Searching for a Debreu's Open Gap Lemma for semiorders

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Abstract

In 1956 R.D. Luce introduced the notion of a semiorder to deal with indifference relations in the representation of a preference. During several years the problem of finding a utility function was studied until a representability characterization was found. However, there was almost no results on the continuity of the representation. A similar result to Debreu's Lemma, but for semiorders was never achieved. In the present paper we propose a characterization for the existence of a continuous representation (in the sense of Scott-Suppes) for bounded semiorders. As a matter of fact, the weaker but more manageable concept of ε-continuity is properly introduced for semiorders. As a consequence of this study, a version of the Debreu's Open Gap Lemma is presented (but now for the case of semiorders) just as a conjecture, which would allow to remove the open-closed and closed-open gaps of a subset $S \subseteq \mathbb{R}$, but now keeping the constant threshold, so that $x+1<y$ if and only if $g(x)+1<g(y)\,(x,y \in S)$.

1 Introduction

The concepts of an interval order and a semiorder were introduced by Wiener, but under a different nomenclature. [28, 35, 36] The notion of a semiorder is usually attributed to Luce (1956), who developed this field when dealing with applications in Economics and Psychology. On the other hand, the idea of an interval order was studied in depth by Fishburn in the 1970's. [23, 24, 27, 26, 25]

The use of those concepts was due to the need of developing mathematical models of measurements related to situations of intransitive indifference.

If X is a nonempty set endowed with an interval order $\prec$, the classical numerical representation (if any) consists of two real-valued functions $u, v\colon X \to \mathbb{R}$ such that $x \prec y \Leftrightarrow v(x) < u(y)$ holds for all $x, y \in X$.

In the special case of a semiorder, the representation *in the sense of Scott-Suppes* is defined by means of a single function $u\colon X \to \mathbb{R}$ such that $x \prec y \Leftrightarrow u(x)+1 < u(y)$, for every $x, y \in X$. Notice that this is actually a special kind of interval order representation through a pair (u, v) in which $v(x) = u(x)+1$ for every $x \in X$. [15, 30, 33, 34]

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In addition, when the set X is also endowed with a topology τ, it may be interesting to study the semicontinuity or continuity of the numerical representations (if any). [10, 11, 12, 16]

The problem of finding a characterization of the continuous representability of a total preorder $\precsim$ defined on a topological space (X,τ) was solved dy Gerard Debreu. Given any subset S of the real line $\mathbb{R}$, Debreu defined in 1964 a *lacuna* of S as a non-degenerate interval disjoint from S and having a lower bound and an upper bound in S, and a *gap* of S as a maximal lacuna of S. The famous *Debreu's Open Gap Lemma* allows us to construct a continuous representation of a representable and continuous total preorder. [20] It reads as follows:

Lemma 1.1. ***(Open Gap Lemma)*** *If $S \subseteq \mathbb{R}$, then there is a strictly increasing function $g\colon S \to \mathbb{R}$ such that all the gaps of $g(S)$ are open.*

However, the analogous problem for interval orders and semiorders remains still open. Important results have been obtained whenever the topology τ is *natural* as regards the interval order through an ordinal condition called *interval order separability* (see Section 2). [3] Other results on continuous representability of interval orders were achieved in [2, 4, 8, 7, 18, 19].

There is no characterization of the continuous Scott-Suppes representability. Here, the idea of a natural topology does not fit well for the general case. [3, 22] Some results about continuous representability of semiorders may be found in [13, 22, 29].

Although a semiorder is a particular case of an interval order, the difference between them is critical. In the case of a SS-representation (u,k) for a semiorder, there is a positive threshold $k > 0$ (we may assume $k = 1$ without loss of generality) such that $x \prec y$ if and only if $u(x) + k < u(y)$, whereas in the case of interval orders we have two functions (u,v) such that $x \prec y$ if and only if $v(x) < u(y)$. Therefore, the SS-representation implies a *geometrical* structure further than the topological one. To see this, notice that, for any representation (u,v) of an interval order and for any (any!) strictly increasing function $g\colon \mathbb{R} \to \mathbb{R}$, the composition $(g \circ u, g \circ v)$ is also a representation. Therefore, we are allowed to make contractions or expansions of the functions at different points. This technique was successfully used in order to achieve continuity (assuming some necessary conditions for the existence of a continuous representation). [6, 21] A similar situation holds with total preorders, for which increasing functions are used in order to achieve continuity, as it is well known since Debreu's work. [20] It is known that Debreu's procedure cannot be applied to the case of semiorders. [22] In fact, the procedure applied in [6] cannot be applied neither to SS-representations, due the fact that in this kind of representations there is a single function u used for comparisons (instead of two functions u and v).

Thus, it is not possible to compose (arbitrarily) a SS-representation with increasing functions for continuinty purposes. However, this is not possible (in general) for SS-representations. Of course, any linear contraction or expansion (that is, by means of a function $g\colon \mathbb{R} \to \mathbb{R}$ such that $g(r) = a \cdot r + b$, with $a > 0$) on $u(X)$ would keep the representation, thus, achieving another SS-representation $(g \circ u, a \cdot k)$ but now with the corresponding new threshold $k' = a \cdot k$. Nevertheless, in general any other kind of transformation w of the pair (u,k) would not represent the order structure, such that

$x \prec y$ if and only if $w(u(x)) + k' < w(u(y))$. The changes made in each threshold interval must be the same. Thus, the semiorders endow X with a rigid structure at a large scale: small changes in a neighborhood of a point may imply changes too far from the point. To see that, we retrieve the following example from [22]:

Example 1.2. Let $X = \mathbb{Q} \bigcap \big([0, 0.5) \cup [1, +\infty)\big)$. Endow X with the topology τ defined by means of the subbasis $\{z \in X : z < x\}_{x \in X} \bigcup \{y \in X : x < y\}_{x \in X}$, where $<$ is the usual strict order of the real line[1] $\mathbb{R}$. Let $\prec$ denote the semiorder defined on X by declaring that $a \prec b \iff a + 1 < b$ $(a, b \in X)$. Thus, the inclusion i (where $i(x) = x$, for any $x \in X$) is a SS-representation. It can be proved that i fails to be continuous at $x = 1$. To see that, notice that the sequence $(x_n) = (0.5 - \frac{1}{n+1})$ converges to 1 in (X, τ). If we try to modify the function i at the point 1 (in a neighborhood of 1) in order to achieve continuity, this modification implies strong changes in the subset $[1'5, 2]$ too.

To see that, notice that if we want to warrant continuity (of our new function u) at 1, then it should happen that the limit $\lim_{n \to +\infty} u(x_n)$ equals $u(1)$. Now, observe that $x_n \prec 1.5 \precsim 1$, and also $x_n \prec 1.6 \precsim 1$, so that $u(x_n) + 1 < u(1.5) \leq u(1) + 1$ and also $u(x_n) + 1 < u(1.6) \leq u(1) + 1$, for every $n \in \mathbb{N}$. Since u is, by hypothesis, continuous, taking limits we get $u(1) + 1 = u(1.5) = u(1.6)$. Thus, this modification implies the contraction of the interval $[1'5, 2]$ to a point, which makes impossible the construction of the desired representation (here, notice that $1'5 \prec_0 1'6$, so it must hold for any representation u that $u(1'5) < u(1'6)$.

The aforementioned limitation in the (arbitrary) use of increasing functions is a huge handicap in the search of a continuous SS-representation. Hence, some advances in the study of continuous representability of interval orders cannot be translated to semiorders.

