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ON A PARTICULAR CLASS OF MEIJER'S G FUNCTIONS APPEARING IN FRACTIONAL CALCULUS

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Abstract: In this paper we investigate the Meijer G-function $G^{p,1}_{p+1,p+1}$ which, for certain parameter values, represents the Riemann-Liouville fractional integral of the Meijer-Nørlund function $G^{p,0}_{p,p}$. The properties of this function play an important role in extending the multiple Erdélyi-Kober fractional integral operator to arbitrary values of the parameters which is investigated in a separate work, in *Fract. Calc. Appl. Anal.*, Vol. **21**, No 5 (2018). Our results for $G^{p,1}_{p+1,p+1}$ include: a regularization formula for overlapping poles, a connection formula with the Meijer-Nørlund function, asymptotic formulas around the origin and unity, formulas for the moments, a hypergeometric transform and a sign stabilization theorem for growing parameters.

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1. Introduction

Throughout the paper we will use the standard notation ${}_pF_q$ for the generalized hypergeometric function (see [1, Section 2.1], [15, Section 5.1], [18, Sections 16.2-16.12] or [2, Chapter 12]) and $G^{m,n}_{p,q}$ for the Meijer's G-function (see [15, Section 5.2], [18, 16.17], [19, 8.2] or [2, Chapter 12]). The role that the Meijer G-function plays in integral representations of the generalized hypergeometric functions was probably first recognized by Kiryakova in [13, Chapter 4] and [14], through the use of successive fractional order integrations. In a series of papers [3, 4, 7, 9, 11] the first author jointly with Kalmykov, Prilepkina and Sitnik extended Kiryakova's results and applied them to discover numerous new facts about the generalized hypergeometric functions. This work was continued in our recent article [5], where the properties of the Meijer-Nørlund function $G^{p,0}_{p,p}$ were employed to investigate the connections of the generalized hypergeometric functions with topics like: inverse factorial series, radial positive definite functions, Luke's inequalities and zero-free regions.

This paper is a detailed investigation of the function $G^{p,1}_{p+1,p+1}$ which emerges as the kernel of the recently developed regularization of the multiple Erdélyi-Kober fractional integral operator presented in our concurrent work [6]. Under certain restrictions on the parameters, the function $G^{p,1}_{p+1,p+1}$ equals the Riemann-Liouville fractional integral of the Meijer-Nørlund function $G^{p,0}_{p,p}$. However, its properties are less studied than those of the Meijer-Nørlund function $G^{p,0}_{p,p}$. We further remark that this particular Meijer's G-function with $p = 2$ appears in the kernel of the composition of two Hankel transforms with different indices, as it can be seen by carefully looking at the corresponding particular case of [20, Theorem 4.7].

Our results can be summarized as follows: we derive an identity relating $G^{p,1}_{p+1,p+1}$ with $G^{p+1,0}_{p+1,p+1}$, a regularization formula for $G^{p,1}_{p+1,p+1}$ when the poles of the integrand of different types superimpose, an expression for the moments of the function $G^{p,1}_{p+1,p+1}$ and a formula for its hypergeometric transform which incorporates generalized Stieltjes, Laplace and Hankel transforms. Furthermore, we prove a proposition on sign stabilization for $G^{p,1}_{p+1,p+1}$ when all but one of the parameters grow infinitely.

2. Main results

Let us fix some notation and terminology first. The standard symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ will be used to denote the natural, integer, real and complex numbers, respectively; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In what follows we will use the shorthand notation for products and sums:

$$\Gamma(\mathbf{a}) := \Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_p), \quad (\mathbf{a})_n := (a_1)_n(a_2)_n\cdots(a_p)_n,$$
$$\mathbf{a}+\mu := (a_1+\mu, a_2+\mu, \ldots, a_p+\mu);$$

inequalities like $\Re(\mathbf{a}) > 0$ and properties like $-\mathbf{a} \notin \mathbb{N}_0$ will be understood element-wise (i.e. $-\mathbf{a} \notin \mathbb{N}_0$ means that no element of $\mathbf{a}$ is non-positive integer). The main character of this paper is the function

$$\widehat{G}_n(t) := G^{p,1}_{p+1,p+1}\left(t \,\middle|\, \begin{matrix} n, \mathbf{b}+n-1 \\ \mathbf{a}+n-1, 0 \end{matrix}\right), \tag{1a}$$

or, equivalently,

$$\widehat{G}_n(t) := \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\Gamma(\mathbf{a}+n-1+s)\Gamma(1-n-s)}{\Gamma(\mathbf{b}+n-1+s)\Gamma(1-s)} t^{-s} ds, \tag{1b}$$

where $\mathbf{a} := (a_1, \ldots, a_p)$, $\mathbf{b} := (b_1, \ldots, b_p)$ are (generally complex) parameter vectors. For $|t| < 1$ the contour $\mathcal{L}$ is a left loop that separates the poles of the integrand of the form $a_{jl} = 1 - a_j - n - l$, $l \in \mathbb{N}_0$, leaving them on the left, from the poles of the form $1 - n + k$, $k \in \mathbb{N}_0$, leaving them on the right. By definition, the two types of poles must not superimpose, which translates into the condition $-a_j \notin \mathbb{N}_0$, $j = 1, \ldots, p$. If they do, the definition can still be repaired by the regularization given in Proposition 2 below. Further details regarding the choice of the contour and the convergence of the integral can be found, for instance, in [12, Section 1.1] or in [5, Section 2]. Definition (1) certainly works for any complex n, but in this work we confine ourselves to $n \in \mathbb{N}_0$. Note, that due to the shifting property (see [19, 8.2.2.15] or [18, 16.19.2])

$$t^{\alpha} G^{m,n}_{p,q}\left(t \,\middle|\, \begin{matrix} \mathbf{b} \\ \mathbf{a} \end{matrix}\right) = G^{m,n}_{p,q}\left(t \,\middle|\, \begin{matrix} \mathbf{b}+\alpha \\ \mathbf{a}+\alpha \end{matrix}\right),$$

any function $G^{p,1}_{p+1,p+1}(t)$ can be written as (1) times some power of t if n is allowed to be complex. The restriction $n \in \mathbb{N}_0$ means that the top left parameter must be greater than the bottom right parameter by a nonnegative integer. We also recall the next definition from [5, (34)]:

$$\widetilde{G}_n(t) := G^{p+1,0}_{p+1,p+1}\left(t \,\middle|\, \begin{matrix} \mathbf{b}-1+n, n \\ \mathbf{a}-1+n, 0 \end{matrix}\right). \tag{2}$$

It is straightforward to see that $\widehat{G}_0(t) = \widetilde{G}_0(t) = G_0(t)$, where

$$G_0(t) := G_{p,p}^{p,0}\left(t \,\middle|\, \begin{matrix}\mathbf{b}-1\\ \mathbf{a}-1\end{matrix}\right).$$

Define

$$a := \min(\Re a_1, \Re a_2, \ldots, \Re a_p) \quad \text{and} \quad \psi := \sum\nolimits_{k=1}^{p}(b_k - a_k). \tag{3}$$

If $\Re(\mathbf{a}) > 0$, the function $\widehat{G}_n(t)$ can be computed as the $n-$th primitive of $G_0(t)$ that satisfies $\widehat{G}_n^{(k)}(0) = 0$ for $k = 1, 2, \ldots, n$ (see details in Proposition 5 below):

$$\widehat{G}_n(t) = \frac{1}{(n-1)!}\int_0^t G_0(x)(t-x)^{n-1}dx. \tag{4}$$

As mentioned above, the function $\widehat{G}_n(t)$ is not defined if any component of $\mathbf{a}$ is a non positive integer, since the basic separation condition for the contour $\mathcal{L}$ in (1) is violated. However, as we will see, the function $\widehat{G}_n(t)/\Gamma(\mathbf{a})$ is entire in $\mathbf{a}$. If $\widehat{G}_n(t)/\Gamma(\mathbf{a})$ is viewed as a function of one parameter (say a_1) and all elements of $\mathbf{a}_{[1]} = (a_2, \ldots, a_p)$ are different modulo integers, then this claim follows from the representation [18, (16.17.2)]

$$\frac{\widehat{G}_n(t)}{\Gamma(\mathbf{a})} = \sum_{k=1}^{p}\frac{\Gamma(\mathbf{a}_{[k]} - a_k)t^{a_k+n-1}}{\Gamma(\mathbf{b}-a_k)\Gamma(a_k+n)\Gamma(\mathbf{a}_{[k]})} \times {}_{p+1}F_p\left(\begin{matrix}a_k, 1+a_k-\mathbf{b}\\ a_k+n, 1+a_k-\mathbf{a}_{[k]}\end{matrix} \,\middle|\, t\right), \tag{5}$$

where here and in the sequel $\mathbf{a}_{[k]} := (a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_p)$. However, in case of multiple poles (i.e. when some of the differences $a_i - a_k \in \mathbb{Z}$) the situation becomes more delicate. In order to treat the general case we will need the following statement which shows that $\widetilde{G}_n(x)$ defined in (2) and $(-1)^n\widehat{G}_n(x)$ differ by a polynomial.

