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Generation of Fuzzy Mathematical Morphologies

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Abstract

Fuzzy Mathematical Morphology aims to extend the binary morphological operators to grey-level images. In order to define the basic morphological operations fuzzy erosion, dilation, opening and closing, we introduce a general method based upon fuzzy implication and inclusion grade operators, including as particular case, other ones existing in related literature

In the definition of fuzzy erosion and dilation we use several fuzzy implications (Annexe A, Table of fuzzy implications), the paper includes a study on their practical effects on digital image processing. We also present some graphic examples of erosion and dilation with three different structuring elements B(i,j)=1, B(i,j)=0.7, B(i,j)=0.4, $i,j \in \{1,2,3\}$ and various fuzzy implications.

Keywords: Fuzzy Mathematical Morphology, Inclusion Grades, Erosion and Dilation.

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1 Introduction

Mathematical morphology provides a set of operations to process and analyse images and signals based on shape features. These morphological transformations are based upon the intuitive notion of “fitting” a structuring element. It involves the study of the different ways in which a structuring element interacts with the image under study, modifies its shape, measures and reduces it to one other image, which is more expressive than the initial one. Mathematical morphology was initially developed to analysing binary images. Binary images are maps $A: \mathbf{U} \longrightarrow \{0, 1\}$, where U represents the Euclidean plane $\mathbf{R}^2$ or the Cartesian grid $\mathbf{Z}^2$ indistinctly, and in each point $x \in \mathbf{U}$ the value of its image only can be 0 or 1, which represents the black colour and the white.

Essentially, mathematical morphology is a theory on morphological transformations. Serra [17] characterises the binary morphological transformations with four principles: Compatibility under translation, compatibility under change of scale, local knowledge and semicontinuity.

The algebraic model that represents the study of the transformation of binary images (crisp subset of **U**) is based on increasing maps Ψ: (P(**U**), $\subseteq$) $\longrightarrow$ (P(**U**), $\subseteq$), such that $A \subseteq B \Rightarrow \Psi(A) \subseteq \Psi(B)$. In particular, it has special relevant the transformations named erosions and dilations of an image A by another B, A and B belonging to P(**U**). Given A and B$\in$P(U) and denoting $A_b=\{a+b \mid a\in A\}$ and $-B=\{-b \mid b\in B\}$, then the erosion and dilation are defined as $\xi(A, B) = \cap\{A_b \mid b\in(-B)\}$ and $\mathcal{D}(A, B) = (\xi(A^C, -B))^C$. These operators are the basis to build others operators used in the theory, as the operators called opening [$@(A, B) = \mathcal{D}(\xi(A, B), B)$], closing [$\zeta(A, B) = \xi(\mathcal{D}(A, -B), -B)$], etc..

Grayscale morphology, as originally formulated by Sternberg, extends binary morphology by treating the space beneath a gray-scale image (A: U$\longrightarrow$G, where G represents $R\cup\{-\infty, \infty\}$ or $Z\cup\{-\infty, \infty\}$) as a binary image, called umbra. The umbra of a image A is defined as $U(A) = \{(x, t)\in UxG \mid t \le A(x)\}$), for an n-dimensional image the umbra is an (n+1)-dimensional set. The customary binary operations are applied to the umbrae of both the image and structuring element. As recognised by Serra, the appropriate framework for mathematical morphology is the set of images whose gray-values are drawn from a complete lattice. The subsequent algebraic development is due to Heijmans and Ronse.

In this paper the binary morphological operations are extended to grayscale images. The gray levels are assumed to belong to the fuzzy subset of Cartesian grid or the Euclidean plane, that is, an image (fuzzy subset) is a map A: **U**$\longrightarrow$[0, 1], where **U** can be any of the planes before mentioned. The grayscale images are interpreted as fuzzy sets in order to be able to apply the operations from fuzzy set theory. For x$\in$U the quantity A(x) denotes the grade which x belongs to fuzzy subset A or the gray level of image A in the point pixel x. On the ordinal scale [0, 1], 0 represents "black" or background and 1 represents "white" or foreground. The usual fuzzy set-theoretic interpretation is also valid: the higher the value, the more that pixel belongs to the image and vice versa.

In the literature on Fuzzy Mathematical Morphology there are different approaches of the operators erosion and dilation [2, 4, 13-16, 19]. For instance, Sinha and Dougherty define the basic operations of Fuzzy Mathematical Morphology by means of the inclusion grade for fuzzy subsets. These authors define the inclusion grade operator as a fuzzy relation, R: F(U)xF(U)$\longrightarrow$[0, 1], such that $R(A, B) = \inf_{y\in Supp\, B} \{\min(1, \lambda(A(x))+\lambda(1-B(x)))\}$ $\forall A, B\in F(U)$, verifying conditions that allow them to consider that for two fuzzy subsets A and B of U, R(A, B) is the degree to which A is subset of B, being λ a function from [0, 1] to [0, 1] verifying some fixed conditions [13-16, 3, 9].

Given A, B$\in$F(U), z$\in$U and denoting –B the fuzzy subset such that $(-B)(x) = B(-x)$ $\forall x\in U$ and B_z the fuzzy subset such that $(B_z)(x) = B(x-z)$ $\forall x\in U$, then they define the erosion of a fuzzy subset A by another one B, called structuring element, as the fuzzy subset $\xi(A, B)$ whose membership function is $\xi(A, B)(z) = R(B_Z, A)$. Dilatation is defined by duality $\mathcal{D}(A, B)(z) = (\xi(A^C, -B)(z))^C$, opening and closing are defined similarly to the binary case in terms of erosion and dilation.

What we do in this paper is to define erosion and dilation with the inclusion grade operators as postulated by Bandler and Kohout [1]. For these authors, given A, B fuzzy subset of U and I a fuzzy implication operator, the degree in which A is a subset of B is given by $R(A, B) = \inf_{x \in U} \{I(A(x), B(x))\}$. Now, fuzzy erosion and dilatation, of A by B, are defined as the fuzzy subset of **U** such that $\xi(A,B)(z) = R(B_z,A) = \{\inf_{x \in U} I(B_z(x),A(x))\}$, and $\mathcal{D}(A, B)(z) = 1\text{-}R((-B)_z, 1\text{-}A)$ $\forall z \in U$ respectively. The opening and closing operators are defined in the usual way, in terms of erosion and dil+ation.

