# Some bipolar-preferences-involved aggregation methods for a sequence of OWA weight vectors

LeSheng Jin[a], Ronald R. Yager[b], Zhen-Song Chen[c,*], Jana Špirkovà[d], Daniel Paternain[e], Radko Mesiar[f], Humberto Bustince[g]

[a]*Business School, Nanjing Normal University, Nanjing, China;*
[b]*Machine Intelligence Institute, Iona College, New Rochelle, NY 10801;*
[c]*School of Civil Engineering, Wuhan University, Wuhan 430072, China;*
[d]*Faculty of Economics, Matej Bel University, Tajovského 10, Sk-975 90 Banská Bystrica, Slovakia;*
[e]*Department of Statistics, Computer Science and Mathematics, Public University of Navarre, Campus Arrosadia sn, 31006 Pamplona, Spain;*
[f]*Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, Sk-810 05 Bratislava, Slovakia; Palacky Univ Olomouc, Fac Sci, Dept Algebra & Geometry, 17 Listopadu 12, Olomouc 77146, Czech Republic;*
[g]*Department of Statistics, Computer Science and Mathematics, Public University of Navarre, Campus Arrosadia sn, 31006 Pamplona, Spain.*

## Abstract

The ordered weighted averaging (OWA) operator and its associated weight vectors have been both theoretically and practically verified to be powerful and effective in modeling the optimism/pessimism preference of decision makers. When several different OWA weight vectors are offered, it is necessary to develop certain techniques to aggregate them into one OWA weight vector. This study firstly details several motivating examples to show the necessity and usefulness of merging those OWA weight vectors. Then, by applying the general method for aggregating OWA operators proposed in a recent literature, we specifically elaborate the use of OWA aggregation to merge OWA weight vectors themselves. Furthermore, we generalize the normal preference degree in the unit interval into a preference sequence, and introduce subsequently the preference aggregation for OWA weight vectors with given preference sequences. Detailed steps in related aggregation procedures and corresponding numerical examples are also provided in the current study.

*Key words:* Aggregation functions, Decision making, Evaluation, OWA operators, Preference-involved aggregation

*Corresponding author

*Email addresses:* `jls1980@163.com` (LeSheng Jin), `yager@panix.com` (Ronald R. Yager), `zschen@whu.edu.cn` (Zhen-Song Chen), `jana.spirkova@umb.sk` (Jana Špirkovà), `daniel.paternain@unavarra.es` (Daniel Paternain), `radko.mesiar@stuba.sk` (Radko Mesiar), `bustince@unavarra.es` (Humberto Bustince)

## 1. Introduction

Aggregation functions (Grabisch et al., 2009, Klement et al., 2000)(also known as aggregation operators) are very powerful and important in a great variety of decision making and evaluation theories and applications (Jin et al., 2019b, Liu & Da, 2005, Ouyang, 2015, Paternain et al., 2012, Yager et al., 2011). Their theories have also been deeply studied during the last decades (Amarante, 2017, Chen et al., 2019, Choquet, 1954, Jin et al., 2018a, 2019a, Lizasoain & Moreno, 2013, Mesiar et al., 2018, 2015, Paternain et al., 2019, Torra, 1997, Yager, 2004). In general, in numerous evaluation problems, the information for aggregation is represented by a real finite vector (or a sequence) $\mathbf{x} = (x_i)_{i=1}^{n}$ , and the aim of aggregation is to return a final result that is still a real number through the implementation of strictly defined aggregation functions. An aggregation function can be classified into three different categories, (i) disjunctive functions, (ii) averaging functions, (iii) conjunctive functions, according to the output value compared against the input vector (Grabisch et al., 2009).

Averaging functions always return an output that is definitely located between the largest and the smallest of values in input vector. Due to this property, averaging functions are very suitable to be used in multi-criteria decision making and evaluation. Some famous averaging functions can be found in a wide range from mean, median, max and min, to ordering statistic (OS) and weighted averaging (WA) (Grabisch et al., 2009). In contrast, a conjunctive function always outputs a value not larger than the smallest value in input vector. Moreover, a disjunctive function can be actually understood as the dual cases of a conjunctive function, thereby leaving with a value as aggregation result that is not smaller than the largest value in input vector.

Within averaging functions, a practical type of significant importance is the preference-involved functions, which consider the optimism/pessimism preference of decision makers or managers involved in the evaluation problem and embody that preference in both of its aggregation process and result. A renowned and well-established category of preference-involved functions was introduced by Yager, called Ordered Weighted Averaging (OWA) operators (Yager, 1988). Instead of having fixed weights for each input value according to its position in the input vector as used in WA operators, an OWA operator always assigns different weights to those input values according to their magnitudes. The weight vectors used in OWA operators have been assigned a measurement called orness/andess (Yager, 1988), indicating the degree to which the optimism/pessimism attitudes are expressed and communicated from certain decision makers.

The most important and immediate extension of OWA operators is the Induced Ordered Weighted Averaging (IOWA) operators (Yager, 2003, Yager & Filev, 1999). This extension features on an additional type of index information called inductive vector $\mathbf{c} = (c_i)_{i=1}^n$ from which the weights allocation process is directed, rather than from the input value vector itself as in OWA operators. An inductive vector can take on some common and concrete variables such as time and importance variables, and thus the weights are correspondingly allocated to all the input values according to their chronological orders and relative importance to one another.

