MINIMAL BOOKS OF RATIONALES

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ABSTRACT. Kalai, Rubinstein, and Spiegler (2002) propose the rationalization of choice functions that violate the "independence of irrelevant alternatives" axiom through a collection (book) of linear orders (rationales). In this paper we present an algorithm which, for any choice function, gives (i) the minimal number of rationales that rationalizes the choice function, (ii) the composition of such rationales, and (iii) information on how choice problems are related to rationales. As in the classical case, this renders the information given by a choice function completely equivalent to that given by a minimal book of rationales. We also study the structure of several choice procedures that are prominent in the literature.

Keywords: Rationalization, Independence of irrelevant alternatives, Order partition, Computational effort.

1. INTRODUCTION

The classical account for the rationalization of choice functions states that the property known as "independence of irrelevant alternatives"[1] (IIA) is a necessary and sufficient condition for the existence of a linear order (i.e., a rationale) that is consistent with choice behavior.[2] In a seminal paper Kalai, Rubinstein, and Spiegler (2002) study the problem of rationalizing choice behavior when axiom IIA does not necessarily hold. Their proposal is to use a *book of rationales*, such that for every set A in the universal set of alternatives X, the choice $c(A)$ is maximal in A for some page of the book. Clearly, there are multiple books that can rationalize a given choice function. The authors propose to

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[1]If an element x is chosen from a set, x should be also chosen from any subset that contains it.

[2]See Arrow (1959); for reviews see Moulin (1985) and Suzumura (1983).

focus on those books that use the *minimal* number of rationales. Of course, in the classical case, that is, when IIA holds, the minimal book is composed of only one page. Kalai, Rubinstein, and Spiegler study the minimal number of rationales for the second-best and the median choice procedures.

In this paper we offer an algorithm that, for any universal set of alternatives X and any choice function c on X, gives the minimal number of rationales that rationalizes c, and the composition of each of such rationales. Furthermore, the algorithm avoids any loss of information as the result of switching from choice behavior to the book of rationales. That is, given a book of rationales, we want which page to turn to in order to find the maximal element for a given choice set A. As in the classical setup, we want the choice function and the book of rationales to be completely equivalent in terms of behavioral information. Therefore, for any pair (X, c), it will be our aim to give:

- the minimal number of rationales that rationalizes c,
- the composition of such rationales, and
- information on how choice problems are associated to rationales.

We will use the term *minimal rationalization by multiple rationales* (minimal RMR) of (X, c) to refer to a solution of these three points.

We will start by showing (section 2) that the problem of finding a minimal RMR is equivalent to that of finding a certain type of minimal partition of a set associated to the space of all choice problems (subsets of X). In particular, we will concentrate on the set of choice problems of which there is no superset with the same chosen element. In a sense, these are the choice problems that make a difference, since the remaining choice problems can be derived from these sets and the application of the IIA property. The type of partition of such a space that we study directly addresses the third point in the above statement: the association between choice problems and rationales. That is, an element of the partition is equivalent to a rationale of the book, but, in the former case, there is detailed information on the choice problems being rationalized by the rationale. This constitutes a formalization of the intuition given by Kalai, Rubinstein, and Spiegler: the decision maker has in mind a partition of the space of choice problems, and applies a rationale to each element of the partition. It is as if each element of the partition constituted a state of the world that the decision maker needs to internalize in order to rationalize her behavior.

The rest of the paper is organized as follows. Section 3 presents and analyzes the properties of the algorithm that we design. Section 4 studies several prominent choice procedures and attempts to reduce the computational effort required by the algorithm. Finally, section 5 concludes and relates our work to recent developments in the literature.

2. Rationalization by Multiple Rationales and Complete Preorder Partitions

Let X be a finite set of objects. We denote by $P(X)$ and $P_2(X)$ the set of all non-empty subsets of X and the set of all non-empty subsets of X with cardinality greater than or equal to 2, respectively. A choice function c on X assigns to every $A \in P(X)$ a unique element $c(A) \in A$.

Definition 2.1 (Kalai, Rubinstein, and Spiegler)**.** A K-tuple of strict preference relations $(\succ_k)_{k=1,\ldots,K}$ on X is a rationalization by multiple rationales (RMR) of c if for every A, the element $c(A)$ is $\succ_k$-maximal in A for some k.