Let us recall that fuzzy implications are maps I: [0, 1] x [0,1] ⟶[0, 1] which extend the implications of Boolean logic, and that just three types of fuzzy implications are commonly studied [5-8, 17, 18]:

I(a, b) = S(n(a), b), where S is a t-conorm and n is a strong negation

I(a, b) = Sup{c∈[0,1] | T(a, c) ≤ b}, where T is a t-norm

I(a, b) = S(n(a), T(a, b)), where S is a t-conorm, n is a strong negation and T is the n-dual of S

However, Willmot, Yager, etc. have proposed other operations that do not belong to the previous mentioned families.

We want to study the properties that characterise the map I:[0,1]x[0,1] ⟶[0,1] for which fuzzy erosion and dilation are relevant to image processing

So we start assuming that the fuzzy implication has the widest meaning, as a map I: [0, 1] x [0,1] ⟶[0, 1] such that I(1, 1) = 1 and I(1, 0) = 0. Note that the implications used here are given in Annexe A, and they are frequently mentioned in the literature [11].

In the next section we discuss some of the properties of these operators erosion and dilation for fuzzy implications of the Annexe A and we also start the analysis of the practical effects of fuzzy erosions and dilations obtained through the different fuzzy implications. The paper includes some examples to illustrate the effects of different operators. The image used for these examples is an image with different gray levels (the Fig. 1 of the Annexe B).

2 Preliminaries

Lemma 1. *Let A and B be two fuzzy subsets of U, I a fuzzy implication of the Annexe A and let R be the corresponding inclusion grade operator. Then the following properties are verified:*

$$R(A, B) = R(-A, -B) \text{ and } R(A, B) = R(A_z, B_z) \quad \forall z \in \mathbf{U}.$$

Proof: $R(-A,-B)= \inf_{x \in U} \{I((-A)(x),(-B)(x))\}= \inf_{x \in \mathbf{U}} \{I(A(-x),B(-x)\} = \inf_{x \in \mathbf{U}} \{I(A(x), B(x)\}$

$$R(A_z,B_z)=\inf_{x\in \mathbf{U}}\{I(A_v(x), B_v(x)\}=\inf_{x\in \mathbf{U}}\{I(A(x\text{-}v), B(x\text{-}v)\}=\inf_{x\in \mathbf{U}}\{I(A(x), B(x)\}=R(A,B).$$

Theorem 1. *Let A and B be two fuzzy subsets of U, I a fuzzy implication of the Annexe A, and* ξ $\mathcal{D}$ *the corresponding fuzzy erosion and dilation. Then the following properties hold:*

1) $\xi(A_V, B) = \xi(A, B)_V$, ***2)*** $\xi(A, B_V) = \xi(A, B)(-v)$,

3) $\mathcal{D}(A_V, B) = \mathcal{D}(A, B)_V$ ***4)*** $\mathcal{D}(A, B_v) = \mathcal{D}(A, B)$

Proof. **1)** $\xi(A_v, B)(z) = R(B_z, A_v) = R(B_{z\text{-}v}, A) = \xi(A, B)(z\text{-}v) = \xi(A, B)_v(z) \quad \forall z\in \mathbf{U}$

2) $\xi(A, B_v)(z) = R((B_z)_v, A) = R(B_{z+v}, A) = \xi(A, B)(z+v) = \xi(A, B)_{(-v)}(z) \quad \forall z\in \mathbf{U}$

3) $\mathcal{D}(A_v,B)(z) = 1\text{-}\xi((A_v)^c, \text{-}B)(z) = 1\text{-}\xi((A^C)_v,\text{-}B)(z) = 1\text{-}\xi(A^C,B)_v(z) = \mathcal{D}(A,B)_v(z) \ \forall z\in \mathbf{U}.$

4) $\mathcal{D}(A, B_v)(z) =1\text{-}\xi(A^C, \text{-}(B_v))(z) =1\text{-}\xi(A^C, (\text{-}B)_{(-v)})(z) =1\text{-}\xi(A^C, \text{-}B)_v(z) = 1\text{-}\xi(A^C, \text{-}B)(z\text{-}v) = \mathcal{D}(A, B)_v(z) \ \forall z\in \mathbf{U}$. .

This theorem proves that for fuzzy dilations and erosions defined with the inclusion grade operator postulated by Bandler and Kohout [1], invariability for translation holds as it happens in ordinary Mathematical Morphology (First principle settled by Serra [15] for the morphological transformations).

The second principle established by Serra [12] for the ordinary morphological transformations deals to the invariability of morphological transformations under changes of scale: $\xi(\alpha A, B) = \alpha\xi(A, B)$ (we define $(\alpha A)(x) = \alpha\, A(x)\ \forall x\in U$), and α is a non negative real number.

In the next theorem it is proved that for the erosion, defined with the usual 18 fuzzy implications listed in Annexe A, the principle of compatibility with changes of scale given by Serra [12] is not satisfied, even restricting ourselves to the values $\alpha\in[0, 1]$.

Theorem 2. *The fuzzy erosions and dilations defined with any fuzzy implication of the Annexe A are not invariant under changes of scale (given A, B fuzzy subset of U, there exits* $\alpha\in(0, 1)$ *such that* $\xi(\alpha A, B) \neq \alpha\xi(A, B)$.*)*

Proof. For each one of the 18 fuzzy implications of the Annexe A, we can find fuzzy subsets A and B, $\alpha\in[0, 1]$ and x, $z\in U$ such that $\xi(\alpha A, B) \neq \alpha\xi(A, B)$. Indeed:

For the fuzzy implications 1, 2, 4, 5, 10, 11, 12, 13, 15, 16, 18, that verify the property $I(0, b) = 1\ \forall b\in[0, 1]$, if we take $\alpha = 0.5$, $B(x) = 0$ and $A(x) = 1\ \forall x\in U$, we have $\alpha\, I(B_z(x), A(x)) = 0.5$ and $I(B_z(x), \alpha A(x)) = 1$.

For the fuzzy implications 6, 7, 8 and 9, if we take $\alpha = 0.5$, $B(x)=0$ and $A(x)=1$ $\forall x\in U$, we have $\alpha\, I(B_z(x), A(x)) \neq I(B_z(x), \alpha A(x))$.

For the fuzzy implications 3, 14, 17, let us just consider the values:

Case 3: $\alpha = 0.5$, $A(x) = 0.5$ and $B(x) = 0.2\ \forall x\in U$

Case 14: $\alpha = 0.5$, $A(x) = 1/3$ and $B(x) = 0.5\ \forall x\in U$

Case 17: $\alpha = 0.5$, $A(x) = 1$ and $B(x) = 0.5$ $\forall x \in U$. .