Proposition 1. *The following identity holds true*

$$\widetilde{G}_n(x) - (-1)^n\widehat{G}_n(x) = \frac{(-x)^{n-1}\Gamma(\mathbf{a})}{(n-1)!\,\Gamma(\mathbf{b})}\,{}_{p+1}F_p\left(\begin{matrix}1-n, \mathbf{a}\\ \mathbf{b}\end{matrix} \,\middle|\, \frac{1}{x}\right), \tag{6}$$

where $\widetilde{G}_n$ and $\widehat{G}_n$ are defined in (2) and (1), respectively.

Proof. Assuming that $\Re(\mathbf{a})$ and $\Re(\psi)$ are positive and substituting the definitions of $\widetilde{G}_n$ and $\widehat{G}_n$ into the left hand side of (6), in view of representation [5, (41)], we get

$$
\begin{aligned}
&G^{p+1,0}_{p+1,p+1}\left(x\left|\begin{matrix}n,\mathbf{b}+n-1\\ \mathbf{a}+n-1,0\end{matrix}\right.\right)-(-1)^nG^{p,1}_{p+1,p+1}\left(x\left|\begin{matrix}n,\mathbf{b}+n-1\\ \mathbf{a}+n-1,0\end{matrix}\right.\right)\\
&\qquad=\frac{1}{(n-1)!}\left\{\int_0^x(t-x)^{n-1}G^{p,0}_{p,p}\left(t\left|\begin{matrix}\mathbf{b}-1\\ \mathbf{a}-1\end{matrix}\right.\right)dt\right.\\
&\qquad\qquad\left.+\int_x^1(t-x)^{n-1}G^{p,0}_{p,p}\left(t\left|\begin{matrix}\mathbf{b}-1\\ \mathbf{a}-1\end{matrix}\right.\right)dt\right\}\\
&=\frac{1}{(n-1)!}\left\{\sum_{j=0}^{n-1}\binom{n-1}{j}(-x)^{n-1-j}\int_0^1t^jG^{p,0}_{p,p}\left(t\left|\begin{matrix}\mathbf{b}-1\\ \mathbf{a}-1\end{matrix}\right.\right)dt\right\}\\
&\qquad=\frac{\Gamma(\mathbf{a})(-x)^{n-1}}{\Gamma(\mathbf{b})(n-1)!}\sum_{j=0}^{n-1}\frac{(-n+1)_j(\mathbf{a})_j}{(\mathbf{b})_jj!}x^{-j}\\
&\qquad\qquad=\frac{(-x)^{n-1}\Gamma(\mathbf{a})}{(n-1)!\Gamma(\mathbf{b})}{}_{p+1}F_p\left(\left.\begin{matrix}-n+1,\mathbf{a}\\ \mathbf{b}\end{matrix}\right|1/x\right),
\end{aligned}
$$

where the Mellin transform of Meijer's G-function [5, (16)] has been used in the pre-ultimate equality. The positivity restrictions $\Re(\mathbf{a}),\Re(\psi)>0$ can now be removed by analytic continuation. □

The above proposition leads immediately to the next statement.

Proposition 2. *The function $\widehat{G}_n(t)/\Gamma(\mathbf{a})$ is entire in each component of $\mathbf{a}$ (all apparent singularities are removable). If $a_i=-m_i$, $m_i\in\mathbb{N}_0$, for $i=1,\dots,r$, $r\le p$, then*

$$
\frac{\widehat{G}_n(x)}{\Gamma(\mathbf{a})}=\frac{x^{n-1}}{(n-1)!\Gamma(\mathbf{b})}{}_{p+1}F_p\left(\left.\begin{matrix}1-n,-\mathbf{m},\mathbf{a}'\\ \mathbf{b}\end{matrix}\right|1/x\right), \tag{7}
$$

where $\mathbf{m}=(m_1,\dots,m_r)$ and $\mathbf{a}=(-\mathbf{m},\mathbf{a}')$.

Proof. The only potential singularities of $\mathbf{a}\to\widehat{G}_n(t)/\Gamma(\mathbf{a})$ are those points, where some components of $\mathbf{a}$ are non-positive integers, since these points violate the separation condition necessary for existence of the contour defining $\widehat{G}_n(t)$. Suppose $\mathbf{a}=(-\mathbf{m},\mathbf{a}')$, where $\mathbf{m}=(m_1,\dots,m_r)$ are nonnegative integers and no component of $\mathbf{a}'$ is equal to a non-positive integer. Using this notation we need to calculate

$$\lim_{\boldsymbol{\epsilon}\to 0}\frac{1}{\Gamma(-\mathbf{m}-\boldsymbol{\epsilon})\Gamma(\mathbf{a}')}G^{p,1}_{p+1,p+1}\left(t\,\middle|\,\begin{matrix}n,\mathbf{b}+n-1\\ -\mathbf{m}-\boldsymbol{\epsilon}+n-1,\mathbf{a}'+n-1,0\end{matrix}\right),$$

where $\boldsymbol{\epsilon} := (\epsilon_1,\ldots,\epsilon_r)$. Dividing (6) by $\Gamma(\mathbf{a})$ and taking the limit we get (7) which shows that all singularities are indeed removable. □

Before we turn to the next proposition, we need to recall some properties of the Meijer-Nørlund function $G^{p,0}_{p,p}$ elaborated in [5, Section 2] and [10]. First, we will need Nørlund's expansion

$$G^{p,0}_{p,p}\left(z\,\middle|\,\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right)=\frac{z^{a_k}(1-z)^{\psi-1}}{\Gamma(\psi)}\sum_{j=0}^{\infty}\frac{g_j(\mathbf{a}_{[k]};\mathbf{b})}{(\psi)_j}(1-z)^j,\quad k=1,2,\ldots,p, \tag{8}$$

which holds in the disk $|1-z|<1$ for all $-\psi=-\sum_{i=1}^{p}(b_i-a_i)\notin\mathbb{N}_0$ and each $k=1,2,\ldots,p$. The coefficients $g_n(\mathbf{a}_{[k]};\mathbf{b})$ are given by [17, (1.28), (2.7), (2.11)]:

$$g_j(\mathbf{a}_{[p]};\mathbf{b})=\sum_{0\le j_1\le j_2\le\cdots\le j_{p-2}\le j}\ \prod_{m=1}^{p-1}\frac{(\psi_m+j_{m-1})_{j_m-j_{m-1}}}{(j_m-j_{m-1})!}(b_{m+1}-a_m)_{j_m-j_{m-1}}, \tag{9}$$

where $\psi_m:=\sum_{i=1}^{m}(b_i-a_i)$, $j_0=0$, $j_{p-1}=j$. The coefficient $g_j(\mathbf{a}_{[k]};\mathbf{b})$ is obtained from $g_j(\mathbf{a}_{[p]};\mathbf{b})$ by exchanging a_p and a_k. These coefficients satisfy two different recurrence relations (in p and j) also discovered by Nørlund. Details can be found in [10, section 2.2]. Taking the limit $\psi\to -l$, $l\in\mathbb{N}_0$ in (8) we obtain

$$G^{p,0}_{p,p}\left(z\,\middle|\,\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right)=z^{a_k}\sum_{j=0}^{\infty}\frac{g_{j+l+1}(\mathbf{a}_{[k]},\mathbf{b})}{j!}(1-z)^j,\quad k=1,2,\ldots,p, \tag{10}$$

where $\psi=-l$, $l\in\mathbb{N}_0$ (see [17, formula (1.34)]). Hence, $G^{p,0}_{p,p}$ is analytic in the neighborhood of $z=1$ for non-positive integer values of ψ. The Mellin transform of $G^{p,0}_{p,p}$ exists if either $\Re(\psi)>0$ or $\psi=-m\in\mathbb{N}_0$. In the former case,

$$\int_0^{\infty}x^{s-1}G^{p,0}_{p,p}\left(x\,\middle|\,\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right)dx=\int_0^{1}x^{s-1}G^{p,0}_{p,p}\left(x\,\middle|\,\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right)dx=\frac{\Gamma(\mathbf{a}+s)}{\Gamma(\mathbf{b}+s)} \tag{11}$$

is valid in the intersection of the half-planes $\Re(s+a_i)>0$, $i=1,\ldots,p$. If $\psi=-m\in\mathbb{N}_0$, then

$$\int_0^\infty x^{s-1}G^{p,0}_{p,p}\left(x\,\middle|\,\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right)dx=\int_0^1 x^{s-1}G^{p,0}_{p,p}\left(x\,\middle|\,\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right)dx=\frac{\Gamma(\mathbf{a}+s)}{\Gamma(\mathbf{b}+s)}-q(s) \qquad (12)$$

in the same half-plane, where $q(s)$ is a polynomial of degree m given by

$$q(s)=\sum_{j=0}^{m}g_{m-j}(\mathbf{a}_{[k]};\mathbf{b})(s+a_k-j)_j,\quad k=1,2,\ldots,p. \qquad (13)$$