However, as a counterpart, we are able to introduce a new concept that generalizes and approximates the idea of continuity for semiorders: the *ε-continuity*. This new concept is useless for interval orders, but it seems crucial in the study of some other representation with a *geometrical* component (e.g. a threshold) such as in the case of SS-representations.

Dealing with real functions, it is easy to measure the length of a jump-discontinuity, so that we may aspire to construct functions such that the length of the jump-discontinuities is smaller than a desired constant. However, working with representations of total preorders or interval orders, this is trivially satisfied.

Example 1.3. Let $\prec$ be the interval order on $S = [0, 1] \cup (1'5, 3] \subseteq \mathbb{R}$ defined by $x \prec y \iff 2x < y$. Endow now the set S with the topology $\tau_{\leq}$ defined by the Euclidean order $\leq$ on S. Then, the pair of fuctions (u, v) where $u(x) = x$ and $v(x) = 2x$ is a representation of the interval order and both are continuous on the whole set with the exception of the point $x = 1$. In fact, at $x = 1$ the function u has a jump-discontinuity of length $0'5$ and v has another one of length 1.

However, for any $n \in \mathbb{N}$, it is possible to find a representation (u', v') whose biggest gap is smaller than $\frac{1}{n}$. For that, it is enough to define the pair $(u' = \frac{1}{n} \cdot u, v' = \frac{1}{n} \cdot v)$. Nevertheless, as we said before, this is quite trivial.

[1]Observe that τ does not coincide with the induced topology $\tau_{u|_X}$ inherited from the usual topology τ_u on $\mathbb{R}$. As a matter of fact $\tau \subsetneq \tau_{u|_X}$.

On the other hand, given the semiorder $x \prec y \iff x+1 < y$ on the same space $(S, \tau_{\leq})$, the pair $(u,1)$ is a SS-representation that fails to be continuous at $x=1$. The length of that jump-discontinuity is $0'5$ (i.e. the ratio with respect to the constant threshold $k=1$ is $\frac{1}{2}$). But now, if we argue as before and construct a new function $u' = \frac{1}{n} \cdot u$ in order to reduce the length of the jump-discontinuity to $\frac{0'5}{n}$, then the threshold is reduced too, so that the ratio between the jump and the threshold is the same: $\frac{0'5/n}{1/n} = \frac{1}{2}$.

In the case of SS-representations there is a threshold k (we may assume that $k=1$), it is possible to compare the length of each jump-discontinuity with the value $k = 1$. Therefore, it makes perfect sense to say that a semiorder is r-continuous (for a positive value $r \in \mathbb{R}$) if there exists a SS-representation $(u,1)$ such that the length of each jump-discontinuity is bounded by this constant r. It order to approximate the idea of continuity, we may say that a semiorder is ε-continuous if for any $\varepsilon > 0$ there exists a SS-representation $(u,1)$ such that the length of each junp-discontinuity is bounded by this value ε.

In the present paper we introduce the concept of ε-continuity as a tool when dealing with semiorders that fail to be continuously representable. As a matter of a fact, through this idea we propose a characterization of the bounded semiorders which are continuously representable.

The structure of the paper goes as follows: After a section of preliminaries, necessary conditions for the existence of a continuous SS-representation are introduced. Then, in Section 4, the image subset $u(X)$ is studied for a given SS-representation $(u,1)$ of a semiorder that satisfies the aforementioned necessary conditions. In next Section 5, the new concept of ε-continuity for semiorders is defined and justified. By means of this new concept, some conjectures on the continuous SS-representability of semiorders are presented.

2 Preliminaries

From now on X will denote a nonempty set.

Definition 2.1. A *binary relation* $\mathscr{R}$ on X is a subset of the Cartesian product $X \times X$. Given two elements $x, y \in X$, the notation $x\mathscr{R}y$ expresses that the pair (x,y) belongs to $\mathscr{R}$.

Associated to a binary relation $\mathscr{R}$ on a set X, we consider its *negation* (respectively, its *transpose*) as the binary relation $\mathscr{R}^c$ (respectively, $\mathscr{R}^t$) on X given by $(x,y) \in \mathscr{R}^c \Leftrightarrow (x,y) \notin \mathscr{R}$ for every $x, y \in X$ (respectively, given by $(x,y) \in \mathscr{R}^t \Leftrightarrow (y,x) \in \mathscr{R}$, for every $x, y \in X$). We also define the *adjoint* $\mathscr{R}^a$ of the given relation $\mathscr{R}$, as $\mathscr{R}^a = (\mathscr{R}^t)^c$.

A binary relation $\mathscr{R}$ defined on a set X is said to be:

(i) *reflexive* if $x\mathscr{R}x$ holds for every $x \in X$,

(ii) *irreflexive* if $\neg(x\mathscr{R}x)$ holds for every $x \in X$,

(iii) *symmetric* if $\mathscr{R}$ and $\mathscr{R}^t$ coincide,

(iv) *antisymmetric* if $\mathscr{R} \cap \mathscr{R}^t \subseteq \Delta = \{(x,x) : x \in X\}$,

(v) *asymmetric* if $\mathscr{R} \cap \mathscr{R}^t = \varnothing$,

(vi) *total* if $\mathscr{R} \cup \mathscr{R}^t = X \times X$,

(vii) *transitive* if $x\mathscr{R}y \wedge y\mathscr{R}z \Rightarrow x\mathscr{R}z$ for every $x,y,z \in X$.

Definition 2.2. A *preorder* $\precsim$ on X is a binary relation which is reflexive and transitive. An antisymmetric preorder is said to be an *order*. A *total preorder* $\precsim$ on a set X is a preorder such that if $x,y \in X$ then $(x \precsim y) \vee (y \precsim x)$ holds. If $\precsim$ is a preorder on X, then as usual we denote the associated *asymmetric* relation by $\prec$ and the associated *equivalence* relation by $\sim$ and these are defined by $x \prec y \Leftrightarrow (x \precsim y) \wedge \neg(y \precsim x)$ and $x \sim y \Leftrightarrow (x \precsim y) \wedge (y \precsim x)$.

Definition 2.3. An *interval order* $\prec$ is an asymmetric binary relation on X such that $(x \prec y) \wedge (z \prec t) \Rightarrow (x \prec t) \vee (z \prec y)$ $(x,y,z,t \in X)$. Its symmetric part is denoted by $\precsim$, so that $a \precsim b \Leftrightarrow \neg(b \prec a)$. The binary relation $\sim$ defined by $a \sim b \Leftrightarrow (a \precsim b) \wedge (b \precsim a)$ is said to be the *indifference* associated to $\prec$.

Remark 2.4. It is well known that given an interval order $\prec$ on a set X, the associated relations $\precsim$ and $\sim$ may fail to be transitive. [23, 24, 27, 30, 33]

Definition 2.5. An interval order $\prec$ is said to be a *semiorder* if $(x \prec y) \wedge (y \prec z) \Rightarrow (x \prec w) \vee (w \prec z)$ $(x,y,z,w \in X)$. A semiorder $\prec$ is said to be *typical* if $\precsim$ is *not* a total preorder on X.

Through the next definition, we introduce the notion of representability for different kinds of orderings, that makes possible to convert qualitative scales into quantitative ones.

Definition 2.6. A total preorder $\precsim$ on X is called *representable* if there is a real-valued function $u\colon X \to \mathbb{R}$ that is order-preserving, so that, for every $x,y \in X$, it holds that $x \precsim y \Leftrightarrow u(x) \leq u(y)$. The map u is said to be a *utility function* for $\precsim$.