3 Properties depending on the implication operator

Theorem 3. *Given a fuzzy implication I such that $I(0, b) = 1$ $\forall b \in [0, 1]$ (numbers 1, 2, 4, 5, 10, 11, 12, 13, 15, 16, 18 of the Annexe A), the fuzzy erosion for fuzzy subsets A, B of U defined as $\xi(A, B)(z) = \inf_{x \in U} \{I(B_z(x), A(x))\}$ extends the binary erosion, that is, for crisps subsets A, B the fuzzy erosion $\xi(A, B)$ is equal to the binary erosion $E(A, B) = \cap\{A_b \mid b \in (-B)\}$.*

Proof. For each $z \in U$ we have that $E(A, B)(z) = \begin{cases} 1 & \text{if} \quad A(z+x)=1 \quad \forall x \ni B(x)=1 \\ 0 & \text{otherwise} \end{cases}$

In the case of the fuzzy erosion we have that $\xi(A, B)(z) = \inf_{x \in U} \{I(B_z(x), A(x))\}$ $= \inf_{x \in U} \{I(B(x), A(z+x))\}$. The expression $I(B(x), A(z + x))$ has one of the following values :

1) If $B(x) = 1$ and $A(x + z) = 1$, then $I(1, 1) = 1$ for all I of the Annexe A
2) If $B(x) = 0$ and $A(x + z) = 1$, then $I(0, 1) = 1$ for all I of the Annexe A that verify $I(0, b) = 1$ $\forall b \in [0, 1]$
3) If $B(x) = 0$ and $A(x + z) = 0$, then $I(0, 0) = 1$ for all I of the Annexe A that verify $I(0, b) = 1$ $\forall b \in [0, 1]$
4) If $B(x) = 1$ and $A(x + z) = 0$, then $I(1, 0) = 0$ for all I of the Annexe A

But if $E(A, B)(z) = 1$, $z \in U$, it is impossible to find $x \in U$ such that $B(x) = 1$ and $A(x+z) = 0$. Therefore, $\xi(A, B)(z) = 1$.

If $E(A, B)(z) = 0$, $z \in U$, then there is an $x \in U$, such that $B(x) = 1$ and $A(x + z) = 0$ and therefore, $\xi(A, B)(z) = 0$.

Theorem 4. *Let A, B and K be fuzzy subsets of U, and let I be a fuzzy implication from Annexe A, such that $b \le b' \Rightarrow I(a, b) \le I(a, b')$ (numbers 1, 2, 4, 5, 10, 11, 12, 13, 15, 16, 18 of the Annexe A). Then:*

1) $\xi(A \cap K, B) = \xi(A, B) \cap \xi(K, B)$

2) $\mathcal{D}(A \cup K, B) = \mathcal{D}(A, B) \cup \mathcal{D}(K, B)$.

Proof. 1) Let $z \in U$, $\xi(A \cap K, B)(z) = \inf_{x \in U} \{I(B_z (x), (A \cap K)(x))\} \le \min(\xi(A, B)(z), \xi(K, B)(z))$ since I is increasing in the second variable.

If we suppose that $\xi(A\cap K, B)(z) < \min(\ \xi(A, B)(z), \xi(K, B)(z)\ )$, then there exists $y\in U$ such that $I(B_z(y), (A\cap K)(y)) < \min(\ \xi(A, B)(z), \xi(K, B)(z)\ ) \Rightarrow I(B_z(y), (A\cap K)(y)) < \min(I(B_z(x), A(x)), I(B_z(x), K(x)))\ \ \forall x\in U$.

In particular $I(B_z(y), (A\cap K)(y)) < \min(\ I(B_z(y), A(y)), I(B_z(y), K(y))\ )$, but

- If $\min(A(y), K(y)) = A(y)$, then $I(B_z(y), A(y)) < I(B_z(y), A(y))$ which is a contradiction
- If $\min(A(y), K(y)) = K(y)$, then $I(B_z(y), K(y)) < I(B_z(y), K(y))$ which is also a contradiction.

Therefore $\xi(A\cap K, B)(z) = \min(\ \xi(A, B)(z), \xi(K, B)(z)\ )$.

Let $z\in U$, $\mathcal{D}(A\cup K, B)(z) = [\xi((A\cup K)^C, -B)\,]^C(z) = 1-\inf_{x\in U}\{I((-B)_z(x), \min(A^C(x), K^C(x)))\} = \sup_{x\in U}\ \{1 - I((-B)_z(x), \min(A^C(x), K^C(x)))\} \geq$

$\sup_{x\in U}\{\max(1-I((-B)_z(x), A^C(x)), 1 - I((-B)_z(x), K^C(x)))\} =$

$\max(\mathcal{D}(A, B)(z), \mathcal{D}(K, B)(z)\ ) = [\mathcal{D}(A, B) \cup \mathcal{D}(K, B)](z)$.

If we suppose that $\mathcal{D}(A\cup K, B)(z) > \max(\ \mathcal{D}(A, B)(z), \mathcal{D}(K, B)(z)\ )$, then $y\in U$ can be found, such that

$1 - I((-B)_z(y), \min(A^C(y), K^C(y))) > \max(\ \mathcal{D}(A, B)(z), \mathcal{D}(K, B)(z)\ ) \Rightarrow$

$1 - I((-B)_z(y), \min(A^C(y), K^C(y))) > \max(1 - I((-B)_z(y), A^C(y)), 1 - I((-B)_z(y), K^C(y))\ ) \Rightarrow$

$I((-B)_z(y), \min(A^C(y), K^C(y))) < \min(\ I((-B)_z(y), A^C(y)), I((-B)_z(y), K^C(y))\ )$. But:

- If $\min(A(y)^C, K^C(y)) = A^C(y)$, then $I(B_z(y), A^C(y)) < I(B_z(y), A^C(y))$ which is a contradiction
- If $\min(A^C(y), K^C(y)) = K(y)$, then $I(B_z(y), K^C(y)) < I(B_z(y), K^C(y))$ which is also a contradiction.

Therefore, $\mathcal{D}(A\cup K, B) = \mathcal{D}(A, B) \cup \mathcal{D}(K, B)$.

Here we have the equivalent, in the area we are working on, to the third principle, established by Serra [12], for morphological transformations: the local knowledge principle.

Let A and B be two fuzzy subsets of **U**, fuzzy images, let K be any bounded crisp set of U in which $A \cap K$ is known. Then fuzzy erosion and dilation satisfy the local knowledge principle if there exits a bounded crisp set K' of **U** which only depends on K, such that $\xi(A \cap K, B) \cap K' = \xi(A, B) \cap K'$ and $\mathcal{D}(A \cap K, B) \cap K' = \mathcal{D}(A, B) \cap K'$.

