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On the nonlinear stability of the triangular points in the circular spatial restricted three-body problem

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Abstract—The well-known problem of the nonlinear stability of L_4 and L_5 in the circular spatial restricted three-body problem is revisited. Some new results under the light of the concept of Lie (formal) stability are presented. In particular, we provide stability and asymptotic estimates for three specific values of the mass ratio that remained uncovered. Moreover, in many cases we improve the estimates found in the literature.

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INTRODUCTION

The nonlinear stability analysis of the points L_4 and L_5 is completely solved in the planar case. The spatial case has also been widely studied, mostly first by the Russian school [19, 21, 22, 28] and more recently by the Italian current [3, 8, 14, 27]. There is a vast literature on the subject and we do not intend to give all the existing references, but only the most relevant from the point of view of our study. Despite the popularity of the topic, there are still some open questions and part of the existing results can be sharpened.

We will look at this system from the point of view of Lie stability, that is a kind of formal stability. It was Khazin [16] who introduced the concept of Lie stability, although he named it Birkhoff stability in the case of elliptic equilibria. In [11], dos Santos *et al.* started calling it Lie stability. We will use a powerful criterion given in [7] that enlarges previous criteria on Lie stability and allows us, not only to recover all the formally stable cases reported in previous references, but also to decide on the formal stability of the system for values of the mass parameter that remained pending up to now.

The bound estimates of the solutions over exponentially long times in the Lie stable cases are obtained through a theorem based in the determination of error bounds for adiabatic invariants in Hamiltonian systems [9]. The theorem appears in [7], see also [6]. In many cases this result allows us to get better estimates than the existing ones and even to achieve bounds for the three aforementioned values of the mass ratio. We will make it more precise in sections 4 and 5.

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The paper is structured as follows. In section 1 we establish the equations of motion of the system. Section 2 is devoted to recall the concepts and the central theorems to establish Lie stability and estimates on the solution. The main results of the manuscript are contained in sections 3 and 4, the first one including our analysis on the Lie stability of L_4 and L_5. Section 4 presents the asymptotic estimates on the solutions in the Lie stable cases. In section 5 we compare our results with the ones existing in the literature and outline our main contributions. In section 6 we establish the existence of invariant 3-tori encasing L_4 and L_5. At the end of the paper there is an appendix containing some useful tables.

The main contributions of our approach are two. First, Theorem 3, where we establish the Lie stability of L_4 and L_5 in terms of the mass parameter μ. Second, Theorem 4, together with Corollary 1, where the asymptotic bounds for the Lie stable cases are provided. Furthermore, the analysis performed in section 5 makes clear that our estimates enhance those obtained in [3] and other references excepting a few situations, namely: (a) when the frequency vector associated to the formal first integrals defined in (2.1) is not Diophantine and (b) when the parameter μ lies in the interval (μ_1, μ_2), with μ_i given in Notation 2 and Notation 3, and such that the corresponding frequency vector given in Definition 1 is a Pythagorean triple, see Definition 2. Details on the comparison appear in section 5. Finally, the cases not considered in [3], namely the values μ_3, $\mu_{(3,3,-2)}$ and $\mu_{(0,3,1)}$, have been proved to be Lie stable, see section 3. Their asymptotic estimates are given in Corollary 1. The case $\mu_{(1,3,0)}$, also not treated in [3], is unstable, as it is proved in section 3.

1. SETTING OF THE PROBLEM

We consider the motion in the three-dimensional space of an infinitesimal particle under the gravitational attraction of two bodies with masses m_1 and m_2 that describe circular orbits around their common centre of mass (see for example [22] or [29] for details). The Hamilton function associated to this system in rectangular coordinates (x, y, z, X, Y, Z) in a rotating reference frame is:

$$\mathcal{H} = \frac{1}{2}\left(X^2 + Y^2 + Z^2\right) - (xY - Xy) - \frac{\mu}{\sqrt{(x+\mu-1)^2 + y^2 + z^2}} - \frac{1-\mu}{\sqrt{(\mu+x)^2 + y^2 + z^2}}. \quad (1.1)$$

It represents an autonomous system with three degrees of freedom depending on the parameter μ, that stands for the quotient $m_2/(m_1 + m_2)$. Assuming $m_1 \geq m_2$, then $\mu \in (0, 1/2]$. The masses m_1 and m_2 are located at the points $(-\mu, 0, 0)$ and $(1-\mu, 0, 0)$, respectively, in the coordinate space. The Hamiltonian system has five equilibria, the Euler points L_1, L_2 and L_3, which are unstable for all μ, and the Lagrangian (also called triangular) points L_4 and L_5, whose stability depends on the parameter μ. The coordinates of the Lagrangian equilibrium points L_4 and L_5 in the six-dimensional phase space are $\left(1/2-\mu,\ \pm\sqrt{3}/2,\ 0,\ \mp\sqrt{3}/2,\ 1/2-\mu, 0\right)$, where the upper sign applies for L_4 and the lower sign does for L_5. The stability of both equilibria is the same, so from now on we only refer to the point L_4, although the same conclusions are valid for L_5.

We translate the equilibrium solution L_4 to the origin by means of the linear change of coordinates given by $x = x_1 + 1/2 - \mu$, $y = y_1 + \sqrt{3}/2$, $z = z_1$, $X = X_1 - \sqrt{3}/2$, $Y = Y_1 + 1/2 - \mu$, $Z = Z_1$. Then, Hamiltonian function (1.1) is expanded in Taylor series around $\mathbf{0}$, constant terms are dropped and we get a Hamiltonian of the form

$$H = H_2 + H_3 + \cdots + H_j + \cdots, \quad (1.2)$$

where

$$\begin{aligned}
H_2 &= \tfrac{1}{8}(x_1^2 - 5y_1^2 + 4z_1^2) + \tfrac{1}{2}(X_1^2 + Y_1^2 + Z_1^2) - (x_1Y_1 - X_1y_1) - \tfrac{3}{4}\sqrt{3}x_1y_1(1-2\mu),\\
H_3 &= \tfrac{1}{16}\left[3\sqrt{3}y_1(x_1^2 + y_1^2 - 4z_1^2) - x_1(7x_1^2 - 33y_1^2 + 12z_1^2)(1-2\mu)\right],\\
H_4 &= \tfrac{1}{128}(37x_1^4 - 3y_1^4 - 48z_1^4) + \tfrac{3}{64}(-41x_1^2y_1^2 + 4x_1^2z_1^2 + 44y_1^2z_1^2)\\
&\quad + \tfrac{5}{32}\sqrt{3}x_1y_1(5x_1^2 - 9y_1^2 + 12z_1^2)(1-2\mu).
\end{aligned}$$

The linearisation matrix associated to L_4 is

$$B = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{1}{4} & \frac{3}{4}\sqrt{3}(1-2\mu) & 0 & 0 & 1 & 0 \\ \frac{3}{4}\sqrt{3}(1-2\mu) & \frac{5}{4} & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of matrix B read $\pm\lambda_1, \pm\lambda_2, \pm\lambda_3$, and are pure imaginary whenever $0 < \mu < \mu_R = \frac{1}{2}(1 - \sqrt{69}/9)$ with $\lambda_1 = i\omega_1$, $\lambda_2 = i\omega_2$, $\lambda_3 = i\omega_3$ and

$$\omega_1 = \frac{\sqrt{1 + \sqrt{1 - 27\mu + 27\mu^2}}}{\sqrt{2}}, \ \omega_2 = \frac{\sqrt{1 - \sqrt{1 - 27\mu + 27\mu^2}}}{\sqrt{2}}, \ \omega_3 = 1. \tag{1.3}$$

The value μ_R is the so-called Gascheau's or Routh's critical value. Note that when $0 < \mu < \mu_R$ one has

$$0 < \omega_2 < \frac{\sqrt{2}}{2} < \omega_1 < 1 \quad \text{and} \quad \omega_1^2 + \omega_2^2 = 1. \tag{1.4}$$

It is also convenient to express μ in terms of ω_1, ω_2, leading to

$$\mu = \frac{1}{2}\left(1 - \frac{\sqrt{27 - 16\omega_1^2 + 16\omega_1^4}}{3\sqrt{3}}\right) = \frac{1}{2}\left(1 - \frac{\sqrt{27 - 16\omega_2^2 + 16\omega_2^4}}{3\sqrt{3}}\right). \tag{1.5}$$

Observe that, taking into account (1.3), $\omega_1 = \sqrt{2}/2$ corresponds to $\mu = \mu_R$ while $\omega_1 = 1$ is associated to $\mu = 0$. When $\mu > \mu_R$ the equilibrium L_4 is of focus-centre type, therefore unstable as it comes from a symplectic system and the eigenvalues are $\pm\lambda \pm i\nu$ with λ, ν positive numbers. When $\mu = \mu_R$ the linear system is not diagonalisable, thus the equilibrium points are not of elliptic nature. This case was proved to be Liapunov stable in the planar case in [17, 24] and formally stable in the spatial case, see [22]. In the context of the planar restricted problem with $\mu = \mu_R$ it is also worth mentioning papers [28], where a first attempt to study nonlinear stability in the sense of Liapunov is performed, and [2], where the author shows that, in a neighbourhood of μ_R, for most initial conditions, trajectories are conditionally periodic. An account of the stability achievements when $\mu \in (0, \mu_R]$, also for the planar case, appears in [23]. In our analysis we focus on the interval $(0, \mu_R)$ for μ, equivalently $(\sqrt{2}/2, 1)$ for ω_1 and $(0, \sqrt{2}/2)$ for ω_2.

There are several ways of performing the normal form transformation of Hamiltonian H in (1.2). The most standard one consists in introducing a real linear symplectic change of coordinates to put H_2 in linear normal form by using the eigenvalues and eigenvectors of matrix B. We call the transformed variables $\mathbf{x} = (q_1, q_2, q_3, p_1, p_2, p_3)$, where the q_i stand for coordinates and p_i do for their conjugate momenta. Then, a complex linear change is introduced to express H_2 in complex diagonal form. Next, the two changes are applied to the higher-order terms. Finally, a procedure based on Lie transformations is applied to normalise the terms from H_3 on so that the resulting Hamiltonian at each step commutes with H_2. This process is executed up to a finite order and in most of the cases in this problem order two is enough, which means, including the polynomials of degree four that define H_4.

In general we denote by $\mathcal{H}^p$ the normal form truncated at order $p-2$, in other words, at degree p in terms of rectangular coordinates, thus

$$\mathcal{H}^p = H_2 + \mathcal{H}_3 + \cdots + \mathcal{H}_p, \tag{1.6}$$

and the Poisson brackets $\{H_2, \mathcal{H}_k\} = 0$ for $k = 2, \ldots, p$. In the setting of elliptic equilibria the normal form is the so-called Birkhoff normal form, see for instance Theorem 5.5 in [1].

