\sqrt{c} t^{1/(2c)}} [1 + \mathcal{O}(t^{-1/c})],$$

and

$$e_c(t) = \frac{(2\pi)^{(c-1)/2}}{\sqrt{c}} \frac{e^{-ct^{1/c}}}{t^{(c-1)/(2c)}}[1 + \mathcal{O}(t^{-1/c})].$$

□

Proof of Corollary 2.3. We apply the Krein criterion for S-indeterminacy of probability densities concentrated on the half-line, using a version given in [5, Theorem 5.1]. It states that if

$$\int_0^\infty \frac{\log e_c(t)\, dt}{\sqrt{t}(1+t)} > -\infty, \tag{29}$$

then $\tau_c = e_c(t)\, dt$ is S-indeterminate. We shall see that (29) holds for $c > 2$.

From Theorem 2.1 combined with the fact that $e_c(t)$ is decreasing when $c > 1$, we see that the inequality in (29) holds if and only if

$$\int_0^\infty \frac{\log((2\pi)^{(c-1)/2}/\sqrt{c}) - ct^{1/c} - ((c-1)/(2c)) \log t}{\sqrt{t}(1+t)}\, dt > -\infty,$$

and the latter holds precisely for $c > 2$. This shows that τ_c is S-indeterminate for $c > 2$. □

Proof of Theorem 2.4.

Since we are studying the behaviour for $t \to 0$, we assume that $0 < t < 1$ so that $\Lambda := \log(1/t) > 0$.

We will need integration along the vertical lines

$$V_a := \{a + iy \mid y = -\infty \ldots \infty\}, \quad a \in \mathbb{R}, \tag{30}$$

and we can therefore express (1) as

$$e_c(t) = \frac{1}{2\pi i} \int_{V_{-1}} t^z \Gamma(-z)^c dz. \tag{31}$$

By the functional equation for Γ we get

$$e_c(t) = \frac{1}{2\pi i} \int_{V_{-1}} (-z)^{-c} t^z \Gamma(1-z)^c dz. \tag{32}$$

To ease the writing we define

$$\varphi(z) := t^z \Gamma(1-z)^c, \quad g(z) := (-z)^{-c} = \exp(-c\, \mathrm{Log}(-z)),$$

and note that φ is holomorphic in $\mathbb{C} \setminus [1, \infty)$, while g is holomorphic in $\mathbb{C} \setminus [0, \infty)$. Here Log is the principal logarithm in the cut plane $\mathcal{A}$, cf. (2).

Note that for $x > 0$

$$g_\pm(x) := \lim_{\varepsilon \to 0^+} g(x \pm i\varepsilon) = x^{-c} e^{\pm i\pi c}.$$

Formula (32) can now be written

$$e_c(t) = \frac{1}{2\pi i}\int_{V_{-1}} g(z)\varphi(z)\,dz. \tag{33}$$

Case 1. We will first treat the case $0 < c < 1$.

We fix $0 < s < 1$, choose $0 < \varepsilon < s$ and integrate $g(z)\varphi(z)$ over the contour $\mathcal{C}$

$$\{-1+iy \mid y = \infty \ldots 0\} \cup [-1,-\varepsilon] \cup \{\varepsilon e^{i\theta} \mid \theta = \pi \ldots 0\} \cup [\varepsilon, s] \cup \{s+iy \mid y = 0 \ldots \infty\}$$

and get 0 by the integral theorem of Cauchy. On the interval $[\varepsilon, s]$ we use the values of g_+.

Similarly we get 0 by integrating $g(z)\varphi(z)$ over the complex conjugate contour $\overline{\mathcal{C}}$, and now we use the values of g_- on the interval $[\varepsilon, s]$.

Subtracting the second contour integral from the first leads to

$$\int_{V_s} - \int_{V_{-1}} - \int_{|z|=\varepsilon} g(z)\varphi(z)\,dz + \int_\varepsilon^s \varphi(x)(g_+(x) - g_-(x))\,dx = 0,$$

where the integral over the circle is with positive orientation. Note that the two integrals over $[-1,-\varepsilon]$ cancel. Using that $0 < c < 1$ it is easy to see that the integral over the circle $|z| = \varepsilon$ converges to 0 for $\varepsilon \to 0$, and we finally get for $\varepsilon \to 0$

$$e_c(t) = \frac{1}{2\pi i}\int_{V_s} g(z)\varphi(z)\,dz + \frac{\sin(\pi c)}{\pi}\int_0^s x^{-c}\varphi(x)\,dx := I_1 + I_2.$$

We claim that the first integral I_1 is $o(t^s)$ for $t \to 0$. To see this we insert the parametrization of V_s and get

$$I_1 = \frac{t^s}{2\pi}\int_{-\infty}^{\infty} (-s-iy)^{-c} t^{iy}\Gamma(1-s-iy)^c\,dy$$

and the integral is $o(1)$ by Riemann-Lebesgue's Lemma, so $I_1 = o(t^s)$.