An interval order $\prec$ defined on X is said to be *representable* (as an interval order) if there exist two real valued maps $u,v\colon X \longrightarrow \mathbb{R}$ such that $x \prec y \Leftrightarrow v(x) < u(y)$ $(x,y \in X)$. The pair (u,v) is called a *utility pair* representing $\prec$.

A semiorder $\prec$ defined on X is said to be *representable in the sense of Scott and Suppes* if there exists a real-valued map $u\colon X \to \mathbb{R}$ (again called a *utility function*) such that $x \prec y \Leftrightarrow u(x) + 1 < u(y)$ $(x,y \in X)$. [33]

In this case, the pair $(u,1)$ is said to be a *Scott-Suppes representation* of $\prec$.

Remark 2.7. If (u,v) represents an interval order $\prec$ defined on a set X, it is straightforward to see that $u(x) \leq v(x)$ for every $x \in X$. And the non-negative real number $v(x) - u(x)$ is said to be the *discrimination threshold for the element* $x \in X$. In the case of a semiorder that is representable in the sense of Scott and Suppes, the discrimination thresholds are all equal to 1.

There exist interval orders that fail to be representable (as interval orders). Also, there exist semiorders that are not representable in the sense of Scott and Suppes, not even as an interval order. [17, 32]

Definition 2.8. Associated to an interval order $\prec$ defined on a nonempty set X, we shall consider three new binary relations. [1, 25, 23]

These binary relations are said to be the *traces* of $\prec$. They are respectively denoted by $\prec^*$ (*left trace*), $\prec^{**}$ (*right trace*) and $\prec^0$ (*main trace*), and defined as follows: $x \prec^* y \Leftrightarrow x \prec z \precsim y$ for some $z \in X$, and similarly $x \prec^{**} y \Leftrightarrow x \precsim z \prec y$ for some $z \in X$ $(x, y \in X)$. In addition, $x \prec^0 y \Leftrightarrow (x \prec^* y) \vee (x \prec^{**} y)$ $(x, y \in X)$.

Remark 2.9. We denote $x \precsim^* y \Leftrightarrow \neg(y \prec^* x)$, $x \sim^* y \Leftrightarrow x \precsim^* y \precsim^* x$, $x \precsim^{**} y \Leftrightarrow \neg(y \prec^{**} x)$ and $x \sim^{**} y \Leftrightarrow x \precsim^{**} y \precsim^{**} x$, and finally $x \precsim^0 y \Leftrightarrow (x \precsim^* y) \wedge (x \precsim^{**} y)$ and $x \sim^0 y \Leftrightarrow (x \precsim^0 y) \wedge (y \precsim^0 x)$ $(x, y \in X)$. Both the binary relations $\precsim^*$ and $\precsim^{**}$ are total preorders on X. Moreover, the indifference relation $\sim$ associated to the interval order $\prec$ is transitive if and only if $\precsim^*$, $\precsim^{**}$ and $\precsim$ coincide. In this case $\precsim$ is actually a total preorder on X. [5, 23, 25, 31, 32]

Furthermore, given an interval order $\prec$ on X, it holds that $\prec$ is actually a semiorder if and only if $\precsim^0$ is a total preorder on X.

Let us recall now a characterization of the numerical representability of semiorders.

Definition 2.10. Let X be a nonempty set. An interval order (e.g. also a semiorder) $\prec$ defined on X is said to be *interval order separable* if there exists a countable subset $D \subseteq X$ such that for every $x, y \in X$ with $x \prec y$ there exists $d \in D$ such that $x \precsim^* d \prec y$.

A semiorder $\precsim$ defined on X is said to be *regular with respect to sequences* if for any $x, y \in X$, and sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subseteq X$, none of the situations $x \prec \cdots \prec x_{n+1} \prec x_n \prec \cdots \prec x_1$ and $y_1 \prec \cdots \prec y_n \prec y_{n+1} \prec \cdots \prec y$ may occur. It is said to be *bounded* if there is no strictly increasing or decreasing infinite sequences, i.e. there is no sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $\cdots \prec x_{n+1} \prec x_n \prec \cdots \prec x_1$ or $x_1 \prec \cdots \prec x_n \prec x_{n+1} \prec \cdots$.

The following result is proved. [14, 15]

Theorem 2.11. *Let X be a nonempty set. Let $\prec$ be a typical semiorder defined on X. Then, $\prec$ is representable in the sense of Scott and Suppes if and only if it is both interval order separable and regular with respect to sequences.*

Remark 2.12. If $(u, 1)$ is a SS-representation of a semiorder $\prec$, then u is strictly increasing with respect to the main trace $\precsim_0$ (see Lemma 4 in [6]).

3 Continuous SS-representability of semiorders

Let (X, τ) stand for a topological space (a set X with a topology τ).

Definition 3.1. Let $\prec$ denote an asymmetric binary relation on X. Given $a \in X$ the sets $L(a) = \{t \in X : t \prec a\}$ and $U(a) = \{t \in X : a \prec t\}$ are called, respectively, the *lower and upper contours* of a relative to $\prec$. We say that $\prec$ is *τ-continuous* if for each $a \in X$ the sets $L(a)$ and $U(a)$ are τ-open.

The following facts are proved. [6, 22]

Theorem 3.2. *Let $\prec$ be an interval order defined on a set X. Then the indifference $\sim^0$ associated to the main trace is an equivalence relation.*

Definition 3.3. Let (X,τ) be a topological space. Let $\prec$ be an interval order on X. The topology τ is said to be *compatible with respect to the indifference of the main trace of* $\prec$ if $x \sim^0 y \Rightarrow (x \in \mathscr{O} \iff y \in \mathscr{O})$ holds true for every $x, y \in X$ and every τ-open subset $\mathscr{O} \in \tau$.

Remark 3.4. Notice that, according to the idea before, elements that are indistinguishable with respect to $\prec$ (because they play the same role on $(X,\prec)$) should also be indistinguishable from a topological point of view.

In particular, in the main case in which $x \sim_0 y \iff x = y$, i.e. when X coincides with the quotient set $X/\sim_0$, the topology is always compatible.

With respect to continuity, the following result was introduced in [22]:

Lemma 3.5. *Let (X,τ) be a topological space endowed with a semiorder $\prec$. Assume that $\prec$ is representable in the sense of Scott and Suppes by means of a pair $(u,1)$ with u continuous. Then the following properties hold true:*

(a) *The semiorder $\prec$ is τ-continuous.*

(b) *If a net $(x_j)_{j\in J} \subseteq X$[1] converges to two points $a, b \in X$, then $a \sim^0 b$.*

(c) *If a net $(x_j)_{j\in J} \subseteq X$ converges to $a \in X$, and $b, c \in X$ are such that $x_j \prec b \precsim a$ and also $x_j \prec c \precsim a$ for every $j \in J$, then $b \sim^0 c$.*

(d) *If a net $(x_j)_{j\in J} \subseteq X$ converges to $a \in X$, and $b, c \in X$ are such that $a \precsim b \prec x_j$ and also $a \precsim c \prec x_j$ for every $j \in J$, then $b \sim^0 c$.*

Throughout the paper, we shall refer to these conditions $(a)-(d)$ by (NC) (*necessary conditions*).

Next result was also proved in [6] (see Remark 8 in [6]).

