Pohlke Theorem: Demonstration and Graphical Solution.

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Abstract. It is known that the axonometric defined by Pohlke, is geometrically known as a means of representing the figures of space using a cylindrical projection and proportions. His theorem says that the three unit vectors orthogonal axes of the basis in the space can be transformed into three arbitrary vectors with common origin located in the frame plane. Another way of expressing this theorem is given in three segments mismatched and incidents at one point in a plane, there is a trirectangular unitary thriedra in the space that can be transformed in these three segments. This paper presents a graphical procedure to demonstrate a solution of Pohlke's theorem. To do this, we start from previous work by the authors on the axonometric perspective. Graphic constructions that allow a single joint invariant description of relationships between an orthogonal axonometric oblique axonometric system and systems associated thereby. At a same time of the geometric locus generated by the diagonal magnitude positioned at any direction in the plane of the picture. This magnitude is the square root of the sum of the squares of the projection of the three segments representing axonometric on arbitrary magnitude.

Keywords: Axonometric system, Descriptive geometry, Pohlke theorem.

1 Introduction

This research deals with the oblique cylindrical projection in order to express graphically both Pohlke´s theorem as well as the more general axonometric stablishment [1]. *Main axonometric system related views as tilt of the coordinate planes* [2] was employed as a starting point and was stablished one arbitrary orthogonal projection to project. In *New constructions in axonometric system fundamentals* [3] the former study was extended, presenting new operations of construction of the perspective starting from the singular position of the ortogonal views that are used in this work. In *Intrinsic relations between the orthogonal axonometric system and its associated obliques. Analytical proposal and graphic operations* [4] was pretended to extend the approach of constructive and analytical aspects of the axonometric system, from the orthogonal to the oblique. The intrinsic triangle of the axonometry was defined and its geometrical properties were enforced. Projective and metric relationships were used on the studied figures and this focus permitted, the development of new axonometric constructions. In this paper, authors demonstrate and solute graphically the Pholke´s Theorem and give a graphical sense operative procedure to the general axonometric approach. This procedure enables the comprenhension and application of graphical engineering techniques.

2 Orthogonal and oblique axonometric system

An ortogonal trihedral is chosen as a reference system $Oxyz$. Taken from the vertex $O=[0,0,0]$ on the lines x, y y z respectively a unitary magnitude u, points $\mathbf{I}=[u,0,0]$, $\mathbf{J}=[0,u,0]$ y $\mathbf{K}=[0,0,u]$ are determined. The frame plane is $\pi_p \equiv a_x x + a_y y + a_z z = 0$ whose direction $d_\pi \equiv [a_x, a_y, a_z]$, verifying $\sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2} = u$. The ortogonal projection of the points $\mathbf{I}$, $\mathbf{J}$ y $\mathbf{K}$ on the frame plane determines the points $I = \left[u^2 - a_x{}^2, -a_x a_y, -a_x a_z\right]/u$, $J = \left[-a_x a_y, u^2 - a_y{}^2, -a_y a_z\right]/u$ y $K = \left[-a_x a_z, -a_y a_z, u^2 - a_z{}^2\right]/u$.

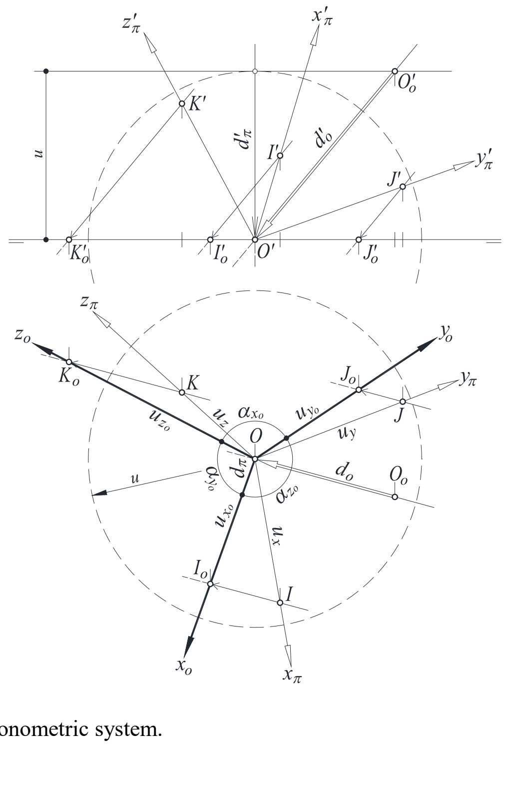

Fig. 1. Axonometric system.