In both OWA and IOWA operators, some weight vectors are always involved to channel the preferences of decision makers into the aggregation process and to express explicitly the degrees of those preferences, not being related to magnitudes of input vector or the inductive vector. Hence, the determination of such weight vectors (also known as OWA weight vector or preference vector in some researches (Jin, 2016, Jin et al., 2018b, Yager et al., 2011)) plays a very crucial and deciding role in related aggregation processes. In the past decades, researchers proposed and developed a large variety of weights determination methods, with some derived and adapted from known mathematical results and others derived by theoretical or practical optimization objectives (Ouyang, 2015, Yager et al., 2011). No matter which method is applied, different decision makers may have different opinions and preferred weights generated from their own or others. In consequence, when different weight vectors are provided as candidates for OWA aggregation, the problem remains in which one is the appropriate or how to reasonably merge them together into one final accepted weight vector. With these elaborated backgrounds, this study will regard weight vectors as the objects for implementing a brand-new process of OWA aggregation, and therefore, it analyzes and proposes some reasonable and effective aggregation methods to address the raised problem in the foregoing.

The remainder of this work is organized as follows. In Section 2, we review and rephrase some basic concepts about OWA operators and some related extensions. In Section 3, we firstly introduce a practical example to show the applicability of OWA weight vectors in evaluation problem, and then we apply the general aggregation method for OWA operators to specifically use preference aggregation to merge OWA weight vectors. In Section 4, we generalize the normal preference provided within the unit interval to become a preference sequence using special techniques, which provides much more diversity to model bipolar preferences of decision makers. Then, by some other special techniques, we elaborately introduce the Preference aggregation for OWA weight vectors with given preference sequence. Section 5 summarizes and concludes this study.

## 2. Some reviews and discussions for OWA operators and related extensions

Throughout this study, without loss of generality, any real inputs (with dimension $n$) for aggregation is expressed either by a real sequence $\mathbf{x} = (x_i)_{i=1}^n$ or by a real vector $\mathbf{x} = (x_1, \cdots, x_n)$, equivalently. The space of all such sequences/vectors is conventionally denoted by $[0,1]^n$. An $n$-dimensional normalized weight vector used for aggregation is presented by $\mathbf{w} = (w_1, \cdots, w_n)$ or with a sequence form $\mathbf{w} = (w_i)_{i=1}^n$. In addition, the space of all such weight vectors with dimension $n$ is denoted by $\mathcal{W}^{<n>}$. Furthermore, a vector formed by $m$ weight vectors of dimension $n$ is consistently expressed by $W = (\mathbf{w}_1, \cdots, \mathbf{w}_m)$ or by a sequence form $W = (\mathbf{w}_j)_{j=1}^m$, and the space of all such vectors is correspondingly denoted by $(\mathcal{W}^{<n>})^m$.

In the sequel, we recap some basic concepts of aggregation functions in a strict and consistent way.

**Definition 1** (aggregation function) (Grabisch et al., 2009) An aggregation function (of dimension $n$) $F : [0,1]^n \to [0,1]$ is a mapping satisfying the following two conditions:

(i) (boundary conditions) $F(\mathbf{0}) = 0$ and $F(\mathbf{1}) = 1$ (where $\mathbf{0} = (0, \cdots, 0)$ and $\mathbf{1} = (1, \cdots, 1)$);

(ii) (monotonicity) For any $\mathbf{x}, \mathbf{y} \in [0,1]^n$, if $\mathbf{x} < \mathbf{y}$ (meaning $x_i \le y_i$ for all $i \in \{1, \cdots, n\}$ and there exist some $k \in \{1, \cdots, n\}$ such that $x_k < y_k$), then $F(\mathbf{x}) \le F(\mathbf{y})$

In what follows, we briefly review OWA operators and the RIM quantifier expressions.

**Definition 2** (OWA operator) (Yager, 1988) An OWA operator (of dimension $n$) with weight vector $\mathbf{w} \in \mathcal{W}^{<n>}$, $\mathsf{OWA}_{\mathbf{w}} : [0,1]^n \to [0,1]$, is defined as follows

$$\mathsf{OWA}_{\mathbf{w}}(\mathbf{x}) = \sum\nolimits_{i=1}^{n} w_i \cdot x_{\sigma(i)}, \tag{1}$$

where $\sigma : \{1, \cdots, n\} \to \{1, \cdots, n\}$ is any suitable permutation on $\{1, \cdots, n\}$ such that $x_{\sigma(i)} \geq x_{\sigma(j)}$ whenever $i < j$.

With the introduction of OWA operators, Yager also defined the orness/andness degree of any weight vector used in the OWA aggregation. Orness/andness can effectively measure the optimism/pessimism degree embodied in the whole OWA aggregation procedures. In general, the larger orness, the more extent of optimism, and vice versa.

**Definition 3** (orness/andness) (Yager, 1988) The orness of any weight vector $\mathbf{w} \in \mathcal{W}^{<n>}$ used in OWA aggregation is defined as a mapping $orness : \mathcal{W}^{<n>} \to [0, 1]$ such that

$$orness(\mathbf{w}) = \sum\nolimits_{i=1}^{n} \frac{n-i}{n-1} \cdot w_i. \tag{2}$$

In duality, the andness of any weight vector $\mathbf{w} \in \mathcal{W}^{<n>}$ is defined as a function $andness : \mathcal{W}^{<n>} \to [0, 1]$ such that

$$andness(\mathbf{w}) = 1 - orness(\mathbf{w}). \tag{3}$$

Soon afterwards, Yager introduced an ingenious and convenient way to generate weight vector used in OWA operator. The method is based on a well defined non-decreasing function on unit interval to efficiently model bipolar preference (e.g., optimism/pessimism preference) of decision makers.