Proposition 3.6. *Let (X,τ) be a topological space endowed with a semiorder $\prec$. Assume that τ is compatible with respect to the indifference of the main trace of $\prec$. Suppose also that $\prec$ is representable in the sense of Scott and Suppes by means of a pair $(u,1)$ with u continuous. Then the total preorder $\precsim^0$ is τ-continuous.*

A partial result was achieved in [6] (namely Theorem 3.7 below), that guarantees the continuous representability of a semiorder but as an interval order (that is, through a pair (u,v) of continuous real-valued functions) whenever the condition of compatibility between the topology τ and the indifference $\sim^0$ is satisfied.

Theorem 3.7. *Let (X,τ) be a topological space. Let $\prec$ be a representable semiorder on X. Assume that $\prec$ satifies the aforementioned necessary conditions (NC) and, in addition, the topology τ is compatible with respect to the indifference of the main trace of $\prec$. If $\precsim^0$ is τ-continuous, then $\prec$ admits a representation as an interval order, through a pair (u,v) of continuous real-valued functions.*

[1] J denotes here a directed set of indices. Since this does not lead to confusion, we will use the same notation '$<$' of the order on the real numbers than for the partial order on the set of indices J.

3.1 A new necessary condition for the existence of a continuous SS-representation

From now on, we shall assume that the topology of the space is compatible with respect to the indifference of the main trace of the semiorder (e.g. the quotient set $X/\sim_0$ coincides with X).

There is a simple reason that clarifies why the semiorder in Example 5 in [22] fails to be continuously representable: there is, at least, another necessary condition that must be satisfied. This new condition tries to explain the *rigid* structure of the semiorder and its representation. In order to study that, first we introduce the new concept of *adjoint nets*.

Definition 3.8. Let (X,τ) be a topological space endowed with a semiorder $\prec$. Let $(x_j)_{j\in J}$ and $(y_k)_{k\in K}$ be two nets on X. We shall say that these nets are *adjoint nets*, and we denote it by $(x_j) \preccurlyeq (y_k)$, if one of the following conditions hold:

Condition 1: If none of these two nets is constant, then the following both conditions are satisfied:

$(1.i)$ for each $j_0 \in J$ there exists $k_0 \in K$ s.t. $x_{j_0} \prec y_k$ for any $k > k_0$,

$(1.ii)$ for each $k_0 \in K$ there exists $j_0 \in J$ s.t. $y_{k_0} \precsim x_j$ for any $j > j_0$.

Condition 2: If one (and only one) of the nets is constant, that is $y_k = b$ for all $k \in K$, where b is called *adjoint point*, then any of the following conditions is satisfied:

$(2.i)$ $x_j \prec b$ for each $j \in J$ and the net converges to $a \in X$ such that $b \precsim a$,

$(2.ii)$ $b \prec x_j$ for each $j \in J$ and the net converges to $a \in X$ such that $a \precsim b$.

Analogously, for each $n \in \mathbb{N}$ we define the *n-adjoint nets*, and we denote them by $(x_j) \preccurlyeq^n (y_k)$, if there exists a chain of length n of adjoint nets: $(x_j) \preccurlyeq (a_{i_1}) \preccurlyeq \ldots \preccurlyeq (a_{i_{n-1}}) \preccurlyeq (y_k)$. For any m such that $-m \in \mathbb{N}$, we also say that $(x_j)_{j\in J}$ and $(y_k)_{k\in K}$ are *m-adjoint nets*, and we denote them by $(x_j) \preccurlyeq^m (y_k))$ if $(y_k) \preccurlyeq^{-m} (x_j)$.

Remark 3.9. In the case in which both nets are constant, i.e. $(x_j)_{j\in J} = (a)$ and $(y_k)_{k\in K} = (b)$, then it holds that $a \prec b$ or $b \precsim a$ (or, dually, $b \prec a$ or $a \precsim b$). In any case, given a SS-representation $(u,1)$, nothing can say about the distance on $\mathbb{R}$ between $u(a)$ and $u(b)$. This distance is what motivates Definition 3.8, as the following Lemma 3.10 shows:

Lemma 3.10. *Let $\prec$ be a semiorder defined on a topological space (X,τ) and let $(u,1)$ be a continuous representation. If $(x_j)_{j\in J}$ and $(y_k)_{k\in K}$ are n-adjoint nets, then* $\lim_{j\in J} u(x_j) + n = \lim_{k\in K} u(y_k)$.

Proof. We prove it by induction on $n \in \mathbb{N}$. For $n = 1$, if $(x_j) \preccurlyeq (y_k)$ and none of them is a constant, then for any $j_0 \in J$ there exists $k_0 \in K$ such that $x_{j_0} \prec y_k$ for each $k > k_0$, so $u(x_{j_0}) + 1 < \lim(u(y_k))$ for each $j_0 \in J$. Hence, $\lim(u(x_j)) + 1 \leq \lim(u(y_k))$.

Similarly, for any $k_0 \in K$ there exists $j_0 \in J$ such that $y_{k_0} \precsim x_j$ for each $j > j_0$, so $u(y_{k_0}) \leq \lim(u(x_j)) + 1$ for each $k_0 \in K$. Hence, $\lim(u(y_k)) \leq \lim(u(x_j)) + 1$. So we have proved that $\lim(u(x_j)) + 1 = \lim(u(y_k))$.

If one of them is a constant net (suppose $y_k = b$ for all $k \in K$), then $x_j \prec b$ for any $j \in J$ and there exists $\lim(x_j) \in X$ such that $b \precsim \lim(x_j)$. So it holds that $\lim(u(x_j)) + 1 \leq \lim u(b) \leq (u(x_j)) + 1$. Hence, $\lim(u(x_j)) + 1 = \lim(u(y_k))$. Similarly if the case $(2.ii)$ of Definition 3.8 holds.

Now, suppose that the lemma is true for a fixed $n \in \mathbb{N}$. If $(x_j)_{j\in J}$ and $(y_k)_{k\in K}$ are two $n+1$-adjoint nets, there exists another net $(z_r)_{r\in R}$ such that $(x_j) \preccurlyeq^n (z_r) \preccurlyeq (y_k)$. So, by the induction hipothesis, it holds that $\lim(u(x_j)) + n - 1 = \lim(u(z_r))$ and $\lim(u(z_r)) + 1 = \lim(u(y_k))$. Hence, it holds that $\lim(u(x_j)) + n = \lim(u(y_k))$.

We proceed analogously if one of them is a constant net. □

Remark 3.11. Notice that, in the proof, we only used the continuity in case of constant nets. Thus, for any two n-adjoint nets $(x_j)_{j\in J}$ and $(y_k)_{k\in K}$ (connected by adjoint nets such that none of them is constant) it holds that $\lim_{j\in J} u(x_j) + n = \lim_{k\in K} u(y_k)$, even without requiring continuity for u.

Through the following theorem we introduce a new necessary condition (we will denote it by (e)) for the continuous SS-representability of semiorders. This condition tries to describe the rigid structure of semiorders.

Theorem 3.12. *Let $\prec$ be a continuously representable semiorder defined on a topological space (X, τ). Let $(x_j)_{j\in J}$ and $(y_k)_{k\in K}$ be two nets such that they converge to the same point a in X and $(w_r)_{r\in R}$ (respectively $(z_s)_{s\in S}$) two n-adjoint nets of $(x_j)_{j\in J}$ (respectively, of $(y_k)_{k\in K}$) for some $n \in \mathbb{Z} - \{0\}$.*

If there are two elements $b, c \in X$ such that $w_r \prec^0 b, c \prec^0 z_s$ (for each $r \in R, s \in S$), then $b \sim^0 c$.