The coefficients $g_i(\mathbf{a}_{[k]};\mathbf{b})$ depend on k. The resulting polynomial $q(s)$, however, is the same for each k. Given a nonnegative integer k suppose that $\Re(\psi)>-k$ and $\Re(\mathbf{a})>0$. Then we have

$$\int_0^1 G^{p,0}_{p,p}\left(x\,\middle|\,\begin{matrix}\mathbf{b}-1\\ \mathbf{a}-1\end{matrix}\right)(1-x)^k dx=\frac{\Gamma(\mathbf{a})}{\Gamma(\mathbf{b})}{}_{p+1}F_p\left(\begin{matrix}-k,\mathbf{a}\\ \mathbf{b}\end{matrix}\,\middle|\,1\right). \qquad (14)$$

The asymptotic properties of $\widehat{G}_n$ are summarized in the next two propositions.

Proposition 3. *Suppose that $n\in\mathbb{N}$, and $\mathbf{a}$ and $\mathbf{b}$ are arbitrary complex vectors. If $\Re(\psi)+n-1>0$ or if $\mathbf{a}$ contains non-positive integers, then*

$$\frac{\widehat{G}_n(x)}{\Gamma(\mathbf{a})}=\frac{{}_{p+1}F_p(-n+1,\mathbf{a};\mathbf{b};1)}{(n-1)!\Gamma(\mathbf{b})}+o(1)\quad\text{as}\quad x\to1. \qquad (15)$$

If $\psi=-m$ with integer $m\geq n-1$ and $-\mathbf{a}\notin\mathbb{N}_0$, then

$$\frac{\widehat{G}_n(x)}{\Gamma(\mathbf{a})}=\frac{{}_{p+1}F_p(-n+1,\mathbf{a};\mathbf{b};1)}{\Gamma(\mathbf{b})(n-1)!}-\frac{1}{\Gamma(\mathbf{a})}\sum_{j=0}^{n-1}\frac{(-1)^j q(j)}{(n-1-j)!j!}+o(1)\quad\text{as}\quad x\to1, \qquad (16)$$

where $q(\cdot)$ is given in (13). *If $\Re(\psi)+n-1<0$, $-\psi\notin\mathbb{N}_0$, and $-\mathbf{a}\notin\mathbb{N}_0$, then*

$$\frac{\widehat{G}_n(x)}{\Gamma(\mathbf{a})}=\frac{(-1)^n(1-x)^{\psi+n-1}}{\Gamma(\mathbf{a})\Gamma(\psi+n)}(1+o(1))\quad\text{as}\quad x\to1. \qquad (17)$$

Proof. Indeed, according to (8)

$$\widetilde{G}_n(x) = \frac{x^{\widetilde{a}_k}(1-x)^{\psi+n-1}}{\Gamma(\psi+n)} \sum_{j=0}^{\infty} \frac{g_j(\widetilde{\mathbf{a}}_{[k]};\widetilde{\mathbf{b}})}{(\psi+n)_j}(1-x)^j, \quad k=1,2,\ldots,p+1,$$

where $\widetilde{\mathbf{a}} := (\mathbf{a}+n-1,0)$ and $\widetilde{\mathbf{b}} := (\mathbf{b}+n-1,n)$. The asymptotic relation (15) for $\Re(\psi)+n-1>0$, as well as formula (17), follow by substituting the above formula into (6) and letting $x \to 1$. In deducing (17) we also used that $g_0(\widetilde{\mathbf{a}}_{[k]};\widetilde{\mathbf{b}}) = 1$. If $\mathbf{a}$ contains non-positive integers, then (15) follows directly from (7). Finally, assume that $\psi = -m$ with integer $m \geq n-1$ and $\mathbf{a}$ does not contain non-positive integers. Then by (4) and (12),

$$\begin{aligned}
\widehat{G}_n(1) = {} & \frac{\int_0^1 G_0(x)(1-x)^{n-1}dx}{(n-1)!} \\
& = \frac{1}{(n-1)!}\int_0^1 G_0(x)dx \sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j x^j \\
& = \frac{1}{(n-1)!}\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j \int_0^1 x^j G_0(x)dx \\
& = \frac{1}{(n-1)!}\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j \left[\frac{\Gamma(\mathbf{a}+j)}{\Gamma(\mathbf{b}+j)} - q(j)\right] \\
& = \frac{\Gamma(\mathbf{a})_{p+1}F_p(-n+1,\mathbf{a};\mathbf{b};1)}{\Gamma(\mathbf{b})(n-1)!} - \frac{1}{(n-1)!}\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}q(j),
\end{aligned}$$

which is a rewriting of (16). □

Note that the poles of the numerator of the integrand

$$t^{-s}\frac{\Gamma(\mathbf{a}+n-1+s)\Gamma(1-n-s)}{\Gamma(\mathbf{b}+n-1+s)\Gamma(1-s)}$$

in the definition of $G^{p,1}_{p+1,p+1}(t)$ may cancel out with the poles of the denominator. Suppose that $b_k = a_i + l$ for some $k = 1,\ldots,p$ and $l \in \mathbb{Z}$. If $l \leq 0$, then all the poles of the function $\Gamma(a_i+n-1+s)$ cancel out with the poles of $\Gamma(b_k+n-1+s)$. We will call the index i and the corresponding component a_i normal if at least one pole of $\Gamma(a_i+n-1+s)$ does not cancel (if such pole is single then it is necessarily the rightmost one). We will say that $\mathbf{a}$ is normal if all its components are normal. In general situation we can "normalize" $\mathbf{a}$ by deleting the exceptional (= not normal) components.

Proposition 4. *Suppose that* $\mathbf{a} \in \mathbb{C}^{p'}$ *is normal or normalized and* $-\mathbf{a} \notin \mathbb{N}_0$. *Set*

$$a' := \min(\Re(a_1), \Re(a_2), \ldots, \Re(a_{p'})), \quad \mathcal{A} = \{a_i : \Re(a_i) = a'\}. \tag{18}$$

($\mathcal{A}$ is generally a multiset, i.e. it may contain repeated elements.) Write $r \in \mathbb{N}$ *for the maximal multiplicity among the elements of* $\mathcal{A}$ *and* $\widehat{a}_1, \ldots, \widehat{a}_l$ *for the distinct elements of* $\mathcal{A}$ *each having multiplicity* r. *Then*

$$\frac{\widehat{G}_n(x)}{\Gamma(\mathbf{a})} = \sum_{k=1}^{l} \alpha_k x^{\widehat{a}_k+n-1} [\log(1/x)]^{r-1} [1 + o(1)] \quad \text{as } x \to 0, \tag{19}$$

where

$$\alpha_k := \frac{\prod_{a_i \neq \widehat{a}_k} \Gamma(a_i - \widehat{a}_k)}{(r-1)!\Gamma(\mathbf{a})(\widehat{a}_k)_n \prod_{i=1}^{p} \Gamma(b_i - \widehat{a}_k)}. \tag{20}$$

If the normalized vector $\mathbf{a}$ *does contain non-positive integers, i.e.* $\mathbf{a} = (-\mathbf{m}, \tilde{\mathbf{a}})$ *with* $\mathbf{m} \in \mathbb{N}_0^j$ *and* $\tilde{\mathbf{a}} \in \mathbb{C}^{p'-j}$, $-\tilde{\mathbf{a}} \notin \mathbb{N}_0$, *then*

$$\frac{\widehat{G}_n(x)}{\Gamma(\mathbf{a})} = x^{n-1-m} \left[\frac{(-1)^m (n-m)_m (\mathbf{a})_m}{\Gamma(\mathbf{b}+m)(n-1)!m!} + \mathcal{O}(x) \right] \quad \text{as } x \to 0, \tag{21}$$

where $m = \min(m_1, \ldots, m_j, n-1)$.