If axis of the trihedra $Oxyz$ are projected on the plane π_p on an arbitrary direction d_o , the axonometric oblique perspective is obtained. This oblique direction is $d_o \equiv [a_{xo}, a_{yo}, a_{zo}]$, verfying $\sqrt{a_x a_{xo} + a_y a_{yo} + a_z a_{zo}} = u$. The final point associated to the project direction is O_o, and its ortogonal projection onto the frame plane is $O_o = [a_{xo} - a_x, a_{yo} - a_y, a_{zo} - a_z]$. Figure 1 shows that the measure of this point is allways the unity. The distance from this point to the origin defines the first fundamental lenght $l_a = |\overline{OO_o}| = u\sqrt{e_o{}^2 - 1}$. The obliqual projection of the points $\mathbf{I}$, $\mathbf{J}$ and $\mathbf{K}$ onto the frame plane determines the the points $I_o = [u^2 - a_x a_{xo}, -a_x a_{yo}, -a_x a_{zo}]/u$, $J_o = [-a_y a_{xo}, u^2 - a_y a_{yo}, -a_y a_{oz}]/u$ and $K_o = [-a_z a_{xo}, -a_z a_{yo}, u^2 - a_z a_{zo}]/u$. The obliqual axonometric scales are $|\overline{OI_o}| = u_{xo}$, $|\overline{OJ_o}| = u_{yo}$ y $|\overline{OK_o}| = u_{xo}$. Next expression between axonometric scales defines the second fundamental lenght $l_b = \sqrt{u_{xo}{}^2 + u_{yo}{}^2 + u_{zo}{}^2} = u\sqrt{e_o{}^2 + 1}$. Here some geometric elements are noted configuring the axonometric perspective. To broaden this study see the reference Gimena *et al*. 2015 [4].

3 Construction of the obliqual axonometric perspective taken as a starting point the orthogonal axonometry

Here we show how to build an obliqual axonometric perspective from the orthogonal one. Obliqual straight line d_o is presented as a project direction and its point O_o associated to the unitary measure. A radiation of straight lines is built which relates this limit point O_o with the intrinsic points I_x , J_y and K_z .

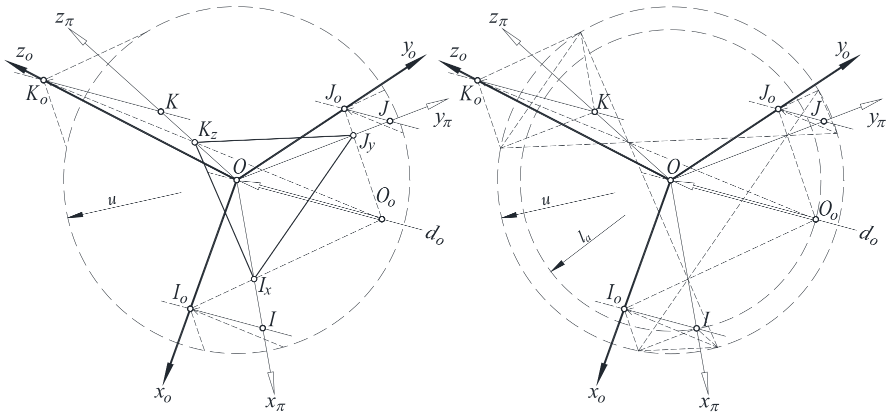

Fig. 2. Construction of the obliqual axonometry.

