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Interval Subsethood Measures with Respect to Uncertainty for the Interval-Valued Fuzzy Setting

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ABSTRACT

In this paper, the problem of measuring the degree of subsethood in the interval-valued fuzzy setting is addressed. Taking into account the widths of the intervals, two types of interval subsethood measures are proposed. Additionally, their relation and main properties are studied. These developments are made both with respect to the regular partial order of intervals and with respect to admissible orders. Finally, some construction methods of the introduced interval subsethood measures with the use interval-valued aggregation functions are examined.

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1. INTRODUCTION

Since fuzzy sets were introduced by Zadeh [1], many new approaches and theories have arisen to treat imprecision and uncertainty in the information theory schema. Particularly, many works can be found in the literature where different types of transitivities, distance measures, similarity measures and subsethood, inclusion or equivalence measures between fuzzy sets have been proposed ([2–11] or [12,13]). Focusing on subsethood measures, different axiomatizations have been proposed [14–17] and they have been adapted and applied in different settings [18,19].

On the other hand, as extensions of classical fuzzy set theory, intuitionistic fuzzy sets [20] and interval-valued fuzzy sets [21,22] are very useful in dealing with imprecision and uncertainty (cf. [23] for more details). In this setting, different proposals for subsethood measures between interval-valued fuzzy sets have been proposed [24,25]. However, these proposals failed to consider the width of the intervals as an important feature in the axiomatization. In this regard, recent works in the literature have proposed this property to be taken into account [26,27].

Thus, the motivation of the present paper is to propose a more natural tool for estimating the degree of subsethood between interval-valued fuzzy sets taking into account the widths of the intervals and to explore their properties. In this attempt, we introduce two types of interval subsethood measures, that is, operators that measure the *grade of subsethoodness* of an interval in another, to end with a new definition of subsethood measure for interval-valued fuzzy sets.

In the interval-valued fuzzy setting, we assume that the precise membership degree of an element in a given set is a number included in the membership interval. For such interpretation, the width of the membership interval of an element reflects the lack of precise membership degree of that element. Hence, the fact that two elements have the same membership intervals does not necessarily mean that their corresponding membership values are the same. Similarly, this interpretation requires that the uncertainty regarding the membership degrees is translated to subsethood measures between interval-valued fuzzy sets, resulting in interval-valued subsethood measures. This is why we have taken into account the importance of the notion of width of intervals while defining new types of subsethood measures. Additionally, these developments are made according to the standard partial order between intervals, but also with respect to admissible orders [28], which are linear.

The paper is organized as follows. In Section 2, basic information on interval-valued fuzzy sets is recalled and the notion of interval-valued aggregation function is presented. In Section 3, two types of interval subsethood measures for the interval-valued fuzzy setting by using partial and linear orders are proposed. Then, some properties and construction methods are examined. Finally, some conclusions are presented.

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2. INTERVAL-VALUED FUZZY SETS

We use the following notation for the set of intervals

$$L^I = \{[\underline{x}, \overline{x}] : \underline{x}, \overline{x} \in [0,1] \ \ and \ \ \underline{x} \leqslant \overline{x}\},$$

which are the basis of interval-valued fuzzy sets introduced by Zadeh [21].

Definition 1. [22,21] An interval-valued fuzzy set A over the universe U is a mapping $A : U \to L^I$ such that

$$A(u) = \left[\underline{A}(u), \overline{A}(u)\right] \quad \text{for all} \quad u \in U,$$

where $\underline{A}, \overline{A}$ are fuzzy sets that satisfy $\underline{A}(u) \leqslant \overline{A}(u)$ for all $u \in U$. The class of all interval-valued fuzzy sets in U is denoted by $IVFS(U)$.

2.1. Orders in the Interval-Valued Fuzzy Setting

The standard partial order between intervals that is used in the interval-valued fuzzy setting [20] is of the form

$$[\underline{x}, \overline{x}] \leqslant_{L^I} \left[\underline{y}, \overline{y}\right] \Leftrightarrow \underline{x} \leqslant \underline{y} \ \ and \ \ \overline{x} \leqslant \overline{y},$$

and $[\underline{x}, \overline{x}] <_{L^I} \left[\underline{y}, \overline{y}\right]$ with strict inequalities. Thus, the operations joint and meet are defined, respectively:

$$[\underline{x}, \overline{x}] \vee \left[\underline{y}, \overline{y}\right] = \left[\max(\underline{x}, \underline{y}), \max(\overline{x}, \overline{y})\right],$$
$$[\underline{x}, \overline{x}] \wedge \left[\underline{y}, \overline{y}\right] = \left[\min(\underline{x}, \underline{y}), \min(\overline{x}, \overline{y})\right].$$

The structure $(L^I, \vee, \wedge)$ is a complete lattice with the partial order $\leqslant_{L^I}$ and $\mathbf{1} = [1,1]$ and $\mathbf{0} = [0,0]$ are the greatest and smallest elements, respectively (see [28]).

We are interested in extending the partial order $\leqslant_{L^I}$ to a linear order, solving the problem of existence of incomparable elements. We recall the notion of an admissible order, which was introduced in [28] and studied, for example, in [29] and [30]. The linearity of the order is needed in many applications of real problems, in order to be able to compare any two interval data [31].

Definition 2. [28] An order $\leqslant_{Adm}$ in L^I is called admissible if it is linear and satisfies that for all $x, y \in L^I$, such that $x \leqslant_{L^I} y$, then $x \leqslant_{Adm} y$.

Plainly, an order $\leqslant_{Adm}$ on L^I is admissible if it is linear and refines the standard partial order $\leqslant_{L^I}$. Admissible orders can be constructed in terms of aggregation functions [28].

Proposition 1. *[28] Let $B_1, B_2 : [0,1]^2 \to [0,1]$ be two continuous aggregation functions, such that, for all $x = [\underline{x}, \overline{x}], y = \left[\underline{y}, \overline{y}\right] \in L^I$, the equalities $B_1(\underline{x}, \overline{x}) = B_1(\underline{y}, \overline{y})$ and $B_2(\underline{x}, \overline{x}) = B_2(\underline{y}, \overline{y})$ hold if and only if $x = y$. Thus, if the order $\leqslant_{B_{1,2}}$ on L^I is defined by*

$$x \leqslant_{B_{1,2}} y \Leftrightarrow B_1\left(\underline{x}, \overline{x}\right) < B_1\left(\underline{y}, \overline{y}\right) \ \ or$$
$$\left(B_1\left(\underline{x}, \overline{x}\right) = B_1\left(\underline{y}, \overline{y}\right) \ \ and \ B_2\left(\underline{x}, \overline{x}\right) \leqslant B_2\left(\underline{y}, \overline{y}\right)\right),$$

then $\leqslant_{B_{1,2}}$ is an admissible order on L^I.

Example 1 (28)**.** The following are special cases of admissible linear orders on L^I:

- The Xu and Yager order:

$$[\underline{x}, \overline{x}] \leqslant_{XY} \left[\underline{y}, \overline{y}\right] \Leftrightarrow \underline{x} + \overline{x} < \underline{y} + \overline{y} \ or$$
$$\left(\overline{x} + \underline{x} = \overline{y} + \underline{y} \ and \ \overline{x} - \underline{x} \leqslant \overline{y} - \underline{y}\right).$$

- The first lexicographical order (with respect to the first variable), $\leqslant_{Lex1}$ defined as:

$$[\underline{x}, \overline{x}] \leqslant_{Lex1} \left[\underline{y}, \overline{y}\right] \Leftrightarrow \underline{x} < \underline{y} \ or \ \left(\underline{x} = \underline{y} \ \ and \ \ \overline{x} \leqslant \overline{y}\right).$$

- The second lexicographical order (with respect to the second variable), $\leqslant_{Lex2}$ defined as:

$$[\underline{x}, \overline{x}] \leqslant_{Lex2} \left[\underline{y}, \overline{y}\right] \Leftrightarrow \overline{x} < \overline{y} \ or \ \left(\overline{x} = \overline{y} \ and \ \underline{x} \leqslant \underline{y}\right).$$

- The $\alpha\beta$ order, $\leqslant_{\alpha\beta}$ defined as:

$$[\underline{x}, \overline{x}] \leqslant_{\alpha\beta} \left[\underline{y}, \overline{y}\right] \Leftrightarrow K_\alpha\left(\underline{x}, \overline{x}\right) < K_\alpha\left(\underline{y}, \overline{y}\right) \ \ or$$
$$\left(K_\alpha\left(\underline{x}, \overline{x}\right) = K_\alpha\left(\underline{y}, \overline{y}\right) \ \ and \right.$$
$$\left. K_\beta\left(\underline{x}, \overline{x}\right) \leqslant K_\beta\left(\underline{y}, \overline{y}\right)\right),$$

  for some $\alpha \neq \beta \in [0,1]$ and $x, y \in L^I$, where $K_\alpha : [0,1]^2 \to [0,1]$ is defined as $K_\alpha(x, y) = \alpha x + (1 - \alpha)y$.