To achieve the normal form we need to calculate the associated generating function that is used to define the symplectic transformation. Since the expressions of the two terms of the generating transformation are lengthy, we do not include them here, but they are available upon request from the authors. The calculations have been performed in terms of the parameter ω_1 using symbolic arithmetic.

Now, we introduce the usual action-angle coordinates, say

$$(\mathbf{I}, \theta) = (I_1, I_2, I_3, \theta_1, \theta_2, \theta_3)$$

where $I_j = \frac{1}{2}(q_j^2 + p_j^2)$ are the actions conjugate to the angles $\theta_j = \tan^{-1}(p_j/q_j)$ with $j = 1, 2, 3$. Then H_2 is converted into

$$H_2 = \omega_1 I_1 - \omega_2 I_2 + \omega_3 I_3.$$

Notice that H_2 is indefinite when $\mu \in (0, \mu_R)$.

The normalised Hamiltonian in action-angle coordinates reads as

$$H = H_2 + \mathcal{H}_4 + \cdots, \tag{1.7}$$

where

$$\mathcal{H}_4 = c_{200} I_1^2 + c_{110} I_1 I_2 + c_{101} I_1 I_3 + c_{020} I_2^2 + c_{011} I_2 I_3 + c_{002} I_3^2 \tag{1.8}$$

and

$$\begin{aligned}
c_{200} &= \frac{\omega_2^2(124\omega_1^4 - 696\omega_1^2 + 81)}{144(1-2\omega_1^2)^2(1-5\omega_1^2)}, & c_{110} &= -\frac{\omega_1\omega_2(64\omega_1^2\omega_2^2 + 43)}{6(1-5\omega_1^2)(1-2\omega_1^2)(1-5\omega_2^2)(1-2\omega_2^2)},\\
c_{101} &= -\frac{8\omega_1\omega_2^2}{3(1-2\omega_1^2)(4-\omega_1^2)}, & c_{020} &= \frac{\omega_1^2(124\omega_1^4 + 448\omega_1^2 - 491)}{144(1-2\omega_1^2)^2(1-5\omega_2^2)},\\
c_{011} &= \frac{8\omega_2\omega_1^2}{3(1-2\omega_2^2)(4-\omega_2^2)}, & c_{002} &= -\frac{\omega_1^2\omega_2^2}{3(4-\omega_1^2)(4-\omega_2^2)}.
\end{aligned}$$

If we discard possible higher-order resonances, odd terms in the normal form Hamiltonian, say $\mathcal{H}_3$, $\mathcal{H}_5$, ..., are zero. Coefficients c_{200}, c_{110}, c_{020} were already calculated in [10] in the context of the planar case.

Taking into consideration (1.4), there are only two resonant cases where the above normal form does not apply. Specifically,

(i) $\omega_1 = \frac{2}{\sqrt{5}}$, $\omega_2 = \frac{1}{\sqrt{5}}$ thus, $\omega_1/\omega_2 = 2$ and

$$\mu = \mu_{(1,2,0)} = \frac{1}{2}\left(1 - \frac{\sqrt{1833}}{45}\right). \tag{1.9}$$

(ii) $\omega_1 = \frac{3}{\sqrt{10}}$, $\omega_2 = \frac{1}{\sqrt{10}}$ thus, $\omega_1/\omega_2 = 3$ and

$$\mu = \mu_{(1,3,0)} = \frac{1}{2}\left(1 - \frac{\sqrt{213}}{15}\right). \tag{1.10}$$

For these two sets of values some denominators of the generating function vanish and, as a consequence, the generic normal form Hamiltonian already calculated is not valid. Thus, in these cases we should compute specific normal forms that we shall show below. Notice that, apart from these two cases, $\mathcal{H}^4$ does not contain any resonant terms under conditions (1.4).

2. ON LIE STABILITY

In this section we review the main concepts and results related to Lie stability with the aim of applying them in section 3. We start by recalling the notion of resonance.

Definition 1. *The system related with Hamiltonian (1.7) presents a resonance relation if there exists an integer vector* $\mathbf{k} = (k_1, k_2, k_3) \neq \mathbf{0}$ *such that*

$$k_1\omega_1 - k_2\omega_2 + k_3\omega_3 = 0.$$

The 1*-norm of vector* $\mathbf{k}$*,* $|\mathbf{k}|_1 = |k_1| + |k_2| + |k_3|$*, is called the order of the resonance, while* $\mathbf{k}$ *is known as the resonance vector and* $\omega = (\omega_1, \omega_2, \omega_3)$ *stands for the frequency vector.*

For instance, resonance vectors corresponding to

$$(\omega_1, \omega_2, \omega_3) = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 1\right) \text{ and } \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}, 1\right),$$

which are associated to (1.9) and (1.10), are $(k_1, k_2, k_3) = (1, 2, 0)$ and $(1, 3, 0)$, respectively.

Notation 1. *The notation* $\mu_{(k_1,k_2,k_3)}$ *means taking the value of* $\mu \in (0, \mu_R)$ *associated to the resonance vector* $\mathbf{k} = (k_1, k_2, k_3)$*. In fact, along the paper we will use the same notation for the* μ_i *and* $\mu_{(k_1,k_2,k_3)}$ *as in reference [3].*

Definition 2. *Consider the frequency vector* $\omega = (m/n, \sqrt{n^2 - m^2}/n, 1)$*, with* $m, n \in \mathbb{Z}^+$*,* $0 < m < n$ *and* m/n *irreducible. Vector* $(m, \sqrt{n^2 - m^2}, n)$ *is a Pythagorean triple if* $n^2 - m^2$ *is a perfect square or, equivalently,* $\omega_2 \in \mathbb{Q}$*. In this case we say that vector* ω *is associated with a Pythagorean triple.*

We will notice that the previous concept is crucial to decide on Lie stability.

Definition 3. *We say that the origin of* $\mathbb{R}^6$ *in (1.2) is Lie stable if there exists* $m > 2$ *such that the truncated Hamiltonian system in Birkhoff normal form associated to* $\mathcal{H}^j$ *is stable in the sense of Liapunov for any (arbitrary)* $j \geq m$*.*

Definition 4. *We say that the origin of* $\mathbb{R}^6$ *in (1.2) is formally stable if there exists a real formal power series* $G(\mathbf{x})$*, possibly divergent, which is an integral of* H *in the formal sense, and is positive definite near* $\mathbf{x} = \mathbf{0}$*.*

Remark 1. The generalisation to n degrees of freedom of the definitions of Lie and formal stability is straightforward [11]. Lie stability is a type of formal stability, see for instance [12]. As the normal form transformation is carried out only to a finite order, checking Lie stability for the system in normal form is equivalent to checking Lie stability for any system previous to the normal form calculations.

Statement 1. *Suppose* $\{\mathbf{k}^1, \ldots, \mathbf{k}^s\}$ *is a basis of the* $\mathbb{Z}$*-module* M_ω *associated to the possible resonances of* H_2*, where* $0 \leq s \leq 2$*. The null space of* M_ω *is a vector subspace of* $\mathbb{R}^3$ *spanned by the vectors* $\{\mathbf{a}_1, \ldots, \mathbf{a}_d\}$ *with* $d = 3 - s$ *that satisfy* $\mathbf{a}_i \cdot \mathbf{k}^j = 0$*, see details in [11], [12]. By setting* $F_l = \mathbf{a}_l \cdot \mathbf{I}$ *with* $l = 1, \ldots, d$*, we get the independent (formal) first integrals of the normalised Hamiltonian (1.7).*

Definition 5. *We define the set*

$$S = \left\{\mathbf{I} \mid F_1(\mathbf{I}) = \ldots = F_d(\mathbf{I}) = 0\right\},$$

which was first introduced in [11].

The set S contains the essential vectors to evaluate the Hamiltonian and decide on the Lie stability of the system. We note that $0 \leq \dim S \leq s$.

Statement 2. *The quadratic part of H in terms of the formal first integrals F_k assumes the form*

$$H_2(\mathbf{I}) = \sum_{k=1}^{d} \sigma_k F_k(\mathbf{I}), \tag{2.1}$$

where σ_k are linear combinations of ω_j, see for instance [13]. The coefficients σ_k are rationally independent.

Definition 6. *It is said that vector $\sigma = (\sigma_1, \dots, \sigma_d)$ satisfies a Diophantine condition when there are fixed constants $c > 0$ and $\nu \geq d - 1$ such that*

$$\forall\, \mathbf{k} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}, \quad |\mathbf{k} \cdot \sigma| \geq c|\mathbf{k}|_1^{-\nu}. \tag{2.2}$$

The symbol $|\ |$ stands for the Euclidean norm.

At this point we recall two results that will be applied in the present paper in order to analyse the nonlinear stability of L_4 and give time estimates in the stable cases. We start with the theorem on Lie stability, as stated in [6, 7], that will be applied in section 3.

Assume that Hamiltonian (1.1) is expressed as H in (1.2) after sufficient manipulations and also that its normal form, up to an order p high enough, is given in (1.6). Then, the following result applies.

Theorem 1. *(A) Suppose there is an integer $j \geq 3$ with $\mathcal{H}^j(\mathbf{I}, \phi_1, \dots, \phi_s) \neq 0$ for all $\mathbf{I} \in S \setminus \{\mathbf{0}\}$, $\{\phi_1, \dots, \phi_s\} \in \mathbb{T}^s$, $\phi_i = \mathbf{k}_i \cdot \theta$ such that for all i with $3 \leq i < j$, $\mathcal{H}^i(\mathbf{I}, \phi_1, \dots, \phi_s)$ does not change sign for $\mathbf{I} \in S \setminus \{\mathbf{0}\}$, $\{\phi_1, \dots, \phi_s\} \in \mathbb{T}^s$, where $|\mathbf{I}|$ is small enough. Then, the origin of $\mathbb{R}^6$ is Lie stable for the Hamiltonian system (1.2).*

(B) Suppose there is an integer $i \geq 3$ such that $\mathcal{H}^i(\mathbf{I}, \phi_1, \dots, \phi_s)$ changes sign for some $\mathbf{I} \in S \setminus \{\mathbf{0}\}$, $\{\phi_1, \dots, \phi_s\} \in \mathbb{T}^s$, where $|\mathbf{I}|$ is small enough. Then, there is no index $j > i$ such that $\mathcal{H}^j(\mathbf{I}, \phi_1, \dots, \phi_s) \neq 0$ for $\mathbf{I} \in S \setminus \{\mathbf{0}\}$, $\{\phi_1, \dots, \phi_s\} \in \mathbb{T}^s$ with $|\mathbf{I}|$ sufficiently small.

