Freedom of choice: the leximax criterion in economic environments.*

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Abstract

Many recent works have investigated the question of extending a preference over a set of alternatives to its power set, as a way to provide a formal representation of the notion of freedom of choice. In general, the results are limited to the finite case, which excludes the case of economic environments. This paper deals with the possibility of extending those results to the context where the basic set of alternatives is the n-dimensional Euclidean space. We present an extension of the leximax criterion of Bossert, Pattanaik and Xu (1994) on this more general framework. This characterization result opens the possibility of application of the literature on freedom of choice to standard economic environments.

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1 Introduction

The aim of this work is to investigate an extension of a preference over a set (possibly infinite) of alternatives to its power set. The motivation for that extension fits in the freedom of choice framework. According to this approach, the level of well-being enjoyed by an individual is determined by the utility achieved, given the available set of alternatives (instrumental value) and by the degree of freedom achieved from that opportunity set (intrinsic value).

There has been many relevant works on this field in recent times, starting from the very notion of freedom of choice which harks back to Hicks (1959), Buchanan (1986), Dasgupta (1986) and Sen (1988, 1991a, 1991b). The slightly different and closely related notion of preference for flexibility appears first in Koopmans (1964), Kreps (1979) and Arrow (1994) and is developed axiomatically in Arlegi and Nieto (2000a,2000b).

In Bossert, Pattanaik and Xu (1994) several rules to compare subsets of alternatives (opportunities) of a given set are defined and characterized by axioms; one of theses rules is called *leximax*. According to this rule, any pair of sets of alternatives are compared by looking first at the best alternative in each set; if this comparison is not decisive, then the rule looks at the second-best alternative, and the procedure continues up to the point in which there are no more alternatives to be compared in one or both sets. In the first case, the set with more alternatives is declared to be better; in the second one, both sets are indifferent.

Unfortunately, the leximax rule, as established in Bossert, Pattanaik and Xu (1994), is defined only when the universe of alternatives is finite. This domain restriction leaves no room for economic environments, in which a): the universe of alternatives is the positive orthant of the n-fold cartesian product, and b): individual preferences over this set is a continuos and complete preorder. In this case, all of the different rules presented in Bossert, Pattanaik and Xu -including the leximax rule- do not apply. Furthermore, in the consumer theory, comparisons among budget sets are made on the sole basis of the indirect utility, and this leaves apart any kind of freedom of choice considerations. This paper tries to fill up this gap extending the notion of freedom of choice to the case in which the basic set of alternatives is the Euclidean Space. We first establish some independent axioms (which are equivalent in spirit to the ones in Bossert, Pattanaik and Xu), applied to economic environments. We provide also a definition of the leximax criterion extended to the continuum case and we prove a characterization theorem in economic environments.

The plan of the paper goes as follows: In Section 2 there are notations and definitions. Section 3 contains the axioms and some relevant facts deduced from them. Section 4 establishes the main result as well as the independence of the axioms, and Section 5 concludes the paper with some comments and remarks.

2 Notation and definitions

$\mathbb{R}$ will denote the set of all real numbers, being $\mathbb{R}^n$ its n-fold cartesian product. Let $X \subset \mathbb{R}^n$ be a nonempty set of alternatives. In order to ensure that the axioms used in the characterization are independent, it is assumed that in X there are at least three elements.

Let R be a complete, reflexive, transitive and continuous ordering on X, being I the indifference relation and P the strict preference relation associated to R.

The set of all subsets of X is denoted by 2^X, and $\neg$ will denote the logical negation.

Let $\succ \subset 2^X \times 2^X$ be a preference relation defined on 2^X. We write $A \succ B$ to indicate that the set A is preferred to B. We assume that $\succ$ is asymmetric and negatively transitive, so $\succeq$ defined by $A \succeq B$ iff $\neg(B \succ A)$ is an ordering (complete and transitive) on 2^X, and the associated relation $\sim$ is an equivalence relation. We will assume $A \succ \emptyset$, for all $A \subset X$, $A \neq \emptyset$.

We investigate possible preferences over sets of alternatives consistent with a given preference structure over the basic alternatives. The formal meaning of consistency will be given by the axioms contained in the next section.

3 The axioms

On the relation between the preference structure on X and the ordering over 2^X we impose the following properties

Total-Freedom Dominance (TFD)

Let two A, B be two infinite sets, $A, B \subset X$, if for all $a \in A$ there exists $b \in B$ such that bRa, then $B \succeq A$.