The orders $\leqslant_{XY}$, $\leqslant_{Lex1}$ and $\leqslant_{Lex2}$ are special cases of the order $\leqslant_{\alpha\beta}$ with $\leqslant_{0.5\beta}$ (for $\beta > 0.5$), $\leqslant_{1,0}$, $\leqslant_{0,1}$, respectively. The orders $\leqslant_{XY}$, $\leqslant_{Lex1}$, $\leqslant_{Lex2}$ and $\leqslant_{\alpha\beta}$ are admissible linear orders $\leqslant_{B_{1,2}}$ defined by pairs of aggregation functions (see Proposition 1), namely weighted means. In the case of the orders $\leqslant_{Lex1}$ and $\leqslant_{Lex2}$, the aggregations that are used are the projections P_1, P_2 and P_2, P_1, respectively.

Remark 1

Throughout the paper we use the notation $\leqslant$ both for partial and admissible orders, with $\mathbf{0}$ and $\mathbf{1}$ as minimal and maximal element of L^I, respectively. Regarding the results for the partial order, the previously introduced notation $\leqslant_{L^I}$ is used, whereas for the results for a general admissible order the notation $\leqslant_{Adm}$ is used.

With respect to the order between interval-valued fuzzy sets, that is, for $A, B \in IVFS(U)$ and $card(U) = n, n \in N$ we use the following notion of partial order

$$A \preceq B \Leftrightarrow a_i \leqslant b_i \quad for \quad i = 1, ..., n,$$

where $\leqslant$ is the same kind of order (partial or linear) for each i and $a_i = A(u_i)$, $b_i = B(u_i)$. Let us note that if for $i = 1, ..., n$ we consider the same linear order $a_i \leqslant b_i$, then the order $A \preceq B$ between interval-valued fuzzy sets A, B is the partial one but it need not be the linear one.

We consider the following notion of strict order between interval-valued fuzzy sets

$$A \prec B \Leftrightarrow a_i < b_i \quad for \quad i = 1, ..., n.$$

2.2. Interval-Valued Aggregation Functions

Let us now recall the concept of an interval-valued aggregation function, or an aggregation function on L^I, which is an important notion for many applications. We consider interval-valued aggregation functions both with respect to $\leqslant_{L^I}$ and $\leqslant_{Adm}$.

Definition 3. [32,33] Let $n \in \mathbb{N}, n \geqslant 2$. A function $\mathcal{A} : (L^I)^n \to L^I$ is called an interval-valued aggregation function if it is increasing with respect to the order $\leqslant$ (partial or linear (see Remark 1)), that is,

$$\forall x_i, y_i \in L^I \qquad x_i \leqslant y_i \Rightarrow \mathcal{A}(x_1, \dots, x_n) \leqslant \mathcal{A}(y_1, \dots, y_n),$$

and it satisfies

$$\mathcal{A}(\mathbf{0}, \dots, \mathbf{0}) = \mathbf{0}, \text{ and}$$
$$\mathcal{A}(\mathbf{1}, \dots, \mathbf{1}) = \mathbf{1}.$$

A special class of interval-valued aggregation functions is the one formed by the so-called representable interval-valued aggregation functions.

Definition 4. [34,35] An interval-valued aggregation function $\mathcal{A} : (L^I)^n \to L^I$ is said to be representable if there exist aggregation functions $A_1, A_2 : [0,1]^n \to [0,1]$ such that

$$\mathcal{A}(x_1, \dots, x_n) = \left[A_1\left(\underline{x}_1, \dots \underline{x}_n\right), A_2\left(\overline{x}_1, \dots, \overline{x}_n\right)\right],$$

for all $x_1, \dots, x_n \in L^I$, provided that $A_1\left(\underline{x}_1, \dots \underline{x}_n\right) \leqslant A_2\left(\overline{x}_1, \dots, \overline{x}_n\right)$.

Remark 2

Lattice operations $\wedge$ and $\vee$ on L^I are examples of representable aggregation functions on L^I with respect to the partial order $\leqslant_{L^I}$, with $A_1 = A_2 = \min$ in the first case and $A_1 = A_2 = \max$ in the second one. However, $\wedge$ and $\vee$ are not interval-valued aggregation functions with respect to $\leqslant_{Lex1}$, $\leqslant_{Lex2}$ or $\leqslant_{XY}$.

Indeed, note that

$$x = [0.2, 0.8] \leqslant_{Lex1} y = [0.3, 0.7] \leqslant_{Lex1} z = [0.5, 0.6],$$

and we obtain a contradiction with isotonicity of $\vee$ with respect to $\leqslant_{Lex1}$, that is,

$$[0.5, 0.8] = x \vee z \geqslant_{Lex1} y \vee z = [0.5, 0.7].$$

Similarly, in the case of $\wedge$ and $\leqslant_{Lex2}$ (or $\leqslant_{XY}$) that for

$$x = [0.4, 0.6] \leqslant_{Lex2} y = [0.2, 0.8] \leqslant_{Lex2} z = [0.1, 0.9],$$

we obtain a contradiction with isotonicity of $\wedge$ with respect to $\leqslant_{Lex2}$, that is,

$$[0.2, 0.6] = x \wedge y \geqslant_{Lex2} x \wedge z = [0.1, 0.6].$$

Example 2. The following are examples of representable interval-valued aggregation functions with respect to $\leqslant_{L^I}$.

- The projections:

$$\mathcal{A}_L\left(\left[\underline{x}, \overline{x}\right], \left[\underline{y}, \overline{y}\right]\right) = \left[\underline{x}, \overline{x}\right],$$
$$\mathcal{A}_R\left(\left[\underline{x}, \overline{x}\right], \left[\underline{y}, \overline{y}\right]\right) = \left[\underline{y}, \overline{y}\right].$$

- The representable product:

$$\mathcal{A}_p\left(\left[\underline{x}, \overline{x}\right], \left[\underline{y}, \overline{y}\right]\right) = \left[\underline{xy}, \overline{xy}\right].$$

- The representable arithmetic mean:

$$\mathcal{A}_{mean}\left(\left[\underline{x}, \overline{x}\right], \left[\underline{y}, \overline{y}\right]\right) = \left[\frac{\underline{x} + \underline{y}}{2}, \frac{\overline{x} + \overline{y}}{2}\right].$$

- The representable geometric mean:

$$\mathcal{A}_{gmean}\left(\left[\underline{x}, \overline{x}\right], \left[\underline{y}, \overline{y}\right]\right) = \left[\sqrt{\underline{x}\,\underline{y}}, \sqrt{\overline{xy}}\right].$$

- The representable harmonic mean:

$$\mathcal{A}_H\left(\left[\underline{x}, \overline{x}\right], \left[\underline{y}, \overline{y}\right]\right) = \begin{cases} [0,0], & \text{if } x = y = [0,0], \\ \left[\dfrac{2\underline{xy}}{\underline{x} + \underline{y}}, \dfrac{2\overline{xy}}{\overline{x} + \overline{y}}\right], & \text{otherwise.} \end{cases}$$

- The representable power mean:

$$\mathcal{A}_{power}\left(\left[\underline{x}, \overline{x}\right], \left[\underline{y}, \overline{y}\right]\right) = \left[\sqrt{\frac{\underline{x}^2 + \underline{y}^2}{2}}, \sqrt{\frac{\overline{x}^2 + \overline{y}^2}{2}}\right].$$

Representability is not the only possible way to build interval-valued aggregation functions with respect to $\leqslant_{L^I}$ or $\leqslant_{Adm}$.