Proof. Let $(u, 1)$ be a continuous SS-representation. From Lemma 3.10 it follows that $\lim(u(x_j)) + n = \lim(u(w_r))$ and $\lim(u(y_k)) + n = \lim(u(z_s))$ in $\mathbb{R}$. The nets $(x_j)_{j\in J}$ and $(y_k)_{k\in K}$ converge to the same point a, so if u is a continuous representation it holds that $\lim(u(z_s)) = \lim(u(y_k)) + n = u(a) + n = \lim(u(x_j)) + n = \lim(u(w_r))$. Hence, for any $b, c \in X$ such that $w_r \prec^0 b, c \prec^0 z_s$ (for each $r \in R, s \in S$) it holds that $\lim(u(w_r)) \leq u(b), u(c) \leq \lim(u(z_s)) = \lim(u(w_r))$, so $u(b) = u(c)$ and then, (see Lemma 1 in [22]) $b \sim^0 c$. □

Remark 3.13. As a matter of a fact, conditions (c) and (d) are now particular cases of this new condition (e), in which $n = 1$. Besides, this new condition is independent of (a) and (b) (as well as it generalizes conditions (c) and (d)). To see that, just notice that Examples 2, 3 and 4 introduced in [22] satisfy this new contidion (e), but that is not the case of Example 5 in [22]. From now on, we refer to all these necessary conditions as (NC).

Hence, these necessary conditions imply a rigid structure on the set, with a geometrical component due to the existence of that constant threshold $k = 1$.

4 The implications of the necessary conditions on the SS-representation

This section is devoted to show that, under the assumption of the necessary conditions, the function u of the SS-representations has a particular structure. This holds for any representation, continuous or not. Again, we will assume that the topology of the space is compatible with respect to the indifference of the main trace $\precsim^0$. Since we are imposing some conditions on the semiorder structure $(X, \precsim)$, these conditions are reflected in the subset $u(X)$ of $(\mathbb{R}, \leq)$ too. In the present section we shall take advantage of these features of $u(X)$ in order to try to achieve continuity.

First, we shall take into account the following results, which are well known [9, 20] (see also Corollary 1 and Proposition 1 in [21]).

Definition 4.1. Let X be a nonempty set and $u\colon X \to \mathbb{R}$ a real function on X. Given a gap $(a,b]$ (respectively, $[a,b)$) of $u(X)$, we will say that a is the *end-point of* $(a,b]$ *with respect to u* (respectively, b is the *end-point of* $[a,b)$ *with respect to u*). For an open gap (a,b), a and b are the *end-points with respect to u*, whereas a closed gap $[a,b]$ has no *end-points with respect to u*.

Corollary 4.2. *Let (X,τ) be a topological space endowed with a τ-continuous total preorder $\precsim$. Given any representation u of the total preorder, then u is continuous at every point of X, excluding the inverse images of the end-points with respect to u of some gaps (of $u(X)$) that are not closed neither open, that is, excluding some $x \in X$ such that $(u(x) = a, b]$ or $[b, u(x) = a)$ is a gap of $u(X) \subseteq \mathbb{R}$.*

Definition 4.3. Let (X,τ) be a topological space endowed with a τ-continuous total preorder $\precsim$. Given a representation u of the total preorder, we will say that a gap of $u(X) \subseteq \mathbb{R}$ is a *bad gap* if the function u is not continuous at the inverse images of the end-point of the gap.

We will refer to the length of the bad gap as the *length of the jump-discontinuity*.

Remark 4.4. From Definition 4.3 above and using Corollary 4.2, notice that given any representation u of a τ-continuous total preorder on (X,τ), then any closed or open gap $([a,b],(a,b) \subseteq \mathbb{R})$ of $u(X)$ is not a bad gap.

Proposition 4.5. *Let $\precsim$ be a τ-continuous total preorder and u a representation. If u is discontinuous at $a \in X$, then there is a gap $(u(a), r]$ or $[r, u(a))$.*

Now, we retrieve Proposition 2 of [21]:

Proposition 4.6. *Let (X,τ) be a topological space endowed with a τ-continuous total preorder $\precsim$. Let u be a representation of $\precsim$, $x \in X$, $\varepsilon > 0$ such that $(u(x), u(x)+\varepsilon]$ is a gap of $u(X)$ and $u(x)$ is not the right end-point of a gap of $u(X)$. If u fails to be continuous at x then there exists a net $(x_i)_{i\in I}$ in X convergent to x and such that $u(x)+\varepsilon < u(x_i)$ for all $i \in I$.*

Let us focus now on condition (a).

Proposition 4.7. *Let $\prec$ be a τ-continuous semiorder on (X,τ). Let $(u,1)$ be a SS-representation. If u is discontinuous at $a \in X$, then one of the following situations holds:*

(i) $(u(a),r]$ *is a gap and* $u(X) \cap (u(a)+1, r+1] = \emptyset$,

(ii) $[r,u(a))$ *is a gap and* $u(X) \cap [r-1, u(a)-1) = \emptyset$.

Proof. We argue by contradiction. If there was an element $u(b) \in (u(a)+1, r+1]$, then it would hold that $u(a)+1 < u(b)$, thus $a \prec b$. Since $L_{\prec}(b)$ is open, that implies that there is no discontinuity at a, arriving at a contradiction.

We use a dual argument for a gap $[r,u(a))$. □

Let us focus now on conditions $(c)-(d)$.

Proposition 4.8. *Let $\prec$ be a semiorder on (X,τ) satisfying the necessary conditions (c) and (d). Let $(u,1)$ be a SS-representation. If u is discontinuous at $a \in X$, then one of the following situations holds:*

(i) $[r,u(a))$ *is a gap and* $u(X) \cap [r+1, u(a)+1]$ *has at most one point* $s \in \mathbb{R}$,

(ii) $(u(a),r]$ *is a gap and* $u(X) \cap [u(a)-1, r-1]$ *has at most one point* $s \in \mathbb{R}$.

Proof. Suppose there is a gap $[r,u(a))$ such that u fails to be continuous at a. Then, by Proposition 4.6, there is a net $(x_i)_{i\in I}$ in X, convergent to a and such that $u(x_i) < r < u(a)$ for all $i \in I$. For any two points $u(b),u(c) \in [r+1, u(a)+1]$, it holds that $u(x_i)+1 < r+1 < u(b), u(c) \leq u(a)+1$. Thus $x_i \prec b, c$ for any $i \in I$ as well as $b, c \precsim a$. Hence, by condition (c) it holds that $b \sim_0 c$ and, since u also represent the trace $\precsim_0$, we conclude that $u(b) = u(c)$, that is $u(X) \cap [r+1, u(a)+1]$ has at most one point.

A similar argument holds for a gap $(u(a),r]$. □

The following result is well known in literature. [9]

Proposition 4.9. *Let $\precsim$ be a continuous and perfectly separable total preorder. Let f be a utility function of $\precsim$. Then, f has at most a countably infinite number of discontinuities (which are jump-discontinuities).*

Finally, let us study now the implications of condition (e).