**Definition 4** (Yager, 1996) A function $Q : [0, 1] \to [0, 1]$ is called a Regular Increasing Monotone (RIM) quantifier if $Q$ satisfies $Q(0) = 0$, $Q(1) = 1$ and $Q(a) \geq Q(b)$ whenever $a > b$. We also denote by $\mathcal{Q}$ the space of all RIM quantifiers.

Based on given RIM quantifiers, a large diversity of weight vectors can be easily generated by the following method.

**Definition 5** (Yager, 1996) Given a RIM quantifier $Q \in \mathcal{Q}$, a weight vector $\mathbf{w}^{<Q>} = (w_i^{<Q>})_{i=1}^{n} \in \mathcal{W}^{<n>}$ is called the *Q-generated OWA weight vector (of dimension n)* if it satisfies for all $i \in \{1, \cdots, n\}$,

$$w_i^{<Q>} = Q\left(i/n\right) - Q\left((i-1)/n\right). \tag{4}$$

Similar to orness/andness of weight vectors, the orness/andness degree can effectively measure the extent to which a bipolar preference (e.g., optimism/pessimism preference) is communicated by relevant decision makers. Moreover, the weights allocation methods derived from Definitions 5 and 6 together with some special proposals in Pu et al. (2019) are extremely useful in generating a required continuous system of OWA weight vectors for later use in this study. In addition, recall that any RIM quantifier $Q$ is monotonic and defined on a closed interval, and thus from basic real analysis they guarantees that $Q$ is necessarily Riemann integrable.

**Definition 6** (Yager, 1996) The orness of any RIM quantifier $Q$ is defined by $orness : \mathcal{Q} \to [0, 1]$ such that

$$orness(Q) = \int_0^1 Q(y)dy. \tag{5}$$

Dually, the andness of any RIM quantifier $Q$ can be defined by $andness : \mathcal{Q} \to [0, 1]$ such that

$$andness(Q) = 1 - orness(Q) = 1 - \int_0^1 Q(y)dy. \tag{6}$$

Above definitions of orness/andness for RIM quantifiers are strictly based on this deduction: $orness(Q) = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{n-i}{n-1} \left[Q(\frac{i}{n}) - Q(\frac{i-1}{n})\right] = \lim\limits_{n\to\infty} \frac{1}{n-1} \sum\limits_{i=1}^{n-1} Q(\frac{i}{n}) = \int_0^1 Q(y)dy$. Moreover, if $Q$ is absolutely continuous, then there necessarily exists a normalized density function $q : [0, 1] \to [0, +\infty)$ such that $Q(y) = \int_0^y q(t)dt$ and $\int_0^1 q(t)dt = 1$. Hence, in the foregoing situation, Equation (4) can be also equivalently expressed by

$$w_i^{<Q>} = Q\left(i/n\right) - Q\left((i-1)/n\right) = \int_{(i-1)/n}^{i/n} q(t)dt. \tag{7}$$

## 3. Merging methods for OWA weight vectors using bipolar preference aggregation

In Section 1, we have stressed that effectively merging some methods for a sequence of OWA weight vectors $W = (\mathbf{w}_j)_{j=1}^m \in (\mathcal{W}^{<n>})^m$ is important. Note that such methods can be directly transferred to the merging methods for other types of bipolar preferences. For example, the linearly ordered evaluation scale {very important, important, less important, not important} also serves as a base on which an OWA weight vector of dimension 4 can be defined. To evaluate the importance of a certain object, some experts are invited and divided into three group based on their expertise such as economics for group 1, technology for group 2 and politics for group 3. When the three different groups of experts hand in three opinions based on their respective statistic in group, they may be, say, with $\mathbf{w}_1 = (0.1, 0.4, 0.3, 0.2)$ from group 1, $\mathbf{w}_2 = (0.3, 0.3, 0.1, 0.3)$ from group 2 and $\mathbf{w}_3 = (0.5, 0.2, 0.2, 0.1)$ from group 3. These bipolar preference information indicate that in group 1 there are 10% of experts submitted their opinion with "very important", 40% with "important", 30% with "less important", and 20% with "not important"; and so forth for group 2 and 3. How to devise some reasonable methods to merge these three pieces of OWA weight vectors into a final representative one in management practices to evaluate objects and make decisions is helpful.

In this section, we discuss some feasible techniques to handle the above-mentioned type of aggregation using OWA aggregation involving preferences.

As we know in OWA aggregation, the successful weights allocation cannot be carried out without a linear ordering of magnitudes of each input values. However, when those input real values are replaced with OWA weight vectors, we also need to try to find a suitable linear ordering (or with the environment of *poset*) for them before weights being further allocated to them. For any two OWA weight vectors, the following rule allows to decide whether they can be compared or not.

**Definition 7** (Jin & Mesiar, 2017) For any OWA weight vector $\mathbf{w} = (w_i)_{i=1}^n \in \mathcal{W}^{<n>}$, we define $\tilde{\mathbf{w}} = (\vec{w}_i)_{i=1}^n \in [0,1]^n$ to be the *accumulation* of $\mathbf{w}$ such that $\vec{w}_i = \sum_{k=1}^i w_k$.