Independence (In)

For all $A, B \subset X$, for all $y, z \in X$, such that yIz, $z \notin A$, and $y \notin B$, then

$$A \succeq B \Longleftrightarrow A \cup \{z\} \succeq B \cup \{y\}.$$

Robustness (Rb)

For all $A, B, C \subset X$, such that $C \cap (A \cup B) = \emptyset$ and verifing that $\forall a \in A$, $\forall b \in B$ and $\forall c \in C$, aPc and bRc, then

$$A \succ B \quad \Longrightarrow \quad A \succ B \cup C$$

Interpretation

(TFD) guarantees, for any pair of sets A and B with an infinite number of elements, and such that for all $a \in A$ there exists an element $b \in B$ verifying bRa, then b is almost as prefered as A. That is, in the case where A and B provide with the same degree of freedom because both contain an infinite number of elements, then Pareto Dominance applies.

(In) shares the same spirit as the Independence axiom of Pattanaik and Xu (1990), but in a weaker form. It says in words that by adding to (or dropping from) sets A, B two indifferent alternatives, the preference over those sets will not be reversed. When $y = z$, then the axiom looks very much like one of Pattanaik and Xu (1990), or Bossert (1992).

Finally, according to (Rb), it is ensured that when a set whose alternatives are worse than all the alternatives in $A \cup B$ is added to the worse set, B, then the ordering between A and B will not change. This is very similar to the axiom Robustness of Strict Preference used by Bossert (1992) and Bossert-Pattanaik-Xu (1994) extended to the case in which what we add is not a single alternative but a set of alternatives.

Consequences

As a consequence of the definition of (In) it is easy to observe that for all $A, B \in 2^X$, for all $y_1, \ldots, y_k \in X$, $z_1, \ldots, z_k \in X$ such that $y_i I z_i$, $z_i \notin A$, and $y_i \notin B$, $i = 1, \ldots k$, then

$$A \succeq B \Longleftrightarrow A \cup \{z_1, \ldots, z_k\} \succeq B \cup \{y_1, \ldots, y_k\}.$$

As a consequence of the definition of (Rb), we obtain the property of **Extension (E)**: for all $x, y \in X$, $xPy \Longleftrightarrow \{x\} \succ \{y\}$. In order to prove it, consider $A = \{x\}$, $B = \emptyset$ and $C = \{y\}$.

(E) is a very standard axiom in the field. It simply says that the preference over alternatives is extended to singletons when the quality of the alternatives matters in comparing opportunity sets. This is not the case in some approaches where only the number of alternatives matters in the comparison of opportunity sets. Thus, Pattanaik and Xu (1990) studied a case in which the freedom of choice attached to a set of opportunities is measured simply by the number of its alternatives. They assume that there will be no distinction between sets of alternatives in which there is no freedom of choice at all, the singletons, for instance, and then they establish that for all $x, y \in X$, $\{x\} \sim \{y\}$. In our paper it is assumed that the quantity of the alternatives, as well as their quality, is taken into account when comparing opportunity sets, and then the property (E) becomes very natural.

Also from the axioms we obtain the following lemmata.

Lemma 1 *Let $A \subset X$, $b \in X$, if for all $a \in A$, bPa, then $\{b\} \succ A$.*

Proof. As $\{b\} \succ \emptyset$, thus, by (Rb), $\{b\} \succ A$

Lemma 2 *Let $A, B \subset X$, A finite and $B \neq \emptyset$, then $A \cup B \succ A$.*

Proof. As $B \succ \emptyset$, thus, by (In), $A \cup B \succ A$.

4 A characterization result

We are now prepared to propose an ordering of opportunity sets that satisfies the above axioms, and that is the only one that accomplishes such a list of properties. We call that ordering the leximax ordering on 2^X and it will be denoted by $\succeq_L$. Both, this name, *leximax*, and the notation $\succeq_L$, appear before in Bossert(1992) and Bossert-Pattanaik-Xu (1994), but unlike these works in our context refer to an ordering on 2^X, where X can be infinite.

In order to define the criterion $\succeq_L$, a piece of adittional notation will be useful. Let $u_R : X \to [0, 1]$ be such that for all $x, y \in X$, $u_R(x) \geq u_R(y)$ iff

xRy. (The assumptions on R are sufficient conditions to guarantee that u_R does exist as a utility representation for R).

Let $A \subset X$, as $u(A)$ is a bounded set, there exists $\sup u(A)$. If there exists $a \in A$ such that $u(a) = \sup u(A)$, this a will be denoted by a_1. In this case, if we consider the bounded set $u(A \setminus \{a_1\})$, then there exists $\sup u(A \setminus \{a_1\})$; again, if there exists $a \in A \setminus \{a_1\}$ such that $u(a) = \sup u(A \setminus \{a_1\})$, this a will be denoted by a_2, and so on.