Example 3. Let $A : [0,1]^2 \to [0,1]$ be an aggregation function.

- The function $\mathcal{A}_1 : (L^I)^2 \to L^I$, where

$$\mathcal{A}_1(x, y) = \begin{cases} [1,1], & \text{if } (x, y) = (\mathbf{1}, \mathbf{1}), \\ \left[0, A\left(\underline{x}, \overline{y}\right)\right], & \text{otherwise,} \end{cases}$$

is a nonrepresentable interval-valued aggregation function with respect to $\leqslant_{L^I}$.

- The functions $\mathcal{A}_2, \mathcal{A}_3 : (L^I)^2 \to L^I$ [36], where

$$\mathcal{A}_2(x, y) = \begin{cases} [1,1], & \text{if } (x, y) = (\mathbf{1}, \mathbf{1}) \\ \left[0, A\left(\underline{x}, \underline{y}\right)\right], & \text{otherwise,} \end{cases}$$

$$\mathcal{A}_3(x, y) = \begin{cases} [0,0], & \text{if } (x, y) = (\mathbf{0}, \mathbf{0}) \\ \left[A\left(\underline{x}, \underline{y}\right), 1\right], & \text{otherwise,} \end{cases}$$

are nonrepresentable interval-valued aggregation functions with respect to $\leqslant_{Lex1}$.

- The functions $\mathcal{A}_4, \mathcal{A}_5 : (L^I)^2 \to L^I$ [36], where

$$\mathcal{A}_4(x,y) = \begin{cases} [1,1], & \text{if } (x,y) = (\mathbf{1},\mathbf{1}) \\ [0, A(\overline{x},\overline{y})], & \text{otherwise}, \end{cases}$$

$$\mathcal{A}_5(x,y) = \begin{cases} [0,0], & \text{if } (x,y) = (\mathbf{0},\mathbf{0}) \\ [A(\overline{x},\overline{y}),1], & \text{otherwise}, \end{cases}$$

are nonrepresentable interval-valued aggregation functions with respect to $\leqslant_{Lex2}$.

- $\mathcal{A}_{mean}$ is an aggregation function with respect to $\leqslant_{\alpha\beta}$ (cf. [29]).
- The following function

$$\mathcal{A}_\alpha(x,y) = \left[\alpha\underline{x} + (1-\alpha)\underline{y}, \alpha\overline{x} + (1-\alpha)\overline{y}\right],$$

is an interval-valued aggregation function on L^I with respect to $\leqslant_{Lex1}$, $\leqslant_{Lex2}$ and $\leqslant_{XY}$ for $x, y \in L^I$ and $\alpha \in [0,1]$ (cf. [30]).

There exist sufficient conditions for a representable interval-valued aggregation function with respect to the partial order to be so with respect to the orders $\leqslant_{Lex1}$ or $\leqslant_{Lex2}$.

Proposition 2. *[37] Let $\mathcal{A} : (L^I)^n \to L^I$ be a representable interval-valued aggregation function with component functions A_1, A_2. If the component aggregation function A_1 is a strictly increasing aggregation function on $[0,1]$, then $\mathcal{A}$ is an interval-valued aggregation function with respect to the linear order $\leqslant_{Lex1}$.*

Proposition 3. *[37] Let $\mathcal{A} : (L^I)^n \to L^I$ be a representable interval-valued aggregation function with component functions A_1 and A_2. If the component aggregation function A_2 is a strictly increasing aggregation function on $[0,1]$, then $\mathcal{A}$ is an interval-valued aggregation function with respect to the linear order $\leqslant_{Lex2}$.*

The following is an example of interval-valued aggregation function with respect to both $\leqslant_{Lex1}$ and $\leqslant_{Lex2}$.

Example 4. [37] Let $0 < r < s$, $r, s \in \mathbb{R}$ and $w_1, \dots, w_n \in [0,1]$ such that $\sum_{k=1}^n w_k = 1$.

Then, the function $\mathcal{A}$, given by

$$\mathcal{A}(x_1,\dots,x_n) = \left[\sqrt[r]{\sum_{k=1}^n w_k \underline{x}_k^r}, \sqrt[s]{\sum_{k=1}^n w_k \overline{x}_k^s}\right],$$

is an interval-valued aggregation function with respect to the linear order $\leqslant_{Lex1}$ and $\leqslant_{Lex2}$.

In the subsequent part of this paper we use the following properties of aggregation functions with respect to partial or linear orders.

Definition 5. (cf. [38]) An interval-valued aggregation function $\mathcal{A} : (L^I)^2 \to L^I$ is said to be:

- symmetric, if

$$\mathcal{A}(x,y) = \mathcal{A}(y,x),$$

- bisymmetric, if

$$\mathcal{A}(\mathcal{A}(x,y), \mathcal{A}(z,t)) = \mathcal{A}(\mathcal{A}(x,z), \mathcal{A}(y,t)),$$

- idempotent, if

$$\mathcal{A}(x,x) = x,$$

- subidempotent, if

$$\mathcal{A}(x,x) \leqslant x,$$

for every $x, y, z, t \in L^I$.

Moreover,

- $\mathcal{A}$ has an absorbing (zero) element $z \in L^I$, if for all $x \in L^I$,

$$\mathcal{A}(x,z) = \mathcal{A}(z,x) = z.$$

3. SUBSETHOOD MEASURES

Subsethood, or inclusion, measures have been studied mainly from constructive and axiomatic approaches and have been introduced successfully into the theory of fuzzy sets and their extensions. Many researchers have tried to relax the rigidity of Zadeh's definition of subsethood to get a soft approach which is more compatible with the spirit of fuzzy logic. For instance [39], defended that quantitative methods were the main approaches in uncertainty inference, a key problem in artificial intelligence, so they presented a generalized definition for subsethood measures, called *i*ncluding degrees. There also exist several works regarding subsethood measures in the interval-valued fuzzy setting [24,30,40–42], however the condition regarding the width of the intervals, with which we deal in this paper, has not been so far considered, to our knowledge.

3.1. Interval Subsethood Measures

We introduce the notion of an interval subsethood measure for a pair of intervals the partial and admissible orders and the width of intervals w, where $w(x) = \overline{x} - \underline{x}$ for $x \in L^I$.

3.1.1. Interval subsethood measure I

First, we consider the notion of an interval subsethood measure where strong inequalities between inputs give the same values of the interval subsethood measure (see Definition 6, axiom (IM2)).

Definition 6. A function $\sigma : (L^I)^2 \to L^I$ is said to be an interval subsethood measure, if it satisfies the following conditions for intervals $x = [\underline{x}, \overline{x}]$, $y = \left[\underline{y}, \overline{y}\right]$, $z = [\underline{z}, \overline{z}] \in L^I$:

(IM1) If $x = \mathbf{1}$, $y = \mathbf{0}$, then $\sigma(x,y) = \mathbf{0}$;
(IM2) If $x < y$, then $\sigma(x,y) = \mathbf{1}$;
(IM3) $\sigma(x,x) = [1 - w(x), 1]$ (reflexivity);
(IM4) If $x \leqslant y \leqslant z$ and $w(x) = w(y) = w(z)$, then $\sigma(z,x) \leqslant \sigma(y,x)$ and $\sigma(z,x) \leqslant \sigma(z,y)$ for $x, y, z \in L^I$.

Axioms (IM1)-(IM4) are inspired in the usual properties that subsethood measures satisfy and, in order to take into account the width of intervals, a similar approach to those in [26,27] has been taken.

