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General grouping functions

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Abstract. Some aggregation functions that are not necessarily associative, namely overlap and grouping functions, have called the attention of many researchers in the recent past. This is probably due to the fact that they are a richer class of operators whenever one compares with other classes of aggregation functions, such as t-norms and t-conorms, respectively. In the present work we introduce a more general proposal for disjunctive n-ary aggregation functions entitled general grouping functions, in order to be used in problems that admit n dimensional inputs in a more flexible manner, allowing their application in different contexts. We present some new interesting results, like the characterization of that operator and also provide different construction methods.

Keywords: Grouping functions · n-dimensional grouping functions · General grouping functions · General overlap functions.

1 Introduction

Overlap functions are a kind of aggregation functions [3] that are not required to be associative, and they were introduced by Bustince et al. in [4] to measure the degree of overlapping between two classes or objects. Grouping functions are the dual notion of overlap function. They were introduced by Bustince et al. [5] in order to express the measure of the amount of evidence in favor of either of two alternatives when performing pairwise comparisons [1] in decision making based on fuzzy preference relations [6]. They have also been used as the disjunction operator in some important problems, such as image thresholding [17] and the construction of a class of implication functions for the generation of fuzzy subsethood and entropy measures [8].

Overlap and grouping have been largely studied since they are richer than t-norms and t-conorms [18], respectively, in many aspects, considering, for example, some properties like idempotency, homogeneity, and, mainly, the self-closeness feature with respect to the convex sum and the aggregation by generalized composition of overlap/grouping functions [9,10,12,7]. For example, there is just one idempotent t-conorm (namely, the maximum t-conorm) and two homogeneous t-conorms (namely, the maximum and the probabilistic sum of t-conorms). On the contrary, there are uncountable numbers of idempotent, as well as homogenous, grouping functions [2,13]. For comparisons among properties of grouping functions and t-conorms, see [2,5,17]

However, grouping functions are bivariate functions. Since they may be non associative, they can only be applied in bi-dimensional problems (that is, when just two classes or objects are considered). In order to solve this drawback, Gómez et al. [16] introduced the concept of n-dimensional grouping functions, with an application to fuzzy community detection.

Recently, De Miguel et al. [20] introduced general overlap functions, by relaxing some boundary conditions, in order to apply to an n-ary problem, namely, fuzzy rule based classification systems, more specifically, in the determination of the matching degree in the fuzzy reasoning method. Then, inspired on the paper by De Miguel et al. [20], the objective of this present paper is to introduce the concept of general grouping functions, providing their characterization and different construction methods. The aim is to define the theoretical basis of a tool that can be used to express the measure of the amount of evidence in favor of one of multiple alternatives when performing n-ary comparisons in multi-criteria decision making based on n-ary fuzzy heterogeneous, incomplete preference relations [14,19,26], which we let for future work.

The paper is organized as follows. Section 2 presents some preliminary concepts. In Sect. 3, we define general grouping functions, studying some properties. In Sect. 4, we study the characterization of general grouping functions, providing some construction methods. Section 5 is the Conclusion.

2 Preliminary concepts

In this section, we highlight some relevant concepts used in this work.

Definition 1. *A function* $N\colon [0,1] \to [0,1]$ *is a fuzzy negation if it holds:* $(N1)$ N *is antitonic, i.e.* $N(x) \leq N(y)$ *whenever* $y \leq x$ *and* $(N2)$ $N(0) = 1$ *and* $N(1) = 0$.

Definition 2. *[3] An* n*-ary aggregation function is a mapping* $A\colon [0,1]^n \to [0,1]$ *satisfying:* $(A1)$ $A(0,\ldots,0) = 0$ *and* $A(1,\ldots,1) = 1$*;* $(A2)$ *increasingness: for each* $i \in \{1,\ldots,n\}$*, if* $x_i \leq y$ *then* $A(x_1,\ldots,x_n) \leq A(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_n)$.

Definition 3. *An n-ary aggregation function* $A\colon [0,1]^n \to [0,1]$ *is called conjunctive if, for any* $\overrightarrow{x} = (x_1,\ldots,x_n) \in [0,1]^n$*, it holds that* $A(\overrightarrow{x}) \leq \min(\overrightarrow{x}) = \min\{x_1,\ldots,x_n\}$*. And* A *is called disjunctive if, for any* $\overrightarrow{x} = (x_1,\ldots,x_n) \in [0,1]^n$*, it holds that* $A(\overrightarrow{x}) \geq \max(\overrightarrow{x}) = \max\{x_1,\ldots,x_n\}$.