**Proposition 1** (Jin & Mesiar, 2017) The orness of OWA weight vector $\mathbf{w}$, $orness(\mathbf{w})$ , can be equivalently expressed by

$$orness(\mathbf{w}) = \frac{1}{n-1} \sum\nolimits_{i=1}^{n-1} \vec{w}_i. \tag{8}$$

Based on accumulations of OWA weight vectors, $\mathcal{W}^{<n>}$ can be extended to become a complete lattice $(\mathcal{W}^{<n>}, \preceq)$, which is defined such that for any two OWA weight vectors $\mathbf{w}_1 = (w_{1i})_{i=1}^n \in \mathcal{W}^{<n>}$ and $\mathbf{w}_2 = (w_{2i})_{i=1}^n \in \mathcal{W}^{<n>}$, $\mathbf{w}_1 \preceq \mathbf{w}_2$ if and only if $\vec{w}_{1i} \leq \vec{w}_{2i}$ holds for all $i \in \{1, \cdots, n\}$ (Jin & Mesiar, 2017). This fact together with Equation (8) tells from a clearer way that larger entries in the accumulation of $\mathbf{w}$ lead to larger orness of its original OWA weight vector $\mathbf{w}$ and vice versa, and thereby indicating a more optimist preference from decision makers.

If any two OWA weight vectors can be comparable, then we can easily allocate weights for all sequences of such vectors, just as it is done in the weights allocation in OWA operators. Consider aforementioned three OWA weight vectors $\mathbf{w}_1 = (0.1, 0.4, 0.3, 0.2)$, $\mathbf{w}_2 = (0.3, 0.3, 0.1, 0.3)$ and $\mathbf{w}_3 = (0.5, 0.2, 0.2, 0.1)$. Their accumulations are $\tilde{\mathbf{w}}_1 = (0.1, 0.5, 0.8, 1)$, $\tilde{\mathbf{w}}_2 = (0.3, 0.6, 0.7, 1)$ and $\tilde{\mathbf{w}}_3 = (0.5, 0.7, 0.9, 1)$, respectively. We find that $\tilde{\mathbf{w}}_1 \preceq \tilde{\mathbf{w}}_3$ and $\tilde{\mathbf{w}}_2 \preceq \tilde{\mathbf{w}}_3$, but $\tilde{\mathbf{w}}_1$ and $\tilde{\mathbf{w}}_2$ are not comparable. Since there are infinitely many pairs of OWA weight vectors that cannot be compared, we need to use some other aggregation techniques to perform preference aggregation for OWA weight vectors.

Before introducing the detail aggregation procedures, in what follows we observe some facts about the relations between OWA weight vectors and their accumulations. A vector $\mathbf{a} = (a_i)_{i=1}^n \in [0,1]^n$ is called non-decreasing if $a_n = 1$ and for any $i \in \{1, \cdots, n-1\}$, $a_i \leq a_{i+1}$. Denote by $\vec{\mathcal{W}}^{<n>}$ the space of all such non-decreasing vectors of dimension $n$. Observe that for any $\mathbf{w} \in \mathcal{W}^{<n>}$, there is a bijective transformation $T : \mathcal{W}^{<n>} \to \vec{\mathcal{W}}^{<n>}$ such that $T(\mathbf{w}) = \tilde{\mathbf{w}}$, and its inverse transformation $T^{-1} : \vec{\mathcal{W}}^{<n>} \to \mathcal{W}^{<n>}$ is such that for any $i \in \{1, \cdots, n\}$, $w_i = \vec{w}_i - \vec{w}_{i-1}$ ($\vec{w}_0 \triangleq 0$). Hence, sometimes we can circumvent a difficulty of aggregating weight vectors by trying to aggregate their accumulations into a single resulting accumulation and then transform it back into a final weight vector by $T^{-1}$. This method based on transformation as a general aggregation method for OWA operators has been discussed in Pu et al. (2019). One method discussed in what follows borrowing accumulations of those vectors is a special application of the general aggregation technique for OWA operators.

The *preference aggregation for OWA weight vectors based on accumulations* contains the following steps.

*Step 1*: collect a sequence of $m$ OWA weight vectors $W = (\mathbf{w}_j)_{j=1}^m = ((w_{ji})_{i=1}^n)_{j=1}^m$ for further aggregation.

*Step 2*: use $T$ to transform those m OWA weight vectors into a sequence of $m$ accumulations of themselves $\vec{W} = (\tilde{\mathbf{w}}_j)_{j=1}^m = ((\vec{w}_{ji})_{i=1}^n)_{j=1}^m$.

*Step 3*: using an orness degree $\alpha$ to indicate a given fixed extent of preference, and determine an OWA weight vector of dimension m $\mathbf{v} \in \mathcal{W}^{<m>}$ with $orness(\mathbf{v}) = \alpha$, representing the optimism/pessimism preference of certain decision making for the aggregation of $(\mathbf{w}_j)_{j=1}^m$.

*Step 4*: for each $i \in \{1, \cdots, n\}$, using OWA operator $\mathsf{OWA}_\mathbf{v} : [0,1]^m \to [0,1]$

to aggregate sequence $(\vec{w}_{ji})_{j=1}^{m}$ and obtain result $\mathsf{OWA}_{\mathbf{v}}((\vec{w}_{ji})_{j=1}^{m})$.