This radiation of straight lines is cutted by the parallel to the direction d_o that intersec throught the unitary points of the ortogonal axonometry I , J and K in the unitary point of the obliqual axonometry I_o , J_o and K_o . In this manner axis of the obliqual axonometric perspective are determined x_o , y_o and z_o . In Figure 2 it is also presented the geometric properties that are necessary for the graphical deduction of Pohlke Theorem.

4 Diagonal magnitude

In this section we start from three segments of the points I_o , J_o and K_o joint with the origin. Its segment is projected onto an arbitrary direction $b \equiv [b_x, b_y, b_z]/u$ (verifying $\sqrt{b_x^2 + b_y^2 + b_z^2} = u$) in the frame plane and the square root of the sum of the second exponential of this projections or diagonal magnitude is determined. The projection of the unit axonometric segments is $\overline{OB_{xo}} = I_o \cdot b = b_{xo}$, $\overline{OB_{yo}} = J_o \cdot b = b_{yo}$ and $\overline{OB_{zo}} = K_o \cdot b = b_{zo}$.

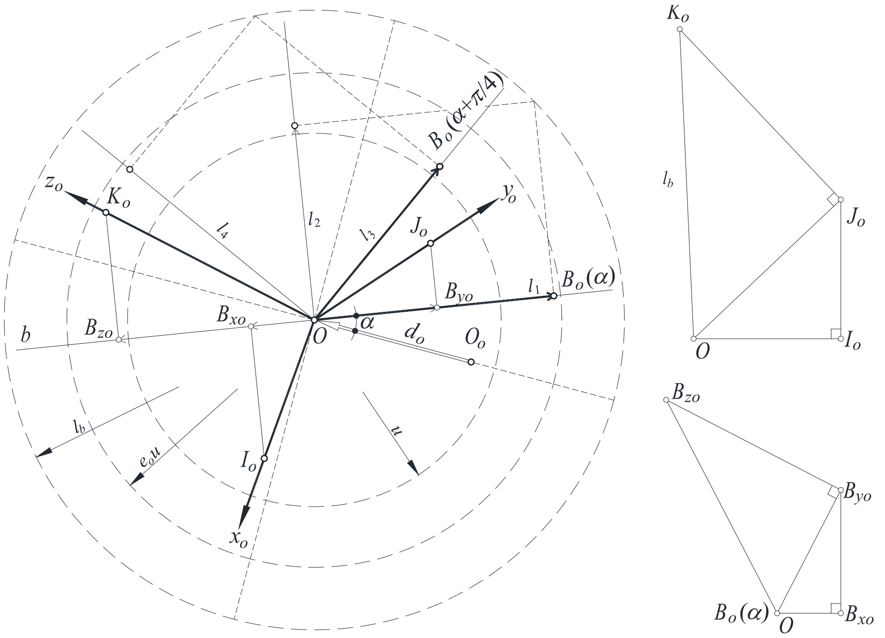

Fig. 3. Project direction and diagonal magnitude.

From the coordinate origin and in the direction used, the diagonal magnitude is built and its expression is as follows $\overline{OB_o}=\sqrt{b_{xo}{}^2+b_{yo}{}^2+b_{zo}{}^2}=b_o$ .In the Figure 3 project direction is express graphically: three axonometric unitary segments and its diagonal magnitude associated.

Whatever project direction can be expressed in function of the principal directions, which in this work are, first $b(\pi/2)$ perpendicular to d_π and d_o, secondly $b(0)$ perpendicular to the former direction. Diagonal magnitudes associated to these two principal directions are $\overline{OB_o(0)}=e_o u$ and $\overline{OB_o(\pi/2)}=u$.