Theorem 5. *Let A and B fuzzy subsets of* ***U****, let K be any bounded crisp set of U in which $A \cap K$ is known and let I be a fuzzy implication decreasing with respect to the first variable and increasing in the second variable and verifying $I(0, b) = 1\ \forall b\in[0, 1]$. Then the following two equalities are satisfied:*

1) $\xi(A\cap K, B) \cap \xi(K, B_{o\alpha}) = \xi(A, B) \cap \xi(K, B_{o\alpha})$
2) $\mathcal{D}((A\cap K)\cup K^C, B) \cap \xi(K, (-B)_{o\alpha})$. $= \mathcal{D}(A\cap K, B) \cap \xi(K, (-B)_{o\alpha})$.

Being $B_{o\alpha} = \bigcup_{\alpha \in]0,1]} B_\alpha$

Proof. From theorem 4, it is easily verified that for A, B and K fuzzy subsets of U:

3) $\xi(A \cap K, B) \cap \xi(K, B) = \xi(A, B) \cap \xi(K, B)$

4) $\mathcal{B}((A \cap K) \cup K^C, B) \cap \xi(K, -B). = \mathcal{B}(A \cap K, B) \cap \xi(K, -B).$

- If B is not a crisp subset, then $\xi(K, B)$ and $\xi(K, -B)$ neither are crisp subsets, but if we take a crisp subset B as structuring element, then $\xi(K, B)$ and $\xi(K, -B)$ are crisp subsets, and the theorem is proved.
- If B is fuzzy subset, we define $B o\alpha = \bigcup_{\alpha \in]0,1]} B\alpha$, for all $z \in U$. We want to see that

  $\min(\xi(A \cap K, B)(z), \xi(K, B\, o\alpha)(z)) = \min(\xi(A, B)(z), \xi(K, B\, o\alpha)(z))$.

The following inequality is trivial $\min(\xi(A \cap K, B)(z), \xi(K, B\, o\alpha)(z)) \le \min(\xi(A, B)(z), \xi(K, Bo\alpha)(z))$. Let us prove $\min(\xi(A \cap K, B)(z), \xi(K, B\, o\alpha)(z)) \ge \min(\xi(A, B)(z), \xi(K, B\, o\alpha)(z))$.

It is easily verified when

1) $\xi(A \cap K, B)(z) = \xi(A, B)(z) = \xi(K, B_{o\alpha})(z)$,

2) $\xi(A \cap K, B)(z) = \xi(A, B)(z) < \xi(K, B_{o\alpha})(z)$

3) $\xi(A \cap K, B)(z) = \xi(A, B)(z) > \xi(K, B_{o\alpha})(z)$,

4) $\xi(A \cap K, B)(z) < \xi(A, B)(z)$ and $\xi(K, B_{o\alpha})(z) = \xi(A \cap K, B)(z)$

In this four cases we have $\min(\xi(A \cap K, B)(z), \xi(K, B_{o\alpha})(z)) = \min(\xi(A, B)(z), \xi(K, B_{o\alpha})(z))$.

But besides, we have to see it when $\xi(A \cap K, B)(z) < \xi(K, B_{o\alpha})(z)$ and $\xi(A \cap K, B)(z) < \xi(A, B)(z)$.

- If $\xi(A \cap K, B)(z) < \xi(K, B_{o\alpha})(z)$, then there is $y \in U$ such that $I(B_z(y), (A \cap K)(y)) < I(B_{o\alpha}(x-z), A(x))\ \forall x \in U$ (**i**). In particular, $I(B_z(y), (A \cap K)(y)) < I(B_{o\alpha}(y-z), A(y))$ for every y that verifies (**i**),
- If $\xi(A \cap K, B)(z) < \xi(A, B)(z)$, then there is $h \in U$ such that $I(B_z(h), (A \cap K)(h)) < I(B_z(x), A(x))\ \forall x \in U$ (**ii**). In particular, $I(B_z(h), (A \cap K)(h)) < I(B_z(h), A(h))$ for every h that verifies (**ii**), and $I(B_z(h), (A \cap K)(h)) < I(B_z(y), A(y))$ for every y and h that verifies (**i**), (**ii**) respectively.

Let $\xi(A \cap K, B)(z) < \xi(K, B_{o\alpha})(z)$ and $\xi(A \cap K, B)(z) < \xi(A, B)(z)$. Then there is $y \in U$ such that $I(B_z(y), (A \cap K)(y)) < I(B_{o\alpha}(y-z), K(y))$. The four following situations should be considered:

- $B_{o\alpha}(y-z) = 0$ and $A(y) < K(y) \Rightarrow B_z(y) = 0$, $I(B_z(y), A(y)) = I(0, A(y)) = 1 < I(B_{o\alpha}(y-z), K(y)) = I(0, K(y)) = 1$, which is a contradiction.

- $B_{o\alpha}(y\text{-}z) = 0$ and $K(y) \leq A(y) \Rightarrow B_z(y) = 0$, $I(B_z(y), K(y)) = I(0, K(y)) = 1 < I(B_{o\alpha}(y\text{-}z), K(y)) = I(0, K(y)) = 1$, which is also a contradiction.
- $B_{o\alpha}(y\text{-}z)= 1$ and $K(y) \leq A(y) \Rightarrow B_z(y) \neq 0$, $I(B_z(y), K(y)) < I(B_{o\alpha}(y\text{-}z), K(y)) = I(1, K(y))$, also is a contradiction since I is decreasing in the first variable.
- $B_{o\alpha}(y\text{-}z) = 1$ and $A(y) < K(y)$. Then this occurs: $I(B_z(h), K(h)) < I(B_z(y), A(y)) < I(B_{o\alpha}(h\text{-}z), K(h))$ by (**ii**) and (**i**) respectively. But $K(h) = 0$, therefore $I(B_z(h), 0) < I(B_z(y), A(y)) < I(1, 0)$ which is contradictory since I is decreasing in the first variable.

Therefore, $\xi(A \cap K, B)(z) < \xi(K, B_{o\alpha})(z)$ and $\xi(A \cap K, B)(z) < \xi(A, B)(z)$ will never occur simultaneously. And the theorem is proved. .

The fourth principle deals with the semicontinuity of the morphological transformations. A fuzzy implication I: $[0, 1]x[0, 1] \longrightarrow [0, 1]$ is semicontinuous if it is decreasing in the first argument, increasing in the second and $I(\sup_{i \in N} a_i, \inf_{j \in N} b_j) = \inf_{i, j \in N} I(a_i, b_j)$. In the next theorem we can to see that erosion (dilation) is semicontinuous if the fuzzy implication used in this operators is semicontinuous [9].