*Step 5*: form a non-decreasing vector $\tilde{\mathbf{q}} = (\vec{q}_i)_{i=1}^{n}$ with $\vec{q}_i = \mathsf{OWA}_{\mathbf{v}}((\vec{w}_{ji})_{j=1}^{m})$. Note that $\mathsf{OWA}_{\mathbf{v}}$ is non-decreasing and for each $j \in \{1, \cdots, m\}$ $\tilde{\mathbf{w}}_j$ is non-decreasing, then $\tilde{\mathbf{q}}$ is non-decreasing. Thus, it allows for $\tilde{\mathbf{q}}$ to be capable of being transformed back into an OWA weight vector $\mathbf{q}$ as the final desired aggregation result such that $\mathbf{q} = T^{-1}(\tilde{\mathbf{q}})$.

Below, we present a simple numerical example using the three OWA weight vectors aforementioned for the preference aggregation for OWA weight vectors based on accumulations.

*Step 1*: collect a sequence of 3 OWA weight vectors $W = (\mathbf{w}_j)_{j=1}^{3} = ((w_{ji})_{i=1}^{4})_{j=1}^{3}$ for further aggregation with $\mathbf{w}_1 = (0.1, 0.4, 0.3, 0.2)$, $\mathbf{w}_2 = (0.3, 0.3, 0.1, 0.3)$ and $\mathbf{w}_3 = (0.5, 0.2, 0.2, 0.1)$.

*Step 2*: use $T$ to transform those 3 OWA weight vectors into a sequence of 3 accumulations of themselves $\vec{W} = (\tilde{\mathbf{w}}_j)_{j=1}^{3} = ((\vec{w}_{ji})_{i=1}^{4})_{j=1}^{3}$ with $\tilde{\mathbf{w}}_1 = (0.1, 0.5, 0.8, 1)$, $\tilde{\mathbf{w}}_2 = (0.3, 0.6, 0.7, 1)$ and $\tilde{\mathbf{w}}_3 = (0.5, 0.7, 0.9, 1)$.

*Step 3*: determine an orness degree $\alpha = 1/3$ to indicate a given fixed extent of preference, and determine an OWA weight vector of dimension 3, $\mathbf{v} \in \mathcal{W}^{<3>}$, $\mathbf{v} = (1/6, 1/3, 1/2)$ with $orness(\mathbf{v}) = \alpha = 1/3$, representing a moderate pessimism preference for the aggregation of $(\mathbf{w}_j)_{j=1}^{3}$.

*Step 4*: for each $i \in \{1, 2, 3, 4\}$, using OWA operator $\mathsf{OWA}_{\mathbf{v}} : [0,1]^3 \to [0,1]$ to aggregate sequence $(\vec{w}_{ji})_{j=1}^{3}$ and obtain result $\mathsf{OWA}_{\mathbf{v}}((\vec{w}_{ji})_{j=1}^{3})$. With computing we have,

$\mathsf{OWA}_{\mathbf{v}}((\vec{w}_{j1})_{j=1}^{3}) = \mathsf{OWA}_{\mathbf{v}}(0.1, 0.3, 0.5) = (1/6)(0.1)+(1/3)(0.3)+(1/2)(0.5) = 0.3$,

$\mathsf{OWA}_{\mathbf{v}}((\vec{w}_{j2})_{j=1}^{3}) = \mathsf{OWA}_{\mathbf{v}}(0.5, 0.6, 0.7) = (1/6)(0.5)+(1/3)(0.6)+(1/2)(0.7) = 0.633$,

$\mathsf{OWA}_{\mathbf{v}}((\vec{w}_{j3})_{j=1}^{3}) = \mathsf{OWA}_{\mathbf{v}}(0.8, 0.7, 0.9) = (1/6)(0.7)+(1/3)(0.8)+(1/2)(0.9) = 0.833$,

$\mathsf{OWA}_{\mathbf{v}}((\vec{w}_{j4})_{j=1}^{3}) = (1, 1, 1) = 1$.

*Step 5*: form a non-decreasing vector $\tilde{\mathbf{q}} = (\vec{q}_i)_{i=1}^{4}$ with $\vec{q}_i = \mathsf{OWA}_{\mathbf{v}}((\vec{w}_{ji})_{j=1}^{3})$. Then, by $\mathbf{q} = T^{-1}(\tilde{\mathbf{q}})$, obtain the final aggregation result $\mathbf{q} = (0.3, 0.333, 0.2, 0.167)$.

## 4. Preference aggregation for OWA weight vectors with given preference sequence

Note that the OWA weight vector $\mathbf{v} \in \mathcal{W}^{<m>}$ in the forgoing Step 3 just discussed in Section 3 is constant for all the $n$ times of OWA aggregation $\mathsf{OWA}_{\mathbf{v}}$ with $(\vec{w}_{ji})_{j=1}^{m}$ $(i \in \{1, \cdots, n\})$, somewhat eluding the achieving of larger diversity of aggregation methods. In this section, we present a diversified preference aggregation for OWA weight vectors still with given fixed extent of preferences. Recall from the forgoing Step 3, a given fixed extent of preference given by decision maker is expressed using orness degree. Rather than still using such a single fixed orness value $\alpha$ within unit interval [0,1] to represent the given preference, we next express such given preference as a non-decreasing sequence of $n-1$ values $\alpha = (\alpha_i)_{i=1}^{n-1} \in [0,1]^{n-1}$, called *preference sequence (with degree n)* $(n \geq 3)$, such that (i) $\frac{1}{n-1}\sum_{i=1}^{n-1}\alpha_i = \alpha$, and (ii) $\alpha_i \leq \alpha_j$ whenever $1 \leq i < j \leq n-1$.