Proof. The asymptotic approximation as $x \to 0$ for the general Fox's H-function, of which Meijer's G-function is a particular case, is given in [12, Theorem 1.5]. However, the computation of the constant in [12, formula (1.4.6)] seems to contain an error, so we redo this computation here. The result in [12, Theorem 1.5] also excludes the case when $\mathbf{a}$ contains non-positive integers. If $\mathbf{a}$ does non contain non-positive integer components, we see from [12, Theorem 1.5] that the asymptotics of $\widehat{G}_n(x)$ as $x \to 0$ is governed by the rightmost poles of the integrand having maximal multiplicity r, i.e by the numbers $\widehat{a}_1, \ldots, \widehat{a}_l$. Let us consider the contribution of the residue at the pole at $s = 1 - n - \widehat{a}_1$ assuming that $\widehat{a}_1$ is not a non-positive integer. From definition (1) we have:

$$\operatorname*{res}_{s=1-n-\widehat{a}_1} \frac{\Gamma(1-n-s)\prod_{i=1}^{p}\Gamma(a_i+n+s-1)}{x^s\Gamma(1-s)\prod_{i=1}^{p}\Gamma(b_i+n+s-1)} = \frac{1}{(r-1)!} \times$$
$$\lim_{s\to 1-n-\widehat{a}_1} \frac{d^{r-1}}{ds^{r-1}} \left\{ \frac{[(s+\widehat{a}_1+n-1)\Gamma(\widehat{a}_1+s+n-1)]^r}{\prod_{i=1}^{p}\Gamma(b_i+n+s-1)\Gamma(1-s)x^s} \right.$$

$$\Gamma(1-n-s)\prod_{a_i\neq\widehat{a}_1}\Gamma(a_i+n+s-1)\Big\}.$$

Using the straightforward relations

$$(s+\widehat{a}_1+n-1)\Gamma(\widehat{a}_1+s+n-1)=\Gamma(\widehat{a}_1+s+n),$$

$$\frac{\Gamma(1-n-s)}{\Gamma(1-s)}=\frac{(-1)^n}{(s)_n},\ \frac{\partial^{r-1}}{\partial s^{r-1}}x^{-s}=x^{-s}(-\log x)^{r-1}$$

and the fact that the above $(r-1)$-th derivative in the definition of the residue has the form

$$\{f(s)x^{-s}\}^{(r-1)}=\{f(s)\}(x^{-s})^{(r-1)}+\mathcal{O}\left((x^{-s})^{(r-2)}\right)\qquad\text{as } x\to 0,$$

we find that

$$\widehat{G}_n(t)=\frac{\prod_{a_i\neq\widehat{a}_1}\Gamma(a_i-\widehat{a}_1)(-1)^n x^{\widehat{a}_1+n-1}[\log(1/x)]^{r-1}}{(r-1)!\prod_{i=1}^{p}\Gamma(b_i-\widehat{a}_1)(1-n-\widehat{a}_1)_n}$$
$$+\mathcal{O}(x^{\widehat{a}_1+n-1}\log^{r-2}(x))+\text{contributions from other poles.}$$

Finally, applying $(1-n-\widehat{a}_1)_n=(-1)^n(\widehat{a}_1)_n$, we obtain

$$\widehat{G}_n(t)=\tilde{\alpha}_1 x^{\widehat{a}_1+n-1}[\log(1/x)]^{r-1}$$
$$+\mathcal{O}(x^{\widehat{a}_1+n-1}\log^{r-2}(x))+\text{residues at other poles}$$

with

$$\tilde{\alpha}_1:=\frac{\prod_{a_i\neq\widehat{a}_1}\Gamma(a_i-\widehat{a}_1)}{(r-1)!(\widehat{a}_1)_n\prod_{i=1}^{p}\Gamma(b_i-\widehat{a}_1)}.$$

Adding up similar contributions from the poles at $s=1-n-\widehat{a}_i$, $i=1,\dots,l$, and dividing throughout by $\Gamma(\mathbf{a})$, we arrive at formula (19) for $-\widehat{a}_j\notin\mathbb{N}_0$ for $j=1,\dots,l$. If the pole at $s=1-n-\widehat{a}_1$ is simple, then $r=1$ and the above calculation simplifies, namely $\mathcal{O}(x^{\widehat{a}_1+n-1}\log^{r-2}(x))$ disappears and the principal contribution from other poles will have asymptotic order $\mathcal{O}(x^{\tilde{a}_2+n-1})$, where $\tilde{a}_2$ is the element with the second smallest real part. This confirms that (19) remains valid in this case.

If $\mathbf{a}=(-\mathbf{m},\tilde{\mathbf{a}})$ with $\mathbf{m}=(m_1,\dots,m_j)\in\mathbb{N}_0^j$, then the result (21) follows immediately from (7) in view of the identity $(1-n)_m=(-1)^m(n-m)_m$. □

Remark. In what follows, the case of real parameters will play a special role. In this case we necessarily have $l=1$ and $\widehat{a}_1=\min_i(a_i)$. If $\mathbf{a}$ does not contain non-positive integers, then the sign of $\widehat{G}_n(x)/\Gamma(\mathbf{a})$ in the neighborhood of $x=0$ is determined by the sign of the real nonzero constant α_1 from (20). In this case, define $\eta\in\{0,1\}$ implicitly by

$$(-1)^{\eta}=\operatorname{sgn}(\alpha_1)=\operatorname{sgn}\left[\frac{1}{\Gamma(\mathbf{a})(\widehat{a}_1)_n\prod_{i=1}^{p}\Gamma(b_i-\widehat{a}_1)}\right], \tag{22a}$$

where $b_i-\widehat{a}_1$ can not take non-positive integer values as we assume $\mathbf{a}$ to be normalized as explained before Proposition 4. If $\mathbf{a}$ contains non-positive integers, then the sign of $\widehat{G}_n(x)/\Gamma(\mathbf{a})$ in the neighborhood of $x=0$ is determined by the sign of the constant in (21), and we define $\eta\in\{0,1\}$ by

$$(-1)^{\eta}=\operatorname{sgn}\left[\frac{(-1)^m(\mathbf{a})_m}{\Gamma(\mathbf{b}+m)}\right]. \tag{22b}$$

Note that formulas (22) imply that $(-1)^{\eta}\widehat{G}_n(x)/\Gamma(\mathbf{a})$ is positive in the neighborhood of $x=0$ for all real vectors $\mathbf{a}$ and $\mathbf{b}$ and the number η is independent of n for $n>-\widehat{a}_1$.

Next, we show that $\widehat{G}_n(x)$ and $\widehat{G}_m(t)$ are related by the Riemann-Liouville fractional integral.

Proposition 5. *Suppose $n>m\geq 0$ are integers and $\mathbf{a}$, $\mathbf{b}$ are arbitrary complex vectors satisfying $m+\Re(\mathbf{a})>0$. Then*

$$\frac{\widehat{G}_n(x)}{\Gamma(\mathbf{a})}=\frac{1}{(n-m-1)!}\int_0^x\frac{\widehat{G}_m(t)}{\Gamma(\mathbf{a})}(x-t)^{n-m-1}dt. \tag{23}$$

Proof. If $\mathbf{a}$ does not contain non-positive integer components, then the claim follows from a particular case of [19, 2.24.2.2]. If $\mathbf{a}$ contains such components, formula (23) can either be justified by analytic continuation in $\mathbf{a}$ in view of Proposition 2, or confirmed directly by termwise integration of (7). The convergence of the integral under the specified conditions follows from Proposition 4 (for $m>0$) and [5, Property 5] (for $m=0$). $\square$

Proposition 5 implies, in particular, that $\widehat{G}'_{m+1}(x)=\widehat{G}_m(x)$ and $\widehat{G}_m(0)=0$ for $m+\Re(\mathbf{a})>0$.

In the following proposition we compute the moments of $\widehat{G}_n(t)$. For brevity we omit the unit argument in the notation of generalized hypergeometric function, i.e. ${}_pF_q(\mathbf{a};\mathbf{b}):={}_pF_q(\mathbf{a};\mathbf{b}|1)$.