For convenience, hereafter we will use the term ***minimal RMR*** of (X, c) to refer to a complete characterization of a minimal book of rationales. That is, a minimal RMR will give the composition of each of the orderings of a minimal book of rationales that explains choice behavior, and information on how choice problems are associated to rationales.

We are now in a position to introduce an equivalent problem to that of finding the minimal RMR of (X, c): finding a minimal complete preorder partition of a set associated to $P(X)$. Let us first specify what we mean by a complete preorder partition.

Definition 2.2. A partition of set V, $\{V_p\}_{p=1,\ldots,P}$, is said to be a *complete preorder partition (CPP)* according to the binary relation $\succeq$ if, for every class V_p, the restriction of $\succeq$ to V_p is a complete preorder. It is said to be a *minimal* CPP if any other CPP according to $\succeq$ has at least P classes.

Although it is possible to define a minimal CPP problem over the space $P(X)$, we provide a more compact formulation that simplifies further analysis and computations.

Definition 2.3. Given (X, c) a subset $S \in P_2(X)$ is said to be c-maximal if for all $T \subseteq X$, with $S \subset T$, it is the case that $c(T) \neq c(S)$. Denote the family of c-maximal sets by M_c.

We concentrate on the set of c-maximal sets, all other sets in $P(X)$ being trivially associated to at least one c-maximal set. In section 4.1 we will study the composition of M_c for several choice procedures.

An obvious candidate for an RMR of (X, c) is the construction of $|M_c|$ rationales with the respective chosen elements of the different c-maximal sets dominating the pertinent rationale. However, it is easy to see that we can do better. That is, elements of M_c can be placed together, "sharing" a rationale. The following binary relation aims to capture this flavor.

Definition 2.4. Define the binary relation R on M_c by ARB if and only if $c(A) \notin B \setminus \{c(B)\}$, $A, B \in M_c$.

We can now state the main result of this section in terms of the following proposition.

Proposition 2.5. *Solving for a minimal RMR problem in (X, c) is equivalent to solving for a minimal CPP problem in (M_c, R). Furthermore, the minimal number of rationales and classes coincide.*

Proof: Consider a pair (X, c). We first prove that any CPP of (M_c, R) with P classes can be associated to an RMR of (X, c) with P rationales. Suppose that $\{V_p\}_{p=1,\dots,P}$ is a CPP of M_c according to R. For each class V_p consider the following binary relation $\succ_p$ on X:

$$x \succ_p y \text{ if and only if } x = c(A), y = c(B), A, B \in V_p \text{ and } ARB.$$

Clearly, since (V_p, R) is a complete preorder set, $\succ_p$ is a preorder on X. Now, if there are several different elements $x_1, \dots, x_t$ such that $x_i \succ_p x_j$ and $x_j \succ_p x_i$ for all $i, j = 1, \dots, t$, eliminate some pairs in such a way that the resulting binary relation is a linear order on $x_1, \dots, x_t$. Complete $\succ_p$ so that the resulting order, say $\succ_p^*$, constitutes a linear order where the remaining pairs of $\succ_p$ after the above-mentioned eliminated pairs are the top elements of $\succ_p^*$. We will show that $(\succ_p^*)_{p=1,\dots,P}$ is an RMR of c.

Take any choice set D. By definition, there exists a maximal set, $B \in M_c$ such that $c(D) = c(B), D \subseteq B$. We prove that the rationale, say $\succ_p^*$, associated to the class in which B is contained, say V_p, rationalizes the election $c(D)$. Notice that $x \succ_p^* c(B) = c(D)$ is only possible if $x = c(A) \neq c(B)$, $A \in M_c$ and ARB. But in this case, by definition, $c(A) \notin B \setminus \{c(B)\}$ and hence, $c(A) \notin D \setminus \{c(D)\}$. Since $c(A) \neq c(D)$, it is the case that $c(A) \notin D$ and therefore the rationalization is possible.