Remark 3

Note that an interval subsethood measure as in Definition 6, in particular due to axiom (IM3), is consistent with our interpretation. Indeed, in the case that there is no uncertainty, the interval subsethood measure of an interval with respect to itself is certain as well, for example, $\sigma([0.3, 0.3], [0.3, 0.3]) = [1, 1]$. However, in case that the uncertainty is maximum, so is it in the case of interval subsethood measures, for example, $\sigma([0, 1], [0, 1]) = [0, 1]$. We refer the reader to Example 5 for specific examples of such an interval subsethood measure.

Let us denote by

$$\mathcal{S} = \left\{\sigma : (L^I)^2 \to L^I \mid \sigma \quad \text{is a subsethood measure}\right\}.$$

Let us present two construction methods for such an interval subsethood measure. The first one is given in the following result.

Theorem 1. *Let $\sigma_z : (L^I)^2 \to L^I$ be the operation given by*

$$\sigma_z(x, y) = \begin{cases} [1 - w(x), 1], & x = y, \\ \mathbf{1}, & x < y, \\ \mathbf{0}, & \textit{otherwise}, \end{cases}$$

for $x, y \in L^I$. Then, σ_z is an interval subsethood measure ($\sigma_z \in \mathcal{S}$).

Proof. Conditions (IM1)-(IM4) need to be checked. (IM1)-(IM3) are obvious. Let us show (IM4). Assume $w(x) = w(y) = w(z)$. There are four possible cases:

- If $x < y < z$, then $\sigma_z(z, x) = \mathbf{0} \leqslant \sigma_z(y, x)$ and $\sigma_z(z, x) = \mathbf{0} \leqslant \sigma_z(z, y)$.
- If $x = y = z$, then

$$\sigma_z(z, x) = [1 - w(x), 1] \leqslant \sigma_z(y, x) = [1 - w(x), 1],$$

  and

$$\sigma_z(z, x) = [1 - w(x), 1] \leqslant \sigma_z(z, y) = [1 - w(x), 1].$$

- If $x = y < z$, then

$$\sigma_z(z, x) = \mathbf{0} \leqslant \sigma_z(y, x) = [1 - w(x), 1],$$

  and $\sigma_z(z, x) = \mathbf{0} \leqslant \sigma_z(z, y) = \mathbf{0}$.
- If $x < y = z$, then $\sigma_z(z, x) = \mathbf{0} \leqslant \sigma_z(y, x) = \mathbf{0}$ and $\sigma_z(z, x) = \mathbf{0} \leqslant \sigma_z(z, y) = [1 - w(z), 1]$.

As a result $\sigma_z : (L^I)^2 \to L^I$ is an interval subsethood measure.

The second construction method is based on the next theorem. Recall that an interval-valued fuzzy negation N_{IV} is an antytonic operation that satisfies $N_{IV}(\mathbf{0}) = \mathbf{1}$ and $N_{IV}(\mathbf{1}) = \mathbf{0}$ [43,44].

Theorem 2. *Let $\sigma_{\mathcal{A}} : (L^I)^2 \to L^I$ be the operation given by*

$$\sigma_{\mathcal{A}}(x, y) = \begin{cases} [1 - w(x), 1], & x = y, \\ \mathbf{1}, & x < y, \\ \mathcal{A}(N_{IV}(x), y), & \textit{otherwise}, \end{cases}$$

for $x, y \in L^I$, where N_{IV} is an interval-valued fuzzy negation such that, for a fuzzy negation n,

$$N_{IV}(x) = [n(\overline{x}), n(\underline{x})] \leqslant [1 - \overline{x}, 1 - \underline{x}],$$

and $\mathcal{A}$ is a representable interval-valued aggregation function with respect to the order $\leqslant$ such that $\mathcal{A} \leqslant \vee$. Thus, $\sigma_{\mathcal{A}}$ is an interval subsethood measure ($\sigma_{\mathcal{A}} \in \mathcal{S}$).

Proof. Conditions (IM1)-(IM4) need to be checked. (IM1)-(IM3) are obvious. Let us show (IM4). Assume $w(x) = w(y) = w(z)$. There are four possible cases:

- If $x < y < z$, then

$$\begin{aligned} \sigma_{\mathcal{A}}(z, x) &= \mathcal{A}(N_{IV}(z), x) \\ &\leqslant \mathcal{A}(N_{IV}(y), x) = \sigma_{\mathcal{A}}(y, x), \end{aligned}$$

  and

$$\begin{aligned} \sigma_{\mathcal{A}}(z, x) &= \mathcal{A}(N_{IV}(z), x) \\ &\leqslant \mathcal{A}(N_{IV}(z), y) = \sigma_{\mathcal{A}}(z, y). \end{aligned}$$

- If $x = y = z$, then

$$\begin{aligned} \sigma_{\mathcal{A}}(z, x) &= [1 - w(x), 1] \\ &\leqslant \sigma_{\mathcal{A}}(y, x) = [1 - w(x), 1], \end{aligned}$$

  and

$$\begin{aligned} \sigma_{\mathcal{A}}(z, x) &= [1 - w(x), 1] \\ &\leqslant \sigma_{\mathcal{A}}(z, y) = [1 - w(x), 1]. \end{aligned}$$

- If $x = y < z$, then

$$\begin{aligned} \sigma_{\mathcal{A}}(z, x) &= [A_1(n(\overline{z}), \underline{x}), A_2(n(\underline{z}), \overline{x})] \\ &\leqslant \left[(1 - \overline{z}) \vee \underline{x}, (1 - \underline{z}) \vee \overline{x}\right] \\ &\leqslant \left[1 - \overline{x} + \underline{x}, 1\right] \\ &= [1 - w(x), 1] = \sigma_{\mathcal{A}}(y, x), \end{aligned}$$

  and

$$\begin{aligned} \sigma_{\mathcal{A}}(z, x) &= \mathcal{A}(N_{IV}(z), x) \\ &\leqslant \mathcal{A}(N_{IV}(z), y) = \sigma_{\mathcal{A}}(z, y). \end{aligned}$$

- The case $x < y = z$ can be proven similarly.

Hence, $\sigma_{\mathcal{A}} : (L^I)^2 \to L^I$ is an interval subsethood measure.

Using the construction methods from Theorem 2 we obtain the following examples.

Example 5. The following function is an interval subsethood measure with respect to $\leqslant_{L^I}$:

$$\sigma_{\mathcal{A}_{meanL^I}}(x, y) = \begin{cases} [1 - w(x), 1], & x = y, \\ \mathbf{1}, & x <_{L^I} y, \\ \left[\dfrac{1 - \overline{x} + \underline{y}}{2}, \dfrac{1 - \underline{x} + \overline{y}}{2}\right], & \textit{otherwise}, \end{cases}$$

where $N_{IV}(x) = [1-\overline{x}, 1-\underline{x}]$. Moreover, the following function is a subsethood measure with respect to $\leqslant_{Lex2}$:

$$\sigma_{\mathcal{A}_{meanLex2}}(x,y) = \begin{cases} [1-w(x),1], & x = y, \\ \mathbf{1}, & x <_{Lex2} y, \\ \left[\dfrac{\underline{y}}{2}, \dfrac{1-\overline{x}+\overline{y}}{2}\right], & otherwise. \end{cases}$$

Using the interval-valued aggregation function $\mathcal{A}_\alpha$ for $\alpha \in [0,1]$, we get the subsethood measure

$$\sigma_{\mathcal{A}_{\alpha Lex2}}(x,y) = \begin{cases} [1-w(x),1], & x = y, \\ \mathbf{1}, & x <_{Lex2} y, \\ [(1-\alpha)\underline{y}, \\ \quad \alpha(1-\overline{x})+ \\ \quad (1-\alpha)\overline{y}], & otherwise, \end{cases}$$

where

$$N_{IV}(x) = \begin{cases} \mathbf{1}, & x = \mathbf{0}, \\ [0, 1-\overline{x}], & otherwise, \end{cases}$$

is an interval-valued fuzzy negation with respect to $\leqslant_{Lex2}$.

Remark 4

[30] The aggregation $\mathcal{A}_\alpha$ preserves the width of the intervals of the same width.

Let us now analyze some properties of interval subsethood measures constructed by means of Theorems 1 and 2.