Theorem 6. *Let I be a fuzzy implication I: $[0, 1]x[0, 1] \longrightarrow [0, 1]$ that is decreasing in the first argument, increasing in de second argument and semicontinuous. Then the fuzzy erosion is semicontinuous, that is:* $\xi(\sup_{i \in N} A^i, \inf_{j \in N} B^j)(z) = \inf_{i, j \in N} \xi(A^i, B^j)(z)\ \forall z \in U$.

In the next theorems and corollaries we will establish the link between some properties of the fuzzy implications and the fuzzy morphological transformation, as well as the dependence between fuzzy erosions and fuzzy implications.

Theorem 7. *Let I and J be fuzzy implications, such that $I \leq J$ ($I(a, b) \leq J(a, b)\ \forall a,b \in [0, 1]$), then it holds that* $\xi_I(A, B)(z) \leq \xi_J(A, B)(z)$, $\mathcal{D}_I(A, B)(z) \geq \mathcal{D}_J(A, B)(z)$, $@_I(A, B)(z) \geq @_J(A, B)(z)$, $\zeta_I(A, B)(z) \leq \zeta_J(A, B)(z)\ \forall z \in \mathbf{U}$.

Proof. For each $z \in \mathbf{U}$ we have $\xi_I(A, B)(z) = \inf_{x \in U} \{I(B_z(x),A(x))\}$ and $\xi_J(A,B)(z) = \inf_{x \in U} \{J(B_z(x),A(x))\}$, but $I(B_z(x),A(x)) \leq J(B_z(x),A(x))\ \forall x \in U$. Therefore $\xi_I(A,B)(z) \leq \xi_J(A,B)(z)$. In the case of dilation we have $\mathcal{D}_I(A, B)(z) = 1 - \xi_I(A^C, -B)(z)$ and $\mathcal{D}_J(A, B)(z) = 1 - \xi_J(A^C, -B)(z)$, but we have seen that $\xi_I(A^C, -B)(z) \leq \xi_J(A^C, -B)(z)$ implies that $\mathcal{D}_I(A, B)(z) \geq \mathcal{D}_J(A, B)(z)$**.** For both opening and closing, the proof is similar.

Theorem 8. *Let A, A', B and B' be fuzzy images, and I a fuzzy implication that satisfies the following conditions:*

i) $a \le a' \Rightarrow I(a, b) \ge I(a', b)$, a, a' and $b \in [0, 1]$,
ii) $b \ge b' \Rightarrow I(a, b) \ge I(a, b')$ b, b' and $a \in [0, 1]$. *Then:*
1) $A \subseteq A' \Rightarrow R(A', B) \le R(A, B)$,
2) $B \subseteq B' \Rightarrow R(A, B) \le R(A, B')$,
3) $R(A \cup A_1, B) = \min(R(A,B), R(A_1, B))$,
4) $R(A, B \cap B_1) = \min(R(A, B), R(A, B_1))$.

Proof: **1)** $A \subseteq A' \Rightarrow A(x) \le A'(x)\ \forall x \in U \Rightarrow I(A(x), b) \ge I(A'(x), b)\ \forall x \in U$ and $\forall b \in [0, 1] \Rightarrow I(A(x), B(x)) \ge I(A'(x), B(x))\ \forall x \in U \Rightarrow R(A, B) = \inf_{x \in U} \{I(A(x), B(x))\}$ $\ge \inf_{x \in U} \{I(A'(x), B(x))\} = R(A', B)$.

3) $R(A \cup A_1, B) = \inf_{x \in U} \{I(\max(A(x), A_1(x)), B(x))\} =$
$\inf_{x \in U} \{\min(I(A(x), B(x)), I(A_1(x), B(x)))\} = \min (R(A, B), R(A_1, B))$.

The cases 2 and 4 can be proved similarly to cases 1 and 3 respectively.

Corollary 9. *Let A and A' be fuzzy images* $A \subseteq A'$ *and I a fuzzy implication such that:* $a \le a' \Rightarrow I(a, b) \ge I(a', b)$, $\forall a, a'$ *and* $b \in [0, 1]$. *Then it holds that:*
$\xi(A, B)(x) \le \xi(A', B)(x)$, $\mathcal{D}(A, B)(x) \le \mathcal{D}(A', B)(x)$, $@(A, B)(x) \le @(A', B)(x)$,
$\zeta(A, B)(x) \ge \zeta(A', B)(x)\ \ \forall x \in \mathbf{U}$.

Corollary 10. *Let B and B' be fuzzy images, such that* $B \subseteq B'$ *and I a fuzzy implication such that:* $b' \ge b \Rightarrow I(a, b) \ge I(a, b')$ $\forall b, b'$ *and* $a \in [0, 1]$
Then: $\xi(A, B')(x) \le \xi(A, B)(x)\ \ \forall x \in \mathbf{U}$ *and* $\mathcal{D}(A, B)(x) \le \mathcal{D}(A, B')(x)\ \ \forall x \in \mathbf{U}$

Corollary 11. *Let A, A', B and B' be fuzzy images of* **U** *and I a fuzzy implication with:* $a \le a' \Rightarrow I(a, b) \ge I(a', b)$, $\forall a, a'$ *and* $b \in [0, 1]$ *and* $b' \ge b \Rightarrow I(a, b) \ge I(a, b')$ $\forall b, b'$ *and* $a \in [0,1]$. *Then* $\xi(A, B \cup B') = \xi(A, B) \cap \xi(A, B')$, $\xi(A \cap A', B) = \xi(A, B) \cap \xi(A' \cap B)$ *and* $\mathcal{D}(A, B \cup B') = \mathcal{D}(A, B) \cup \mathcal{D}(A \cap B')$.

Theorem 12. *Given n a strong negation and I a fuzzy implication such that* $I(a, b) = I(n(b), n(a))\ \forall a, b \in [0, 1]$, *then* $R(A, B) = R(B^C, A^C)$.

Corollary 13. *Given n a strong negation and I fuzzy implication such that* $I(a, b) = I(n(b), n(a))\ \forall a, b \in [0, 1]$, *then it holds that the associated dilation operator is commutative, that is,* $\mathcal{D}(A, B) = \mathcal{D}(B, A)\ \forall A, B \in F(\mathbf{U})$.

4. Analysis of the different definitions of erosion: practical effects on the image processing

In this section we study the effects of the structuring element B in the grey levels of the output image $\xi(A, B)$ / $\mathscr{D}(A, B)$, when processing a digital image A with the erosion or dilation operators.