Proposition 6. *Suppose* $\mathbb{N}_0 \ni n > -\min(\Re(\psi), \Re(\mathbf{a}))$. *Then the following formulas hold*:

$$m_k := \int_0^1 \widehat{G}_n(t) t^k dt = \frac{\Gamma(\mathbf{a})}{\Gamma(\mathbf{b})(n+k)(n-1)!} {}_{p+2}F_{p+1}\left(\begin{matrix} -n-k, -n+1, \mathbf{a} \\ -n-k+1, \mathbf{b} \end{matrix}\right) + \frac{(-1)^n \Gamma(\mathbf{a}+n+k) k!}{\Gamma(\mathbf{b}+n+k)(n+k)!}, \quad (24)$$

$$\hat{m}_k := \int_0^1 \widehat{G}_n(t)(1-t)^k dt = \frac{\Gamma(\mathbf{a}) k!}{\Gamma(\mathbf{b})(n+k)!} {}_{p+1}F_p\left(\begin{matrix} -n-k, \mathbf{a} \\ \mathbf{b} \end{matrix}\right) \quad (25)$$

for $k \in \mathbb{N}_0$ *and* $1/(-1)! = 0$. *Moreover, define* $\Delta m_k := m_{k+1} - m_k$, $\Delta^r m_k := \Delta(\Delta^{r-1} m_k)$. *Then, for* $k, r \in \mathbb{N}_0$,

$$\Delta^r m_k = \Delta^k \hat{m}_r = \int_0^1 \widehat{G}_n(t)(1-t)^r t^k dt = \frac{\Gamma(\mathbf{a})}{\Gamma(\mathbf{b})} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j (r+j)!}{(n+r+j)!} {}_{p+1}F_p\left(\begin{matrix} -n-r-j, \mathbf{a} \\ \mathbf{b} \end{matrix}\right). \quad (26)$$

Remark. In view of Proposition 2, formulas (24)-(26) remain valid after division by $\Gamma(\mathbf{a})$ for $\mathbf{a}$ containing non-positive integers.

Proof. Assume for a moment that $\Re(\mathbf{a}) > 0$ and $\Re(\psi) > 0$. Then we have

$$\int_0^1 \widehat{G}_n(t) t^k dt = \frac{1}{(n-1)!} \int_0^1 t^k dt \int_0^t G^{p,0}_{p,p}\left(x \,\middle|\, \begin{matrix} \mathbf{b}-1 \\ \mathbf{a}-1 \end{matrix}\right) (t-x)^{n-1} dx = \frac{1}{(n-1)!} \int_0^1 G^{p,0}_{p,p}\left(x \,\middle|\, \begin{matrix} \mathbf{b}-1 \\ \mathbf{a}-1 \end{matrix}\right) dx \int_x^1 (t-x)^{n-1} t^k dt.$$

The inner integral is computed by an application of the binomial expansion to $(t-x)^{n-1}$:

$$\int_x^1 (t-x)^{n-1} t^k dt = \frac{1}{n+k} {}_2F_1\left(\begin{matrix} -n-k, -n+1 \\ -n-k+1 \end{matrix} \,\middle|\, x\right) + \frac{(-1)^n x^{n+k}}{n\binom{n+k}{k}}.$$

Substituting this expression into the formula above, integrating termwise and applying (11) we arrive at (24). To verify (25) write

$$\hat{m}_k = \frac{1}{(n-1)!} \int_0^1 (1-t)^k dt \int_0^t G^{p,0}_{p,p}\left(x \,\middle|\, \begin{matrix} \mathbf{b}-1 \\ \mathbf{a}-1 \end{matrix}\right) (t-x)^{n-1} dx$$

and repeat the above steps with (14) in place of (11) to obtain (25). Finally, to get (26) calculate

$$\int_0^1 (1-t)^r \widehat{G}_n(t) t^k dt$$
$$= \frac{1}{(n-1)!} \int_0^1 G^{p,0}_{p,p}\left(x \,\middle|\, \begin{matrix} \mathbf{b}-1 \\ \mathbf{a}-1 \end{matrix}\right) dx \int_x^1 (t-x)^{n-1} t^k (1-t)^r dt.$$

Now, using

$$\int_x^1 (t-x)^{n-1}(1-t)^l dt = \frac{(1-x)^{n+l} l!(n-1)!}{(n+l)!},$$

we have

$$\begin{aligned}\int_x^1 (t-x)^{n-1} t^k (1-t)^r dt &= \int_x^1 (t-x)^{n-1}(1-(1-t))^k (1-t)^r dt \\ &= \sum_{j=0}^k (-1)^j \binom{k}{j} \int_x^1 (t-x)^{n-1}(1-t)^{r+j} dt \\ &= \sum_{j=0}^k (-1)^j \binom{k}{j} \frac{(1-x)^{n+r+j}(r+j)!(n-1)!}{(n+r+j)!}.\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta^r m_k = \Delta^k \hat{m}_r &= \int_0^1 (1-t)^r \widehat{G}_n(t) t^k dt \\ &= \frac{\Gamma(\mathbf{a})}{\Gamma(\mathbf{b})} \sum_{j=0}^k (-1)^j \binom{k}{j} \frac{(r+j)!}{(n+r+j)!} {}_{p+1}F_p\left(\begin{matrix} -n-r-j, \mathbf{a} \\ \mathbf{b} \end{matrix} \,\middle|\, 1\right).\end{aligned}$$

The restrictions $\Re(\mathbf{a}) > 0$, $\Re(\psi) > 0$ can now be removed by analytic continuation. $\square$

Corollary 1. *For any natural n and nonnegative integer k the following identity holds*:

$$\sum_{j=0}^k \binom{k}{j} \frac{(-1)^j j!}{(n+j)!} {}_{p+1}F_p\left(\begin{matrix} -n-j, \mathbf{a} \\ \mathbf{b} \end{matrix} \,\middle|\, 1\right)$$
$$= \frac{1}{(n+k)\Gamma(n)} {}_{p+2}F_{p+1}\left(\begin{matrix} -n-k, -n+1, \mathbf{a} \\ -n-k+1, \mathbf{b} \end{matrix} \,\middle|\, 1\right) + \frac{(-1)^n (\mathbf{a})_{n+k} k!}{(\mathbf{b})_{n+k}(n+k)!}.$$

Proof. The result follows by comparing (24) with the $r = 0$ case of (26). Note that the claimed identity is not contained in [19] but can be deduced from [19, 15.3.2.12] by the appropriate limit transition. □

The formulas for the moments derived in Proposition 6 play the key role in the following computation of the hypergeometric transform of $\widehat{G}_n(t)$.

Proposition 7. *Suppose that $u \le s+1$ are nonnegative integers, $\mathbf{c} \in \mathbb{C}^u$, $\mathbf{d} \in \mathbb{C}^s$, $\Re(\psi) > -n$ and $\Re(\mathbf{a}) > -n$, where ψ is defined in* (3)*. Then*

$$\frac{1}{\Gamma(\mathbf{a})}\int_0^1 {}_uF_s\left(\begin{matrix}\mathbf{c}\\ \mathbf{d}\end{matrix}\middle| -zt\right)\widehat{G}_n(t)dt$$
$$= \frac{1}{\Gamma(\mathbf{b})}\sum_{j=0}^{\infty}\frac{z^j(\mathbf{c})_j}{(n+j)!(\mathbf{d})_j}{}_uF_s\left(\begin{matrix}\mathbf{c}+j\\ \mathbf{d}+j\end{matrix}\middle| -z\right){}_{p+1}F_p\left(\begin{matrix}-n-j,\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| 1\right), \quad (27)$$

where the series on the right converges for all $z \in \mathbb{C}$ if $u \le s$ and in the half-plane $\Re(z) > -1/2$ if $u = s+1$. Therefore, for $u = s+1$ the integral on the left hand side is an explicit representation of the analytic continuation to the cut plane $\mathbb{C}\backslash(-\infty,-1]$ of the function defined by the right hand side.