Proposition 4. *Let $\mathcal{B} : (L^I)^2 \to L^I$ be subidempotent interval-valued aggregation with respect to $\leqslant_{Adm}$, with zero element $\mathbf{0}$. Thus σ_z is a $\mathcal{B}$-quasi-ordered operation (reflexive and $\mathcal{B}$-transitive with respect to $\leqslant_{Adm}$).*

Proof. Reflexivity is obvious by (IM3). We will prove $\mathcal{B}$-transitivity of σ_z, that is,

$$\mathcal{B}(\sigma_z(x,y), \sigma_z(y,z)) \leqslant_{Adm} \sigma_z(x,z), \quad x,y,z \in L^I.$$

We consider the following cases.

1. If $x <_{Adm} y <_{Adm} z$, then
$$\mathcal{B}(\sigma_z(x,y), \sigma_z(y,z)) = \mathcal{B}(\mathbf{1},\mathbf{1}) \leqslant_{Adm} \mathbf{1} = \sigma_z(x,z).$$

2. If $y <_{Adm} x <_{Adm} z$, then
$$\begin{aligned}\mathcal{B}(\sigma_z(x,y), \sigma_z(y,z)) &= \mathcal{B}(\mathbf{0},\mathbf{1}) = \mathbf{0} \\ &\leqslant_{Adm} \mathbf{1} = \sigma_z(x,z).\end{aligned}$$

3. If $x <_{Adm} y = z$, then
$$\begin{aligned}\mathcal{B}(\sigma_z(x,y), \sigma_z(y,z)) &= \mathcal{B}\left(\mathbf{1}, [1-w(y),1]\right) \\ &\leqslant_{Adm} \mathbf{1} = \sigma_z(x,z).\end{aligned}$$

4. If $x = y <_{Adm} z$, then
$$\begin{aligned}\mathcal{B}(\sigma_z(x,y), \sigma_z(y,z)) &= \mathcal{B}([1-w(x),1], \mathbf{1}) \\ &\leqslant_{Adm} \mathbf{1} = \sigma_z(x,z).\end{aligned}$$

5. If $x = y = z$, then
$$\begin{aligned}\mathcal{B}(\sigma_z(x,y), \sigma_z(y,z)) &= \mathcal{B}([1-w(x),1], [1-w(x),1]) \\ &\leqslant_{Adm} [1-w(x),1] = \sigma_z(x,z).\end{aligned}$$

Similarly we can show the remaining 8 cases. As a result σ_z is a $\mathcal{B}$-quasi-ordered operation.

Remark 5

We may obtain a similar result to Proposition 4 considering the partial order $\leqslant_{L^I}$, that is, $\mathcal{B}$-transitivity with respect to $\leqslant_{L^I}$ and $\mathcal{B} : (L^I)^2 \to L^I$ subidempotent interval-valued aggregation function with respect to $\leqslant_{L^I}$.

Example 6. The functions $\wedge$, A_p and $T_{L_{IV}}$, where

$$T_{L_{IV}}(x,y) = \left[\max\left(0, \underline{x}+\underline{y}-1\right), \ \max\left(0, \overline{x}+\overline{y}-1\right)\right],$$

satisfy Proposition 4.

Moreover, these three functions are interval-valued t-norms, that is, binary operations that are isotonic with respect to each variable, associative, commutative and have neutral element $\mathbf{1}$.

Proposition 5. *Let $\mathcal{A} : (L^I)^2 \to L^I$ be a subidempotent, symmetric, bisymmetric interval-valued aggregation function with respect to $\leqslant_{Adm}$, with neutral element $\mathbf{1}$ and satisfying $\mathcal{A}(x, N_{IV}(x)) = 1$ for an interval-valued fuzzy negation N_{IV} which satisfies $N_{IV}(x) \leqslant_{Adm} x$. Then $\sigma_{\mathcal{A}}$ is a $\mathcal{A}$-quasi-ordered operation (reflexive and $\mathcal{A}$-transitive with respect to $\leqslant_{Adm}$).*

In addition, if $\mathcal{B} \leqslant_{Adm} \mathcal{A}$, then $\sigma_{\mathcal{A}}$ is a $\mathcal{B}$-quasi-ordered operation (reflexive and $\mathcal{B}$-transitive with respect to $\leqslant_{Adm}$).

Proof. Reflexivity is obvious by (IM3). We will prove $\mathcal{A}$-transitivity of $\sigma_{\mathcal{A}}$, that is,

$$\mathcal{A}(\sigma_{\mathcal{A}}(x,y), \sigma_{\mathcal{A}}(y,z)) \leqslant_{Adm} \sigma_{\mathcal{A}}(x,z), \quad x,y,z \in L^I.$$

We consider the following cases.

1. If $x <_{Adm} y <_{Adm} z$, then
$$\mathcal{A}(\sigma_{\mathcal{A}}(x,y), \sigma_{\mathcal{A}}(y,z)) = \mathcal{A}(\mathbf{1},\mathbf{1}) \leqslant_{Adm} \mathbf{1} = \sigma_{\mathcal{A}}(x,z).$$

2. If $y <_{Adm} x <_{Adm} z$, then
$$\begin{aligned}\mathcal{A}(\sigma_{\mathcal{A}}(x,y), \sigma_{\mathcal{A}}(y,z)) &= \mathcal{A}(\mathcal{A}(N_{IV}(x),y),\mathbf{1}) \\ &= \mathbf{0} \\ &\leqslant_{Adm} \mathbf{1} = \sigma_{\mathcal{A}}(x,z).\end{aligned}$$

3. If $x <_{Adm} y = z$, then
$$\begin{aligned}\mathcal{A}(\sigma_{\mathcal{A}}(x,y), \sigma_{\mathcal{A}}(y,z)) &= \mathcal{A}(\mathbf{1}, [1-w(y),1]) \\ &\leqslant_{Adm} \mathbf{1} = \sigma_{\mathcal{A}}(x,z).\end{aligned}$$

4. If $z <_{Adm} y <_{Adm} x$, then

$$\begin{aligned}\mathcal{A}(\sigma_{\mathcal{A}}(x,y),\sigma_{\mathcal{A}}(y,z)) &= \mathcal{A}(\mathcal{A}(N_{IV}(x),y),\mathcal{A}(N_{IV}(y),z))\\ &= \mathcal{A}(\mathcal{A}(N_{IV}(x),z),\mathcal{A}(y,N_{IV}(y)))\\ &= \mathcal{A}(N_{IV}(x),z) = \sigma_{\mathcal{A}}(x,z).\end{aligned}$$

5. If $x = y = z$, then

$$\begin{aligned}\mathcal{A}(\sigma_{\mathcal{A}}(x,y),\sigma_{\mathcal{A}}(y,z)) &= \mathcal{A}([1-w(x),1],[1-w(x),1])\\ &\leqslant_{Adm} [1-w(x),1] = \sigma_{\mathcal{A}}(x,z).\end{aligned}$$

6. If $y <_{Adm} z <_{Adm} x$, then

$$\begin{aligned}\mathcal{A}(\sigma_{\mathcal{A}}(x,y),\sigma_{\mathcal{A}}(y,z)) &= \mathcal{A}(\mathcal{A}(N_{IV}(x),y),\mathbf{1})\\ &= \mathcal{A}(\mathcal{A}(N_{IV}(x),y),\mathcal{A}(z,N_{IV}(z)))\\ &= \mathcal{A}(\mathcal{A}(N_{IV}(x),z),\mathcal{A}(y,N_{IV}(z)))\\ &\leqslant_{Adm} \mathcal{A}(\mathcal{A}(N_{IV}(x),z),\mathcal{A}(y,N_{IV}(y)))\\ &= \mathcal{A}(N_{IV}(x),z) = \sigma_{\mathcal{A}}(x,z).\end{aligned}$$

7. If $z <_{Adm} y = x$, then

$$\begin{aligned}\mathcal{A}(\sigma_{\mathcal{A}}(x,y),\sigma_{\mathcal{A}}(y,z)) &= \mathcal{A}([1-w(x),1],\mathcal{A}(N_{IV}(y),z))\\ &= \mathcal{A}([1-w(x),1],\mathcal{A}(N_{IV}(y),z))\\ &= \mathcal{A}([1-w(x),1],\mathcal{A}(N_{IV}(x),z))\\ &\leqslant_{Adm} \mathcal{A}(\mathbf{1},\mathcal{A}(N_{IV}(x),z))\\ &= \sigma_{\mathcal{A}}(x,z).\end{aligned}$$

Similarly we can show the remaining 6 cases. As a result $\sigma_{\mathcal{A}}$ is a $\mathcal{A}$-quasi-ordered operation. By analogy, we may prove the case of $\mathcal{B}$-quasi-order.