The project direction in function of these two principal directions can be an noted as $b\equiv b(\alpha)=b(0)\cos\alpha+b(\pi/2)\sin\alpha$. Its diagonal magnitude can be also expressed as $\overline{OB_o}=\overline{OB_o(\alpha)}=l_1=u\sqrt{e_o{}^2\cos^2\alpha+\sin^2\alpha}$. Besides the diagonal magnitude associated to the perpendicular direction $b(\alpha+\pi/2)$ can be also obtained $\overline{OB_o(\alpha+\pi/2)}=l_2$. From these diagonal magnitudes l_1 and l_2 the second fundamental length can be obtained:

$$\sqrt{l_1{}^2+l_2{}^2}=u\sqrt{e_o{}^2+1}=l_b \tag{1}$$

The diagonal magnitude associated to the projection direction $b(\alpha+\pi/4)$ is $\overline{OB_o(\alpha+\pi/4)}=l_3=u\frac{\sqrt{2}}{2}\sqrt{\left(e_o{}^2+1\right)-2\left(e_o{}^2-1\right)\sin\alpha\cos\alpha}$.

With this distance and the equation (1) it can be determined the value of the magnitude associated to the project direction $b(\alpha+3\pi/4)$ which expression is $\overline{OB_o(\alpha+3\pi/4)}=l_4=\sqrt{l_b{}^2-l_3{}^2}$.

5 Pohlke's solution theorem

In this section it is presented an analytical and graphical procedure to solve the Pohlke´s Theorem (given three arbitrary segments and coincidents at an origin point, there is a unit tri-rectangular trihedra in the space that proceed from the projection of these segments in a cylindrical projection).

Given three arbitrary segments in the plane in a point $\overline{OI_o}$, $\overline{OJ_o}$ and $\overline{OK_o}$, diagonal magnitudes are determined, l_1, and l_3 associated respectively to the project direction $b(\alpha)$ and $b(\alpha+\pi/4)$ (Figure 3).

With this two magnitudes and using the equation (1) it can be deduced the diagonals l_2 and l_4 associated respectively to the projection direction $b(\alpha+\pi/2)$ and $b(\alpha+3\pi/4)$.

From next relation between the four diagonal magnitudes it can be deduced the first fundamental length $l_a=\sqrt[4]{\left(\left|l_1^2-l_2^2\right|\right)^2+\left(\left|l_3^2-l_4^2\right|\right)^2}=u\sqrt{e_o^2-1}=\left|OO_o\right|$. In the figure 4 it is presented the graphical obtention of the fundamental magnitudes starting from the four diagonal length.

Besides, from the fundamental lengths l_a, l_b it can be determined the values that define the principal direction of projection $\overline{OB_o(0)}=e_o u=\sqrt{\left(l_a^2+l_b^2\right)/2}$ y $\overline{OB_o(\pi/2)}=u=\sqrt{\left(l_b^2-l_a^2\right)/2}$.

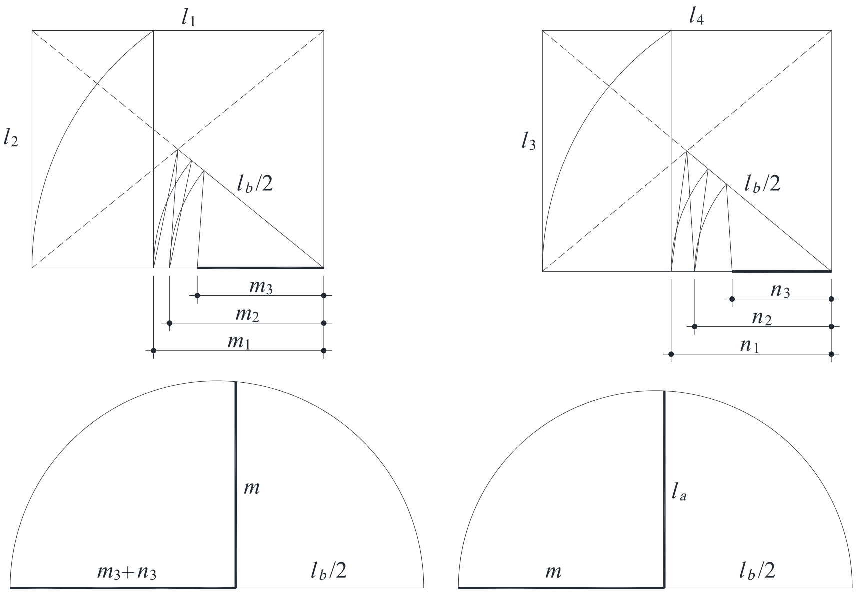

Fig. 4. Graphical operations to determine the fundamental length.