We now show the reverse implication, namely, that any RMR of (X, c) with P rationales can be associated to a CPP of (M_c, R) with, at most, P classes. Let $(\succ_p)_{p=1,\dots,P}$ be an RMR of c. Consider a mapping (there may be several) $f : M_c \to \{1, \dots, P\}$ such that $f(A) = j$ if and only if $\succ_j$ rationalizes the choice in A. Of the partitions of M_c induced by such a map, we consider the one that contains, at most, P

classes. Let $\{A_s^t\}_{t=1,2,\ldots,T}$ be the c-maximal sets associated to class s, and without loss of generality assume that satisfy

$$c(A_s^i) \succ_s c(A_s^j) \text{ implies } i < j$$

We only need to prove that the restriction of R to this class is a complete preorder. We initially prove that A_s^1 R-dominates the other sets in class s. Rationalization requires that $c(A_s^1)$ should not be present in $A_s^j \setminus \{c(A_s^j)\}$, $j = 2, \ldots, T$, which is merely the definition of R. This can be repeated for the rest of the class, to conclude that a CPP with at most P classes can be defined.

Hence, given a minimal CPP of m classes we can give a minimal RMR of m rationales, and vice-versa. Therefore, solving for a minimal RMR problem in (X, c) is equivalent to solving for a minimal CPP problem in (M_c, R). ■

As the proof details, a rationale is equivalent to a class of the partition. That is, the proof shows how to construct a rationale from a class of the partition, and vice-versa. Further, the composition of the classes in the partition provides the information needed to relate choice problems with rationales. The equivalence between the two problems means that, throughout the rest of the paper, we will often refer indistinctly to either of the two.

3. The Algorithm

The algorithm that we present below starts by focusing on the family M_c and the binary relation R. Then, it simply tries to group elements of M_c on the basis of R, in as few disjoint sets as possible. This creates a partition of M_c into complete preordered subsets, where each subset is to be viewed as a rationale that can be computed following the proof of Proposition 2.5. It will then be proved that the algorithm does, indeed, give the minimal CPP of (M_c, R), and consequently the minimal RMR of (X, c).

For the sake of simplicity, we now present the crucial steps of the algorithm. More specific details concerning each of these steps are available upon request. The algorithm makes use of linear orders $<$ over M_c, interpreting $A < B$ by *"A is analyzed before B."*

Algorithm 1. For any (X, c):

(1) Compute M_c associated to c, and R on M_c.
(2) For each possible linear order $<$ on M_c, apply the following subprocess. (Denote by $A_1, \ldots, A_k$ the sets in M_c ordered according to $<$.)
  (a) Define a class C_1 and include A_1 in such a class.
  (b) If $A_1 R A_2$, include A_2 in C_1. If not, define the class C_2 and include A_2 in that class.
  (c) Consider $A_n \in M_c$ and let $C_1, \ldots, C_p$ be the classes defined after considering $A_1, \ldots, A_{n-1}$. Identify the classes C_j for which $A R A_n$ for all $A \in C_j$. Call these classes, *admissible classes* for A_n. Select one of these admissible classes and include A_n in that class. If no admissible class can be identified, define C_{p+1} and include A_n in that class. Iterate this process until $n = k$.
(3) Select one of the linear orders $<$ over M_c that gives a minimal partition.

Notice that the rule, according to which a c-maximal set A is associated to a particular admissible class C_j from the set of admissible classes, has not been specified in the algorithm (see point 2c in Algorithm 1). In the next proposition we show that the algorithm determines a minimal CPP *irrespective* of the particular selection rule from the set of admissible classes.

Proposition 3.1. *For every choice function c Algorithm* 1 *generates a minimal CPP.*

Proof: We will show that, given a minimal CPP of (M_c, R) with m classes, there is a specific linear order $<$ on M_c that generates a (possibly different) CPP with, at most, m classes. Let $\{C_1, \ldots, C_m\}$ be a minimal CPP. Denote by C_i^j, $i = 1, \ldots, m$, $j = 1, \ldots, j(i)$, the j-best c-maximal set in C_i according to R, breaking indifferences as desired. We will show that the following ordering of the elements of M_c generates a CPP with at most m classes:

$$C_{i_1}^{j_1} < C_{i_2}^{j_2} \text{ if and only if } i_1 < i_2 \text{ or } i_1 = i_2, j_1 < j_2$$

To see this, we will show that Algorithm 1 situates any c-maximal set C_i^j in a class h, with $h \leq i$.