Proposition 14. *Given A and B fuzzy subsets of U and I a fuzzy implication satisfying $I(0, x) = 1\ \forall x \in [0, 1]$. Then*

$$\xi(A, B)(z) = \inf_{y \in SuppB} \{I(B(z), A(z + y))\}\ \forall z \in U$$

Proof. Given $x \notin$ Supp B, then $B(x) = 0$ and $I(0, A(x+y)) = 1\ \forall y \in U$. Therefore

$$\xi(A, B)(z) = \inf_{y \in U} \{I(B(z), A(z + y))\} = \inf_{y \in SuppB} \{I(B(z), A(z + y))\}\ \forall z \in U.$$

Proposition 15. *Given A, B fuzzy subsets of U, $B(x) = \alpha \in [0, 1]\ \forall x \in Supp\ B$, and I a fuzzy implication such that $I(0, x) = 1\ \forall x \in [0, 1]$. Then* $\xi(A, B)(z) = \inf_{y \in SuppB} \{I(\alpha, A(z + y))\}$ *and* $\mathscr{D}(A, B)(z) = 1 - \inf_{y \in SuppB} \{I(\alpha, 1 - A(z + y))\}$.

Proof. It is follows immediately from the fuzzy erosion-dilation and Proposition 14. .

Note that the fuzzy implications 1,2,4,5,10,11,12,13,15,16 and 18 of the Annexe A verifies the property $I(0, x) = x\ \forall x \in [0, 1]$.

Proposition 16. *Given A, B fuzzy subset of U, $B(x) = 1\ \forall x \in Supp\ B$, and I a fuzzy implication such that $I(0, x) = 1\ \forall x \in [0, 1]$ and $I(1, x) = x\ \forall x \in [0, 1]$. Then*

$$\xi(A, B)(z) = \inf_{y \in SuppB} \{A(z + y)\} \text{ and } \mathscr{D}(A, B)(z) = \sup_{y \in SuppB} \{A(z + y)\}.$$

Proof. It is follows directly from Proposition 15. .

Note that the fuzzy implications 1,2,5,10,11,13 and 18 of the Annexe A satisfies the properties $I(0, x) = 1\ \forall x \in [0, 1]$ and $I(1, x) = x\ \forall x \in [0, 1]$.

In the next paragraph we delve further into some consequences derived from the fuzzy implication and the structuring element chosen in the operators erosion / dilation. In particular we study erosion-dilation for fuzzy implications 1,2,5,10,11,13 and 18 of the Annexe A. Let A and B be fuzzy subsets of **U,** such that $B(x) = \alpha \in (0, 1)\ \forall x \in$ **U**. Then the following equality and inequality hold for several fuzzy implications:

- Zadeh Implication
  - If $1 > \alpha > 0.5$, then $\xi(A, B)(x) \le \alpha$. If $\alpha \le 0.5$, then $\xi(A, B)(x) = 1 - \alpha$.

- if 1> α >0.5, then $\mathcal{S}$(A, B)(x) = $\begin{cases} 1-a & \text{if } 1-a > A(x+y) \\ \inf\limits_{y\in Supp\,B} \{a, A(x+y)\} & \text{otherwise} \end{cases}$.

  If α ≤ 0.5, $\mathcal{S}$(A, B)(x) = α

- Lukasiewicz Implication
  - If 1>α, then ξ(A,B)(x)= $\inf\limits_{y\in SuppB}$ {min(1.1-α+A(x+y)}> $\inf\limits_{y\in SuppB}$ {A(x+y)}
  - If 1 > α > 0, then ξ(A, B)(x) ≥ $\inf\limits_{y\in SuppB}$ {A(x + y)}, the whole image brightens depending of the value of α.
  - If 1>α >0, then $\mathcal{S}$(A, B)(x)= $\begin{cases} 0 & \text{if } a+A(x+y)\le 1\ \forall y\in D_B \\ \sup\limits_{y\in SuppB} \{a+A(x+y)\} & \text{otherwise} \end{cases}$

- Gödel Implication
  - If α <1, then ξ(A, B)(x) = $\begin{cases} 1 & \text{if } a \le A(y+x)\ \ \forall y\in D_B \\ \inf\limits_{y\in Supp\,B} \{A(y+x)\} & \text{otherwise} \end{cases}$.

Areas, whose greys are brighter than the structuring element, turn into white, otherwise they take the minimum of the elements of the image involved.

- For α < 1, if α ≤ 1-A(x + y) ∀y∈ D_B then $\mathcal{S}$(A, B)(x) = 0 else $\mathcal{S}$(A, B)(x) = max(A(x + y)| y∈D_B)

- Kleene-Dienes Implication
  - If α<1, then ξ(A, B)(x) = $\begin{cases} \inf\limits_{y\in Supp\,B} \{A(y+x)\} & \text{if } 1-a \le A(y+x)\ \forall y\in D_B \\ 1-a & \text{otherwise} \end{cases}$.
  - For 0< α <1, if α < A(x + y) ∀y∈D_B then $\mathcal{S}$(A, B)(x) = α else $\mathcal{S}$(A, B)(x) = 1-max{A(x + y)| y∈D_B}

- Gaines Implication
  - If α<1, then ξ(A, B)(x) = $\begin{cases} 1 & \text{if } a \le A(y+x)\ \forall y\in D_B \\ \inf\limits_{y\in Supp\,B} \left\{\dfrac{A(y+x)}{a}\right\} & \text{otherwise} \end{cases}$.
  - For 0<α<1, $\mathcal{S}$(A,B)(x) = $\begin{cases} 0 & \text{if } A(x+y)\le 1-\alpha\ \forall y\in D_B \\ 1-\min\left\{\dfrac{1-A(x+y)}{\alpha} \mid y\in D_B\right\} & \text{otherwise} \end{cases}$

- Kleene-Dienes-Lukasiewicz Implication
  - If 0<α<1, then ξ(A, B)(x) = $\inf\limits_{y\in SuppB}$ {1-α+A(y+x)α},.
  - For 0<α<1, $\mathcal{S}$(A, B)(x) = max{A(y+x)α | y∈D_B }

- Yager Implication
  - If $0<\alpha<1$, then $\xi(A, B)(x) = \inf_{y\in SuppB} \{A(y+x)^{\alpha}\} > \inf_{y\in SuppB} \{A(y+x)\}$.
  - If $0 < \alpha < 1$, $\xi(A, B)(x) = \inf_{y\in SuppB} \{A(y + x)^{\alpha}\} > \inf_{y\in SuppB} \{A(y + x)\}$.
  - For $0 < \alpha < 1$, if $A(x + y) = 0$ $\forall y\in$ Supp B then $\mathcal{D}(A,B)(x) = 0$ else (if $A(x + y) = 1$ $\forall y\in$ SuppB then $\mathcal{D}(A, B)(x) = 1$ else $\mathcal{D}(A, B)(x) = 1 - \min\{(1-A(y + x))\alpha \mid y\in DB \})$.