3.1.2. Interval subsethood measure II

Definition 6 is satisfactory in situations where the comparisons of subsethood measure values is not required for strongly comparable elements, as there are no differences in these situations (see axiom (IM2) of Definition 6). Consider, for example, the partial order $\leqslant_{L^I}$, thus,

$$\sigma(\mathbf{0},\mathbf{1}) = \sigma([0.1,0.5],[0.3,0.7]) = \mathbf{1}.$$

However if, for application purposes, we needed to distinguish the subsethood values for strongly comparable elements, then we may use the following axiom (IM2') instead of (IM2):

(IM2') If $x < y$, then $\overline{\sigma}(x,y) = 1$.

Thus, we propose another definition of an interval subsethood measure.

Definition 7. A function $\sigma : (L^I)^2 \to L^I$ is said to be a strengthened interval subsethood measure, if it satisfies the following conditions:

(IM1) If $x = \mathbf{1}$, $y = \mathbf{0}$, then $\sigma(x,y) = \mathbf{0}$;
(IM2) If $x < y$, then $\overline{\sigma}(x,y) = 1$;
(IM3) $\sigma(x,x) = [1-w(x),1]$ (reflexivity);
(IM4) If $x \leqslant y \leqslant z$ and $w(x) = w(y) = w(z)$, then $\sigma(z,x) \leqslant \sigma(y,x)$ and $\sigma(z,x) \leqslant \sigma(z,y)$ for $x,y,z \in L^I$.

Let us denote by

$$\mathcal{S}' = \{\sigma : (L^I)^2 \to L^I \mid \sigma \text{ is a strengthened subsethood measure}\}.$$

The dependence between the families $\mathcal{S}$ and $\mathcal{S}'$ is clear:

$$\mathcal{S} \subset \mathcal{S}',$$

as depicted in Figure 1.

Remark 6

Observe that $w(x) < w(y)$ (respectively, $w(x) = w(y)$) if and only if $\sigma(y,y) < \sigma(x,x)$ (respectively, $\sigma(y,y) = \sigma(x,x)$).

Since (IM2') provides only the upper value of an interval, for the partial order $\leqslant_{L^I}$, we may propose the following method to construct the lower value and, as a result, an example of a strengthened interval subsethood measure fulfilling axioms (IM1), (IM2'), (IM3) and (IM4) (Definition 7).

For $x, y \in L^I$ we set

$$r(x,y) = \max\{|\underline{x}-\underline{y}|, |\overline{x}-\overline{y}|\}.$$

Observe that $r(x,y) = r(y,x)$ in any case, and that $x = y$ if and only if $r(x,y) = 0$.

Theorem 3. *For $x, y \in L^I$ the operation $\sigma : (L^I)^2 \to L^I$ is a strengthened interval subsethood measure*

$$\underline{\sigma}(x,y) = 1 - \max(w(x), r(x,y)),$$

and

$$\overline{\sigma}(x,y) = \begin{cases} 1, & x <_{L^I} y,\\ 1 - r(x,y), & \text{otherwise}.\end{cases}$$

Proof. The map σ is well defined as in any case $0 \leqslant \underline{\sigma}(x,y) \leqslant \overline{\sigma}(x,y) \leqslant 1$.

(IM1) $r(\mathbf{1},\mathbf{0}) = 1$ and $w(\mathbf{1}) = 0$, hence $\sigma(\mathbf{1},\mathbf{0}) = \mathbf{0}$.

(IM2) Is satisfies by definition of operation σ.

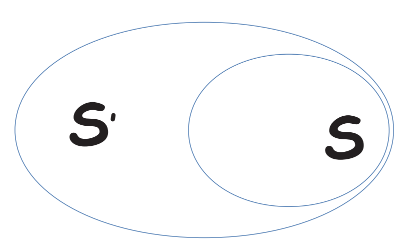

Figure 1 Dependence between the families $\mathcal{S}$ and $\mathcal{S}'$ of interval subsethood measures and strengthened interval subsethood measures, respectively.

(IM3) As $r(x,x)=0$ and so $\max(w(x), r(x,x)) = w(x)$.

(IM4) Assume $x \leqslant_{L^I} y \leqslant_{L^I} z$, $w(x)=w(y)=w(z) := w$. Then we have

$\underline{x} \leqslant \underline{y} \leqslant \underline{z}$ and $\overline{x} \leqslant \overline{y} \leqslant \overline{z}$. So $\underline{z}-\underline{y} \leqslant \underline{z}-\underline{x}$ and $\overline{z}-\overline{y} \leqslant \overline{z}-\overline{x}$,

hence $r(z,y) \leqslant r(z,x)$. Analogously $r(y,x) \leqslant r(z,x)$. Therefore if $x <_{L^I} y <_{L^I} z$ we have

$$\sigma(z,x) = [1-w, 1-r(z,x)] \leqslant_{L^I} [1-w, 1-r(z,y)] = \sigma(z,y),$$

and analogously $\sigma(z,x) \leqslant_{L^I} \sigma(y,x)$. The case $x=y=z$ is trivial and the cases $x=y<_{L^I} z$, respectively $x <_{L^I} y = z$, follow immediately taking in account that then we have $\overline{\sigma}(y,x)=1 \leqslant \overline{\sigma}(z,x)$, respectively $\overline{\sigma}(z,y) = 1 \leqslant \overline{\sigma}(z,x)$.

Considering the construction from Theorem 3, we derive the following results.

Proposition 6. *Let $\sigma \in \mathcal{S}'$ as in Theorem 3. For $x,y \in L^I$, $\overline{\sigma}(x,y) = 1$ if and only if $x \leqslant_{L^I} y$.*

Proposition 7. *Let $\sigma \in \mathcal{S}'$ as in Theorem 3. For $x,y \in L^I$ the following are equivalent:*

1. $\underline{\sigma}(x,y) = 1$,
2. $\sigma(x,y) = \mathbf{1}$,
3. $x=y$ *and* $w(x)=0$.

Proof. As $\underline{\sigma}(x,y) \leqslant \overline{\sigma}(x,y)$ we have 1. ⇔ 2. Further $\underline{\sigma}(x,y)=1$ is equivalent to $w(x)=r(x,y)=0$, that is to $x=y$ and $w(x)=0$, and 1. ⇔ 3.

Proposition 8. *Let $\sigma \in \mathcal{S}'$ as in Theorem 3. For $x,y \in L^I$, $\underline{\sigma}(x,y) = 0$ if and only if either $x=[0,1]$, or $x=\mathbf{1}$ and $\underline{y}=0$, or $y=\mathbf{1}$ and $\underline{x}=0$, or $x=\mathbf{0}$ and $\overline{y}=1$, or $y=\mathbf{0}$ and $\overline{x}=1$.*

Proof. As $w(x)=1$ if and only if $x=[0,1]$, and $r(x,y)=1$ if and only if $x=\mathbf{1}$ and $\underline{y}=0$, or $y=\mathbf{1}$ and $\underline{x}=0$, or $x=\mathbf{0}$ and $\overline{y}=1$, or $y=\mathbf{0}$ and $\overline{x}=1$.

Proposition 9. *Let $\sigma \in \mathcal{S}'$ as in Theorem 3. For $x,y \in L^I$ the following are equivalent:*

1. $\overline{\sigma}(x,y) = 0$,
2. $\sigma(x,y) = \mathbf{0}$,
3. *Either* $x=\mathbf{1}$ *and* $\underline{y}=0$, *or* $y=\mathbf{0}$ *and* $\overline{x}=1$.