We prove this by double induction, both on i and j. We initially see it for $i = 1$. It is obvious for C_1^1. Then, if it is true for $j = 1, \ldots, n-1$ we will prove that it is also true for C_1^n. Since the elements in C_1 are ordered according to R, it must be that $C_1^j R C_1^n$ for all $j = 1, \ldots, n-1$

and hence, the algorithm situates C_1^n in the first class C_1. Suppose this fact is true for C_i, $i = 1, \ldots, r-1$. We will prove it for C_r. Obviously, C_r^1 is situated, by the algorithm, in the r-th class, or before. If this is true for C_r^j, $j = 1, \ldots, n-1$, we show it is true for C_r^n. But, since all the c-maximal sets have been situated in C_r or in a previous class, the class C_r^* (which may be empty) constructed by Algorithm 1, is contained in the original one, that is, in C_r. Thus, if C_r^n was dominated by all the c-maximal sets C_r^h, with $h < n$, in C_r, it is also dominated by all c-maximal sets in C_r^*, when the time comes to consider C_r^n. Hence, this subset may be included in class C_r^* or in a previous one, and the proof is concluded. ■

Algorithm 1 is very general, since it does not need to specify a particular selection rule from the set of admissible classes in order to provide a minimal RMR, and also because it has a very primitive way of identifying the set of admissible classes for every $A \in M_c$. In section 4.1, when we study the second-best choice procedure, we will extend the algorithm in order to expand the set of admissible classes.

Notice that, in order to determine a minimal RMR of a particular choice procedure, we need to analyze all the possible linear orders $<$ over the set M_c. Depending on the size of the set M_c, this task may require a heavy computational effort.[3] For this reason, we proceed within the following section to analyze two different ways of reducing this effort. One is to study specific choice procedures and to analyze the manageability of the associated M_c sets. The other is to investigate general properties on the set of all linear orders over M_c that might reduce the number of linear orders that needs to be considered in order to obtain a minimal RMR.

4. Specific choice procedures and linear orders

In this section, we first study three choice procedures that present very different levels of complexity and, consequently, require very different degrees of computational effort: the rational, the second-best, and the median choice procedures. We then approach the problem of providing properties on the set of linear orders over M_c that reduce the

[3]We conjecture that the problem of finding a minimal RMR for every (X, c) is, in the language of theoretical computer science, an NP-complete problem (for an excellent introduction to the theory of NP-completeness see Garey and Johnson, 1979). We have proved that our problem is in NP, that to partition a directed graph into complete and transitive subgraphs (PICTSG) is an NP-complete problem, and that our problem is a subproblem of PICTSG. We can provide details upon request.

number of cases that need to be considered in order to give a minimal RMR.

4.1. **Some Specific Choice Procedures.** We begin by studying a prominent case, the rational procedure (that is, when property IIA holds). We will observe that rational behavior implies an important contraction of M_c with respect to $P(X)$, thus lending simplicity to our algorithm. We, therefore, will be able to give not only the composition of the set M_c, but also the linear order on M_c that generates a minimal RMR.

Then, we proceed to analyze the second-best and the median procedures. Although they show similar features (choice is made in positional terms, according to a linear order over the alternatives), their complexity is rather dissimilar. While the second-best procedure generates a set M_c with a relatively small number of elements (polynomial in terms of $|X|$), the median procedure generates an exponential number (thus, it does not constitute a significant reduction in size with respect to $P(X)$). The "simplicity" of the second-best procedure allows us to give an exact description of the linear order on M_c that generates the minimal book. In the case of the median procedure, we will give the composition of M_c.

4.1.1. *The Rational Procedure.* By the rational procedure we mean a pair (X, c) such that c satisfies the IIA property.