According to the previous theorem, the Lie stability analysis consists in calculating the set S introduced above, which is a linear subspace of $\mathbb{R}^6$ that is contained into the orthogonal space related to the frequency vector ω. The Hamiltonian in normal form is computed only up to a suitable order, checking whether its truncation vanishes only at the origin of S. When it so happens, then Lie stability is obtained.

Remark 2. When $S = \{\mathbf{0}\}$ there is always Lie stability, see details in [12], [7].

For the Lie stable equilibria we will give an estimate of the solution's evolution according to the following theorem [6, 7]. Assume that $\mathbf{x}(t, \mathbf{x}_0)$ is a solution of the Hamiltonian system associated to H in (1.2) with initial condition $\mathbf{x}_0$. The following result holds.

Theorem 2. *If the real analytic Hamiltonian (1.2) has the origin of $\mathbb{R}^6$ as a formally stable equilibrium according to hypotheses (A) of Theorem 1, while the frequency vector σ satisfies the Diophantine condition (2.2), then there exist $C > 0$, $E > 0$, $a > 1$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, and for all $\mathbf{x}_0$ with $|\mathbf{x}_0| < \varepsilon$ we have*

$$|\mathbf{x}(t, \mathbf{x}_0)| < a\,\varepsilon^{2/j} \quad \text{for all } t \text{ with} \quad 0 \leq t \leq T = C \exp\left(\frac{E}{\varepsilon^{1/(\nu+1)}}\right).$$

The proofs of theorems 1 and 2 appear in [6, 7].

3. LIE STABILITY OF L_4

Let us start by analysing the cases where the normal form (1.8) does not apply. Recall from section 1 that (1.8) is not valid valid for $\omega_1 = 2/\sqrt{5}$, i.e. $\mu = \mu_{(1,2,0)}$, nor for $\omega_1 = 3/\sqrt{10}$, that is, $\mu = \mu_{(1,3,0)}$. A specific normal form is calculated for each of these values concluding instability in both cases, as it is also inferred from the planar problem [19, 20].

Case $\mu_{(1,2,0)}$: For $\omega_1 = 2/\sqrt{5}$ the normal form up to degree 3 is

$$\mathcal{H}^3(\mathbf{I},\phi_1) = \frac{1}{\sqrt{5}}(2I_1 - I_2) + I_3 + \frac{1}{5^{3/4}\cdot 3312\sqrt{109}}\sqrt{I_1}I_2\left(3401\sqrt{5}\cos\phi_1 - 3155\sqrt{611}\sin\phi_1\right),$$

where $\phi_1 = \theta_1 + 2\theta_2$. Notice that $S = \{(I_1, 2I_1, 0) \mid I_1 \geq 0\}$. Then, we take $\mathbf{I}$ in S and get

$$\mathcal{H}^3(\mathbf{I},\phi_1) = \frac{1}{5^{3/4}\cdot 1656\sqrt{109}} I_1^{3/2}\left(3401\sqrt{5}\cos\phi_1 - 3155\sqrt{611}\sin\phi_1\right).$$

As the coefficient of $I_1^{3/2}$ has a simple zero then, by Theorem 3.1 in [11], the equilibrium is unstable.

Case $\mu_{(1,3,0)}$: For $\omega_1 = 3/\sqrt{10}$ the normal form up to terms of degree four is

$$\begin{aligned}\mathcal{H}^4(\mathbf{I},\phi_1) \; = \; & \tfrac{1}{\sqrt{10}}(3I_1 - I_2) + I_3 + \tfrac{309}{2240}I_1^2 + \tfrac{79}{320}I_2^2 - \tfrac{1}{403}I_3^2 - \tfrac{1219}{560}I_1I_2 + \tfrac{\sqrt{10}}{31}I_1I_3 + \tfrac{\sqrt{10}}{13}I_2I_3 \\ & - \tfrac{1}{32560\sqrt{28083}}\sqrt{I_1}I_2^{3/2}\left(24146471\cos\phi_1 + 143827\sqrt{710}\sin\phi_1\right),\end{aligned}$$

with $\phi_1 = \theta_1 + 3\theta_2$. On this occasion $S = \{(I_1, 3I_1, 0) \mid I_1 \geq 0\}$. Evaluating $\mathcal{H}^4$ in S we get

$$\mathcal{H}^4(\mathbf{I},\phi_1) = \tfrac{-3}{4267118240}I_1^2\left(5932056339 + 338050594\sqrt{9361}\cos\phi_1 + 2013578\sqrt{6646310}\sin\phi_1\right).$$

Then, the coefficient of I_1^2 in $\mathcal{H}^4(\mathbf{I},\phi_1)$ has a simple zero and then, by Theorem 3.1 in [11], we achieve instability.

Now we study the rest of the cases starting with the determination of the set S. This passes through the construction of the formal integrals F_i associated to H_2. We can have one, two or three linearly independent integrals. The basic relations to be taken into account are written as

$$\omega_1 I_1 - \omega_2 I_2 + I_3 = 0, \qquad k_1\omega_1 - k_2\omega_2 + k_3 = 0,$$

with $k_1, k_2, k_3 \in \mathbb{Z}$ and $I_1, I_2, I_3 \geq 0$. The following situations are in order:

(a_1) If $\omega_1, \omega_2 \in \mathbb{Q}$, then the frequency vector is associated to a Pythagorean triple. We get $F_1 = \omega_1 I_1 - \omega_2 I_2 + I_3$, $d = 1$, $s = 2$ and

$$S = \left\{\left(I_1, \frac{1}{\omega_2}(\omega_1 I_1 + I_3), I_3\right) \,\middle|\, I_1, I_3 \geq 0\right\}.$$

Considering $\mathbf{I} \in S \setminus \{\mathbf{0}\}$, taking into account (1.4) to express ω_2 as a function of ω_1, and replacing everything in (1.8) we arrive at

$$\mathcal{H}^4(\mathbf{I}) = \beta_1 I_1^2 + \beta_2 I_1 I_3 + \beta_3 I_3^2, \tag{3.1}$$

with

$$\begin{aligned}\beta_1 &= \frac{644\omega_1^8 - 1288\omega_1^6 + 1185\omega_1^4 - 541\omega_1^2 + 36}{16(1-\omega_1^2)(1-2\omega_1^2)^2(1-5\omega_1^2)(4-5\omega_1^2)},\\ \beta_2 &= \frac{\omega_1\left(18580\omega_1^{12} - 67928\omega_1^{10} + 70827\omega_1^8 + 30890\omega_1^6 - 62113\omega_1^4 + 22128\omega_1^2 - 8496\right)}{72(1-\omega_1^2)\left(1-2\omega_1^2\right)^2\left(1-5\omega_1^2\right)\left(3+\omega_1^2\right)\left(4-\omega_1^2\right)\left(4-5\omega_1^2\right)},\\ \beta_3 &= \frac{\omega_1^2\left(960\omega_1^{10} - 7364\omega_1^8 + 29940\omega_1^6 - 48219\omega_1^4 + 24155\omega_1^2 - 444\right)}{144(1-\omega_1^2)\left(1-2\omega_1^2\right)^2(3+\omega_1^2)(4-\omega_1^2)(4-5\omega_1^2)}.\end{aligned} \tag{3.2}$$

We have to determine when there is no sign-change in $\mathcal{H}^4(\mathbf{I})$, keeping in mind that I_1, I_3 should be non-negative. First, notice that there is only one solution of $\beta_1 = 0$ in the interval $(1/\sqrt{2}, 1)$.

Notation 2. *We denote*

$$\omega_1^* = \frac{1}{2}\sqrt{2 + \sqrt{\frac{2}{161}(-219 + \sqrt{199945})}} \approx 0.959622914235418$$

the only solution of $\beta_1 = 0$ *in the interval* $(1/\sqrt{2}, 1)$. *From (1.5) the corresponding value of* μ *is*

$$\mu_1 = \frac{1}{2}\left(1 - \frac{1}{3}\sqrt{\frac{1}{483}(3265 + 2\sqrt{199945})}\right),$$

following the notation in [3].

Second, notice that $\beta_3 = 0$ has only one solution in the interval $(1/\sqrt{2}, 1)$.

Notation 3. *We denote* $\omega_1^\sharp$ *the only solution of* $\beta_3 = 0$ *in the interval* $(1/\sqrt{2}, 1)$. *It is the square root of a root of the fifth-degree polynomial* $960x^5 - 7364x^4 + 29940x^3 - 48219x^2 + 24155x - 444$ *lying in the interval* $(1/\sqrt{2}, 1)$. *An approximation of it is*

$$\omega_1^\sharp \approx 0.935871439168618.$$

From (1.5) the corresponding value of μ *is*

$$\mu_2 \approx 0.016376755355816,$$

following the notation in reference [3].

Using Mathematica version 12, in particular applying the specific routines of solving equations and inequalities and eliminating quantifiers, we have proved that $\mathcal{H}^4$ keeps the same sign for all $\omega_1 \in D \cap \mathbb{Q}$, where

$$D = \left(\frac{1}{\sqrt{2}}, 1\right) \setminus \left(\left\{\frac{2}{\sqrt{5}}\right\} \cup [\omega_1^\sharp,\ \omega_1^*]\right). \tag{3.3}$$

It is stressed that $\mathcal{H}^4(\mathbf{I})$ with $\mathbf{I} \in S \setminus \{\mathbf{0}\}$ does not change sign for $\omega_1 \in D$ regardless of whether ω_1 is rational or not.

At this point it is rather convenient to introduce the *rational root test* [25]: Given a polynomial in the variable x of degree n, say, $r(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0 = 0$, where a_0, a_1, …, a_n are integers, the rational root test says that for the polynomial $r(x)$ to have a rational solution of the form p/q (irreducible fraction), q must divide a_n and p must divide a_0.

Applying the criterion given above we have proved that $\omega_1^\sharp$, $\omega_2^\sharp = (1 - \omega_1^{\sharp 2})^{1/2}$ belong to $\mathbb{R} \setminus \mathbb{Q}$. Moreover, we have checked that there is no integer vector $\mathbf{k} \neq \mathbf{0}$ such that $\mathbf{k} \cdot (\omega_1^\sharp, -\omega_2^\sharp, 1) = 0$. So this case will be tackled in (b_1).

Regarding ω_1^* we have also checked that it is irrational and so is the corresponding $\omega_2^* = (1 - \omega_1^{*2})^{1/2}$. Additionally there is no integer vector $\mathbf{k} \neq \mathbf{0}$ such that $\mathbf{k} \cdot (\omega_1^*, -\omega_2^*, 1) = 0$. Therefore, this case has to be analysed in (b_1) too.

Thence, by virtue of Theorem 1 the equilibrium L_4 is Lie stable for $\omega_1 \in D \cap \mathbb{Q}$ and $\omega_2 \in \mathbb{Q}$ related to a Pythagorean triple.