Definition 4. *[4] A binary function* $O\colon [0,1]^2 \to [0,1]$ *is said to be an overlap function if it satisfies the following conditions, for all* $x, y, z \in [0,1]$*:*

(O1) *$O(x, y) = O(y, x)$;*
(O2) *$O(x, y) = 0$ if and only if $x = 0$ or $y = 0$;*
(O3) *$O(x, y) = 1$ if and only if $x = y = 1$;*
(O4) *if $x \leq y$ then $O(x, z) \leq O(y, z)$;*
(O5) *O is continuous;*

Definition 5. *[5] A binary function $G\colon [0, 1]^2 \to [0, 1]$ is said to be a grouping function if it satisfies the following conditions, for all $x, y, z \in [0, 1]$:*

(G1) *$G(x, y) = G(y, x)$;*
(G2) *$G(x, y) = 0$ if and only if $x = y = 0$;*
(G3) *$G(x, y) = 1$ if and only if $x = 1$ or $y = 1$;*
(G4) *If $x \leq y$ then $G(x, z) \leq G(y, z)$;*
(G5) *G is continuous;*

For all properties and related concepts on overlap functions and grouping functions, see [2,5,9,11,12,21,23,24,25].

Definition 6. *[22] A function $G\colon [0, 1]^2 \to [0, 1]$ is a 0-grouping function if the second condition in Def. 5 is replaced by: (G$2'$) If $x = y = 0$ then $G(x, y) = 0$. Analogously, a function $G\colon [0, 1]^2 \to [0, 1]$ is a 1-grouping function if the third condition in Def. 5 is replaced by: (G$3'$) If $x = 1$ or $y = 1$ then $G(x, y) = 1$.*

Both notions were extended in several ways and some of them are presented in the following definitions.

Definition 7. *[15] An n-ary function $\mathcal{G}\colon [0, 1]^n \to [0, 1]$ is called an n-dimensional grouping function if for all $\vec{x} = (x_1, ..., x_n) \in [0, 1]^n$:*

1. *$\mathcal{G}$ is commutative;*
2. *$\mathcal{G}(\vec{x}) = 0$ if and only if $x_i = 0$, for all $i = 1, \ldots, n$;*
3. *$\mathcal{G}(\vec{x}) = 1$ if and only if there exists $i \in \{1, \ldots, n\}$ with $x_i = 1$;*
4. *$\mathcal{G}$ is increasing;*
5. *$\mathcal{G}$ is continuous.*

Definition 8. *[20] A function $\mathcal{GO}\colon [0, 1]^n \to [0, 1]$ is said to be a general overlap function if it satisfies the following conditions, for all $\vec{x} = (x_1, \ldots, x_n) \in [0, 1]^n$:*

($\mathcal{GO}1$) *$\mathcal{GO}$ is commutative;*
($\mathcal{GO}2$) *If $\prod_{i=1}^{n} x_i = 0$ then $\mathcal{GO}(\vec{x}) = 0$;*
($\mathcal{GO}3$) *If $\prod_{i=1}^{n} x_i = 1$ then $\mathcal{GO}(\vec{x}) = 1$;*
($\mathcal{GO}4$) *$\mathcal{GO}$ is increasing;*
($\mathcal{GO}5$) *$\mathcal{GO}$ is continuous.*

3 General grouping functions

Following the ideas given in [20], we can also generalize the idea of general grouping functions as follows.

Definition 9. *A function $\mathcal{GG}\colon [0,1]^n \to [0,1]$ is called a general grouping function if the following conditions hold, for all $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$:*

($\mathcal{GG}1$) *$\mathcal{GG}$ is commutative;*
($\mathcal{GG}2$) *If $\sum_{i=1}^{n} x_i = 0$ then $\mathcal{GG}(\vec{x}) = 0$;*
($\mathcal{GG}3$) *If there exists $i \in \{1, \ldots, n\}$ such that $x_i = 1$ then $\mathcal{GG}(\vec{x}) = 1$;*
($\mathcal{GG}4$) *$\mathcal{GG}$ is increasing;*
($\mathcal{GG}5$) *$\mathcal{GG}$ is continuous.*

Note that ($\mathcal{GG}2$) is the same of saying that 0 is an anhilator of the general grouping function $\mathcal{GG}$.