Proposition 4.1. *Let (X, c) be a rational procedure. Then, $|M_c| = |X| - 1$, and a concrete order $<$ of the family M_c is provided so that steps 2a, 2b, 2c of Algorithm 1 give a CPP with only one class.*

Proof: It is easy to see that for every family of sets $\mathcal{X}_j = \{A \subset X : |A| = |X| - j\}$, $j = 1, \ldots, n-2$, there is a unique set A_j with $c(X), c(A_i) \notin A_j$, $i < j$. Then, M_c is composed of set X and the collection of sets A_j, $j = 1, \ldots, n-2$. Order M_c by the cardinality of its members, from bigger to smaller. By construction, it is the case that XRA_j, $j = 1, \ldots, n-2$, and that A_iRA_j, $i < j$. Hence, the algorithm integrates all members of M_c in a single class, C_1, in $|X| - 1$ steps. ■

4.1.2. *The Second-Best Procedure.* Let $\succ$ be a linear order on X. The second-best procedure states that for every $A \subseteq X$, c selects the second-best element in A according to $\succ$. Starting with Sen (1993), this procedure has gained the attention of economists (see Baigent and Gaertner, 1996; Gaertner and Xu, 1999a; McFadden, 1999). Kalai, Rubinstein and Spiegler (2002) show that the minimal number of rationales for the second-best procedure is equal to $log_2|X|$ rounded off to the higher integer.

In order to give the minimal RMR of the second-best procedure in a simpler and more intuitive formulation, we need to expand the set of admissible classes, and to specify an admissible class-selection rule.

Definition 4.2. The class $C = \{C^1, \ldots, C^j\}$ is admissible-2 for a set $A \in M_c$ if either

(1) $C^p RA$ for all $p = 1, \ldots, j$, or
(2) There exists a twin set C_i of A in C, i.e., $c(C_i) = c(A)$ and for all C_p, $p = 1, \ldots, j$, $C_p RA \Leftrightarrow C_p RC_i$ and $ARC_p \Leftrightarrow C_i RC_p$.

Definition 4.2 expands the set of admissible classes of $A \in M_c$ by also considering those classes where there is a c-maximal set B with the same choice as A and A has the same relation with the rest of the sets in the class as B has.

On the basis of Definition 4.2, we can now define a version of Algorithm 1 that we will refer to as Algorithm 2. Further, we can obtain the analogous result to Proposition 3.1, that we state below without proof.

Proposition 4.3. *For every choice function c Algorithm 2 generates a minimal CPP.*

We adopt a specific selection rule of associated classes.

Definition 4.4. Select the associated class of $A \in M_c$ as follows: (Let C_i precede C_j if C_i was created before C_j and write $i < j$.)

(1) Identify among the admissible classes those in which there is a set B with $C(B) = C(A)$.
(2) If several are identified, select the first among them. If none is identified, select the first among the admissible classes.

The above rule tries to group a c-maximal set with other c-maximal sets of the same choice.

We are now in a position to present the characterization of the second-best procedure.

Proposition 4.5. *Let (X, c) be a second-best procedure. Then, $|M_c| = |X| \times (|X| - 1)/2$, and a concrete order $<$ of the family M_c is provided so that steps 2a, 2b, 2c of Algorithm 2, implemented by Definition 4.4, give a CPP with $log_2|X|$, rounded off to the higher integer, classes.*

Proof: Let, without loss of generality $X = \{1, 2, \ldots, n\}$, with $1 \succ 2 \succ \cdots \succ n-1 \succ n$. The proof consists in three inductive steps. In the first, we define the family M_c and state its cardinality, in the second, we specify the particular order of the elements of M_c, and in the last, we show that, for such an ordering of M_c, steps 2a, 2b, 2c of Algorithm 2, implemented by Definition 4.4, generate a minimal CPP.

Step 1: We first characterize the set M_c.

If $X = \{1, 2\}$ the only c-maximal set is $\{1, 2\}$. Suppose that for certain $X = \{1, \dots, n\}$, $M_c^n = \{A_k\}_{k=1,\dots,K}$. Consider the set $X \cup \{n+1\}$. It can be easily shown that, given A_k in M_c^n, $A_k^* = A_k \cup \{n+1\} \in M_c^{n+1}$. Consider, furthermore, the additional sets $\{B_r\}_{r=1,\dots,n}$, $B_r = \{r, n+1\}$ with choice $n+1$. Obviously, these sets belong to M_c^{n+1}. It is not hard to see that these are all the sets that compose M_c^{n+1}. Hence, we can inductively obtain the cardinality of M_c as follows. When $X = \{1, 2\}$, $|M_c^2| = 1$, and $|M_c^{n+1}| = |M_c^n| + n$. Hence, it is easy to see that $|M_c^n| = 1+2+\ldots+(n-1) = n \times (n-1)/2$, as stated in the proposition.