Proof. Assume first that $\Re(\mathbf{a}) > 0$. Then using (14) and the $r = 0$ case of (26) we obtain by termwise integration and interchange of the order of summations:

$$\frac{1}{\Gamma(\mathbf{a})}\int_0^1 {}_uF_s\left(\begin{matrix}\mathbf{c}\\ \mathbf{d}\end{matrix}\middle| -zt\right)\widehat{G}_n(t)dt$$
$$= \frac{1}{\Gamma(\mathbf{b})}\sum_{k=0}^{\infty}\frac{(\mathbf{c})_k(-z)^k}{(\mathbf{d})_k k!}\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^j j!}{(n+j)!}{}_{p+1}F_p\left(\begin{matrix}-n-j,\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| 1\right)$$
$$= \frac{1}{\Gamma(\mathbf{b})}\sum_{j=0}^{\infty}\frac{(-1)^j j!}{(n+j)!}{}_{p+1}F_p\left(\begin{matrix}-n-j,\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| 1\right)\sum_{k=j}^{\infty}\binom{k}{j}\frac{(\mathbf{c})_k(-z)^k}{(\mathbf{d})_k k!}$$
$$= \frac{1}{\Gamma(\mathbf{b})}\sum_{j=0}^{\infty}\frac{z^j(\mathbf{c})_j}{(n+j)!(\mathbf{d})_j}{}_{p+1}F_p\left(\begin{matrix}-n-j,\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| 1\right)\sum_{l=0}^{\infty}\frac{(\mathbf{c}+j)_l(-z)^l}{(\mathbf{d}+j)_l l!}.$$

The claims regarding the convergence domains are justified by the following formulas due to Krottnerus. If $u = s+1$, then by [15, 7.3(3)] for $z \in \mathbb{C}\backslash(-\infty,-1]$

$${}_{s+1}F_s\left(\begin{matrix}\mathbf{c}+j\\ \mathbf{d}+j\end{matrix}\middle| -z\right) = (1+z)^{\nu-j}\left(1+\mathcal{O}(j^{-1})\right) \text{ as } j \to \infty$$

with $\nu := \sum_{i=1}^{s} d_i - \sum_{i=1}^{s+1} c_i$. The convergence is uniform in z on every compact subset of $\mathbb{C}\backslash(-\infty,-1]$. If $u=s$, then by [15, 7.3(4)] for all $z \in \mathbb{C}$,

$$ {}_sF_s\left(\begin{matrix}\mathbf{c}+j\\ \mathbf{d}+j\end{matrix}\middle| -z\right) = e^{-z}\left(1+\mathcal{O}(j^{-1})\right) \text{ as } j\to\infty. $$

Finally, if $u<s$ by [15, 7.3(5)] for all $z\in\mathbb{C}$,

$$ {}_uF_s\left(\begin{matrix}\mathbf{c}+j\\ \mathbf{d}+j\end{matrix}\middle| -z\right) = 1+\mathcal{O}(j^{u-s}) \text{ as } j\to\infty. $$

In the last two formulas the convergence is uniform in z on every compact subset of $\mathbb{C}$. It remains to note that ${}_{p+1}F_p(-n-j,\mathbf{a};\mathbf{b}|1)$ cannot grow faster than polynomially by (25) and $|z/(z+1)|<1$ is equivalent to $\Re(z)>-1/2$. As both sides of (27) are analytic in each a_i in the domain $\Re(a_i)>-n$, the formula holds in the region stated in the conclusions of the proposition in view of Proposition 2. □

Particular cases of Proposition 7 lead to the formulas for the generalized Stieltjes, Laplace and (slightly modified) Hankel transform of $\widehat{G}_n(t)$ that we summarize in the next corollary.

Corollary 2. *Suppose $\Re(\psi)>-n$ and $\Re(\mathbf{a})>-n$. Then for any $\sigma\in\mathbb{C}$,*

$$ \frac{1}{\Gamma(\mathbf{a})}\int_0^1 \frac{\widehat{G}_n(t)}{(1+zt)^{\sigma}}dt = \frac{1}{\Gamma(\mathbf{b})(1+z)^{\sigma}}\sum_{j=0}^{\infty}\frac{(\sigma)_j z^j}{(n+j)!(1+z)^j}\,{}_{p+1}F_p\left(\begin{matrix}\mathbf{a},-n-j\\ \mathbf{b}\end{matrix}\middle| 1\right), \quad (28) $$

where the series on the right hand side converges in the half-plane $\Re(z)>-1/2$. Further,

$$ \frac{1}{\Gamma(\mathbf{a})}\int_0^1 \widehat{G}_n(t)e^{-zt}dt = \frac{e^{-z}}{\Gamma(\mathbf{b})}\sum_{j=0}^{\infty}\frac{z^j}{(n+j)!}\,{}_{p+1}F_p\left(\begin{matrix}\mathbf{a},-n-j\\ \mathbf{b}\end{matrix}\middle| 1\right) \quad (29) $$

and

$$ \frac{1}{\Gamma(\mathbf{a})}\int_0^1 \widehat{G}_n(t)\,{}_0F_1(-;\nu;-zt)dt = \frac{1}{\Gamma(\mathbf{b})}\sum_{j=0}^{\infty}\frac{z^j\,{}_0F_1(-;\nu+j;-z)}{(n+j)!(\nu)_j}\,{}_{p+1}F_p\left(\begin{matrix}-n-j,\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| 1\right) \quad (30) $$

for all $z\in\mathbb{C}$ and $\nu\in\mathbb{C}$.

Further, Proposition 7 leads to an expansion discovered by Meijer [16, (113)].

Corollary 3. *The following summation formula holds:*

$$ {}_{u+p}F_{s+p}\left(\begin{matrix}\mathbf{a},\mathbf{c}\\ \mathbf{b},\mathbf{d}\end{matrix}\middle| -z\right)=\sum_{j=0}^{\infty}\frac{(\mathbf{c})_j z^j}{(\mathbf{d})_j j!}{}_uF_s\left(\begin{matrix}\mathbf{c}+j\\ \mathbf{d}+j\end{matrix}\middle| -z\right){}_{p+1}F_p\left(\begin{matrix}-j,\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| 1\right), \tag{31}$$

where the series converges for $z\in\mathbb{C}$ if $u\le s$ and for $\Re(z)>-1/2$ if $u=s+1$.

Proof. Take $n=0$ in (27) and apply [4, Theorem 1]. $\square$

If $u=s+1=1$ in (31) we get Nørlund's formula [17, (1.21)] (see also [19, formula 6.8.1.3]), which generalizes Pfaff's transformation [1, (2.2.6)] for ${}_2F_1$. Taking $u=s=0$ we obtain the summation formula [19, formula 6.8.1.2] generalizing Kummer's transformation [1, (4.1.11)] for ${}_1F_1$. Finally, if $s=u+1=1$ we get the known summation formula [19, formula 6.8.3.4].

The next elementary lemma on sign stabilization of the Riemann-Liouville fractional integral may be of independent interest.

Lemma 1. *Suppose $f:(0,1]\to\mathbb{R}$ is continuous and integrable (possibly in improper sense). If $f>0$ in some neighborhood of zero, then there exists $\alpha>0$ such that the Riemann-Liouville fractional integral*

$$[I_+^{\alpha}f](x)=\frac{1}{\Gamma(\alpha)}\int_0^x f(t)(x-t)^{\alpha-1}dt$$

is positive for all $x\in(0,1]$.

Proof. It follows from the conditions of the lemma that there exist $0<t_0<t_1\le 1$ such that $f(t)\ge\delta$ on $[t_0,t_1]$ for some $\delta>0$. Then if $x\le t_1$ the claim is obviously true for all $\alpha\ge 1$. Assume that $x>t_1$ and estimate

$$\begin{aligned}\int_0^x f(t)(x-t)^{\alpha-1}dt&=x^{\alpha}\int_0^1 f(xu)(1-u)^{\alpha-1}du\\ &>x^{\alpha}\int_{t_0/x}^{t_1/x} f(xu)(1-u)^{\alpha-1}du+x^{\alpha}\int_{t_1/x}^{1} f(xu)(1-u)^{\alpha-1}du\\ &\ge x^{\alpha}\delta\int_{t_0/x}^{t_1/x}(1-u)^{\alpha-1}du+x^{\alpha}f(t_{\alpha})\int_{t_1/x}^{1}(1-u)^{\alpha-1}du\end{aligned}$$

$$=\frac{\delta x^{\alpha}}{\alpha}\left((1-t_0/x)^{\alpha}-(1-t_1/x)^{\alpha}\right)+\frac{x^{\alpha}f(t_{\alpha})}{\alpha}(1-t_1/x)^{\alpha}$$
$$=\frac{x^{\alpha}(1-t_0/x)^{\alpha}}{\alpha}\left\{\delta+(f(t_{\alpha})-\delta)\left(\frac{x-t_1}{x-t_0}\right)^{\alpha}\right\},$$

where we applied the mean value theorem to the second integral on the second line, so that $t_{\alpha}\in[t_1,x]$. The second term inside the braces clearly tends to zero as $\alpha\to\infty$ and positivity follows. □

Remark. This lemma admits an obvious generalization as follows. Since $I_+^{\alpha}f=I_+^{\alpha_2}I_+^{\alpha_1}f$ for $\alpha=\alpha_1+\alpha_2$ it is sufficient to assume that the conditions of the lemma hold for $I_+^{\alpha_1}f$ for some $\alpha_1\geq 0$.