Proof. As above, 1. ⇔ 2. Now by definition $\overline{\sigma}(x,y)=0$ if and only if $x \nleqslant y$ and $r(x,y)=1$, applying Proposition 8.

Let us now present some other construction methods for strengthened interval subsethood measures.

Theorem 4. *For $x,y \in L^I$ the operation $\sigma_{z'} : (L^I)^2 \to L^I$ is a strengthened interval subsethood measure*

$$\sigma_{z'}(x,y) = \begin{cases} [1-w(x), 1], & x \leqslant y, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

Proof. Justification is analogous to Theorem 1.

Proposition 10. *Let $\mathcal{B} : (L^I)^2 \to L^I$ be an interval-valued aggregation function with respect to $\leqslant_{Adm}$ such that $\mathcal{B} \leqslant_{Adm} \mathcal{A}_p$. Then, $\sigma_{z'}$ is a $\mathcal{B}$-quasi-ordered operation (reflexive and $\mathcal{B}$-transitive with respect to $\leqslant_{Adm}$).*

Proof. Reflexivity is obvious by (IM3). We will prove $\mathcal{B}$-transitivity of $\sigma_{z'}$, that is,

$$\mathcal{B}(\sigma_{z'}(x,y), \sigma_{z'}(y,z)) \leqslant_{Adm} \sigma_{z'}(x,z), \quad x,y,z \in L^I.$$

By $\mathcal{B} \leqslant_{Adm} \mathcal{A}_p$ (i.e., $\mathcal{B}$ has element zero $\mathbf{0}$) we consider the following cases:

1. If $x \leqslant_{Adm} y \leqslant_{Adm} z$, then

$$\begin{aligned}\mathcal{B}(\sigma_{z'}(x,y), \sigma_{z'}(y,z)) &= \mathcal{B}([1-w(x),1],[1-w(y),1]) \\ &\leqslant_{Adm} \mathcal{A}_p([1-w(x),1],[1-w(y),1]) \\ &\leqslant_{Adm} [1-w(x),1] = \sigma_{z'}(x,z).\end{aligned}$$

2. If $y <_{Adm} x \leqslant_{Adm} z$, then

$$\begin{aligned}\mathcal{B}(\sigma_{z'}(x,y), \sigma_{z'}(y,z)) &= \mathcal{B}(\mathbf{0},[1-w(y),1]) = \mathbf{0} \\ &\leqslant_{Adm} [1-w(x),1] = \sigma_{z'}(x,z).\end{aligned}$$

3. If $y \leqslant_{Adm} z < x$, then

$$\begin{aligned}\mathcal{B}(\sigma_{z'}(x,y), \sigma_{z'}(y,z)) &= \mathcal{B}(\mathbf{0},[1-w(y),1]) = \mathbf{0} \\ &= \sigma_{z'}(x,z).\end{aligned}$$

Similarly, the remaining 3 cases can be checked. As a result, $\sigma_{z'}$ is a $\mathcal{B}$-quasi-ordered operation.

Remark 7

We may obtain a similar result to Proposition 10 considering the partial order $\leqslant_{L^I}$, that is, $\mathcal{B}$-transitivity with respect to $\leqslant_{L^I}$ and $\mathcal{B} \leqslant_{L^I} \mathcal{A}_p$ which is an interval-valued aggregation with respect to $\leqslant_{L^I}$.

Theorem 5. *Let $x,y \in L^I$ and let the function $\sigma_{\mathcal{A}} : (L^I)^2 \to L^I$ be given by*

$$\sigma_{\mathcal{A}}(x,y) = \begin{cases} [1-w(x), 1], & x = y, \\ [A_1(n(\overline{x}), \underline{y}), 1], & x < y, \\ \mathcal{A}(N_{IV}(x), y), & \text{otherwise,} \end{cases}$$

for an interval-valued fuzzy negation N_{IV} such that

$$N_{IV}(x) = [n(\overline{x}), n(\underline{x})] \leqslant [1-\overline{x}, 1-\underline{x}],$$

where n is a fuzzy negation and $\mathcal{A}$ is a representable interval-valued aggregation function with respect to $\leqslant$ such that $\mathcal{A} = [A_1, A_2] \leqslant \vee$.

Thus, $\sigma_{\mathcal{A}}$ is a strengthened interval subsethood measure.

Proof. Justification is similar to the one in Theorem 2.

Using the construction method given in Theorem 5 we obtain the following example.

Example 7. Let us consider the partial order $\leqslant_{L^I}$. The following is a strengthened interval subsethood measure:

$$\sigma(x,y) = \begin{cases} [1 - w(x), 1], & x = y, \\ \left[\dfrac{1-\overline{x}+\underline{y}}{2}, 1\right], & x <_{L^I} y, \\ \left[\dfrac{1-\overline{x}+\underline{y}}{2}, \dfrac{1-\underline{x}+\overline{y}}{2}\right], & \text{otherwise.} \end{cases}$$

Theorem 6. *Let $x, y \in L^I$ and let the function $\sigma_{\mathcal{A}} : (L^I)^2 \to L^I$ be given by*

$$\sigma_{\mathcal{A}}(x,y) = \begin{cases} [1 - \max(w(x), r(x,y)), 1], & x \leqslant_{L^I} y, \\ \mathcal{A}(N_{IV}(x), y), & \text{otherwise,} \end{cases}$$

where N_{IV} is an interval-valued fuzzy negation such that

$$N_{IV}(x) = [n(\overline{x}), n(\underline{x})] \leqslant_{L^I} [1-\overline{x}, 1-\underline{x}],$$

where n is fuzzy negation and $\mathcal{A}$ is a representable interval-valued aggregation function with respect to the order $\leqslant_{L^I}$, satisfying $\mathcal{A} = [A_1, A_2] \leqslant_{L^I} \vee$.

Thus, $\sigma_{\mathcal{A}}$ is a strengthened interval subsethood measure.

Proof. Justification is analogous to Theorem 2.

Using the construction method given in Theorem 6 we get the following example.

Example 8. Let us consider the partial order $\leqslant_{L^I}$. The following is a strengthened interval subsethood measure

$$\sigma(x,y) = \begin{cases} [1 - \max(w(x), r(x,y)), 1], & x \leqslant_{L^I} y, \\ \left[\dfrac{1-\overline{x}+\underline{y}}{2}, \dfrac{1-\underline{x}+\overline{y}}{2}\right], & \text{otherwise.} \end{cases}$$

3.2. Connection between Interval-Valued Implication Functions and Subsethood Measures

Fuzzy implication operators are an example of functions that are used in many applications. In the literature, the definition of an implication in the interval-valued setting has been provided with respect to the partial order $\leqslant_{L^I}$ (cf. [40,45]), but note that it is possible to build interval-valued implication functions with respect to diverse orders. In [30], the definition and study of an interval-valued implication with respect to a total order was presented.

Definition 8. An interval-valued fuzzy implication with respect to $\leqslant$ is a function $I_{IV} : (L^I)^2 \to L^I$ which verifies the following properties:

i. I_{IV} is a decreasing function in the first component and an increasing function in the second component with respect to the order $\leqslant$,

ii. $I_{IV}(\mathbf{0},\mathbf{0}) = I_{IV}(\mathbf{1},\mathbf{1}) = I_{IV}(\mathbf{0},\mathbf{1}) = \mathbf{1}$,

iii. $I_{IV}(\mathbf{1},\mathbf{0}) = \mathbf{0}$.

We would like to point out the connection between interval-valued implication functions and the examined interval subsethood measures.

Remark 8

Let $x, y, z \in L^I$ and $w(x) = w(y) = w(z)$.

- Let $\sigma \in \mathcal{S}$. Then σ is an interval-valued implication function.
- Let $\sigma \in \mathcal{S}'$. Then σ is an interval-valued implication function if $\sigma(\mathbf{0},\mathbf{1}) = \mathbf{1}$.