An example lying in this class is for instance $\omega_1 = 4/5$, $\omega_2 = 3/5$, which yields Lie stability. However, choosing $\omega_1 = 35/37$, $\omega_2 = 12/37$, as $\omega_1 \in [\omega_1^\sharp, \omega_1^*]$, one has that $\mathcal{H}^4$ changes sign in S and then we cannot decide on its stability. This corresponds to a Pythagorean triple and it is associated to a resonance of order eight, say $(k_1, k_2, k_3) = (1, 6, 1)$. It is indeed the lowest order for a resonance corresponding to a Pythagorean triple in the interval $(\omega_1^\sharp, \omega_1^*)$.

(a_2) If $\omega_1 \in \mathbb{Q}$ and $\omega_2 \in \mathbb{R}\setminus\mathbb{Q}$, then the frequency vector is not associated to a Pythagorean triple, and one gets $F_1 = \omega_1 I_1 + I_3$, $F_2 = I_2$. Thus $d = 2$, $s = 1$ and

$$S = \{(I_1, I_2, I_3) \mid \omega_1 I_1 + I_3 = 0,\ I_2 = 0,\ I_1, I_3 \geq 0\} = \{\mathbf{0}\}.$$

Then, by applying Theorem 1 we conclude that L_4 is Lie stable.

An example in (a_2) is $\omega_1 = 6/7$ and $\omega_2 = \sqrt{13}/7$.

(b_1) If $\omega_1, \omega_2 \in \mathbb{R}\setminus\mathbb{Q}$ and there is no integer vector $\mathbf{k} = (k_1, k_2, k_3) \neq (0,0,0)$ such that $\mathbf{k}\cdot(\omega_1, -\omega_2, 1) = 0$ then there are no resonances among the main frequencies. Hence, $F_j = I_j$ for $j = 1,2,3$, $d = 3$, $s = 0$ and $S = \{\mathbf{0}\}$. So Lie stability holds.

As we have seen in (a_1), the values $\omega_1^\sharp$, ω_1^*, correspondingly μ_2, μ_1, belong to this case, so both of them lead to Lie stability.

Two more examples are $\omega_1 = 6/\sqrt{41}$, $\omega_2 = \sqrt{5/41}$ and $\omega_1 = 2/e$, $\omega_2 = \sqrt{e^2-4}/e$.

(b_2) If $\omega_1, \omega_2 \in \mathbb{R}\setminus\mathbb{Q}$ and there is an integer vector $\mathbf{k} = (k_1, k_2, k_3) \neq (0,0,0)$ such that $\mathbf{k}\cdot(\omega_1, -\omega_2, 1) = 0$, then

$$\omega_1 = \frac{k_2}{k_1}\omega_2 - \frac{k_3}{k_1}, \tag{3.4}$$

with $k_1 \neq 0$. Notice that $k_1 = 0$ would imply $-\omega_2 k_2 + k_3 = 0$ and then, either $k_2 = 0$, in which case $k_1 = k_2 = k_3 = 0$, that is impossible, or $\omega_2 = k_3/k_2 \in \mathbb{Q}$, contradicting the hypotheses of (b_2). Analogously $k_2 \neq 0$ because $k_2 = 0$ would lead to $\omega_1 = -k_3/k_1 \in \mathbb{Q}$, that is not feasible. However, $k_3 = 0$ is possible and this implies $k_1\omega_1 = k_2\omega_2$. We have to exclude the particular value $\omega_1 = 3/\sqrt{10}$, as it leads to instability.

Using (3.4) we get

$$H_2 = \left(\frac{k_2}{k_1} I_1 - I_2\right)\omega_2 - \frac{k_3}{k_1} I_1 + I_3,$$

from where we deduce that $F_1 = k_2 I_1/k_1 - I_2$, $F_2 = -k_3 I_1/k_1 + I_3$, $d = 2$ and $s = 1$.

Consider the set

$$K = \left\{(k_1, k_2, k_3) \in \mathbb{Z}^3 \mid k_1 \neq 0,\ k_2/k_1 > 0,\ k_3/k_1 \geq 0\right\}.$$

If $\mathbf{k} \in K$ then one has $\dim S = 1$ with

$$S = \left\{\left(I_1, \frac{k_2}{k_1} I_1, \frac{k_3}{k_1} I_1\right) \,\middle|\, I_1 \geq 0\right\}.$$

Taking $\mathbf{I} \in S$ and using (1.4) and (3.4) we get

$$\mathcal{H}^4(\mathbf{I}) = \left(\beta_1 + \beta_2\frac{k_3}{k_1} + \beta_3\frac{k_3^2}{k_1^2}\right) I_1^2,$$

where β_1, β_2 and β_3 are given in (3.2). In fact $\mathcal{H}^4(\mathbf{I})$ can be obtained by replacing $I_3 = k_3 I_1/k_1$ in (3.1). Due to the form acquired by $\mathcal{H}^4$, we notice this is a particular situation of $\mathcal{H}^4$ in (a_1) and we can conclude that $\mathcal{H}^4(\mathbf{I})$ does not change sign in $S\setminus\{\mathbf{0}\}$ when $\omega_1 \in D\cap(\mathbb{R}\setminus\mathbb{Q})$ with D given in (3.3) and $\omega_2 \in \mathbb{R}\setminus\mathbb{Q}$. Therefore, Lie stability is accomplished when $\mathbf{k} \in K$.

As in (a_1), we have verified with Mathematica that in the interval $[\omega_1^\sharp, \omega_1^*]$ there is no $\mathbf{k} \in K$ such that $\mathcal{H}^4(\mathbf{I})$ vanishes for $\mathbf{I}$ in $S\setminus\{\mathbf{0}\}$, so for the study of this interval we go back to (b_1), concluding Lie stability.

Finally when $\mathbf{k} \notin K$ then, from the first integrals F_1, F_2 given a few lines above one deduces that $S = \{\mathbf{0}\}$, concluding Lie stability.

An example of the previous situation is $\mu_{(3,3,-2)}$ that will be mentioned in the forthcoming sections. In this case

$$\omega_1 = \tfrac{1}{6}(2+\sqrt{14}), \quad \omega_2 = \tfrac{1}{6}(-2+\sqrt{14}),$$

which are irrational although resonant, the resonance vector being $\mathbf{k} = (3,3,-2) \notin K$. Then, Lie stability holds.

(b_3) If $\omega_1 \in \mathbb{R}\setminus\mathbb{Q}$ and $\omega_2 \in \mathbb{Q}$, then we take $F_1 = I_1$, $F_2 = -\omega_2 I_2 + I_3$, from where we get $d = 2$, $s = 1$ and

$$S = \left\{ \left(0, \frac{1}{\omega_2} I_3, I_3\right) \;\middle|\; I_3 \geq 0 \right\}.$$

We consider $\mathbf{I} \in S \setminus \{\mathbf{0}\}$, write $\omega_2 = (1-\omega_1^2)^{1/2}$ and replace it in (1.8), ending up with $\mathcal{H}^4(\mathbf{I}) = \beta_3 I_3^2$, with β_3 given in (3.2). Notice that we arrive at the same expression replacing $I_1 = 0$ in (3.1). The domain where the normal form is properly defined and where $\beta_3 \neq 0$ is $\left(1/\sqrt{2}, 1\right) \setminus \{2/\sqrt{5}, \omega_1^\sharp, 3/\sqrt{10}\}$. Since the values of ω_1 that we are discarding are in correspondence with irrational values of ω_2, we do not care about them. Therefore, applying Theorem 1, the equilibrium L_4 is Lie stable whenever $\omega_1 \in \left(1/\sqrt{2}, 1\right)$ with $\omega_1 \in \mathbb{R}\setminus\mathbb{Q}$ and $\omega_2 \in \mathbb{Q}$.

An example borrowed from reference [3] is $\mu_{(0,3,1)}$. On this occasion $\omega_1 = 2\sqrt{2}/3 \in \mathbb{R}\setminus\mathbb{Q}$, $\omega_2 = 1/3 \in \mathbb{Q}$, concluding Lie stability.

In Table 1 we have put the resonant cases corresponding to $\dim S = 0$ until the eighth-order of resonance whereas in Table 2 we have written the resonant cases corresponding to $\dim S = 1$ also to order 8. Note that σ is always a Diophantine vector. Both tables appear in the Appendix.

Taking into account Notation 2, Notation 3 and $\mu_{(1,2,0)}$, $\mu_{(1,3,0)}$ introduced respectively in (1.9) and (1.10), we summarise our main result on the Lie stability of L_4 in the spatial case of the restricted circular three body problem as follows.

Theorem 3. *For $0 < \mu < \mu_R$ the equilibrium point L_4 is Lie stable for the Hamiltonian system related to (1.1), excepting the unstable situations $\mu_{(1,2,0)}$, $\mu_{(1,3,0)}$ and the values $\mu \in (\mu_1, \mu_2)$ leading to a Pythagorean triple.*

4. ASYMPTOTIC ESTIMATES

In this section we apply Theorem 2 to bound the solutions of the restricted circular three-body problem near the equilibrium (in case it is Lie stable) over exponentially long times.

First we have to take into account the order j in the normal form (1.7) that determines the Lie stability. When $S = \{\mathbf{0}\}$, then $j = 2$ and in the rest of situations $j = 4$. Second, we have to consider the number of independent first integrals, that is d, which ranges from 1 to 3 in the problem at hand.

In our study the solution is expressed as $\mathbf{I}(t)$, that is a function of the order of $|\mathbf{x}|^2$. So, $|\mathbf{x}(0)| < \varepsilon$ implies $|\mathbf{I}(0)| < \varepsilon^2$. Introducing $\epsilon = \varepsilon^2$ and setting $\mathbf{I}_0 = \mathbf{I}(0)$, then the thesis of Theorem 2 will read as

$$|\mathbf{I}(t)| < \alpha\, \epsilon^{2/j} \quad \text{for all } \ t \ \text{ with } \quad 0 \leq t \leq T = \mathcal{C} \exp\left(\frac{\mathcal{E}}{\epsilon^{1/(2(\nu+1))}}\right),$$

where α, $\mathcal{C}$, $\mathcal{E}$ are obtained respectively from a, C and E.

Usually for the estimates in terms of action-angle coordinates, the norm used to bound the actions is the 1-norm, but here we use the Euclidean norm as both are equivalent.

The following considerations are in order:

1. If $S = \{\mathbf{0}\}$, then either $d = 2$ (cases (a_2) and (b_2) for $\mathbf{k} \notin K$) or $d = 3$ (case (b_1)). Thus, the parameter ν involved in the Diophantine condition satisfies $\nu \geq 1$ or $\nu \geq 2$, respectively.