Step 2: We now describe a particular ordering of M_c.

If $X = \{1, 2\}$, the issue is trivial. Suppose that, for certain $X = \{1, \dots, n\}$, $M_c = \{A_k\}_{k=1,\dots,K}$ is, without loss of generality, ordered according to the natural order of positive integers, and that this order generates the minimal CPP of (M_c, R). In this case, establish the following order for M_c^{n+1}. First, put $\{A_k^*\}_{k=1,\dots,K}$. We now proceed to describe the order of sets $\{B_r\}_{r=1,\dots,n}$. To this end, we need to identify three parameters: t, s, and p. Parameter t gives information on the number of classes (equivalently , number of rationales); s gives information on the number of classes needed to rationalize all choice sets $\{B_r\}_{r=1,\dots,n}$; p gives information on which particular classes are selected to rationalize all $n+1$ choices.[4]

Determine the positive integer t such that $n+1 \in (2^t, 2^{t+1}]$. Calculate the positive integer s, $0 \leq s \leq t$ such that $n+1 \in (2^t + \binom{t}{0} + \ldots + \binom{t}{s-1}, 2^t + \binom{t}{0} + \ldots + \binom{t}{s-1} + \binom{t}{s}]$. Compute the number $p = (n+1) - (2^t + \binom{t}{0} + \ldots + \binom{t}{s-1})$.

Consider the set $T = \{2^{t-1}, 2^{t-1}+2, \dots, 2^t - 1\}$. Now consider all possible sets of s elements that can be formed from the elements of set T. Establish a leximax ordering over the latter sets. (Where leximax is, $A \geq_L B$ if and only if the largest number in $A \setminus B$ is greater than the largest number in $B \setminus A$). Select the set that occupies position p in the leximax ordering. Denote this set by $\{x_1, \dots, x_s\}$, where the elements are ordered in descending order with respect to the positive integers that they represent.

[4] The following reasoning is only valid for $t > 2$. The ordering for $t = 1, 2$ is almost trivial.

Now, to complete the ordering of the c-maximal sets write the set $\{2^t, n+1\}$, followed by sets $\{x_1, n+1\}, \ldots, \{x_s, n+1\}$. The remaining sets follow in any order.

Step 3: We now show that the application of steps 2a, 2b, and 2c of Algorithm 2, implemented by Definition 4.4, over this ordering on M_c generates a minimal CPP.

This is obviously true for $X = \{1, 2\}$. If it is true for $X = \{1, \ldots, n\}$ we will prove that it is also true when adding $n + 1$. Let $\{C_i\}_{i=1,\ldots,P}$ denote the family of classes of $X = \{1, \ldots, n\}$, arranged in order of appearance. Clearly, $\{A_k^*\}_{k=1,\ldots,K}$ generates the same CPP as $\{A_k\}_{k=1,\ldots,K}$ $= M_c^n$, and hence we concentrate on c-maximal sets $\{B_r\}_{r=1,\ldots,n}$. We consider two cases.

If $n+1 = 2^t+1$ for some positive integer t, a new class C_{P+1} is created to which set $\{2^t, n+1\}$ is associated. This is so because by construction there are c-maximal sets with choice 2^t in every C_i, $i = 1, \ldots, P$ (note that $t = s$ when $n+1 = 2^t$) Then, given the selection rule in Definition 4.4, all $\{B_r\}_{r=1,\ldots,n}$ are associated to class C_{P+1}. Now, it is easy to see that $P+1 = t+1 = log_2(n+1) = log_2(2^t+1)$ rounded off to the higher integer. Hence, Kalai, Rubinstein and Spiegler's (2002) Proposition 3 guarantees that this is the minimal CPP.