We see that (IM1) implies $\sigma(\mathbf{1},\mathbf{0}) = \mathbf{0}$, (IM2) implies $\sigma(\mathbf{0},\mathbf{1}) = \mathbf{1}$ and (IM3) implies $\sigma(\mathbf{0},\mathbf{0}) = \sigma(\mathbf{1},\mathbf{1}) = \mathbf{1}$ because $w(x) = 0$. Moreover, by (IM4), we observe that σ is a decreasing function in the first component and an increasing function in the second component with respect to the order $\leqslant$. Thus, $\sigma \in \mathcal{S}$ is an interval-valued implication function.

Condition (IM2'), the weaker version of (IM2), implies that we need to add the assumption $\sigma(\mathbf{0},\mathbf{1}) = \mathbf{1}$ to recover an interval-valued implication function from σ.

3.3. Subsethood Measures of Interval-Valued Fuzzy Sets

Subsethood measures may be also defined to give an estimation of "how included" an interval-valued set is in another.

We use the notion of interval-valued aggregation function to define subsethood measures and strengthened subsethood measures of interval-valued fuzzy sets.

Definition 9. Let $\mathcal{M} : (L^I)^n \to L^I$ be an interval-valued aggregation function and σ be an interval subsethood measure (respectively, a strengthened interval subsethood measure). The mapping $\sigma^{\mathcal{M}} : IVFS(U) \times IVFS(U) \to L^I$ given by

$$\sigma^{\mathcal{M}}(A,B) = \mathcal{M}(\sigma(A(u_1), B(u_1)), \ldots, \sigma(A(u_n), B(u_n))),$$

is a subsethood measure (respectively, a strengthened subsethood measure) on $IVFS(U)$ defined by σ and $\mathcal{M}$.

Definition 9 presents the concept of subsethood measure (and strengthened subsethood measure) between interval-valued fuzzy sets providing a method for constructing such a measure from an interval subsethood measure (or a strengthened interval subsethood measure). In what follows, we present two theorems that describe the properties that a so-constructed subsethood measure between interval-valued fuzzy sets satisfy. Note that there is concordance between these properties and the ones of interval subsethood measures and strengthened interval subsethood measures in Section III.A. Additionally, the properties presented in the next theorems are in accordance with a possible axiomatic definition of subsethood measure for interval-valued fuzzy sets, which justifies Definition 9.

Given $A \in IVFS(U)$, we use the following notation

$$w(A) = (w(a_1), \ldots, w(a_n)).$$

Moreover, $0, 1 : U \to L^I$ are defined by $0(u_i) = \mathbf{0}$, $1(u_i) = \mathbf{1}$ for each $i = 1, \dots, n$.

Theorem 7. *Let U be a nonempty set such that $card(U) = n \in \mathbb{N}$ and $\sigma^{\mathcal{M}}$ be a subsethood measure on $IVFS(U)$ defined by an interval subsethood measure σ and an interval-valued aggregation function $\mathcal{M}$. Then, for $A, B, C \in IVFS(U)$, the following hold:*

(IMV1) $\sigma^{\mathcal{M}}(1, 0) = \mathbf{0}$,

(IMV2) *if $A \prec B$, then* $\sigma^{\mathcal{M}}(A, B) = \mathbf{1}$,

(IMV3) $\sigma^{\mathcal{M}}(A, A) = \mathcal{M}([1 - w(A(u_1)), 1], \dots, [1 - w(A(u_n)), 1])$,

(IMV4) *if $A \preceq B \preceq C$ and $w(A) = w(B) = w(C)$, then $\sigma^{\mathcal{M}}(C, A) \leqslant \sigma^{\mathcal{M}}(C, B)$ and $\sigma^{\mathcal{M}}(C, A) \leqslant \sigma^{\mathcal{M}}(B, A)$.*

Proof. Let us set $a_i = A(u_i)$, $b_i = B(u_i)$, $c_i = C(u_i)$, $i = 1, \dots, n$.

(IMV1) By (IM1) we get

$$\begin{aligned}\sigma^{\mathcal{M}}(1, 0) &= \mathcal{M}(\sigma(\mathbf{1}, \mathbf{0}), \dots, \sigma(\mathbf{1}, \mathbf{0})) \\ &= M(\mathbf{0}, \dots, \mathbf{0}) = \mathbf{0}.\end{aligned}$$

(IMV2) Assume that $A \prec B$, then $a_i < b_i$ for $i = 1, \dots, n$ and, by (IM2), it holds that

$$\begin{aligned}\sigma^{\mathcal{M}}(A, B) &= \mathcal{M}(\sigma(a_1, b_1), \dots, \sigma(a_n, b_n)) \\ &= M(\mathbf{1}, \dots, \mathbf{1}) = \mathbf{1}.\end{aligned}$$

(IMV3) It follows the fact that, by (IM3), we have $\sigma(a_i, a_i) = [1 - w(a_i), 1]$.

(IMV4) Assume that $A \preceq B \preceq C$ and $w(A) = w(B) = w(C)$. Then, it holds that $a_i \leqslant b_i \leqslant c_i$ and $w(a_i) = w(b_i) = w(c_i)$ for $i = 1, \dots, n$. Thus, by (IM4),

$$\begin{aligned}\sigma^{\mathcal{M}}(C, A) &= \mathcal{M}(\sigma(c_1, a_1), \dots, \sigma(c_n, a_n)) \\ &\leqslant \mathcal{M}(\sigma(c_1, b_1), \dots, \sigma(c_n, b_n)) \\ &= \sigma^{\mathcal{M}}(C, B).\end{aligned}$$

Similarly, it can be shown that $\sigma^{\mathcal{M}}(C, A) \leqslant \sigma^{\mathcal{M}}(B, A)$, which proves (IMV4).

Theorem 8. *Let U be a nonempty set such that $card(U) = n \in \mathbb{N}$ and $\sigma^{\mathcal{M}}$ be a strengthened subsethood measure on $IVFS(U)$ defined by a strengthened interval subsethood measure σ and a representable interval-valued aggregation function $\mathcal{M} = [M_1, M_2]$. Then, for $A, B, C \in IVFS(U)$, conditions (IMV1), (IMV3), (IMV4) are fulfilled. Moreover, the following condition holds:*

(IMV2') $A \prec B$, *then* $\overline{\sigma^{\mathcal{M}}}(A, B) = 1$.

Proof. By Theorem 7, it suffices to show (IMV2'). Setting $a_i = A(u_i)$ and $b_i = B(u_i)$ for $i = 1, \dots, n$, we have that if $A \prec B$, then $a_i < b_i$ for $i = 1, \dots, n$. Consequently, by (IM2'), it holds that

$$\begin{aligned}\overline{\sigma^{\mathcal{M}}}(A, B) &= M_2(\overline{\sigma}(a_1, b_1), \dots, \overline{\sigma}(a_n, b_n)) \\ &= M_2(1, \dots, 1) = 1.\end{aligned}$$

As we can observe by Theorems 7 and 8 the subsethood measures or the strengthened subsethood measure have similar properties to their corresponding generators, or interval subsethood measures.

4. CONCLUSIONS

In this paper, we have discussed two possible axiomatical definitions of interval subsethood measures for the interval-valued fuzzy setting taking into account the widths of the intervals involved. Specifically, we have introduced interval subsethood measures (Definition 6) and strengthened interval subsethodd measures (Definition 7). The relationships among the proposed subsethood measures of intervals have been examined.

Since the inclusion of the width of intervals has been proven to be useful in image processing [26,27] and so have fuzzy subsethood measures [19], our plan for future works is to apply the introduced subsethood measures in constructions of width-based indistinguishability measures and to use them in image processing problems.

CONFLICT OF INTEREST

The authors declare no conflict of interests.

AUTHORS' CONTRIBUTIONS

Barbara Pekala, Urszula Bentkowska and Mikel Sesma-Sara developed the main mathematical results and were in charge of writing. Javier Fernandez, Julio Lafuente and Humberto Bustince conceived the idea and developed some of the theoretical results. Abdulrahman Altalhi, Maksymilian Knap and Jesús M. Pintor checked the correctness of the theory and the writing.

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