Proposition 4.10. *Let $\prec$ be a semiorder on (X,τ) satisfying the necessary condition (e). Let $(u,1)$ be a SS-representation. Assume that u is discontinuous at $a \in X$, i.e. there is a gap $[r,u(a))$ (or $(u(a),r]$). Thus, there is a net $(u(y_i))_{i\in I}$ converging to r in $\mathbb{R}$ and there is also another net $(u(x_j))_{j\in J}$ (it may be constant, i.e. $u(x_j) = u(a)$ for any $j \in J$) converging to $u(a)$ in $\mathbb{R}$.*

If there exist two nets $(w_r)_{r\in R}$ and $(z_s)_{s\in S}$ such that $(w_r)_{r\in R}$ is n-adjoint to $(y_i)_{i\in I}$ and $(z_s)_{s\in S}$ is n-adjoint to $(x_j)_{j\in J}$ (for some $n \in \mathbb{Z} - \{0\}$), then $u(X) \cap [r+n, u(a)+n]$ has at most one point $s \in \mathbb{R}$.

Proof. Suppose there is a gap $[r, u(a))$ such that u fails to be continuous at a. Then, by Proposition 4.6, there is a net $(y_i)_{i\in I}$ in X convergent to a and such that $u(y_i) < r < u(a)$ for all $i \in I$. Thus, we have two nets $(y_i)_{i\in I}$ and $(x_j)_{j\in J}$ both converging to a such that $(u(y_i))_{i\in I}$ converges to r and $(u(x_j))_{j\in J}$ converges to $u(a)$.

If there exist two nets $(w_r)_{r\in R}$ and $(z_s)_{s\in S}$ such that $(w_r)_{r\in R}$ is n-adjoint to $(y_i)_{i\in I}$ and $(z_s)_{s\in S}$ is n-adjoint to $(x_j)_{j\in J}$ (for some $n \in \mathbb{Z} - \{0\}$), then by Lemma 3.10, it holds that $\lim u(y_i) + n = r + n = \lim u(w_r)$ as well as $\lim u(x_j) + n = u(a) + n = \lim u(z_s)$.

For any two points $u(b), u(c) \in [r+n, u(a)+n]$, it holds that $w_r \prec_0 b, c \prec_0 z_s$ for any $r \in R$ and $s \in S$. Hence, by Theorem 3.12 $b \sim_0 c$ is satisfied, so $u(b) = u(c)$. Thus, we conclude that $u(X) \cap [r+n, u(a)+n]$ has at most one point $s \in \mathbb{R}$.

A similar argument holds for a gap $(u(a), r]$. □

The following corollary tries to summarize the structure of any SS-representation (i.e.not necessarily continuous) of a continuously representable semiorder. That will be made using the necessary conditions. For the sake of brevity, we include a sketch of proof instead of a more detailed one.

Corollary 4.11. *Let $\prec$ be a continuously representable semiorder on (X, τ), i.e. satisfying the necessary conditions (NC). Let $(u, 1)$ be a SS-representation. Suppose that there is a discontinuity at a point a such that $[r, u(a))$ (or $(u(a), r]$) is a gap.*

Then, $u(X) \cap [r+1, u(a)+1]$ (or $u(X) \cap [u(a)-1, r-1]$, respectively) has at most one point $s \in \mathbb{R}$ and one of the following situations holds:

(a_1) *Depending on the existence of that point s, it holds that $[r+1, u(a)+1]$ (there is no s), $(r+1, u(a)+1]$ ($s = r+1$) or $[r+1, u(a)+1)$ ($s = u(a)+1$) is a gap, or $[r+1, u(a)+1]$ is the union of two consecutive gaps $[r+1, s) \cup (s, u(a)+1]$. In that case, $u(X) \cap [r+2, u(a)+2]$ (resp. $u(X) \cap [u(a)-2, r-2]$) has at most one point $s' \in \mathbb{R}$ and we will continue applying these cases (a_1) or (b_1), but now on $[r+2, u(a)+2]$ (resp. $[u(a)-2, r-2]$).*

(b_1) *If the case (a_1) does not hold, then there exist $\gamma_l, \gamma_r \geq 0$ such that at least one of them is strictly positive, such that $u(X) \cap [r+1-\gamma_l, u(a)+1+\gamma_r]$ (or $u(X) \cap [u(a)-1-\gamma_l, r-1+\gamma_r]$, respectively) has at most that point s'. In that case, $[r+2, u(a)+2]$ (resp. $[u(a)-2, r-2]$) may contain more than one point. In fact, if there exists that point s' then $(s'+1, u(a)+2]$ may be nonempty if $\gamma_r > 0$, dually, $[r+2, s'+1)$ may be nonempty if $\gamma_l > 0$.*

Moreover, $u(X) \cap [r-1, u(a)-1) = \emptyset$ (or $u(X) \cap (u(a)+1, r+1] = \emptyset$, resp.). Here, again, one of the following situations holds:

(a_2) *If $[r-1, u(a)-1]$ or $[r-1, u(a)-1)$ (resp. with $[u(a)+1, r+1]$ or $[u(a)+1, r+1)$) is a gap (depending on the existence of that adjoint point $u(a)-1$ (resp. $u(a)+1$)), then $u(X) \cap [r-2, u(a)-2]$ (resp. $u(X) \cap [u(a)+2, r+2]$) has at most the adjoint point $u(a)-2$ (resp. $u(a)+2$), and we will continue applying these cases (a_2) or (b_2) but now on $[r-2, u(a)-2]$ (resp. $[u(a)+2, r+2]$).*

(b_2) *If the case (a_2) before does not hold, then there exist $\gamma_l, \gamma_r \geq 0$ such that at least one of them is strictly positive, such that $u(X) \cap [r-1-\gamma_l, u(a)-1+\gamma_r]$ (or*

$u(X)\cap[u(a)+1-\gamma_l, r+1+\gamma_r]$, respectively) has at most that point $u(a)-1$. In that case, $[r-2,u(a)-2]$ (resp. $[u(a)+2,r+2]$) may contain more than one point. In fact, if there exists that point $u(a)-1$, then $[r-2,u(a)-2]$ may be nonempty if $\gamma_l>0$.

Proof. First, if there is a discontinuity at a point a such that $[r,u(a))$ (or $(u(a),r]$) is a gap, then there is a net $(u(y_i))_{i\in I}$ converging to r in $\mathbb{R}$ and there is another net $(u(x_j))_{j\in J}$ (it may be constant, i.e. $u(x_j)=u(a)$ for any $j\in J$) converging to $u(a)$ in $\mathbb{R}$.

The first statement before points (a_1) and (b_1), relative to the possible existence of a unique point in $u(X)\cap[r+1,u(a)+1]$ (or $u(X)\cap[u(a)-1,r-1]$, respectively), is proved in Proposition 4.8.

If $[r+1,u(a)+1]$ (resp. $[u(a)-1,r-1]$) is as described in case (a_1), then notice that there exist adjoint nets $(z_s)_{s\in S}$ and $(w_r)_{r\in R}$ such that $(y_i)\precsim(z_s)$ and $(x_j)\precsim(w_r)$. Hence, by Proposition 4.10 we deduce that $u(X)\cap[r+2,u(a)+2]$ (resp. $[u(X)\cap u(a)-2,r-2]$) contains at most one point s'. We will continue arguing on $[r+2,u(a)+2]$ (resp. $[u(a)-2,r-2]$).