On the other hand, if $n + 1 \neq 2^t + 1$ for any positive integer t, then we show that no new class is created, and hence the result follows. Set $\{2^t, n + 1\}$ can only be associated to class C_P, since, by construction, all other classes contain c-maximal sets A such that $c(A) = \{2^t\}$. Further, the ordering on M_c^{n+1} has been designed in such a way that, by the selection rule in Definition 4.4, sets $\{x_1, n + 1\}, \ldots, \{x_s, n + 1\}$ are associated to the classes $C_{i_1}, \ldots, C_{i_s}$ where C_{i_j} is the first class in which x_j is not selected for any c-maximal set in the class. Finally, the remaining subsets do not generate any disturbance since they can be associated to one of the classes $C_{i_1}, \ldots, C_{i_s}$. This completes the proof of the proposition.■

4.1.3. *The Median Procedure.* Gaertner and Xu (1999b) provide a characterization of the median choice procedure, and Kalai, Rubinstein, and Spiegler (2002) give lower and upper bounds for the minimal number of rationales required to rationalize it. The basic idea of such a rule is to select, from those subsets with an odd number of alternatives, the element that is situated in an intermediate position with respect to an original linear order over the alternatives. When considering subsets with an even number of alternatives, we adopt the rule of selecting the first intermediate element.

We show here that the cardinality of M_c for the median procedure is *not* polynomially bounded in terms of $|X|$, and thus does not admit a significant contraction of set $P(X)$. Proposition 4.6 characterizes M_c when N is odd, being the case when N is even analogous.

Proposition 4.6. *Let (X, c) be a median procedure. Then, M_c is not polynomially bounded in terms of $|X| = N$. When N is odd, the number of elements is:*
$|M_c| = \sum_{j=1}^{j=(N-1)/2} [\binom{((N-1)/2)+j}{2j-1} + \binom{((N-1)/2)+j}{2j}]$

Proof: Suppose that N is an odd number. First, we proceed to calculate the number of sets in M_c for which some element in $\{(N+1)/2 - k\}_{k=1,2,\ldots,(N-1)/2}$ is the choice. Notice that, for any of these elements to be chosen in a c-maximal set, two conditions must be fulfilled:

- the element must be in the c-maximal set, and
- for every $(N+1)/2 - k$ there are exactly $(N+1)/2 - k - 1$ elements in the c-maximal set *before* $(N+1)/2 - k$. That is, all the elements $1, 2, \ldots, (N+1)/2 - k$ are in the set, and this constitutes exactly half the cardinality of the c-maximal set.

This is equivalent to extracting exactly $2k - 1$ elements from a set of $(N-1)/2 + k$, i.e., $\binom{(N-1)/2+k}{2k-1}$. This constitutes the first summation.

Secondly, we proceed to calculate the number of sets in M_c for which some element in $\{(N+1)/2 + k\}_{k=1,\ldots,(N-1)/2-1}$ is the choice.[5] By reasoning similar to above, for every $(N+1)/2+k$ it is sufficient to consider the number of sets for which all the elements $(N+1)/2 + k, \ldots, N$ are in the set, constituting half the cardinality of the set plus one, or equivalently, to consider the possible extractions of $2k$ elements from a set of $(N-1)/2 + k$, i.e., $\binom{(N-1)/2+k}{2k}$. The c-maximal set $\{1, 2, \ldots, N\}$ in which $(N+1)/2$ is selected completes the list, and, since $1 = \binom{N}{N}$, it may be considered within the second summation. Clearly, the value of $|M_c|$ is not polynomially bounded in terms of $|X|$. A similar reasoning applies for N being even, concluding the proof. ■

Hence, the task of computing a minimal CPP for the median procedure still remains complicated. This divergence between the second-best and the median procedures was also present in the work of Kalai, Rubinstein, and Spiegler (2002), where the possibility of providing the exact minimal number of rationales in the case of the second-best was accompanied with a bounded solution in the case of the median procedure.

[5] Notice that the element N is never selected except in the singleton $\{N\}$ which does not belong to M_c by definition.

4.2. **Reducing the set of linear orders over M_c to be considered.** We start by giving a very partial answer to the question of reducing the number of linear orders over M_c that must be analyzed in order to obtain a minimal RMR. Proposition 4.7 shows that it is sufficient to evaluate the linear orders over M_c that situate X (this is always an element of M_c) as the first choice problem.