The substitution $u = x\log(1/t) = x\Lambda$ in the integral I_2 leads to

$$I_2 = \frac{\sin(\pi c)}{\pi}\Lambda^{c-1}\int_0^{s\Lambda} u^{-c}e^{-u}\Gamma(1-u/\Lambda)^c\,du. \tag{34}$$

We split the integral in (34) as

$$\Gamma(1-c) + \int_0^{s\Lambda} u^{-c}e^{-u}\left[\Gamma(1-u/\Lambda)^c - 1\right]\,du - \int_{s\Lambda}^{\infty} u^{-c}e^{-u}\,du, \tag{35}$$

and by the mean-value theorem and $\Psi = \Gamma'/\Gamma$ we have

$$\Gamma(1-u/\Lambda)^c - 1 = -\frac{u}{\Lambda}c\Gamma(1-\theta u/\Lambda)^c\Psi(1-\theta u/\Lambda)$$

for some $0 < \theta < 1$, but this implies that

$$|\Gamma(1-u/\Lambda)^c - 1| \leq \frac{cu}{\Lambda} M(s), \quad 0 < u < s\Lambda,$$

where

$$M(s) := \max\{\Gamma(x)^c |\Psi(x)| \mid 1-s \leq x \leq 1\},$$

so the first integral in (35) is $\mathcal{O}(\Lambda^{-1})$. The second integral is an incomplete Gamma function, and by known asymptotics for this, see [9], we get that the second integral is $\mathcal{O}(\Lambda^{-c}t^s)$. Putting things together and using Euler's reflection formula for Γ, we see that

$$e_c(t) = \frac{\Lambda^{c-1}}{\Gamma(c)} + \mathcal{O}(\Lambda^{c-2}),$$

which is (16).

Case 2. We now assume $1 < c < 2$.

The Gamma function decays so rapidly when $z = -1 + iy \in V_{-1}, y \to \pm\infty$, that we can integrate by parts in (32) to get

$$e_c(t) = -\frac{1}{2\pi i} \int_{V_{-1}} \frac{(-z)^{-(c-1)}}{c-1} \frac{d}{dz}(t^z \Gamma(1-z)^c)\, dz. \tag{36}$$

Defining

$$\varphi_1(z) := \frac{d}{dz}(t^z\Gamma(1-z)^c) = t^z\Gamma(1-z)^c(\log t - c\Psi(1-z)),$$

and using the same contour technique as in case 1 to the integral in (36), where now $0 < c-1 < 1$, we get for $0 < s < 1$ fixed

$$e_c(t) = -\frac{1}{c-1}\frac{1}{2\pi i}\int_{V_s} (-z)^{-(c-1)}\varphi_1(z)\, dz - \frac{\sin(\pi(c-1))}{(c-1)\pi}\int_0^s x^{-(c-1)}\varphi_1(x)\, dx.$$

The first integral is $o(t^s\Lambda)$ by Riemann-Lebesgue's Lemma, and the substitution $u = x\Lambda$ in the second integral leads to

$$\begin{aligned}
&\int_0^s x^{-(c-1)}\varphi_1(x)\, dx \\
&= \Lambda^{c-2}\int_0^{s\Lambda} u^{-(c-1)}\varphi_1(u/\Lambda)\, du \\
&= -\Lambda^{c-1}\int_0^{s\Lambda} u^{-(c-1)}e^{-u}\, du - \Lambda^{c-1}\int_0^{s\Lambda} u^{-(c-1)}e^{-u}\left(\Gamma(1-u/\Lambda)^c - 1\right) du \\
&- c\Lambda^{c-2}\int_0^{s\Lambda} u^{-(c-1)}e^{-u}\Gamma(1-u/\Lambda)^c\Psi(1-u/\Lambda)\, du \\
&= -\Lambda^{c-1}\Gamma(2-c) + \mathcal{O}(\Lambda^{c-2}).
\end{aligned}$$

Using that

$$\left(-\frac{\sin(\pi(c-1))}{(c-1)\pi}\right)\left(-\Lambda^{c-1}\Gamma(2-c)\right) = \frac{\Lambda^{c-1}}{\Gamma(c)}$$

by Euler's reflection formula, we see that (16) holds.

Case 3. We now assume $c > 2$.

We perform the change of variable $w = \Lambda z$ in (32) and obtain

$$e_c(t) = \frac{\Lambda^{c-1}}{2\pi i}\int_{V_{-\Lambda}} (-w)^{-c} e^{-w}\Gamma(1-w/\Lambda)^c\, dw.$$

Using Cauchy's integral theorem, we can shift the contour $V_{-\Lambda}$ to V_{-1} as the integrand is holomorphic in the vertical strip between both paths and exponentially small at both extremes of that vertical strip. Then,

$$e_c(t) = \frac{\Lambda^{c-1}}{2\pi i}\int_{V_{-1}} (-w)^{-c} e^{-w}\Gamma\left(1-w/\Lambda\right)^c\, dw.$$