In the Figure 5 it is exposed a way to proceed of obtaining the diagonal magnitudes $\overline{OB_o(0)}$, $\overline{OB_o(\pi/2)}$ associated to these two principal directions.

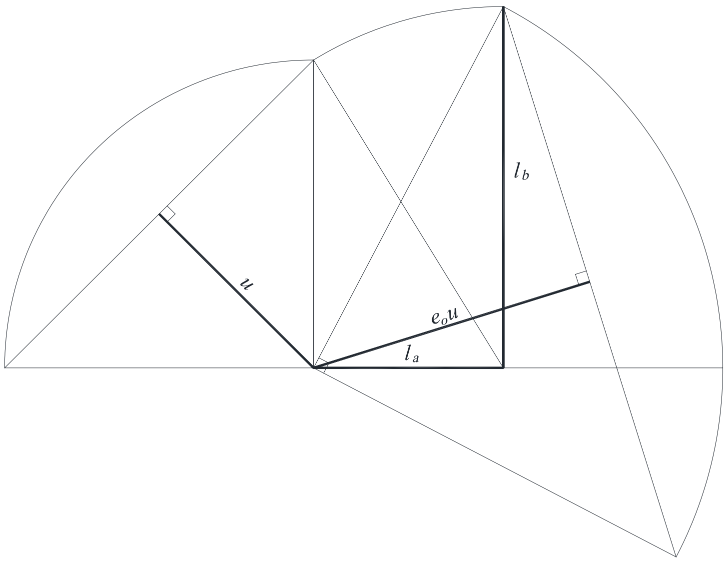

Fig. 5. Graphical procedure to determine the principal diagonal magnitudes.

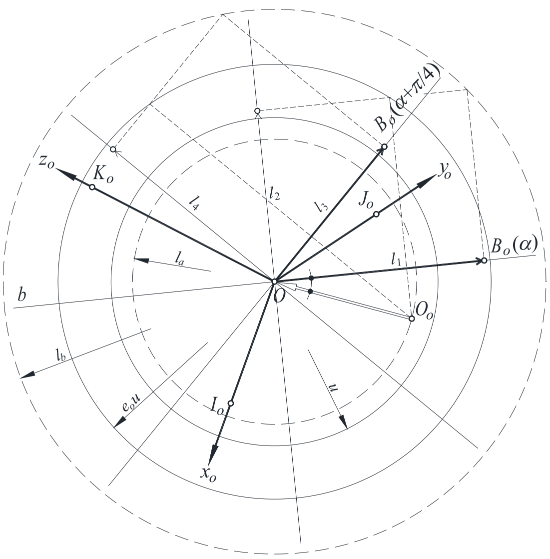

Fig. 6. Graphical solution of Pohlke´s Theorem.

In the Figure 6 it is presented the graphical solution of Pohlke´s Theroem consisting in determining a point associated to the oblicual project direction O_o .

The relations presented in the section of the construction of an oblicual axonometry starting from an orthogonal can be applied to obtain all the elements that define the trihedra reference system $Oxyz$. With this presented procedure it has been demonstrated and solved the Pohlke´s Theorem analytically as well as graphically.

6 Conclusions

Pohlke´s theorem can be stated as follows: given three arbitrary segments and coincidents at an origin point, there is a unit tri-rectangular trihedra in the space that proceed from the projection of these segments in a cylindrical projection. This paper presents the graphical demonstration and solves the theorem. No graphical complexity determining the diagonal magnitude associated with any direction in the frame plane. There is not complexity determining the fundamental lengths and obtention of the values that define the principal projection directions. From these graphs measures and constructions through on the invariants relations between orthogonal axonometric system and its associated oblique axonometric systems found a simple and practical way to solve Pohlke´s theorem.

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