Partial results of the effects of performing of fuzzy erosion and dilation can be seen in the Annexe B.

5 Conclusions

The fuzzy erosion and dilation operators defined with the grade inclusion operator $R(A, B) = \inf_{x\in U} \{I(A(x), B(x))\}$, are relevant to Fuzzy Mathematical Morphology if the map $I: [0, 1]x[0, 1] \longrightarrow [0, 1]$ used satisfies the following properties:

1) $I: [0, 1]x[0, 1] \longrightarrow [0, 1]$ with $I(1, 1) = 1$ and $I(1, 0) = 0$
2) If $a \le a'$ then $I(a, b) \ge I(a', b)$ and If $b \ge b'$ then $I(a, b) \ge I(a, b')$ $\forall a, a', b, b'\in [0, 1]$
3) $I(0, b) = 1$ $\forall b\in [0, 1]$

In this case, the fuzzy erosions (dilations) for fuzzy subsets extend the binary erosion (dilation) and these operators satisfy some of the properties that holds in ordinary mathematical morphology. In particular, erosion (dilation) verifies the first principle established by Serra [12] (the invariability of translations for morphological transformations); the third principle, the sufficiency of the local knowledge. The fulfilment of the second and fourth principles of the same author depends on the semicontinuity and invariability under homothetics of the used fuzzy implication.

For binary images and binary structuring elements, erosions and dilations, such as we have defined here, produce the same results as the analogous operators in ordinary mathematical morphology.

We can finish these conclusions saying that the morphological operators proposed in this work, extend the correspondent concepts of binary operators and that the effects of performing of these operators are of interest in image processing. For instance, in the computerised treatment of sonography or radiology pictures, where grey, black and white levels are apt to appear.

References

[1] Bandler, W. and Kohout, L., Fuzzy Power sets and Fuzzy Implication Operator. *Fuzzy Sets and Systems*, **4** (1980) 13-30.

[2] Bloch, I. and Maître, H., *Constructing a Fuzzy Mathematical Morphology: Alternative ways*, Telecom Paris 92 C 002, September 1992.

[3] Burillo, P., Frago, N. and Fuentes, R., Generación de Morfologías Matematicas Borrosas, *Actas del VIII Congreso Español sobre Tecnologías y Lógica Fuzzy,* Pamplona 1998.

[4] De Baets, Bernad, Fuzzy morphology: a logical approach, Uncertainty Analysis in *Engineering and Sciences: Fuzzy Logic, Statistics and Neural Network Approach* (B. Ayyub and M. Gupta, eds.), Kluwer Academic Publishers, 1997, pp. 53-67.

[5] Dubois D..and Prade H., Fuzzy Logics and the generalised modus ponens revisited. *Cybernetics and Systems: An International Journal*, **15** (1984) 293-331.

[6] Dubois, D and. Prade, H., *Fuzzy Sets and Systems: Theory and Applications, Mathematics in Science and Engineering*, (Academic Press, New York 1980).

[7] Dubois, D and. Prade, H., Fuzzy sets in approximate reasoning, Part 1: Inference with possibility distributions. *Fuzzy Sets and Systems*, **40** (1991) 143-202.

[8] Dubois, D., Jérôme, L. and. Prade, H., Fuzzy Sets in approximate reasoning, Part 2: Logical approaches. *Fuzzy Sets and Systems*, **40** (1991) 203-244.

[9] Frago, N., *Morfología Matemática Borrosa basada en operadores generalizados de Lukasiewicz: Procesamiento de imagenes*. Ph.D Thesis, Universidad Pública de Navarra, Spain 1996.

[10] Giardina, R. G. and. Dougherty, E. R , *Morphological Methods in Image and Signal Processing*, (Prentice-Hall, New Yersy 1988).

[11] Ruan, D. and. Kerre, E.E., Fuzzy implications operators and generalised fuzzy method of cases, *Fuzzy set and Systems*, **54** (1993) 23-37

[12] Serra, J., *Image Analysis and Mathematical Morphology*, Vol. 1 y 2, Academic Press, London 1982.

[13] Sinha, D and Dougerty, E. R., Characterization of Fuzzy Minkowski Algebra, *SPIE Vol. 1769 Image Algebra and Morphological Image Processing III* (1992) 59-69.

[14] Sinha, D and Dougerty, E. R., Fuzzification of Set inclusion, *SPIE Vol. 1708 Applications of Artificial Intelligence X: Machine Vision and Robotics* (1992) 440-449.

[15] Sinha, D and Dougerty, E. R., Fuzzification of set inclusion: Theory and applications, *Fuzzy Sets and Systems*, **55** (1993) 15-42.

[16] Sinha, D and Dougerty, E. R., Fuzzy Mathematical Morphology, *Journal of Visual Communication and Image Representation*, Vol. **3**, No. 3, September 1992, 286-302.

[17] Trillas, E. and Valverde, L., On mode and implication in approximate reasoning, in: *Aproximate Reasoning in Expert System* (M. M Gupta et al., Eds.), North-Holland, Amsterdam, (1985) 105-112.

[18] Trillas, E., del Campo, C. and Cubillo, S., Nota sobre el concepto de Implicación borrosa, *Actas del Congreso Español sobre Tecnologías y Lógica Fuzzy*, Pamplona, 8-10-1998.

[19] Xu , J. and Giardina, C. R., *The algebraic structure of the generalisation of binary Morphology using Fuzzy set concept*. Inst. Graphic Commun Waltham, MA, USA, 1988, 292-297