Then, given a set X (possibily infinite), the leximax ordering on 2^X is defined as follows: let $A, B \subset X$ be two sets of alternatives, there could be three possibilities:

1) There exists a_1 and there does not exist b_1.

If for all $x \in B$ $a_1 P x$, then $A \succ_L B$.
If there exists $x \in B$ such that $x P a_1$, then $B \succ_L A$.

2) There do not exist both a_1 and b_1.

If $\sup u(A) > \sup u(B)$, then $A \succ_L B$.
If $\sup u(B) > \sup u(A)$, then $B \succ_L A$.
If $\sup u(A) = \sup u(B)$ then $A \sim_L B$.

3) There exist a_1 and b_1.

If $a_1 P b_1$, then $A \succ_L B$.
If $b_1 P a_1$, then $B \succ_L A$.
If $a_1 I b_1$, we repeat the same procedure for a_2 and b_2, that is, we consider the sets $A \setminus \{a_1\}$ and $B \setminus \{b_1\}$, and so on. In the case where there exist $a_k \in A$, $b_k \in B$ such that $a_k I b_k$, for all $k \in \mathbb{N}$, then $A \sim_L B$.

Note that if each set A is identified with the sequence in $[0, 1]$, $U(A) = \{u(a_1), u(a_2), \ldots, u(a_{k_o}), \sup u(A \setminus \{a_1, \ldots a_{k_o}\}), 0, \ldots\}$, in order to relate a pair of sets A and B, we are, in fact, using the lexicografhic ordering between $U(A)$ and $U(B)$, with the restriction that if there exist a_k and b_k, for $k = 1, \ldots k_o$, $a_k I b_k$, and there exists a_{k_o+1} but there do not exist b_{k_o+1} and $u(a_{k_o+1}) = \sup u(B \setminus \{b_1, \ldots, b_{k_o}\})$, then $A \succ_L B$, instead of checking the next elements of both $U(A)$ and $U(B)$.

The main result of the paper is the following:

Theorem 1 *Let $\succeq$ be an ordering on 2^X.*
$\succeq$ satisfies (TFD), (In) and (Rb) if and only if $\succeq = \succeq_L$.

Proof. It is easy to prove that $\succeq_L$ satisfies (TFD), (In) and (Rb). We will prove that, if $\succeq$ is an ordering on 2^X verifying (TFD), (In) and (Rb), then $\succeq = \succeq_L$.

We start supposing that $A \sim_L B$ and then we will prove that $A \sim B$. There could be three cases:

(a) there exist $a_k \in A, b_k \in B$ for all $k \in I\!N$ with $a_k I b_k$, for all $k \in I\!N$, or

(b) there exists $k_o \in I\!N$ such that there exist $a_k \in A, b_k \in B$, for all $k < k_o$, verifing $a_k I b_k$, there does not exist $a_{k_o} \in A, b_{k_o} \in B$, and $\sup u(A \setminus \{a_1, \ldots, a_{k_o-1}\}) = \sup u(B \setminus \{b_1, \ldots, b_{k_o-1}\})$.

(c) There do not exist a_1 and b_1 and $\sup u(A) = \sup u(B)$.

In the case (a), by (E) $\{a_1\} \sim \{a_1\}$, and by (In) $A \sim B$ In the case (b), by (TFD) $A \setminus \{a_1, \ldots, a_{k_o-1}\} \sim B \setminus \{b_1, \ldots, b_{k_o-1}\}$, and by (In) $A \sim B$. In the case (c), applying directly (TFD) we have $A \sim B$.

We will suppose now that $A \succ_L B$, and we will proof that $A \succ B$. Suppose that there exists $k_o \in I\!N$ such that there exist $a_k \in A, b_k \in B$, for all $k < k_o$, with $a_k I b_k$. There could be the next possibilities,
(a) there exist $a_{k_o} \in A, b_{k_o} \in B$ with $a_{k_o} P b_{k_o}$;
(b) there exists $a_{k_o} \in A$, there do not exist $b_{k_o} \in B$ and $a_{k_o} P x$, for all $x \in B \setminus \{b_1, \ldots, b_{k_o-1}\}$;
(c) there does not exist $a_{k_o} \in A$, there exists $b_{k_o} \in B$ and there exists $a \in A \setminus \{a_1, \ldots, a_{k_o-1}\}$ such that $a P b_{k_o}$;
(d) there does not exist $a_{k_o} \in A, b_{k_o} \in B$ and $\sup u(A \setminus \{a_1, \ldots, a_{k_o-1}\}) > \sup u(B \setminus \{b_1, \ldots, b_{k_o-1}\})$.