The above lemma leads to the following statement regarding the sign stabilization of $\widehat{G}_n/\Gamma(\mathbf{a})$ as n grows to infinity.

Proposition 8. *For arbitrary real vectors* $\mathbf{a}$, $\mathbf{b}$ *there exists* $N\in\mathbb{N}_0$ *such that*

$$\frac{(-1)^{\eta}}{\Gamma(\mathbf{a})}G^{p,1}_{p+1,p+1}\left(t\,\middle|\,\begin{matrix}n,\mathbf{b}+n-1\\ \mathbf{a}+n-1,0\end{matrix}\right)>0$$

for all $n\geq N$, $t\in(0,1]$ *and* η *given in* (22).

Proof. By Proposition 5 for $n=k+m$ we have

$$\frac{1}{\Gamma(\mathbf{a})}G^{p,1}_{p+1,p+1}\left(t\,\middle|\,\begin{matrix}n,\mathbf{b}+n-1\\ \mathbf{a}+n-1,0\end{matrix}\right)$$
$$=I_+^{k}\left[\frac{1}{\Gamma(\mathbf{a})}G^{p,1}_{p+1,p+1}\left(\cdot\,\middle|\,\begin{matrix}m,\mathbf{b}+m-1\\ \mathbf{a}+m-1,0\end{matrix}\right)\right](t).$$

On the other hand, according to Proposition 4 there exists $m\in\mathbb{N}_0$ such that

$$\frac{(-1)^{\eta}}{\Gamma(\mathbf{a})}G^{p,1}_{p+1,p+1}\left(t\,\middle|\,\begin{matrix}m,\mathbf{b}+m-1\\ \mathbf{a}+m-1,0\end{matrix}\right)$$

satisfies the conditions of Lemma 1 and the claim follows. □

Recall that a sequence $\{f_k\}_{k\geq 0}$ is completely monotonic if $(-1)^m\Delta^m f_k\geq 0$ for all integer $m,k\geq 0$. By the celebrated result of Hausdorff, the necessary and sufficient conditions for a sequence to be completely monotone is that it is equal to the moment sequence of a nonnegative measure supported on $[0,1]$. In view of this fact we get the next corollary of Proposition 8.

Corollary 4. *For arbitrary real vectors* $\mathbf{a}$, $\mathbf{b}$ *there exists* $N \in \mathbb{N}_0$*, such that for all* $n \geq N$ *both sequences* $(-1)^{\eta} m_k/\Gamma(\mathbf{a})$ *and* $(-1)^{\eta} \hat{m}_k/\Gamma(\mathbf{a})$ *defined in* (24) *and* (25)*, respectively, are completely monotonic. Here* η *is given in* (22)*.*

It is then natural to formulate the following
Open problem. *How to find or estimate* N *in Proposition 8 and Corollary 4?*

3. Examples

We conclude this paper with some examples of explicit representations of the function $\widehat{G}_n(t)$ in terms of other special functions for small p.

Example 1. Take the simplest case $p = 1$. According to [19, 8.4.49.19]

$$G^{1,1}_{2,2}\left(t \,\middle|\, \begin{matrix} n, b+n-1 \\ a+n-1, 0 \end{matrix}\right) = \frac{t^{a+n-1}}{\Gamma(\psi)(a)_n} {}_2F_1\left(\begin{matrix} a, 1-\psi \\ a+n \end{matrix} \,\middle|\, t\right)$$

for $0 < t < 1$ and all values of a, where $\psi = b - a$. In particular, this function vanishes for $-\psi \in \mathbb{N}_0$. Substituting the above expression in (6) and using (32) for $G^{2,0}_{2,2}$ we arrive at the next identity:

$$\frac{x^{a+n-1}}{\Gamma(\psi)(a)_n} {}_2F_1\left(\begin{matrix} a, 1-\psi \\ a+n \end{matrix} \,\middle|\, x\right) = \frac{(-1)^n (1-x)^{\psi+n-1}}{\Gamma(\psi+n)} {}_2F_1\left(\begin{matrix} 1-a, \psi \\ \psi+n \end{matrix} \,\middle|\, 1-x\right) + \frac{x^{n-1}\Gamma(a)}{(n-1)!\Gamma(b)} {}_2F_1\left(\begin{matrix} a, 1-n \\ b \end{matrix} \,\middle|\, \frac{1}{x}\right).$$

Certainly, there are many other ways to prove this identity, but it does not seem to appear in the literature in this form.

Example 2. For $p = 2$ and $-\psi = a_1 + a_2 - b_1 - b_2 \notin \mathbb{N}_0$ we have by [19, 8.4.49.22]:

$$G^{2,0}_{2,2}\left(x \,\middle|\, \begin{matrix} b_1, b_2 \\ a_1, a_2 \end{matrix}\right) = \frac{x^{a_2}(1-x)_+^{\psi-1}}{\Gamma(\psi)} {}_2F_1\left(\begin{matrix} b_1 - a_1, b_2 - a_1 \\ \psi \end{matrix} \,\middle|\, 1-x\right), \tag{32}$$

where a_1 and a_2 may be interchanged on the right hand side. If $\psi = -m$, $m \in \mathbb{N}_0$, an easy calculation based on (32) leads to

$$G^{2,0}_{2,2}\left(x \,\middle|\, \begin{matrix} b_1, b_2 \\ a_1, a_2 \end{matrix}\right) = \frac{x^{a_2}(b_1-a_1)_{m+1}(b_2-a_1)_{m+1}}{(m+1)!}$$

$$\times {}_2F_1\left(\begin{matrix} b_1-a_1+m+1, b_2-a_1+m+1 \\ m+2 \end{matrix}\middle| 1-x\right)$$

for $t \in (0,1)$. Further, using a variation of Euler's integral representation

$$\int_0^t x^{a_2-1}(1-x)^{\psi-1+k}(t-x)^{n-1}dx \\ = \frac{\Gamma(n)\Gamma(a_2)}{\Gamma(a_2+n)} t^{a_2-1+n} {}_2F_1\left(\begin{matrix} a_2, 1-\psi-k \\ a_2+n \end{matrix}\middle| t\right), \quad (33)$$

we get for $-\psi \notin \mathbb{N}_0$ by termwise integration:

$$G^{2,1}_{3,3}\left(t \middle| \begin{matrix} n, \mathbf{b}+n-1 \\ \mathbf{a}+n-1, 0 \end{matrix}\right) = \frac{1}{(n-1)!}\int_0^t (t-x)^{n-1} G^{2,0}_{2,2}\left(x \middle| \begin{matrix} \mathbf{b}-1 \\ \mathbf{a}-1 \end{matrix}\right) dx \\ = \frac{t^{a_2-1+n}}{\Gamma(\psi)(a_2)_n} \sum_{k=0}^{\infty} \frac{(b_1-a_1)_k(b_2-a_1)_k}{(\psi)_k k!} {}_2F_1\left(\begin{matrix} a_2, 1-\psi-k \\ a_2+n \end{matrix}\middle| t\right), \quad (34a)$$

where we utilized (32) and (33). Invoking [15, 7.2(24)] it is easy to see that the above series converges in the unit disk if $\Re(a_1) < 1$. Exchanging the order of summations in the last series and applying the Euler transformation [1, (2.2.7)] we get an alternative expression:

$$G^{2,1}_{3,3}\left(t \middle| \begin{matrix} n, \mathbf{b}+n-1 \\ \mathbf{a}+n-1, 0 \end{matrix}\right) = \frac{t^{a_2-1+n}(1-t)^{\psi+n-1}}{\Gamma(\psi)(a_2)_n} \sum_{k=0}^{\infty} \frac{(a_2+\psi+n-1)_k}{(a_2+n)_k} \\ \times \frac{(n)_k t^k}{k!} {}_3F_2\left(\begin{matrix} b_1-a_1, b_2-a_1, a_2+\psi+n+k-1 \\ \psi, a_2+\psi+n-1 \end{matrix}\middle| 1-t\right) \quad (34b)$$

converging in the unit disk $|t| < 1$ for all values of parameters. The standard expression for $G^{2,1}_{3,3}$ is the following (see [19, 8.2.2.3] or [18, 16.17.2]):

$$G^{2,1}_{3,3}\left(t \middle| \begin{matrix} n, \mathbf{b}+n-1 \\ \mathbf{a}+n-1, 0 \end{matrix}\right) \\ = \frac{\Gamma(a_2-a_1)t^{a_1+n-1}}{(a_1)_n\Gamma(b_1-a_1)\Gamma(b_2-a_1)} {}_3F_2\left(\begin{matrix} a_1, a_1-b_1+1, a_1-b_2+1 \\ a_1-a_2+1, a_1+n \end{matrix}\middle| t\right) \\ + \frac{\Gamma(a_1-a_2)t^{a_2+n-1}}{(a_2)_n\Gamma(b_1-a_2)\Gamma(b_2-a_2)} {}_3F_2\left(\begin{matrix} a_2, a_2-b_1+1, a_2-b_2+1 \\ a_2-a_1+1, a_2+n \end{matrix}\middle| t\right). \quad (34c)$$