If $[r+1,u(a)+1]$ (resp. $[u(a)-1,r-1]$) is as described in case (b_1), i.e. there is an $\varepsilon>0$ such that $u(X)\cap[r+1-\varepsilon,u(a)+1]$ or $u(X)\cap[r+1,u(a)+1+\varepsilon]$ (or $u(X)\cap[u(a)-1-\varepsilon,r-1]$ or $u(X)\cap[u(a)-1,r-1+\varepsilon]$, respectively) has at most one point, then there are not adjoint nets with respect to $(y_i)_{i\in I}$ and $(x_j)_{j\in J}$. In that case, $[r+2,u(a)+2]$ (resp. $[u(a)-2,r-2]$) may contain more than one point without violating any necessary condition. The last part of this statement (b_1) is deduced arguing in the existence of a continuous SS-representation.

A similar argument holds for the second part $(a_2)-(b_2)$ correspondig to $[r-1,u(a)-1]$ and $[u(a)+1,r+1]$. □

Proposition 4.12. *Let $\prec$ be a bounded semiorder on (X,τ). Let $(u,1)$ be a SS-representation. Then, there is no sequence of gaps $\{g_n\}_{n\in\mathbb{N}}$ such that the length of a gap g_n is strictly smaller than the length of g_{n+1}, for any $n\in\mathbb{N}$. Hence, there always exists a maximal gap, that is, a gap which length is the biggest.*

Proof. Since the semiorder is bounded, so is its representation and, therefore, the sum of the length of the gaps (denoted by $\sum_{n\in\mathbb{N}} L(g_n)$) is finite. Hence, we conclude that the sequence $\{L(g_n)\}_{n\in\mathbb{N}}$ converges to 0. In consecuence, there always exists a maximal gap. □

The next corollary is directly deduced from the proposition before.

Corollary 4.13. *Let $\prec$ be a bounded semiorder on (X,τ). Let $(u,1)$ be a SS-representation. Then, for any gap g there exists another smaller gap g' such that there is no gap which length is between the length of g and that of g'.*

5 A continuity approach: ε-continuity

First of all, we recall an important point: we shall assume that the topology of the space is compatible with respect to the indifference of the main trace of the semiorder.

Therefore, the total preorder $\precsim^0$ is τ-continuous and, besides, we may assume without loss of generality that the function u of a given SS-representation also represents the total preorder $\precsim_0$. [6] The author believes that this case is the most common one, since if there are two elements which are equivalent (as regards to the order structure), then –at first– there is no reason to distinguish them topologically. This is also the case when the topology τ is defined on the quotient set $X/\sim_0$.

Definition 5.1. Let $\prec$ be a semiorder on (X,τ). We shall say that the semiorder is *r-continuous* (for a positive value $r \in \mathbb{R}$) if there exists a SS-representation $(u,1)$ such that the length of each jump-discontinuity is strictly smaller than this constant r.

It order to approximate to the idea of continuity, we may let r tend to 0. This motivates the following definition.

Definition 5.2. Let $\prec$ be a semiorder on X. We shall say that the semiorder is ε-*continuous* if for any $\varepsilon > 0$ there exists a SS-representation $(u_\varepsilon, 1)$ such that the length of each jump-discontinuity is strictly smaller than the value ε.

It is trivial that this new concept is weaker that the usual continuity. Actually, the so called necessary conditions $(a)-(e)$ for the usual continuity are not needed for the existence of an ε-continuous SS-representation.

Example 5.3. Let X be the set $[0,0'5)\cup[1,3]$ endowed with the topology $\tau_\leq$ generated by the usual Euclidean order on X. We define now the semiorder $\prec$ by $x \prec y \iff x+1<y$.

The inclusion function $i\colon (X,\precsim) \to (\mathbb{R},\tau_u)$ is a SS-representation that fails to be continuous at the point $x=1$. To see that, notice that the sequence $\{0'5-\frac{1}{n+1}\}_{n\in\mathbb{N}}$ converges to 1 in $(X,\tau_\leq)$, whereas the image sequence fails to converge to $i(1)$ in $(\mathbb{R},\tau_u)$. In fact, this semiorder does not satisfy condition (b), therefore it is not continuously SS-representable.

However, it is ε-continuous since for any $\varepsilon>0$ there exists a SS-representation $(u_\varepsilon,1)$ such that the length of the jump-discontinuities is bounded by this value ε.

Let us prove that it is actually an ε-continuously representable semiorder. Let $\delta>0$ be the length of the bad gap of the new function we shall define. We construct it stretching the pieces proportionally, keeping the order structure, but since the gap $[0'5,1)$ must be shortened, we have to shrink too the intervals $[1'5,2]$ and $[2'5,3)$, again, proportionally :

$$u_\delta(x) = \begin{cases} x\cdot\frac{1-\delta}{1/2} & ; \quad x\in[0,0'5), \\ x\cdot\frac{1-\delta}{1/2}-(1-2\delta) & ; \quad x\in[1,1'5], \\ x\cdot\frac{\delta}{1/2}+(2-4\delta) & ; \quad x\in[1'5,2), \\ x\cdot\frac{1-\delta}{1/2}+(4\delta-2) & ; \quad x\in[2,2'5), \\ x\cdot\frac{\delta}{1/2}+(3-6\delta) & ; \quad x\in[2'5,3], \end{cases}$$

Hence, we conclude that the semiorder of the present example is ε-continuously representable and, in fact, we know how to construct the corresponding function.

Unfortunately, although for any $n \in \mathbb{N}$ we are able to construct a SS-representation $(u_n, 1)$ such that the length of the jump-discontinuities is less than $\frac{1}{n}$, it is straightforward to see that the limit function $u = \lim_{n\to+\infty} u_n$[2] fails to be a SS-representation, since the intervals $[1'5, 2]$ and $[2'5, 3]$ would be reduced to a point.

We set this idea as a corollary:

Corollary 5.4. *Let $\prec$ be a semiorder on (X, τ). If it is continuously SS-representable, then it is ε-continuous. However, there exist ε-continuous semiorders that fail to be continuously SS-representable.*

Furthermore, there exist semiorders that fail to be ε_0-continuously SS-representable, for a given $\varepsilon_0 > 0$ (with $\varepsilon_0 \leq 1$). To see this we introduce the following example.

Example 5.5. Let X be the set $X = (-10, -0'5) \cup [0, 1] \cup (1'5, 10)$ endowed with the topology $\tau_{\leq}$ defined by the Euclidean order $\leq$ on X. Let $\prec$ be a semiorder on $X \subseteq \mathbb{R}$ defined by $x \prec y \iff x + 1 < y$, for any $x, y \in X$. It is straightforward to see that the identity function is a SS-representation of the semiorder, which has two bad gaps of length $0'5$, a first gap $[-0'5, 0)$ and the second one $(1, 1'5]$.

Notice that the sequences $(-0'5 - \frac{1}{n})_{n\in\mathbb{N}}$, $(0'5 - \frac{1}{n})_{n\in\mathbb{N}}$ and $(1'5 + \frac{1}{n})_{n\in\mathbb{N}}$ converge to $-0'5, 0'5$ and $1'5$, respectively.

Let δ be a value in $(0, 0'5]$. If there exists a δ-continuous SS-representation u, then, since $(-0'5 - \frac{1}{n})_{n\in\mathbb{N}}$ and $(0'5 - \frac{1}{n})_{n\in\mathbb{N}}$ are adjoint sequences, it holds that $\lim\limits_{n\to\infty} u(-0'5 - \frac{1}{n}) + 1 = \lim\limits_{n\to\infty} u(0'5 - \frac{1}{n})$. And it also holds true that $u(0'5) + 1 < u(1'5 + \frac{1}{n})$, for any $n \in \mathbb{N}$.