Another necessary part for the intended aggregation method is to introduce a (*continuous*) *system of OWA weight vectors* (Pu et al., 2019) of dimension $m$ $(\mathbf{v}_t)_{t\in[0,1]} = ((v_{tj})_{j=1}^{m})_{t\in[0,1]}$ such that (i) $orness(\mathbf{v}_t) = t$, and (ii) for each $j \in \{1, \cdots, m\}$, $\vec{v}_{t_1 j} \leq \vec{v}_{t_2 j}$ whenever $0 \leq t_1 < t_2 \leq 1$. Note that with above two conditions, for each $j \in \{1, \cdots, m\}$ , both $v_{tj}$ and $\vec{v}_{tj}$ are surely continuous with respect to variable $t \in [0,1]$ (Pu et al., 2019). In addition, given any system of OWA weight vectors $(\mathbf{v}_t)_{t\in[0,1]}$, if $0 \leq t_1 < t_2 \leq 1$, then $\mathsf{OWA}_{\mathbf{v}_{t_1}} \leq \mathsf{OWA}_{\mathbf{v}_{t_2}}$ (i.e., $\mathsf{OWA}_{\mathbf{v}_{t_1}}(\mathbf{x}) \leq \mathsf{OWA}_{\mathbf{v}_{t_2}}(\mathbf{x})$ holds for all $\mathbf{x} \in [0,1]^n$) (Pu et al., 2019).

With above mentioned two important tools, preference sequence and system of OWA weight vectors, and their related properties, when a fixed extent of optimism $\alpha \in [0,1]$ is communicated by a decision maker, we can create a preference sequence $\alpha$ to model the preference of that decision maker in more diversified ways. Then, we can use a system of OWA weight vectors $(\mathbf{v}_t)_{t\in[0,1]}$ and single out $n-1$ OWA weight vectors of dimension m from it with $\mathbf{v}_{\alpha_1}$, $\mathbf{v}_{\alpha_2}$, …, and $\mathbf{v}_{\alpha_{n-1}}$ so that $\frac{1}{n-1}\sum_{i=1}^{n-1}\alpha_i = \alpha$ . Recall that using a system of RIM quantifiers, one can easily generate a system of OWA weight vectors by Definitions 4-6, together with the properties of continuous system of RIM quantifiers. For more of this special technique, one may refer to (Pu et al., 2019).

Next, using OWA operators $\mathsf{OWA}_{\mathbf{v}_{\alpha_i}} : [0,1]^m \to [0,1]$ to aggregate $(\vec{w}_{ji})_{j=1}^{m}$ for each $i \in \{1, \cdots, n-1\}$. Also note that for any aggregation function $F$ to be used, we have $F((\vec{w}_{jn})_{j=1}^{m}) = F(\mathbf{1}) = 1$ since for any $\tilde{\mathbf{w}} = (\vec{w}_i)_{i=1}^{n} \in \vec{\mathcal{W}}^{<n>}$, it holds $\vec{w}_n = 1$. At last, an accumulation $\tilde{\mathbf{q}} = (\vec{q}_i)_{i=1}^{n}$ is formed with $\vec{q}_i = \mathsf{OWA}_{\mathbf{v}_{\alpha_i}}((\vec{w}_{ji})_{j=1}^{m})$ and thus an OWA weight vector $\mathbf{w}$ can be transformed from it to be as the final aggregation result. It is also noteworthy that the resulted OWA weight vector $\mathbf{w}$ satisfies that $\mathbf{w}_j \preceq \mathbf{w}$ for all $j \in \{1, \cdots, m\}$ where $(\mathbf{w}_j)_{j=1}^m = ((w_{ji})_{i=1}^n)_{j=1}^m$ is an input sequence of $m$ OWA weight vectors prepared for aggregation.

The corresponding *preference aggregation for OWA weight vectors based on accumulations with given preference sequence* is divided into the following detailed steps.

*Step 1*: collect a sequence of $m$ OWA weight vectors $W = (\mathbf{w}_j)_{j=1}^m = ((w_{ji})_{i=1}^n)_{j=1}^m$ for further aggregation.

*Step 2*: use $T$ to transform those $m$ OWA weight vectors into a sequence of $m$ accumulations of themselves $\vec{W} = (\tilde{\mathbf{w}}_j)_{j=1}^m = ((\vec{w}_{ji})_{i=1}^n)_{j=1}^m$.

*Step 3*: form a preference sequence $\alpha = (\alpha_i)_{i=1}^{n-1} \in [0,1]^{n-1}$ with $\frac{1}{n-1}\sum_{i=1}^{n-1} \alpha_i = \alpha$ to model a given fixed extent of preference with $\alpha \in [0,1]$.

*Step 4*: select a system of OWA weight vectors $(\mathbf{v}_t)_{t\in[0,1]} = ((v_{tj})_{j=1}^m)_{t\in[0,1]}$ and single out a collection of $n-1$ OWA weight vectors of dimension $m$ from it with $\mathbf{v}_{\alpha_1}$, $\mathbf{v}_{\alpha_2}$, ..., and $\mathbf{v}_{\alpha_{n-1}}$ .

*Step 5*: for each $i \in \{1, \cdots, n-1\}$, using the OWA operator $\mathsf{OWA}_{\mathbf{v}_{\alpha_i}} : [0,1]^m \to [0,1]$ to aggregate sequence $(\vec{w}_{ji})_{j=1}^m$ and obtain result $\mathsf{OWA}_{\mathbf{v}_{\alpha_i}}((\vec{w}_{ji})_{j=1}^m)$.