   When the vector $\sigma = (\sigma_1, \sigma_2)$ or $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ respectively, satisfies (2.2), there exist $\alpha > 1$, $\mathcal{C} > 0$, $\mathcal{E} > 0$ and $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ with $|\mathbf{I}(0)| < \epsilon$ we get

$$|\mathbf{I}(t)| < \alpha\,\epsilon \quad \text{for all } t \text{ with } \quad 0 \leq t \leq T = \mathcal{C}\exp\left(\frac{\mathcal{E}}{\epsilon^{1/4}}\right), \tag{4.1}$$

   in (a_2) and (b_2) with $\mathbf{k} \notin K$ or

$$|\mathbf{I}(t)| < \alpha\,\epsilon \quad \text{for all } t \text{ with } \quad 0 \leq t \leq T = \mathcal{C}\exp\left(\frac{\mathcal{E}}{\epsilon^{1/6}}\right), \tag{4.2}$$

   in the situation (b_1).

2. If $S \neq \{\mathbf{0}\}$, then $j = 4$.

   - When $d = 2$, $\nu \geq 1$ (cases (b_2) with $\mathbf{k} \in K$ and (b_3)). Thence, if the frequency vector $\sigma = (\sigma_1, \sigma_2)$ satisfies (2.2), there exist $\alpha > 1$, $\mathcal{C} > 0$, $\mathcal{E} > 0$ and $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ and $|\mathbf{I}(0)| < \epsilon$, the following estimate holds

$$|\mathbf{I}(t)| < \alpha\,\epsilon^{1/2} \quad \text{for all } t \text{ with } \quad 0 \leq t \leq T = \mathcal{C}\exp\left(\frac{\mathcal{E}}{\epsilon^{1/4}}\right). \tag{4.3}$$

   - If $d = 1$ (case (a_1)), then $\nu \geq 0$ and since there is only one first formal integral no Diophantine condition is needed. There exist $\alpha > 1$, $\mathcal{C} > 0$, $\mathcal{E} > 0$ and $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ and $|\mathbf{I}(0)| < \epsilon$ we get

$$|\mathbf{I}(t)| < \alpha\,\epsilon^{1/2} \quad \text{for all } t \text{ with } \quad 0 \leq t \leq T = \mathcal{C}\exp\left(\frac{\mathcal{E}}{\epsilon^{1/2}}\right). \tag{4.4}$$

Notice that the constants α, $\mathcal{C}$ and $\mathcal{E}$ are independent of ϵ and are supposed to be obtained using bounds on the normal-form terms.

In summary, we have the following results.

Theorem 4. *For the Hamiltonian system associated with (1.1) we have the subsequent asymptotic estimates around the equilibrium point L_4 in case it is Lie stable:*

1. *When the vector ω is associated to a Pythagorean triple in D, then there exist $\alpha > 1$, $\mathcal{C} > 0$, $\mathcal{E} > 0$ and $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ with $|\mathbf{I}(0)| < \epsilon$:*

$$|\mathbf{I}(t)| < \alpha\,\epsilon^{1/2} \quad \textit{for all } t \textit{ with } \quad 0 \leq t \leq T = \mathcal{C}\exp\left(\frac{\mathcal{E}}{\epsilon^{1/2}}\right).$$

2. *When σ is Diophantine, then there exist $\alpha > 1$, $\mathcal{C} > 0$, $\mathcal{E} > 0$ and $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ with $|\mathbf{I}(0)| < \epsilon$:*

   (a) If either $\omega_1 \in \mathbb{Q}$ and $\omega_2 \in \mathbb{R} \setminus \mathbb{Q}$ or $\omega_1, \omega_2 \in \mathbb{R} \setminus \mathbb{Q}$ and there is $\mathbf{k} \notin K$ such that $\mathbf{k} \cdot (\omega_1, -\omega_2, 1) = 0$, then

$$|\mathbf{I}(t)| < \alpha\,\epsilon \quad \textit{for all } t \textit{ with } \quad 0 \leq t \leq T = \mathcal{C}\exp\left(\frac{\mathcal{E}}{\epsilon^{1/4}}\right).$$

   (b) If $\omega_1, \omega_2 \in \mathbb{R} \setminus \mathbb{Q}$ and there are no resonances among the main frequencies, then

$$|\mathbf{I}(t)| < \alpha\,\epsilon \quad \textit{for all } t \textit{ with } \quad 0 \leq t \leq T = \mathcal{C}\exp\left(\frac{\mathcal{E}}{\epsilon^{1/6}}\right).$$

(c) If $\omega_1 \in \mathbb{R}\setminus\mathbb{Q}$ and either $\omega_2 \in \mathbb{Q}$ or $\omega_2 \in \mathbb{R}\setminus\mathbb{Q}$ and there is $\mathbf{k}\in K$ such that $\mathbf{k}\cdot(\omega_1,-\omega_2,1)=0$, then

$$|\mathbf{I}(t)| < \alpha\,\epsilon^{1/2} \quad \text{for all } t \text{ with} \quad 0 \le t \le T = \mathcal{C}\exp\left(\frac{\mathcal{E}}{\epsilon^{1/4}}\right).$$

We specify the results obtained by our approach for some values handled in previous studies.

Corollary 1. *Given $\epsilon \in (0,\epsilon_0)$ and $|\mathbf{I}(0)| < \epsilon$, the following estimates around the equilibrium point L_4 are satisfied.*

For $\mu = \mu_1, \mu_2$ we obtain

$$|\mathbf{I}(t)| < \alpha\,\epsilon \quad \text{for all } t \text{ with} \quad 0 \le t \le T = \mathcal{C}\exp\left(\frac{\mathcal{E}}{\epsilon^{1/6}}\right).$$

For $\mu = \mu_{(3,3,-2)}$ we achieve

$$|\mathbf{I}(t)| < \alpha\,\epsilon \quad \text{for all } t \text{ with} \quad 0 \le t \le T = \mathcal{C}\exp\left(\frac{\mathcal{E}}{\epsilon^{1/4}}\right).$$

For $\mu = \mu_{(0,3,1)}$ we arrive at

$$|\mathbf{I}(t)| < \alpha\,\epsilon^{1/2} \quad \text{for all } t \text{ with} \quad 0 \le t \le T = \mathcal{C}\exp\left(\frac{\mathcal{E}}{\epsilon^{1/4}}\right).$$

For $\mu = \mu_3 \approx 0.014780913055964$ we get

$$|\mathbf{I}(t)| < \alpha\,\epsilon \quad \text{for all } t \text{ with} \quad 0 \le t \le T = \mathcal{C}\exp\left(\frac{\mathcal{E}}{\epsilon^{1/6}}\right).$$

The case μ_3 will be tackled in section 5.

5. COMPARISON WITH PREVIOUS RESULTS

Regarding related approaches to the nonlinear stability of L_4, we mention that in 1971 Markeev [20] proved the stability of L_4 in $\mu \in (0,\mu_R)\setminus\{\mu_{(1,2,0)},\mu_{(1,3,0)}\}$ for most initial conditions, in the sense of the Lebesgue measure. In other words, he proved the existence of KAM 3-tori around L_4. In 1973, see [21], the same author proved the formal stability of L_4 for $\mu \in (0,\mu_1)\cup(\mu_2,\mu_{(1,2,0)})\cup(\mu_{(1,2,0)},\mu_R)$. This corresponds to the domain D that we have defined in (3.3). The formal stability analysis made in (μ_1,μ_2) is not complete as double resonances are not considered for orders higher than 6.

Remark 3. Theorem 3 recovers the formal stability achievements already obtained in the set D and extends them to μ_1, μ_2 and the values $\mu \in (\mu_1,\mu_2)$ not leading to a Pythagorean triple. Moreover our proof of Lie stability uses different arguments from the one of Markeev in [21]. Indeed, our approach is straightforward since we only need that $\mathcal{H}^4$ restricted to the set S be sign-definite, while Markeev introduced a formal first integral in each specific case.

In 1989, Giorgilli *et al.* [14], applying normal form techniques with floating-point arithmetic, proved that L_4 was Nekhoroshev stable for $\mu \in (0,\mu_R)$, excepting a few values of μ that led to resonances. Furthermore the vector ω should satisfy a Diophantine condition. In 1991, Celletti and Giorgilli [8], along the same line, refined the preceding achievements.

In 1998 Benettin *et al.* [3] extended previous results with the idea of determining Nekhoroshev stability without imposing any Diophantine condition. On the one hand, the requirement of quasi-convexity was relaxed by introducing the concept of *directional quasi-convexity*, that is specific for elliptic equilibria. On the other hand, they enlarged the directional quasi-convexity requirement by proposing a steepness condition on the 3-jet of the sixth order normal form, following the original ideas of Nekhoroshev. They applied their results using normal forms with floating-point arithmetic to establish that $\mathcal{H}^4(\mathbf{I})$ was directionally quasi-convex, and then concluding that L_4 is Nekhoroshev stable in the domain D.

Remark 4. It is worth mentioning that the estimates claimed in Theorem 2 of [3] have not been proved so far, and only very recently bounds for steep elliptic equilibria have appeared [4]. As these estimates depend on the steepness indices, it is not immediate to apply them in our problem, therefore we have preferred to use the ones of [3] in our comparisons.

For $\mu \in D$, $|\mathbf{I}(0)| \le \epsilon$ and ϵ sufficiently small, two estimates were obtained in [3], namely,

$$|\mathbf{I}(t)| \le \epsilon^{1/3} \quad \text{for} \quad 0 < \mathrm{t} \le \exp(\epsilon^{-1/3}) \tag{5.1}$$

and

$$|\mathbf{I}(t)| \le \epsilon^{1/2} \quad \text{for} \quad 0 < \mathrm{t} \le \exp(\epsilon^{-1/6}). \tag{5.2}$$

Several comparisons between our approach and that of [3], when ω_1 is in the domain D, are in order:

(i) According to Theorem 4, when ω is associated to a Pythagorean triple in D, our confinement bound for $|\mathbf{I}(t)|$ is of the same order $\epsilon^{1/2}$ but our time estimate is longer, indeed of the order of $\exp(\epsilon^{-1/2})$, whereas in (5.2) is $\exp(\epsilon^{-1/6})$.

(ii) When the vector σ satisfies the Diophantine condition (2.2) and $S = \{\mathbf{0}\}$ the bounds obtained using Theorem 4 are sharper than the ones of (5.2). Observe that in cases 2.(a) and 2.(b) of Theorem 4 the confinement is of order ϵ, whereas in (5.1) and (5.2) it is either of order $\epsilon^{1/2}$ or $\epsilon^{1/3}$. In fact, the order ϵ is stated in the context of Nekhoroshev estimates only when the 3-jet is computed.

(iii) Under the usual Diophantine condition, in case 2.(c) of Theorem 4, the time estimate is better than it is in (5.2), say $\exp(\epsilon^{-1/4})$ versus $\exp(\epsilon^{-1/6})$, for the same confinement of the solution, that is $\epsilon^{1/2}$, using only the normal form $\mathcal{H}^4$, that is, without calculating higher-order terms.