Thus, if $\lim\limits_{n\to\infty} u(-0'5 - \frac{1}{n}) > u(0) - \delta$, then $u(1'5 + \frac{1}{n}) > u(0) - \delta + 2$, for any $n \in \mathbb{N}$, as well as $u(1) \leq u(0) + 1$. Therefore, $u(1'5 + \frac{1}{n}) > u(1) + 1 - \delta$, for any $n \in \mathbb{N}$. In consecuence, if the length of one gap is reduced from $0'5$ to $\delta \leq 0'5$, then the other gap must increase from $0'5$ to $1 - \delta$.

Hence, we conclude that the semiorder of this example fails to be $0'5$-continuous.

Before we introduce our conjectures, we include the following concept that could allow us to present a constructive method to handle ε-continuity under some appropiate conditions.

Definition 5.6. Let (X, τ) be a topological space and $u \colon X \to \mathbb{R}$ a real function on X. Let $I = [a, b]$ be a bounded interval of the real line. A subset $\mathscr{C} = u(X) \cap I$ is said to be a *discontinuous Cantor set* if it satisfies the following properties:

(i) It has measure 0,

(ii) it has an infinite number of gaps,

(iii) every gap of $\mathscr{C}$ is a bad gap.

If there is a bounded interval I such that $\mathscr{C} = u(X) \cap I$ is a discontinuous Cantor set, then we will say that $u(X)$ *contains a discontinuous Cantor set*.

[2] Here, notice that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the sup norm.

Remark 5.7. Notice that, given a discontinuous Cantor set $\mathscr{C} = I \cap u(X)$, then the sum of all the gaps of $\mathscr{C}$ is the length of the interval I.

We present the following example inspired in the Cantor set, which justifies the name of this new concept.

Example 5.8. Let I be the unit interval $[0,1]$. First we iteratively define the following Cantor set (we denote it by X) deleting the open middle thirds from a set of line segments. First, we delete the middle third $(1/3, 2/3]$ from $[0,1]$, leaving two segments: $[0,1/3] \cup (2/3,1]$. Next, the middle third of each of these remaining segments is deleted, leaving four segments: $[0,1/9] \cup (2/9,1/3] \cup (2/3,7/9] \cup (8/9,1]$. This process is continued, where the nth set is

$$C_n = \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3}\right) \text{ for } n \geq 1, \text{ and } C_0 = [0,1].$$

This Cantor set X contains all points in the interval $[0,1]$ that are not deleted at any step in this infinite process: $X = \bigcap_{n=1}^{\infty} C_n$.

Now we endow X with the Euclidean order topology $\tau_{\leq}$ and apply the inclusion function $i\colon (X, \tau_{\leq}) \to (\mathbb{R}, \tau_u)$. We denote $i(X) \cap [0,1] = i(X)$ by $\mathscr{C}$. It is known that $\mathscr{C}$ measures 0. Furthermore, each gap is a bad gap (with respect to the usual topology of the real line and the Euclidean order topology on $\mathscr{C}$), since the inclusion function $i\colon (X, \tau_{\leq}) \to (\mathbb{R}, \tau_u)$ is discontinuous at every point a of each gap $(a,b]$.

Now, we are ready to present our main conjectures.

First, we introduce the weakest one.

Conjecture 5.9. (**The Weak Conjecture**)
Let $\prec$ be a SS-representable and bounded semiorder on a topological space (X, τ) and $(u,1)$ a SS-representation. If it satifies the necessary conditions (NC) and there is no discontinuous Cantor set contained in $u(X)$, then it is ε-continuously representable.

Now, we present the *Strong Conjecture*.

Conjecture 5.10. (**The Strong Conjecture**)
Let $\prec$ be a SS-representable and bounded semiorder on a topological space (X, τ). If it satifies the necessary conditions (NC), then it is continuously representable.

In case the conjecture above holds true, then we could conclude the following result, that we present as a *Debreu's Open Gap Lemma for Bounded Semiorders*.

Conjecture 5.11. (***Debreu's Open Gap Lemma for Bounded Semiorders***)
Let S be a bounded subset of $\mathbb{R}$. Then, there exists a strictly increasing function $g\colon S \to \mathbb{R}$ such that all the gaps of $g(S)$ are open or closed, and satisfying that $x+1<y \iff g(x)+1<g(y)$ if and only if the following conditions hold:

(i) *There are no open-closed or closed-open gaps which length is bigger than or equal to 1.*

(ii) *For any gap $[a,b)$:*

(a) $[a+n,b+n]\cap S$ *may content one point, for any* $n\in\mathbb{N}$ *with* $n<m_r$*, for some* $m_r\in\mathbb{N}$ *such that there exist* $\gamma_l,\gamma_r\geq 0$ *(with at least one of them different from 0) satisfying that* $S\cap[a+m_r-\gamma_l,b+m_r+\gamma_r]$ *contains at most one point s. If there exists that point s then* $(s+n,b+n]$ *may be nonempty if* $\gamma_r>0$*, for any* $n>m_r$ *and, dually,* $[a+n,s+n)$ *may be nonempty if* $\gamma_l>0$ *(*$n>m_r$*).*

(b) $[a-n,b-n]$ *or* $[a-n,b-n)$ *are gaps, for any* $n\in\mathbb{N}$ *with* $n<m_l$*, for some* $m_l\in\mathbb{N}$ *such that there exist* $\gamma_l,\gamma_r\geq 0$ *(with at least one of them different from 0) satisfying that* $S\cap[a-m_l-\gamma_l,b-m_l+\gamma_r]$ *contains at most the point* $b-m_l+\gamma_r$*.*

(ii) *For any gap* $(a,b]$*:*

(a) $S\cap[a-n,-b-n]$ *contains at most one point, for any* $n\in\mathbb{N}$ *with* $n<m_l$*, for some* $m_l\in\mathbb{N}$ *such that there exist* $\gamma_l,\gamma_r\geq 0$ *(with at least one of them different from 0) satisfying that* $S\cap[a-m_l-\gamma_l,b-m_l+\gamma_r]$ *may contain at most one point s. If there exists that point s, then* $(s-n,b-n]$ *may be nonempty if* $\gamma_r>0$*, for any* $n>m_l$ *and, dually,* $[a-n,s-n)$ *may be nonempty if* $\gamma_l>0$ *(*$n>m_l$*).*

(b) $[a+n,b+n]$ *or* $[a+n,b+n)$ *are gaps, for any* $n\in\mathbb{N}$ *with* $n<m_r$*, for some* $m_r\in\mathbb{N}$ *such that there exist* $\gamma_l,\gamma_r\geq 0$ *(with at least one of them different from 0) satisfying that* $S\cap[a+m_r-\gamma_l,b+m_r+\gamma_r]$ *contains at most the point* $a+m_r-\gamma_l$*.*

6 Concluding remarks

In the present study we have shown that a semiorder implies a rigid structure on the set, with a geometrical component due to the existence of the constant threshold $k=1$. The necessary conditions for the existence of a continuous SS-representation are extremely strict and demanding. Hence, a semiorder rarely would be continuously representable.

Due to this handicap, we introduce the weaker concept of ε-continuity as a tool when dealing with semiorders that fail to be continuously representable. Furthermore, through this idea we present some conjectures on continuous SS-representability of semiorders. In case of a positive answer, these conjectures can be summarized just as a version of the *Debreu's Open Gap Lemma* but with the additional component of a threshold.

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