For any holomorphic function h in a domain G which is star-shaped with respect to 0 we have

$$h(z) = h(0) + z\int_0^1 h'(uz)\, du, \quad z \in G.$$

If this is applied to $G = \mathbb{C}\setminus[1,\infty)$ and $h(z) = \Gamma(1-z)^c$ we find

$$\Gamma(1-w/\Lambda)^c = 1 - \frac{cw}{\Lambda}\int_0^1 \Gamma(1-uw/\Lambda)^c\Psi(1-uw/\Lambda)\, du. \tag{37}$$

Defining

$$R(w) = \int_0^1 \Gamma(1-uw/\Lambda)^c\Psi(1-uw/\Lambda)\, du,$$

we get

$$e_c(t) = \frac{\Lambda^{c-1}}{2\pi i}\int_{V_{-1}} (-w)^{-c} e^{-w} dw + \frac{c\Lambda^{c-2}}{2\pi i}\int_{V_{-1}} (-w)^{1-c} e^{-w} R(w) dw. \tag{38}$$

For any $w \in V_{-1}, 0 \le u \le 1$ and for $\Lambda \ge 1$ we have that $1 - uw/\Lambda \in \Omega$, where Ω is the closed vertical strip located between the vertical lines V_1 and V_2. Because $\Gamma(z)^c\Psi(z)$ is continuous in Ω and exponentially small at the upper and lower limits of Ω, the function $R(w)$ is bounded for $w \in V_{-1}$ by a constant independent of $\Lambda \ge 1$. Therefore,

$$\frac{c\Lambda^{c-2}}{2\pi i}\int_{V_{-1}} (-w)^{1-c} e^{-w} R(w) dw = \mathcal{O}(\Lambda^{c-2}),$$

where we use that $(-w)^{1-c}e^{-w}$ is integrable over V_{-1} because $c > 2$.

On the other hand, in the first integral of (38), the contour V_{-1} may be deformed to a Hankel contour

$$\mathcal{H} := \{x - i \mid x = \infty \ldots 0\} \cup \{e^{i\theta} \mid \theta = -\pi/2 \ldots - 3\pi/2\} \cup \{x + i \mid x = 0 \ldots \infty\}$$

surrounding $[0, \infty)$, and the integral over $\mathcal{H}$ is Hankel's integral representation of the inverse of the Gamma function:

$$\frac{1}{2\pi i} \int_{\mathcal{H}} (-w)^{-c} e^{-w} dw = \frac{1}{\Gamma(c)}.$$

Therefore, when we join everything, we obtain that for $c > 2$:

$$e_c(t) = \frac{(\log(1/t))^{c-1}}{\Gamma(c)} + \mathcal{O}((\log(1/t))^{c-2}), \quad t \to 0.$$

Case 4. $c = 1, c = 2$.
These cases are easy since $e_1(t) = e^{-t}$ and $e_2(t) = 2K_0(2\sqrt{t})$. $\square$

Remark 3.2. The behaviour of $e_c(t)$ for $t \to 0$ can be obtained from (31) using the residue theorem when c is a natural number, so e_c belongs to the Meijer-G family. In fact, in this case $\Gamma(-z)^c$ has a pole of order c at $z = 0$, and a shift of the contour V_{-1} to V_s, where $0 < s < 1$, has to be compensated by a residue, which will give the behaviour for $t \to 0$.

4. Remarks about the Gumbel distribution

The standard Gumbel distribution has the probability density

$$G(x) = \exp\left(-x - e^{-x}\right), \quad x \in \mathbb{R}$$

with respect to Lebesgue measure. It is known to be infinitely divisible, see [16], and hence embeddable in a convolution semigroup $(G_c(x))_{c>0}$ with $G_1 = G$. The image measure of the Gumbel distribution under the group isomorphism $x \mapsto e^{-x}$ of $(\mathbb{R}, +)$ onto $((0, \infty), \cdot)$ is the exponential distribution τ_1 given in (5), and therefore (G_c) is mapped onto the Urbanik semigroup (τ_c), so we obtain

$$G_c(x) = e^{-x} e_c(e^{-x}), \quad x \in \mathbb{R}, c > 0.$$

From the asymptotic behaviour of e_c in Theorem 2.1 and Theorem 2.4 we can obtain the asymptotic behaviour of the Gumbel convolution roots $G_c(x)$:

$$G_c(x) = \frac{x^{c-1} e^{-x}}{\Gamma(c)} \left[1 + \mathcal{O}(\frac{1}{x})\right], \quad x \to \infty, \tag{39}$$

$$G_c(x) = \frac{(2\pi)^{(c-1)/2}}{\sqrt{c}} \exp\left(-x\frac{c+1}{2c} - ce^{-x/c}\right)\left[1+\mathcal{O}(e^{x/c})\right], \quad x \to -\infty. \quad (40)$$

The Gumbel distribution is determinate because the moments

$$s_n = \int_{-\infty}^{\infty} x^n G(x)\,dx = \int_0^{\infty} (-\log t)^n e^{-t}\,dt$$

satisfy $s_{2n} \le 2(2n)!$. This shows that Carleman's condition $\sum 1/\sqrt[2n]{s_{2n}} = \infty$ is satisfied. By [4, Corollary 3.3] it follows that Carleman's condition is satisfied for all Gumbel roots $G_c, c > 0$, so they are all determinate.

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