When $\mu \in [\mu_1, \mu_2]$, Benettin *et al.* [3] calculated the normal form term $\mathcal{H}^6(\mathbf{I})$ (with numerical coefficients) as directional quasi-convexity does not hold in this interval. It was also checked that the normal form of order eight could be computed excepting the values $\mu_{(1,3,0)}$, $\mu_{(0,3,1)}$ and $\mu_{(3,3,-2)}$. They also proved that $\mathcal{H}^6(\mathbf{I})$ was steep except at μ_3. More specifically, they determined when $\mathcal{H}^6$ was non-degenerate, in other words, under which conditions the unique solution of $\mathcal{H}_2(\mathbf{I}) = \mathcal{H}_4(\mathbf{I}) = \mathcal{H}_6(\mathbf{I}) = 0$ was $\mathbf{I} = \mathbf{0}$. To simplify the calculations, it was also required in [3] that the restriction of the Hessian matrix of $\mathcal{H}_4$, say A, to the plane orthogonal to ω is nonsingular, in other words if $\omega \cdot \mathbf{I} = 0$, $A\,\mathbf{I} = \mathbf{0}$ then $\mathbf{I} = \mathbf{0}$. Then, excluding the aforementioned values, steepness (and Nekhoroshev stability) was concluded in $[\mu_1, \mu_2]$ and the estimates were as follows.

For $\mu \in [\mu_1, \mu_2] \setminus \{\mu_{(3,3,-2)}, \mu_{(1,3,0)}, \mu_3, \mu_{(0,3,1)}\}$, $|\mathbf{I}(0)| \le \epsilon$ and ϵ small enough:

$$|\mathbf{I}(t)| < \epsilon \quad \text{for} \quad 0 < \mathrm{t} < \exp(\epsilon^{-1/20}). \tag{5.3}$$

The cases $\mu_{(3,3,-2)}$, $\mu_{(0,3,1)}$ were excluded from the estimates (5.3) because, although for both cases $\mathcal{H}_6$ depends only on the actions, the corresponding $\mathcal{H}_8$ contains resonant terms, and the above bounds require that $\mathcal{H}^8$ depend only on $\mathbf{I}$.

In the interval $[\mu_1, \mu_2]$ we stress the following points of our approach:

(i) For $\mu \in [\mu_1, \mu_2] \setminus \{\mu_{(3,3,-2)}, \mu_{(1,3,0)}, \mu_3, \mu_{(0,3,1)}\}$, when σ satisfies (2.2), and such that the frequencies leading to Pythagorean triples are excluded, the time estimates of Theorem 4 obtained through $\mathcal{H}^4$ are better than the ones in (5.3) deduced from $\mathcal{H}^6$. Nevertheless, the confinement of the actions obtained in [3] is sharper, excepting when $S = \{\mathbf{0}\}$, as then both are of the same order.

(ii) The case of the Pythagorean triples for $\mu \in (\mu_1, \mu_2)$ is a pending issue in our analysis. What happens is that $\mathcal{H}_4$ changes sign in S and then, we cannot conclude Lie stability. According to (B) in Theorem 1, computing higher-order terms in the normal form would not lead to a sign-definite formal integral. From [3] (and from our own analysis) we know that $\mathcal{H}_6$ is steep for all Pythagorean triples, thus bounds for the solution are provided. Although steepness does not imply stability, if instability holds, the diffusion mechanism would be very slow.

(iii) Case $\mu_{(3,3,-2)}$ is Lie stable, as it belongs to (b_2) in section 3. Case $\mu_{(0,3,1)}$ is also Lie stable, as it belongs to (b_3) in section 3. Corollary 1 gives asymptotic estimates in both cases. However they cannot be studied from the point of view of Nekhoroshev theory.

(iv) Cases $\mu_{(1,3,0)}$ and $\mu_{(1,2,0)}$ do not satisfy the necessary conditions leading to Nekhoroshev stability, but they are already known to be unstable.

The case μ_3 corresponds to a degenerate 3-jet. So, when applying Nekhoroshev techniques, one needs to pursue the calculations to get the 4-jet, that is, $\mathcal{H}^8(\mathbf{I})$. Moreover, one has to take into account that for r-jets with $r > 3$ additional conditions apart from the non-degeneracy of the jet are needed to ensure steepness. This case was studied by Schirinzi and Guzzo in 2015 [27], establishing Nekhoroshev stability for $\mu = \mu_3$. It should be noticed that their normal form calculations were performed using floating-point arithmetic. However, the related asymptotic estimates have not been determined so far.

Trying to get the value μ_3 from our normal form $\mathcal{H}^6$ we notice that the corresponding 3-jet becomes zero for μ_3 but then the associated $I_3 < 0$. More precisely, in our analysis μ_3 does not appear as a special value and Lie stability is accomplished without the need of analysing $\mathcal{H}^6$. The value $\mu_3 \approx 0.014780913055963$ is in correspondence through (1.3) with the value ω_1 lying in the interval $(\omega_1^\sharp, \omega_1^*)$ obtained as the square root of a root of the polynomial of degree 52 given by $\sum_{i=0}^{52} c_i x^i$ whose coefficients are given in Table 3 of the Appendix. By applying the rational root test we have proved that the value of ω_1 related to μ_3 is irrational and the same is true with the corresponding ω_2. Therefore this situation is a specific example of (b_1) in section 3.

We do the following considerations regarding μ_3:

(i) For μ_3 the equilibrium is Lie stable and time estimates are given in Corollary 1. This conclusion is obtained just from the analysis of $\mathcal{H}^4$.

(ii) The calculations we have performed in order to achieve Lie stability use symbolic arithmetic. We have carried out the computations up to order 4, that is, determining $\mathcal{H}^6(\mathbf{I})$, although for our analysis only order 2 is required, excepting for the explicit calculation of μ_3. The computations are valid disregarding the resonances that appear in the generating functions and that we have treated separately.

It is not straightforward to compare our asymptotic estimates with those obtained in [3] and other references, but in general our bounds are sharper than those accomplished with the use of Nekhoroshev theory with low order normal forms. For instance, using $\mathcal{H}^4$ in the Lie stable cases where $\dim S = 2$ our time estimate is of the order $\exp(\mathcal{E}\epsilon^{-1/2})$, which is not obtained applying Nekhoroshev theory. Moreover, the exponent of ϵ^{-1} in the time estimates never exceeds $1/6$, while in [3] it can be $1/20$, as we have seen before. Sharper bounds could be obtained in the setting of Nekhoroshev theory by computing higher-order normal forms, see for instance [3, 26], and references therein.

Finally we point out a couple of considerations related to our estimates:

(i) It would be desirable to drop, or at least to relax, the Diophantine condition in Theorem 2, but so far it is not possible since it is an essential requirement for estimating the bounds for formal integrals in [9]. Sometimes given a non-resonant vector σ it is not straightforward to deduce whether it is Diophantine or Liouville, although it is a well-known fact that for a fixed ν the Lebesgue measure of the set of vectors $\sigma \in \mathbb{R}^d$ that does not satisfy the Diophantine condition for any $c > 0$ is zero. In cases of Liouville vectors σ we make use of Benettin *et al.*'s bounds given in (5.1) and (5.2). In this context, perhaps we could apply Lochak's method of averaging [18] by analysing the neighbourhoods of periodic solutions of the unperturbed system, see reference [26], dropping therefore the need of using condition (2.2).

(ii) In case (b_1) it could happen that the frequency vector $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ be Diophantine. If this occurs the best time estimates are of doubly exponential character, see [5].

In Fig. 1 we adapt Fig. 1 from [3] to collect known results about the stability of L_4.

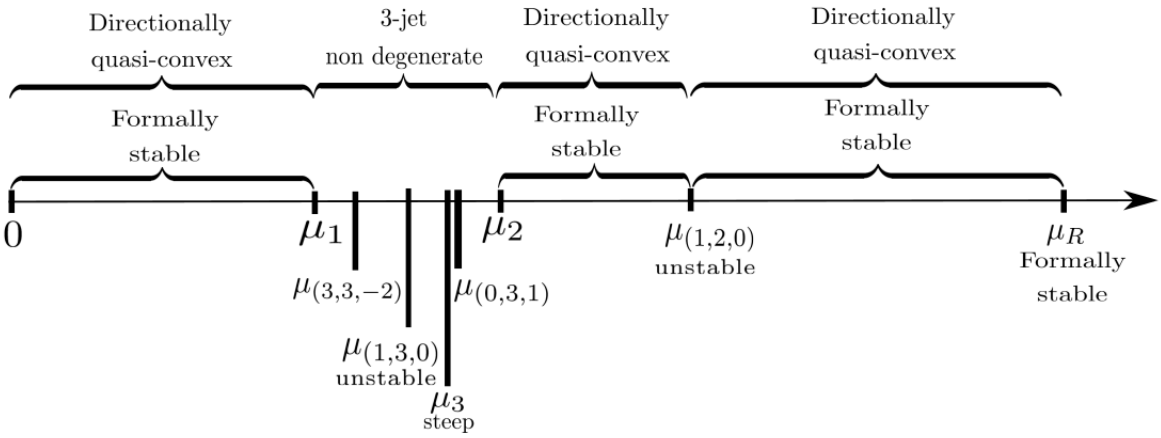

Fig. 1. Type of stability of the Lagrangian points depending on μ in the spatial case.