It is valid if $a_1 - a_2 \notin \mathbb{Z}$. On the other hand we can apply (6) to express $G^{2,1}_{3,3}$ in terms of $G^{3,0}_{3,3}$ and ${}_3F_2$. Further, applying (35) to express $G^{3,0}_{3,3}$ we get the identity:

$$G_{3,3}^{2,1}\left(t\,\middle|\,\begin{matrix}n,\mathbf{b}+n-1\\ \mathbf{a}+n-1,0\end{matrix}\right)=\frac{(-1)^n(1-t)^{\psi+n-1}}{\Gamma(\psi+n)}$$
$$\times\sum_{k=0}^{\infty}\frac{(\psi-b_1+1)_k(\psi-b_2+1)_k}{(\psi+n)_k k!}(1-t)^k$$
$$\times\,{}_3F_2\left(\begin{matrix}-k,1-a_1,1-a_2\\ \psi-b_1+1,\psi-b_2+1\end{matrix}\right)+\frac{t^{n-1}\Gamma(\mathbf{a})}{\Gamma(\mathbf{b})(n-1)!}{}_3F_2\left(\begin{matrix}1-n,\mathbf{a}\\ \mathbf{b}\end{matrix}\,\middle|\,\frac{1}{t}\right),\qquad(34\text{d})$$

where the argument 1 is omitted in ${}_3F_2$ for conciseness. Employing an alternative expression [19, (8.4.51.2)] for $G_{3,3}^{3,0}$ in terms of Appell's hypergeometric function F_3 of two variables [18, 16.13.3] we get

$$G_{3,3}^{2,1}\left(t\,\middle|\,\begin{matrix}n,\mathbf{b}+n-1\\ \mathbf{a}+n-1,0\end{matrix}\right)=\frac{t^{a_1+a_2-b_1+n-2}(1-t)_+^{\psi+n-1}}{(-1)^n\Gamma(\psi+n)}$$
$$\times F_3(b_1-a_2,n;b_1-a_1,b_2-1;\psi+n;1-1/t,1-t)$$
$$+\frac{t^{n-1}\Gamma(\mathbf{a})}{\Gamma(\mathbf{b})(n-1)!}{}_3F_2\left(\begin{matrix}1-n,\mathbf{a}\\ \mathbf{b}\end{matrix}\,\middle|\,\frac{1}{t}\right).\qquad(34\text{e})$$

Equating the right hand sides of the formulas (34a)-(34e) we arrive at transformation formulas for the sums of ${}_3F_2$s and reduction formulas for Appell's F_3 function. Some of these formulas might be new.

Finally, if $-\psi=m\in\mathbb{N}_0$, we have

$$G_{3,3}^{2,1}\left(t\,\middle|\,\begin{matrix}n,\mathbf{b}+n-1\\ \mathbf{a}+n-1,0\end{matrix}\right)=\frac{t^{a_2-1+n}(b_1-a_1)_{m+1}(b_2-a_1)_{m+1}}{(m+1)!(a_2)_n}$$
$$\times\sum_{k=0}^{\infty}\frac{(b_1-a_1+m+1)_k(b_2-a_1+m+1)_k}{(m+1)_k k!}{}_2F_1\left(\begin{matrix}a_2,-k\\ a_2+n\end{matrix}\,\middle|\,t\right).$$

Example 3. For $p=3$ the coefficients in (8) are given by

$$g_n(\mathbf{a}_{[3]};\mathbf{b})=\frac{(\psi-b_1+a_3)_n(\psi-b_2+a_3)_n}{n!}$$
$$\times\,{}_3F_2\left(\begin{matrix}-n,b_3-a_1,b_3-a_2\\ \psi-b_1+a_3,\psi-b_2+a_3\end{matrix}\,\middle|\,1\right).$$

See [17, formula 2.10] or [5, p. 48]. The coefficients $g_n(\mathbf{a}_{[1]};\mathbf{b})$ and $g_n(\mathbf{a}_{[2]};\mathbf{b})$ are obtained from the above by permutation of indices. Hence,

$$G^{3,0}_{3,3}\left(t\,\middle|\,\begin{matrix}\mathbf{b}-1\\ \mathbf{a}-1\end{matrix}\right)=\frac{t^{a_3-1}(1-t)_+^{\psi-1}}{\Gamma(\psi)}\sum_{n=0}^{\infty}\frac{(\psi-b_1+a_3)_n(\psi-b_2+a_3)_n}{(\psi)_n n!}$$
$$\times\,{}_3F_2\left(\begin{matrix}-n,b_3-a_1,b_3-a_2\\ \psi-b_1+a_3,\psi-b_2+a_3\end{matrix}\,\middle|\,1\right)(1-t)^n \quad (35)$$

for $-\psi=a_1+a_2+a_3-b_1-b_2-b_3\notin\mathbb{N}_0$ and

$$G^{3,0}_{3,3}\left(t\,\middle|\,\begin{matrix}\mathbf{b}-1\\ \mathbf{a}-1\end{matrix}\right)=\frac{(b_2+b_3-a_1-a_2)_{m+1}(b_1+b_3-a_1-a_2)_{m+1}}{t^{1-a_3}(m+1)!}$$
$$\times\sum_{n=0}^{\infty}\frac{(1-b_1+a_3)_n(1-b_2+a_3)_n}{(m+2)_n n!}$$
$$\times\,{}_3F_2\left(\begin{matrix}-n-m-1,b_3-a_1,b_3-a_2\\ -m-b_1+a_3,-m-b_2+a_3\end{matrix}\,\middle|\,1\right)(1-t)^n$$

for $-\psi=m\in\mathbb{N}_0$. Termwise integration and application of (33) lead to

$$G^{3,1}_{4,4}\left(t\,\middle|\,\begin{matrix}n,\mathbf{b}+n-1\\ \mathbf{a}+n-1,0\end{matrix}\right)=\frac{t^{a_3-1+n}}{(a_3)_n}\sum_{k=0}^{\infty}\frac{(\psi-b_1+a_3)_k(\psi-b_2+a_3)_k}{\Gamma(\psi+k)k!}$$
$$\times\,{}_3F_2\left(\begin{matrix}-k,b_3-a_1,b_3-a_2\\ \psi-b_1+a_3,\psi-b_2+a_3\end{matrix}\,\middle|\,1\right){}_2F_1\left(\begin{matrix}a_3,1-\psi-k\\ a_3+n\end{matrix}\,\middle|\,t\right),$$

for $-\psi\notin\mathbb{N}_0$ and

$$G^{3,1}_{4,4}\left(t\,\middle|\,\begin{matrix}n,\mathbf{b}+n-1\\ \mathbf{a}+n-1,0\end{matrix}\right)=\frac{(b_2+b_3-a_1-a_2)_{m+1}}{(a_3)_n}$$
$$\times\frac{(b_1+b_3-a_1-a_2)_{m+1}t^{a_3-1+n}}{(m+1)!}\sum_{k=0}^{\infty}\frac{(1-b_1+a_3)_k(1-b_2+a_3)_k}{(m+2)_k k!}$$
$${}_3F_2\left(\begin{matrix}-k-m-1,b_3-a_1,b_3-a_2\\ -m-b_1+a_3,-m-b_2+a_3\end{matrix}\,\middle|\,1\right){}_2F_1\left(\begin{matrix}a_3,-k\\ a_3+n\end{matrix}\,\middle|\,t\right),$$

for $-\psi=m\in\mathbb{N}_0$.

Here, we again can employ (6) to express $G^{3,1}_{4,4}$ in terms of $G^{4,0}_{4,4}$ and ${}_4F_3$. Next, we can write an explicit expansion for $G^{4,0}_{4,4}$ using (8) and [5, formula above (15)] for the coefficients $g_j(\mathbf{a}_{[k]};\mathbf{b})$. Comparing the resulting expression with the one above leads to further transformation formulas for double sums of hypergeometric functions. As these formulas are quite cumbersome we omit them here.

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