Proposition 4.7. *A reformulation of Algorithm* 1 *(or* 2*) that only considers linear orders over M_c starting with set X generates a minimal CPP.*

Proof: We proceed as in the proof of Proposition 3.1, considering a minimal CPP $\{C_1, \ldots, C_m\}$. Note, however, that the selected ordering of classes in that proof was irrelevant, while, here, we will suppose that $X \in C_1$. We only need to prove that $X = C_1^1$, since, in this case, the ordering that starts with X and follows as stated in the proof of the cited proposition, generates a CPP with, at most, m classes. Consider any $A \in C_1$. Since $A \in M_c$ it is the case that $c(A) \neq c(X)$. Therefore, $c(A) \in X \setminus \{c(X)\}$ and it is not the case that ARX. But as long as C_1 endowed with R is a complete preorder, it must be that XRA, implying $X = C_1^1$ and concluding the proof. ■

The above result is due to the fact that the binary relation R is intimately linked to the set inclusion partial order $\subset$. It is not difficult to observe that, for every $A, B \in M_c$, $A \subset B$ implies that it is not true that ARB, which was the key point in the previous proof. Thus, the set X must be the top element within its class. Notice that this was the case of the specific linear orders $<$ given for the rational and second-best procedures. More importantly, a scrutiny of those solutions shows that the linear orders were in fact extensions of the set inclusion partial order $\subset$. This, in conjunction with the link of R with $\subset$ could lead us to venture that we only need to analyze linear orders over M_c that constitute extensions of the partial order $\subset$. This would considerably reduce the number of linear orders to be analyzed, and hence the computational effort. Unfortunately, this conjecture is not correct, as the next proposition shows.

Proposition 4.8. *The set of linear orders over M_c that extend the set inclusion partial order $\subset$ does not necessarily generate a minimal CPP of (M_c, R).*

Proof: Let $X = \{1, 2, 3, 4, 5\}$. Consider the choice function given by $c(\{1, 3, 4, 5\}) = 4$, $c(\{2, 4, 5\}) = 5$ and, otherwise, $c(A) = i$, such that i in A and $i < j$ for all j in A. It is not hard to see that M_c

is composed by $A_1 = X, A_2 = \{2,3,4,5\}, A_3 = \{3,4,5\}$ and the sets $A_4 = \{1,3,4,5\}$ and $A_5 = \{2,4,5\}$.

Any ordering of M_c that respects the inclusion partial order leads to a CPP with 3 classes. However, there are orderings that generate a CPP with 2 classes, which is the minimal number. Let us consider, for instance, the ordering A_1, A_3, A_5, A_2, A_4. This generates the CPP conformed by classes $\{A_1, A_3, A_5\}$ and $\{A_2, A_4\}$. ■

Note that a direct corollary of Proposition 4.8 rejects another "natural" candidate for reducing the set of linear orders to be considered: to order sets in M_c according to their cardinality. Hence, beyond the result in Proposition 4.7, the task of restricting the set of $<$ over M_c remains open.

5. Conclusions

In this paper we have elaborated on the issue of rationalizing choice functions that do not necessarily satisfy the IIA axiom by using multiple rationales, a notion originally proposed by Kalai, Rubinstein, and Spiegler (2002). It has been our aim to provide a complete description of a minimal book of rationales for every choice function. That is, we give the composition of each of the rationales composing a minimal number of rationales that rationalizes choice behavior, and information on how choice problems are associated to rationales. Such a characterization of the problem renders the information given by a choice function completely equivalent to that given by a minimal book of rationales. This feature, which was present in the classical case, is one that we explicitly wanted to restore when IIA does not hold.

A systematic inquiry into the nature of minimal books of rationales may, we believe, constitute a valuable tool for the ultimate goal of understanding the idiosyncrasies of behavior. For example, understanding why the decision-maker opens one page or another of the book in order to rationalize a choice problem may significantly help to advance understanding of the nature of individual behavior.

We will finish by relating our work to recent developments in the literature. Salant (2003) studies two computational aspects of choice when the IIA axiom does not necessarily hold: the amount of memory choice behavior requires, and the computational power needed for the computation of choice. He shows that the rational procedure is favored by these considerations. Manzini and Mariotti (2004) study the nature of choice functions that can be rationalized by sequentially applying a fixed set of asymmetric binary relations. Interestingly, they show that choice procedures like the second-best or median cannot be sequentially rationalized. Finally, Ok (2004) provides an axiomatic characterization of choice correspondences that satisfy axiom IIA.

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