6. KAM TORI

In this section we prove the existence of 3-dimensional KAM tori and quasi-periodic motions encasing the equilibrium point L_4 of the Hamiltonian system related to (1.1) at each energy level. This is indeed Markeev's analysis performed in [20]. For our purpose we apply the classical theorem by Kolmogorov, Arnold and Moser in its isoenergetic version, see, for instance [1]. Thus, an isoenergetic nondegeneracy condition has to be satisfied. More precisely, we get

$$D_4 = \begin{vmatrix} \dfrac{\partial^2 \mathcal{H}^4}{\partial \mathbf{I}^2} & \dfrac{\partial \mathcal{H}^4}{\partial \mathbf{I}} \\ \dfrac{\partial \mathcal{H}^4}{\partial \mathbf{I}} & 0 \end{vmatrix} = \frac{\omega_1^2 \left(\omega_1^2 - 1\right) d_4}{559872 \left(1 - 2\omega_1^2\right)^6 \left(\omega_1^2 - 4\right)^3 \left(\omega_1^2 + 3\right)^3 \left(5\omega_1^2 - 4\right)^3 \left(5\omega_1^2 - 1\right)^3},$$

where

$$\begin{aligned} d_4 = &-856968120000\omega_1^{32} + 6855744960000\omega_1^{30} - 12012443413200\omega_1^{28} - 35888432907600\omega_1^{26} \\ &+156438442275660\omega_1^{24} - 160144630175160\omega_1^{22} + 31637715760125\omega_1^{20} \\ &-362513394226125\omega_1^{18} + 1355837182686882\omega_1^{16} - 2073003173172738\omega_1^{14} \\ &+1782670403156769\omega_1^{12} - 952761177324729\omega_1^{10} + 327497353333812\omega_1^{8} \\ &-74195783114400\omega_1^{6} + 11875034325888\omega_1^{4} - 1435874045184\omega_1^{2} + 95785141248. \end{aligned}$$

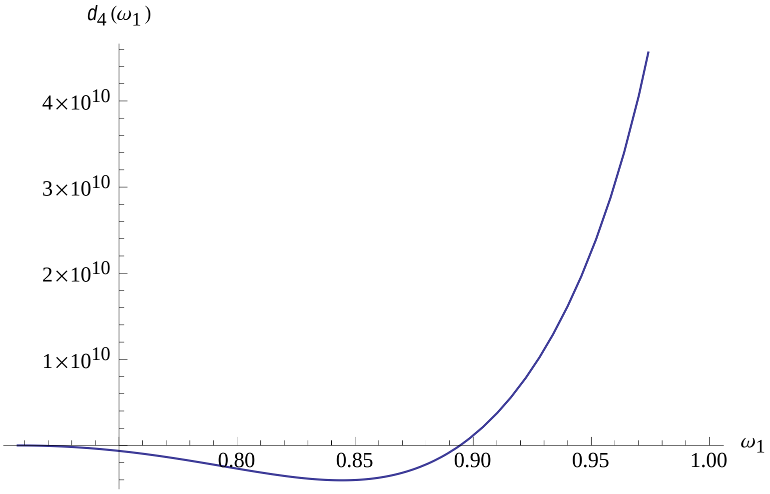

Fig. 2. Graph of the curve $d_4(\omega_1)$.

Note that the function $d_4 = d_4(\omega_1)$ is null only when $\omega_1 = 2/\sqrt{5}$, i.e., $\mu = \mu_{(1,2,0)}$, see Fig. 2.

Thus, D_4 does not vanish for $\omega_1 \in (1/\sqrt{2}, 1) \setminus \{2/\sqrt{5}\}$. Even in the situations where we do not know if Lie stability holds, that is, the ones corresponding with Pythagorean triples such that $\mu \in (\mu_1, \mu_2)$, D_4 is different from zero.

We establish the following result.

Theorem 5. *For $\mu \in (0, \mu_R) \setminus \{\mu_{(1,2,0)}, \mu_{(1,3,0)}\}$, most of the invariant 3-tori corresponding to the term $\mathcal{H}^4$ derived from the Hamiltonian system related to (1.1) will persist slightly deformed for any sufficiently small perturbation of them, that is, for the full system. Moreover, the Lebesgue measure of the complement of the set of tori tends to zero when the perturbation is small. More precisely, the invariant tori form a majority on each energy-level manifold. The measure of the complement of the invariant tori that remain is of the order $\mathcal{O}(\varepsilon^{1/4})$ and can be refined to $\mathcal{O}(\varepsilon^{(l-3)/4})$ when the frequency vector ω does not satisfy resonance relations of order l with $l \leq 4$.*

Remark 5. We have also applied Han-Li-Yi's Theorem [15] dropping the isoenergetic condition. In this way we have obtained invariant 3-tori and quasiperiodic motions for the full Hamiltonian of the spatial circular restricted three body problem (1.1) in the situations where it can be expressed as $H_2(\mathbf{I}) + \varepsilon^2 \mathcal{H}_4(\mathbf{I}) + \mathcal{O}(\varepsilon^4)$. Around L_4, this is possible after introducing a stretching of coordinates as we have done in the course of the paper, discarding the two unstable cases, for a sufficiently small $\varepsilon > 0$. Specifically there are families of invariant 3-tori enclosing the equilibrium point L_4. These invariant tori form a majority in the sense that the measure of the complement of their union is of the order $\mathcal{O}(\varepsilon^{\delta})$ with a fixed value of δ such that $0 < \delta < 1/5$. The invariant tori are organised in Cantor families that depend on the parameter μ.

Notice that the application of Han-Li-Yi's Theorem for high-order proper degeneracy yields similar results to the ones obtained through Thereom 5. Indeed, according to Han *et al.*, the measure of the tori that do not remain after the perturbation can be refined pushing the normal form computation to higher orders in case that resonant terms are not encountered, analogously as in Theorem 5.

APPENDIX

We include two tables containing the single resonant cases up to $|\mathbf{k}|_1 = 8$. Table 1 accounts for the case of $\dim S = 0$. For the case of $\dim S = 1$, see Table 2.

In Table 3 we give the coefficients of the polynomial μ_3 comes from as a root.

$\mathbf{k}$	F_1	F_2	$\sigma = (\sigma_1, \sigma_2)$	Case
$(2, 2, -1)$	$I_2 - I_1$	$I_1 + 2I_3$	$\left(\frac{1-\sqrt{7}}{4}, \frac{1}{2}\right)$	(b_2)
$(3, 2, -1)$	$3I_2 - 2I_1$	$I_1 + 3I_3$	$\left(\frac{2(1-3\sqrt{3})}{39}, \frac{1}{3}\right)$	(b_2)
$(3, 1, -2)$	$3I_2 - I_1$	$2I_1 + 3I_3$	$\left(\frac{2-3\sqrt{6}}{30}, \frac{1}{3}\right)$	(b_2)
$(2, 3, -1)$	$2I_2 - 3I_1$	$I_1 + 2I_3$	$\left(\frac{3-4\sqrt{3}}{26}, \frac{1}{2}\right)$	(b_2)
$(1, -3, -2)$	$3I_1 + I_2$	$2I_1 + I_3$	$\left(\frac{-6+\sqrt{6}}{10}, 1\right)$	(b_2)
$(4, 0, -3)$	I_2	$3I_1 + 4I_3$	$\left(-\frac{\sqrt{7}}{4}, \frac{1}{4}\right)$	(a_2)
$(3, 3, -1)$	$I_2 - I_1$	$I_1 + 3I_3$	$\left(\frac{1-\sqrt{17}}{6}, \frac{1}{3}\right)$	(b_2)
$(2, 4, -1)$	$I_2 - 2I_1$	$I_1 + 2I_3$	$\left(\frac{2-\sqrt{19}}{10}, \frac{1}{2}\right)$	(b_2)
$(1, -4, -2)$	$4I_1 + I_2$	$2I_1 + I_3$	$\left(\frac{-8+\sqrt{13}}{17}, 1\right)$	(b_2)
$(4, 3, -1)$	$4I_2 - 3I_1$	$I_1 + 4I_3$	$\left(\frac{3-8\sqrt{6}}{100}, \frac{1}{4}\right)$	(b_2)
$(4, 1, -3)$	$4I_2 - I_1$	$3I_1 + 4I_3$	$\left(\frac{3-8\sqrt{2}}{68}, \frac{1}{4}\right)$	(b_2)
$(3, 4, -1)$	$3I_2 - 4I_1$	$I_1 + 3I_3$	$\left(\frac{2(2-3\sqrt{6})}{75}, \frac{1}{3}\right)$	(b_2)
$(3, 3, -2)$	$I_1 - I_2$	$2I_1 + 3I_3$	$\left(\frac{-2+\sqrt{14}}{6}, \frac{1}{3}\right)$	(b_2)
$(2, 5, -1)$	$2I_2 - 5I_1$	$I_1 + 2I_3$	$\left(\frac{5-4\sqrt{7}}{58}, \frac{1}{2}\right)$	(b_2)
$(1, -4, -3)$	$4I_1 + I_2$	$3I_1 + I_3$	$\left(\frac{2(-6+\sqrt{2})}{17}, 1\right)$	(b_2)
$(1, -5, -2)$	$5I_1 + I_2$	$2I_1 + I_3$	$\left(\frac{-10+\sqrt{22}}{26}, 1\right)$	(b_2)

Table 1. Resonance vector, first integrals and vector σ in cases of $\dim S = 0$. Hamiltonian $H_2 = \sigma_1 F_1 + \sigma_2 F_2$. In the situation (b_2), $\mathbf{k} \notin K$.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

k	F_1	F_2	$\sigma = (\sigma_1, \sigma_2)$	Case
$(0,2,1)$	I_1	$I_2 - 2I_3$	$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$	(b_3)
$(0,3,1)$	I_1	$I_2 - 3I_3$	$\left(\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right)$	(b_3)
$(2,3,0)$	$3I_1 - 2I_2$	I_3	$\left(\frac{1}{\sqrt{13}}, 1\right)$	(b_2)
$(0,4,1)$	I_1	$I_2 - 4I_3$	$\left(\frac{\sqrt{15}}{4}, -\frac{1}{4}\right)$	(b_3)
$(0,3,2)$	I_1	$2I_2 - 3I_3$	$\left(\frac{\sqrt{5}}{3}, -\frac{1}{3}\right)$	(b_3)
$(1,5,0)$	$5I_1 - I_2$	I_3	$\left(\frac{1}{\sqrt{26}}, 1\right)$	(b_2)
$(0,5,1)$	I_1	$I_2 - 5I_3$	$\left(\frac{2\sqrt{6}}{5}, -\frac{1}{5}\right)$	(b_3)
$(2,5,0)$	$5I_1 - 2I_2$	I_3	$\left(\frac{1}{\sqrt{29}}, 1\right)$	(b_2)
$(2,4,1)$	$2I_1 - I_2$	$I_1 - 2I_3$	$\left(\frac{2+\sqrt{19}}{10}, -\frac{1}{2}\right)$	(b_2)
$(1,6,0)$	$6I_1 - I_2$	I_3	$\left(\frac{1}{\sqrt{37}}, 1\right)$	(b_2)
$(1,4,2)$	$4I_1 - I_2$	$2I_1 - I_3$	$\left(\frac{\sqrt{13}+8\sqrt{17}}{17}, -\sqrt{17}\right)$	(b_2)
$(0,6,1)$	I_1	$I_2 - 6I_3$	$\left(\frac{\sqrt{35}}{6}, -\frac{1}{6}\right)$	(b_3)
$(0,5,2)$	I_1	$2I_2 - 5I_3$	$\left(\frac{\sqrt{21}}{5}, -\frac{1}{5}\right)$	(b_3)
$(3,5,0)$	$5I_1 - 3I_2$	I_3	$\left(\frac{1}{\sqrt{34}}, 1\right)$	(b_2)
$(2,5,1)$	$5I_1 - 2I_2$	$I_1 - 2I_3$	$\left(\frac{5+4\sqrt{7}}{58}, -\frac{1}{2}\right)$	(b_2)
$(1,7,0)$	$7I_1 - I_2$	I_3	$\left(\frac{1}{5\sqrt{2}}, 1\right)$	(b_2)
$(1,5,2)$	$5I_1 - I_2$	$2I_1 - I_3$	$\left(\frac{10+\sqrt{22}}{26}, -1\right)$	(b_2)
$(0,7,1)$	I_1	$I_2 - 7I_3$	$\left(\frac{4\sqrt{3}}{7}, -\frac{1}{7}\right)$	(b_3)

Table 2. Resonance vector, first integrals and vector σ in cases of $\dim S = 1$. Hamiltonian $H_2 = \sigma_1 F_1 + \sigma_2 F_2$. In the situation corresponding to (b_2), $\mathbf{k} \in K$.