ANNEXE A: Fuzzy.Implications

1	$L_m(a, b)=\max(1-a, \min(a, b))$	Zadeh
2	$L_a(a, b)=\min(1, 1-a+b)$	Lukasiewicz
3	$L_c(a, b)=\min(a, b)$	Mamdani
4	$Ls(a, b)= \begin{cases} 1 & \text{if } a \le b \\ 0 & \text{otherwise} \end{cases}$	Standard Strict
5	$L_g(a, b)= \begin{cases} 1 & \text{if } a \le b \\ b & \text{otherwise} \end{cases}$	Standard Star (Gödel)
6	$L_{sg}(a, b)=\min(L_s(a, b), L_g(1-a, 1-b))$	Standard Strict-Star
7	$L_{gs}(a, b)=\min(L_g(a, b), L_s(1-a, 1-b))$	Standard Star-Strict
8	$L_{gg}(a, b)=\min(L_g(a, b), L_g(1-a, 1-b))$	Standard Star -Star
9	$L_{ss}(a, b)=\min(L_s(a, b), L_s(1-a, 1-b))$	Standard Strict - Strict
10	$L_b(a, b)=\max(1-a, b)$	Kleene-Dienes
11	$L_\Delta(a,b) = \begin{cases} 1 & \text{if } a \le b \\ 0 & \text{otherwise} \end{cases}$	Gaines
12	$L_\sigma(a, b)= \begin{cases} \min\left(1, \frac{b}{a}, \frac{1-a}{1-b}\right) & \text{if } a > 0 \text{ and } b < 1 \\ 1 & \text{otherwise} \end{cases}$	Modified Gaines
13	$L_*(a, b) =1-a+ab$	Kleene-Dienes-Lukasiewicz
14	$L_\#(a, b) =\min(\max(1-a, b), \max(a, 1-a), \max(b, 1-b))$	Willmott
15	$L\ (a, b) = \begin{cases} 1 & \text{if } a < 1 \text{ or } b = 1 \\ 0 & \text{otherwise} \end{cases}$	Standard Sharp
16	$L_{1b}(a, b) = \begin{cases} 1 & \text{if } a \le b \\ \min(1-a, b) & \text{otherwise} \end{cases}$	Wul
17	$L_{1e}(a, b) = \begin{cases} 0 & \text{if } a < b \\ b & \text{otherwise} \end{cases}$	Wu2
18	$L_E(a, b) = b^a$	Yager

ANNEXE B

Let A be a gray scale image, Figure 1, then Table 1 and 2 display the effect of performing of fuzzy dilation and erosion, when we use three different fuzzy implications and three distinct structuring element 3 x 3 defined by B(i, j) = α, i, j∈{1,2,3} and α = 1, 0.7, 0.4 respectively.

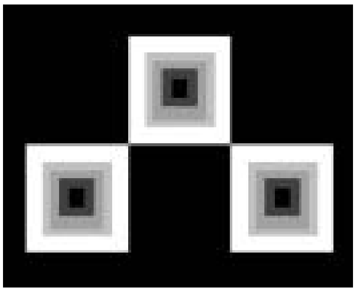


Figure 1: image A

Table 1. Erosions

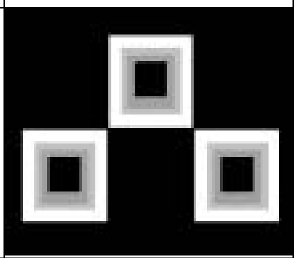
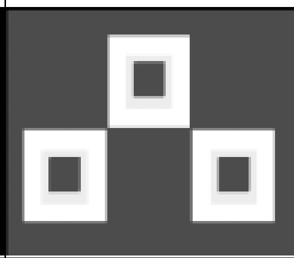
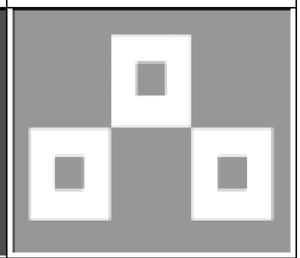
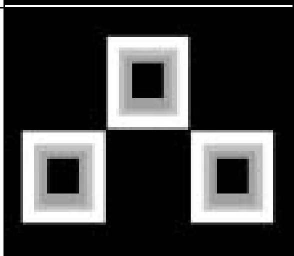
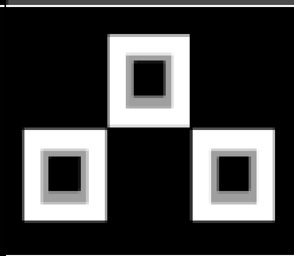
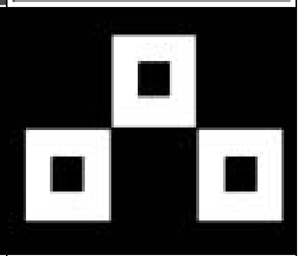
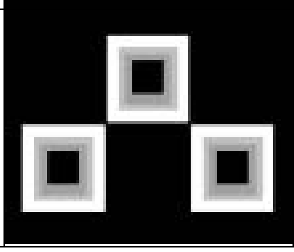
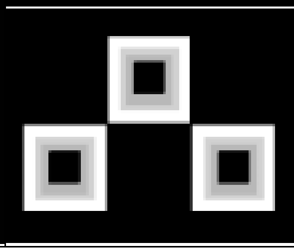
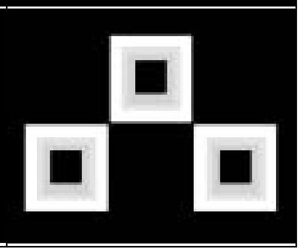
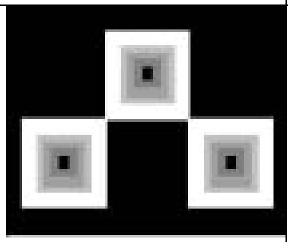
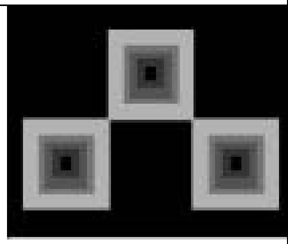
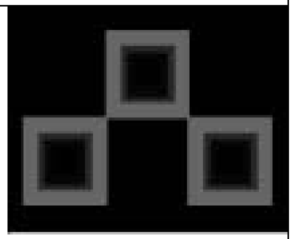
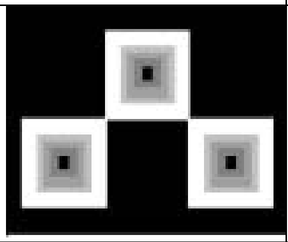
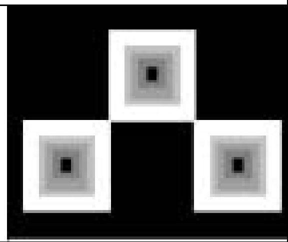
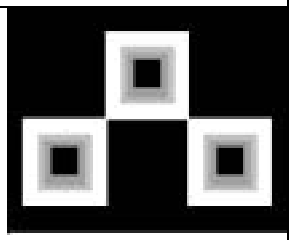
s. element / Implication	B(i,j)=1	B(i,j)=0,7	B(i,j)=0,4
Lukasiewicz I(a,b) = min(1,1-a+b)			
Gödel I(a,b)= $\begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$			
Yager b^a			

Table 2. Dilations

s. element / Implication	B(i,j)=1	B(i,j)=0.7	B(i,j)=0.4
Lukasiewicz I(a,b) = min(1,1-a+b)			
Gödel I(a,b)= $\begin{cases} 1 & \text{if } a \le b \\ b & \text{otherwise} \end{cases}$			
Yager b^a