*Step 6*: this last step forms a non-decreasing vector $\tilde{\mathbf{q}} = (\vec{q}_i)_{i=1}^n$ with $\vec{q}_i = \mathsf{OWA}_{\mathbf{v}_{\alpha_i}}((\vec{w}_{ji})_{j=1}^m)$ when $i \in \{1, \cdots, n-1\}$ and $\vec{q}_n = 1$. Still, for each $j \in \{1, \cdots, m\}$, $\tilde{\mathbf{w}}_j$ is non-decreasing. Recall also that we have shown $\mathsf{OWA}_{\mathbf{v}_{\alpha_i}} \le \mathsf{OWA}_{\mathbf{v}_{\alpha_{i+1}}}$ for any $i \in \{1, \cdots, n-1\}$. Consequently, $\tilde{\mathbf{q}}$ is evidently non-decreasing. Thus, it still allows for $\tilde{\mathbf{q}}$ to be transformed back into an OWA weight vector $\mathbf{q}$ as the final aggregation result such that $\mathbf{q} = T^{-1}(\tilde{\mathbf{q}})$.

Likewise, we present a detailed numerous example to help better illustrate the foregoing detailed procedures.

*Step 1*: collect a sequence of 3 OWA weight vectors $W = (\mathbf{w}_j)_{j=1}^3 = ((w_{ji})_{i=1}^4)_{j=1}^3$ for further aggregation with $\mathbf{w}_1 = (0.1, 0.4, 0.3, 0.2)$, $\mathbf{w}_2 = (0.3, 0.3, 0.1, 0.3)$ and $\mathbf{w}_3 = (0.5, 0.2, 0.2, 0.1)$.

*Step 2*: use $T$ to transform those 3 OWA weight vectors into a sequence of 3 accumulations of themselves $\vec{W} = (\tilde{\mathbf{w}}_j)_{j=1}^3 = ((\vec{w}_{ji})_{i=1}^4)_{j=1}^3$ with $\tilde{\mathbf{w}}_1 = (0.1, 0.5, 0.8, 1)$, $\tilde{\mathbf{w}}_2 = (0.3, 0.6, 0.7, 1)$ and $\tilde{\mathbf{w}}_3 = (0.5, 0.7, 0.9, 1)$.

*Step 3*: using a preference sequence $\alpha = (\alpha_i)_{i=1}^3 = (1/6, 1/3, 1/2) \in [0,1]^3$ with $\frac{1}{3}\sum_{i=1}^3 \alpha_i = \alpha = 1/3$ to model a given fixed extent of preference with $\alpha = 1/3 \in [0,1]$.

*Step 4*: select a system of OWA weight vectors $(\mathbf{v}_t)_{t\in[0,1]} = ((v_{tj})_{j=1}^3)_{t\in[0,1]}$ and single out a collection of 3 OWA weight vectors of dimension 3 from it with $\mathbf{v}_{\alpha_1} = \mathbf{v}_{1/6}$, $\mathbf{v}_{\alpha_2} = \mathbf{v}_{1/3}$, and $\mathbf{v}_{\alpha_3} = \mathbf{v}_{1/2}$. The system $(\mathbf{v}_t)_{t\in[0,1]}$ can be decided as follows:

(i) if $t \in [0, 0.5]$, then $\mathbf{v}_t = (v_{t1}, v_{t2}, v_{t3}) = (t/2, t, 1-(3t/2))$;

(ii) if $t \in (0.5, 1]$, then $\mathbf{v}_t = (v_{t1}, v_{t2}, v_{t3}) = ((3t-1)/2, 1-t, (1-t)/2))$. One may easily check the continuities with $t \in [0,1]$ in each $v_{tj}$, $j \in \{1,2,3\}$, and then verify that $(\mathbf{v}_t)_{t\in[0,1]}$ is indeed a system of OWA weight vectors. With above decided system, we then obtain

$\mathbf{v}_{\alpha_1} = \mathbf{v}_{1/6} = (1/12, 1/6, 3/4)$, $\mathbf{v}_{\alpha_2} = \mathbf{v}_{1/3} = (1/6, 1/3, 1/2)$, and $\mathbf{v}_{\alpha_3} = \mathbf{v}_{1/2} = (1/4, 1/2, 1/4)$.

*Step 5*: for each $i \in \{1,2,3\}$, using OWA operator $\mathsf{OWA}_{\mathbf{v}_{\alpha_i}} : [0,1]^3 \to [0,1]$ to aggregate sequence $(\vec{w}_{ji})_{j=1}^3$ and obtain result $\mathsf{OWA}_{\mathbf{v}_{\alpha_i}}((\vec{w}_{ji})_{j=1}^3)$. With computing, we obtain

$\mathsf{OWA}_{\mathbf{v}_{\alpha_1}}((\vec{w}_{j1})_{j=1}^3) = (1/12)(0.1) + (1/6)(0.3) + (3/4)(0.5) = 0.433$,
$\mathsf{OWA}_{\mathbf{v}_{\alpha_2}}((\vec{w}_{j2})_{j=1}^3) = (1/6)(0.5) + (1/3)(0.6) + (1/2)(0.7) = 0.633$,
$\mathsf{OWA}_{\mathbf{v}_{\alpha_3}}((\vec{w}_{j3})_{j=1}^3) = (1/4)(0.7) + (1/2)(0.8) + (1/4)(0.9) = 0.8$.