REFERENCES

1. Arnol'd, V. I., Kozlov, V. V. and Neishtadt, A. I., *Mathematical aspects of classical and celestial mechanics, Dynamical Systems III Encyclopaedia of Mathematical Sciences 3*, 3rd ed., Berlin: Springer–Verlag, 2006.
2. Bardin, B., On Motions Near the Lagrange Equilibrium Point L_4 in the Case of Routh's Critical Mass Ratio, *Celestial Mech. Dynam. Astronom.*, 2002, vol. 82, no. 2, pp. 163–177.

c_0	−3345543647444698872524094664484096	c_1	3194933800193210510939874224007806976
c_2	−133142852751634975765240694606860910592	c_3	3353229252229449617635906930441300475904
c_4	−59006295355549187747044109281616236118016	c_5	788225115948882812404191577579711057035264
c_6	−8404704358510620010977067016337185034731520	c_7	73695475954794906843167530358753671462404096
c_8	−540576732470886723756654268885633469206699008	c_9	3350560634073938838006800472295740136106555136
c_{10}	−17657500682256413919143297554372385724778097280	c_{11}	79463164160925835810094642601972732662428571232
c_{12}	−306396477115452068307283022024953343053130416596	c_{13}	1015297587226646281157210009476933721162552276229
c_{14}	−2901167315298202702512087053641668436785252459945	c_{15}	7184430538308699052301532759394960483420936688598
c_{16}	−15552532804758882888717376187454633135733711386930	c_{17}	29880956548317259685613622218904331089896620336199
c_{18}	−52201223121743383064233227580596228239865365448051	c_{19}	85510386721294691989215784315942923230159418394628
c_{20}	−134625043875319416149393738742778612287835778168608	c_{21}	203983737378961879679113646763821294842098948776483
c_{22}	−290058963748297439559277326592130302207012032103487	c_{23}	372821687408889124493528843631412931581817725164518
c_{24}	−417681073832456682793620148713018873912445009123434	c_{25}	394397633015250251046398071718980958400736933611921
c_{26}	−301566539207032698480846971131018825843193755572069	c_{27}	173540952413898297286246299380019891424924554663552
c_{28}	−59602931193130186046156398967117200301837265410192	c_{29}	−7679677151122149928520105697347585943096732332960
c_{30}	27613989666597836606695506879391169105432388547168	c_{31}	−20980601034714219218681474857905315558796080240640
c_{32}	9209831878024868593574581627166440847282274760960	c_{33}	−2345627974195951773973005090446131999866171647232
c_{34}	468644966788218481020647383542439007628150846720	c_{35}	−530135429423849146464518754398950077120430964736
c_{36}	639827077256385977476978518239705975042078466048	c_{37}	−472355628562578539510969144175875192178435473408
c_{38}	246133207108175370094381015583635396587927519232	c_{39}	−99872439676762163536197196055142095448444108800
c_{40}	33952315204186600296511625347287194679262576640	c_{41}	−10271715872598767663299743179381342718579507200
c_{42}	2867337640268954204120843075629828505228083200	c_{43}	−737859950885920377593058895165940280852480000
c_{44}	170062953704533146451372328358210842787840000	c_{45}	−34201507579437555095782374809247055872000000
c_{46}	6116590633655045522465092353724350464000000	c_{47}	−1064226850447741909301401361239244800000000
c_{48}	191742496293562199236270016803635200000000	c_{49}	−319954033919892690304444129280000000000000
c_{50}	40691079767327808199912723251200000000000	c_{51}	−32815198130037765397335244800000000000000
c_{52}	126212300500145251528212480000000000000		

Table 3. Coefficients of the polynomial $\sum_{i=0}^{52} c_i x^i$ from where μ_3 comes as a root.

3. Benettin, G., Fassò, F. and Guzzo, M., Nekhoroshev-Stability of L_4 and L_5 in the Spatial Restricted Three-Body Problem, *Regul. Chaotic Dyn.*, 1998, vol. 3, no. 3, pp. 56–71.
4. Bounemoura, A., Fayad, B. and Niederman, L., Nekhoroshev Estimates for Steep Real-Analytic Elliptic Equilibrium Points, *Nonlinearity*, 2020, vol. 33, no. 1, pp. 1–33.
5. Bounemoura, A., Fayad, B. and Niederman, L., Super-Exponential Stability for Generic Real-Analytic Elliptic Equilibrium Points, *hal-01188980v2*, 2019.
6. Cárcamo-Díaz, D., Stability and Estimates Near Elliptic Equilibrium Points in Hamiltonian Systems and Applications, *PhD Thesis*, Universidad del Bío-Bío, Concepción, Chile, 2019.
7. Cárcamo-Díaz, D., Palacián, J. F., Vidal, C. and Yanguas, P., Formal Stability of Elliptic Equilibria in Hamiltonian Systems with Exponential Time Estimates, *Preprint*, 2019.
8. Celletti, A. and Giorgilli, A., On the Stability of the Lagrangian Points in the Spatial Restricted Problem of Three Bodies, *Celestial Mech. Dynam. Astronom.*, 1991, vol. 50, no. 1, pp. 31–58.
9. Chartier, P., Murua, A. and Sanz-Serna, J. M., Higher-Order Averaging, Formal Series and Numerical Integration III: Error Bounds, *Found. Comput. Math.*, 2015, vol. 15, no. 2, pp. 591–612.
10. Deprit, A. and Deprit-Bartholomé, A., Stability of the Triangular Lagrangian Points, *The Astronomical Journal*, 1967, vol. 72, no. 2, pp. 173–179.
11. dos Santos, F., Mansilla, J. E. and Vidal, C., Stability of Equilibrium Solutions of Autonomous and Periodic Hamiltonian Systems with n Degrees of Freedom in the Case of Single Resonance, *J. Dynam. Differential Equations*, 2010, vol. 22, no. 4, pp. 805–821.
12. dos Santos, F. and Vidal, C., Stability of Equilibrium Solutions of Autonomous and Periodic Hamiltonian Systems in the Case of Multiple Resonances, *J. Differential Equations*, 2015, vol. 258, no. 11, pp. 3880–3901.

13. Dumas, H. S., Meyer, K. R., Palacián, J. F. and Yanguas, P., Asymptotic Stability Estimates Near an Equilibrium Point, *J. Differential Equations*, 2017, vol. 263, no. 2, pp. 1125–1139.
14. Giorgilli, A., Delshams, A., Fontich, E., Galgani, L. and Simó, C., Effective Stability for a Hamiltonian System Near an Elliptic Equilibrium Point, with an Application to the Restricted Three Body Problem, *J. Differential Equations*, 1989, vol. 77, no. 1, pp. 167–198.
15. Han, Y., Li, Y. and Yi, Y., Invariant Tori in Hamiltonian Systems with High Order Proper Degeneracy, *Ann. Henri Poincaré*, 2010, vol. 10, no. 8, pp. 1419–1436.
16. Khazin, L. G., On the Stability of Hamiltonian Systems in the Presence of Resonances, *J. Appl. Math. Mech.*, 1971, vol. 35, no. 3, pp. 384–391; see also: *Prikl. Mat. Mekh.*, 1971, vol. 35, no. 3, pp. 423–431.
17. Lerman, L. M. and Markova, A. P., On Stability at the Hamiltonian Hopf Bifurcation, *Regul. Chaotic Dyn.*, 2009, vol. 14, no. 1, pp. 148–162.
18. Lochak, P., Canonical Perturbation Theory Via Simultaneous Approximation, *Russian Math. Surveys*, 1992, vol. 47, no. 6, pp. 57–133.
19. Markeev, A. P., On the Stability of a Nonautonomous Hamiltonian System with Two Degrees of Freedom, *J. Appl. Math. Mech.*, 1969, vol. 33, pp. 550–557; see also: *Prikl. Mat. Mekh.*, 1969, vol. 33, no. 3, pp. 563–569.
20. Markeev, A. P., Stability of the Triangular Lagrangian Solutions of the Restricted Three-Body Problem in the Three-Dimensional Circular Case, *Soviet Astronomy, Astronomical Journal*, 1972, vol. 15, no. 4, pp. 682–686; see also: *Astronomicheskii Zhurnal*, 1971, vol. 48, no. 4, pp. 862–868.
21. Markeev, A. P., On the Stability Problem for the Lagrange Solutions of the Restricted Three-Body Problem, *J. Appl. Math. Mech.*, 1973, vol. 37, no. 4, pp. 713–717; see also: *Prikl. Mat. Mekh.*, 1973, vol. 37, no. 4, pp. 753–757.
22. Markeev, A. P., *Libration points in celestial mechanics and astrodynamics*, Moscow: Moscow Izdatel'stvo Nauka, 1978 (Russian).
23. Meyer, K. R. and Offin, D., *Introduction to Hamiltonian dynamical systems and the n–body problem*, 3rd ed., New York: Springer–Verlag, 2017.
24. Meyer, K. R., Palacián, J. F. and Yanguas, P., Stability of a Hamiltonian System in a Limiting Case, *Regul. Chaotic Dyn.*, 2012, vol. 17, no. 1, pp. 24–35.
25. Niven, I. M., *Numbers: rational and irrational*, New York: Random House, 1961.
26. Pöschel, J., On Nekhoroshev's Estimate at an Elliptic Equilibrium, *Int. Math. Res. Not. IMRN*, 1999, vol. 1999, no. 4, pp. 203–215.
27. Schirinzi, G. and Guzzo, M., Numerical Verification of the Steepness of Three and Four Degrees of Freedom Hamiltonian Systems, *Regul. Chaotic Dyn.*, 2015, vol. 20, no. 1, pp. 1–18.
28. Sokol'skii, A. G., Proof of the Stability of Lagrange Solutions at a Critical Relation of Masses, *Soviet Astronomy Letters*, 1978, vol. 4, no. 2, pp. 79–81; see also: *Pis'ma v Astronomicheskii Zhurnal*, 1978, vol. 4, no. 3, pp. 148–152.
29. Szebehely, V., *Theory of orbits: the restricted problem of three bodies*, New York and London: Academic Press, 1967.