First, note that if $k_o = 1$, the four cases are analogous.

Otherwise, in the case (a), by using (E), (In) and (Rb), we have the chain of consequences, $\{a_{k_o}\} \succ \{b_{k_o}\} \Longrightarrow \{a_1, \ldots, a_{k_o}\} \succ \{b_1, \ldots, b_{k_o}\} \Longrightarrow \{a_1, \ldots, a_{k_o}\} \succ B$, on the other hand, by lemma 2, $A \succ \{a_1, \ldots, a_{k_o}\}$, then $A \succ B$.

In case (b), by using lemma 1, $\{a_{k_o}\} \succ B \setminus \{b_1, \ldots, b_{k_o-1}\}$, and by (In) $\{a_1, \ldots, a_{k_o}\} \succ B$. By using lemma 2 we obtain $A \succ \{a_1, \ldots, a_{k_o}\}$, thus $A \succ B$.

In case (c) by using (E), (In) and (Rb), we obtain $\{a\} \succ \{b_{k_o}\} \Longrightarrow \{a_1, \ldots, a_{k_o-1}, a\} \succ \{b_1, \ldots, b_{k_o}\} \Longrightarrow \{a_1, \ldots, a_{k_o-1}, a\} \succ B$, and by using lemma 2, $A \succ \{a_1, \ldots, a_{k_o-1}, a\}$, then $A \succ B$.

In case (d), $\sup u(A \setminus \{a_1, \ldots, a_{k_o-1}\}) > \sup u(B \setminus \{b_1, \ldots, b_{k_o-1}\})$, implies that there exists $a \in A \setminus \{a_1, \ldots, a_{k_o-1}\}$ such that aPb for all $b \in B \setminus \{b_1, \ldots, b_{k_o-1}\}$ and then, by using the lemma 1 and (In), we have the chain of consequences $\{a\} \succ B \setminus \{b_1, \ldots, b_{k_o-1}\} \Longrightarrow \{a_1, \ldots, a_{k_o-1}, a\} \succeq B$, and with lemma 2 again, $A \succ B$.

5 Remarks

The result presented in Theorem 1 can be seen as an extension of the leximax rule of Bossert, Pattanaik and Xu (1994) to economic environments. An economic environment is defined as the case in which the universal set of alternatives is the Euclidean space and the decision makers preferences over this domain are continuous. Even though the leximax procedure is usually defined for the finite case, this paper presents a suitable extension to the continuous case. In this extension we use the fact that the sets to be compared are bounded having then suprema. The comparison among sets applies the lexicographic procedure to a finite list of (possibly infinite) bounded sets.

Notice that, in some cases, the leximax rule characterized in this paper enables to show the intrinsic value of freedom of choice, while the standard indirect utility rule does not. For example, let A be a classical budget set, with a_1 its best alternative according to relation R (see figure 1). According to the standard criterion of the consumer theory, $A \sim \{a_1\}$, while according with the leximax, $A \succ \{a_1\}$.

By the other hand, let A and B two budget sets, $B \subset A$ such their best alternative according R agrees, $a_1 = b_1$ (see figure 2). According both of them, the standard criterion and the leximax rule, $A \sim B$. Note that this seems to contradict the idea of freedom of choice, but can also be interpreted from other point of view, keeping with the spirit of freedom of choice. In fact the reasons for A be indifferent B are, first, there are the same quantity of alternatives in A than in B, and, second, these alternatives are equally desired, that is, for each alternative in A, different that a_1, there is some alternative in B better than it, and reciprocally.

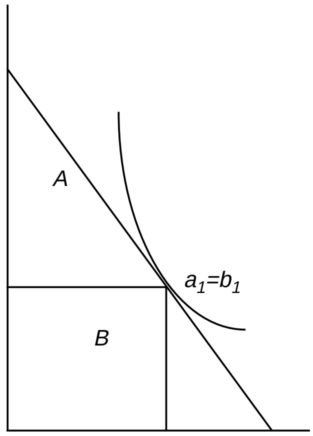

Figure 1

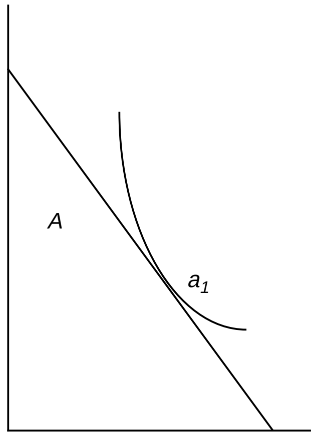

Figure 2

Acknowledgements

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