*Step 6*: this last step forms a non-decreasing vector $\tilde{\mathbf{q}} = (\vec{q}_i)_{i=1}^4$ with $\vec{q}_i = \mathsf{OWA}_{\mathbf{v}_{\alpha_i}}((\vec{w}_{ji})_{j=1}^3)$ when $i \in \{1,2,3\}$ and $\vec{q}_4 = 1$. Then, $\tilde{\mathbf{q}} = (0.433, 0.633, 0.8, 1)$, and therefore $\mathbf{q} = T^{-1}(\tilde{\mathbf{q}}) = (0.433, 0.2, 0.167, 0.2)$.

Observe that the algorithm complexity of all the proposed methods in this study are very practical and only around $mn$ level, without involving any exponent and factorial level computing.

In the below, we summarize some properties as desired in the preference aggregation for OWA weight vectors based on accumulations with given preference sequence. I (i) and (ii) are evident to observe; and II (i) is due to the basic monotonicity of aggregation functions, while II (ii) is obtained directly from the fact that the system of OWA weight vectors of dimension $m$: $(\mathbf{v}_t)_{t\in[0,1]} = ((v_{tj})_{j=1}^m)_{t\in[0,1]}$ is continuous (Pu et al., 2019).

I (Two Idempotencies)

(i) When the $m$ OWA weight vectors are identical, i.e., $\mathbf{w}_j = \mathbf{w}$ for any

$j \in \{1, \cdots, m\}$, then the final aggregation result $\mathbf{q} = \mathbf{w}$ due to the fact that every OWA operator is idempotent.

(ii) When the preference sequence $\alpha = (\alpha_i)_{i=1}^{n-1}$ satisfies $\alpha_i = \alpha$ for each $i \in \{1, \cdots, n-1\}$, then this aggregation degenerate into the preference aggregation for OWA weight vectors based on accumulations discussed in Section 3.

II (Two monotonicities)

(i) For any two sequences of $m$ OWA weight vectors $W = (\mathbf{w}_j)_{j=1}^m = ((w_{ji})_{i=1}^n)_{j=1}^m$ and $U = (\mathbf{u}_j)_{j=1}^m = ((u_{ji})_{i=1}^n)_{j=1}^m$, if $\mathbf{w}_j \preceq \mathbf{u}_j$ for each $j \in \{1, \cdots, m\}$, then $\mathbf{q} \preceq \mathbf{p}$, where $\tilde{\mathbf{q}} = (\vec{q}_i)_{i=1}^n$ with $\vec{q}_i = \mathsf{OWA}_{\mathbf{v}_{\alpha_i}}((\vec{w}_{ji})_{j=1}^m)$ and $\tilde{\mathbf{p}} = (\vec{p}_i)_{i=1}^n$ with $\vec{p}_i = \mathsf{OWA}_{\mathbf{v}_{\alpha_i}}((\vec{u}_{ji})_{j=1}^m)$ for any $i \in \{1, \cdots, n-1\}$.

(ii) For any two preference sequences $\alpha = (\alpha_i)_{i=1}^{n-1}$ and $\beta = (\beta_i)_{i=1}^{n-1}$, if $\alpha_i \leq \beta_i$ for each $i \in \{1, \cdots, n-1\}$, then $\mathbf{q} \preceq \mathbf{p}$, where $\tilde{\mathbf{q}} = (\vec{q}_i)_{i=1}^n$ with $\vec{q}_i = \mathsf{OWA}_{\mathbf{v}_{\alpha_i}}((\vec{w}_{ji})_{j=1}^m)$ and $\tilde{\mathbf{p}} = (\vec{p}_i)_{i=1}^n$ with $\vec{p}_i = \mathsf{OWA}_{\mathbf{v}_{\beta_i}}((\vec{w}_{ji})_{j=1}^m)$ for any $i \in \{1, \cdots, n-1\}$.

## 5. Conclusions

Diversity of aggregation operators provides more possibilities and flexibilities for decision makers to choose and aggregate given inputs in evaluation problems. It also enriches the theories about automated data aggregation and computational intelligence in evaluation. OWA operators have proved to be a category of powerful and instrumental aggregation functions, and the general method for aggregation of OWA operators was also proposed in a recent literature.

We firstly showed the importance and usefulness of OWA weight vectors in decision making and evaluation practices. Then, this study focused on a special case of aggregations for OWA operators, the preference-involved aggregation. We elaborately formulated the aggregation problem and discussed the detailed aggregation procedures.

Furthermore, we introduced a preference induced aggregation method based on the given degree of optimism that has been further expressed as a preference sequence (with degree $n$) rather than a normal value within the unit interval, thereby offering wider diversity of aggregation styles and results. The second method we have proposed in this study also serves as a generalization of the first method discussed. Some numerical examples were provided to facilitate the applying of those proposals in practices. In addition, some mathematical properties with two types of idempotencies and two types of monotonicities were also discussed.

## Acknowledgment

This work was supported by the Scientific Research Start-up Foundation Project (grant no. 184080H202B165), the Science and Technology Assistance Agency (grant no. APVV-18-0052), the project of Grant Agency of the Czech Republic (GAČR) (grant no. 18-06915S), the National Natural Science Foundation of China (grant no. 71801175) and the Fundamental Research Funds for the Central Universities (grant no. 2042018kf0006).

## Compliance with Ethical Standards

- **Conflict of Interest**

  Authors declare that they have no conflict of interest.

- **Informed Consent**

  Informed consent was not required as no human or animals were involved.

- **Human and Animal Rights**

  This article does not contain any studies with human or animal subjects performed by any of the authors.

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