Proposition 1. *If $\mathcal{G}\colon [0,1]^n \to [0,1]$ is an n-dimensional grouping function, then $\mathcal{G}$ is also a general grouping function.*

Proof. Straighforward. □

From this proposition, we can conclude that the concept of general grouping functions is a generalization of n-dimensional grouping functions, which on its turn is a generalization of the concepts of 0-grouping functions and 1-grouping functions.

Example 1.
1. Every grouping function $G\colon [0,1]^2 \to [0,1]$ is a general grouping function, but the converse does not hold.
2. The function $\mathcal{GG}(x,y) = \min\{1, 2-(1-x)^2-(1-y)^2\}$ is a general grouping function, but it is not a bidimensional grouping function, since $\mathcal{GG}(0.5, 0.5) = 1$.
3. Consider $G(x,y) = \max\{1-(1-x)^p, 1-(1-y)^p\}$, for $p > 0$ and $S_{\mathfrak{L}}(x,y) = \min\{1, x+y\}$. Then, the function $\mathcal{GG}^{S_{\mathfrak{L}}}(x,y) = G(x,y)S_{\mathfrak{L}}(x,y)$ is a general grouping function.
4. Take any grouping function G, and a continuous t-conorm S. Then, the generalization of the previous item is the binary general grouping function given by: $\mathcal{GG}(x,y) = G(x,y)S(x,y)$
5. Other examples are:

$$Prod\_S\_Luk(x_1, \ldots, x_n) = \left(1 - \prod_{i=1}^{n}(1-x_i)\right) * \left(\min\left\{\sum_{i=1}^{n} x_i, 1\right\}\right)$$

$$GM\_S\_Luk(x_1, \ldots, x_n) = \left(1 - \sqrt[n]{\prod_{i=1}^{n}(1-x_i)}\right) * \left(\min\left\{\sum_{i=1}^{n} x_i, 1\right\}\right).$$

The generalization of the third item of Example 1 can be seen as follows.

Proposition 2. *Take any grouping function G, and any t-conorm S. Then, the binary general grouping function given by: $\mathcal{GG}(x,y) = G(x,y)S(x,y)$.*

Proposition 3. *Let $F\colon [0,1]^n \to [0,1]$ be a commutative and continuous aggregation function. Then the following statements hold:*

(i) *If F is disjunctive, then F is a general grouping function.*
(ii) *If F is conjunctive, then F is neither a general grouping function nor an n-dimensional grouping function.*

Proof. Consider a commutative and continuous aggregation function $F\colon [0,1]^n \to [0,1]$. It follows that:
(i) Since F is commutative ($\mathcal{GG}1$), continuous ($\mathcal{GG}5$) and clearly increasing ($\mathcal{GG}4$), then it remains to prove the following:
($\mathcal{GG}2$) Suppose that $\sum_{i=1}^{n} x_i = 0$. Then, since F is an aggregation function, it holds that $F(0,\ldots,0) = 0$.
($\mathcal{GG}3$) Suppose that, for some $\vec{x} = (x_1,\ldots,x_n) \in [0,1]^n$, there exists $i \in \{1,\ldots,n\}$ such that $x_i = 1$. Then, since F is disjunctive, then $F(\vec{x}) \geq \max\{x_1,\ldots,1,\ldots,x_n\} = 1$, which means that $F(\vec{x}) = 1$.
(ii) Suppose that F is a conjunctive aggregation function and it is either a general grouping function or an n-dimensional grouping function. Then, by either **($\mathcal{GG}3$)** or **($\mathcal{G}3$)**, if for some $\vec{x} = (x_1 \ldots, x_n) \in [0,1]^n$, there exists $i \in \{1,\ldots,n\}$ such that $x_i = 1$, then $F(\vec{x}) = 1$. Take $\vec{x} = (1,0\ldots,0)$, it follows that $F(1,0\ldots,0) = 1 = \max\{1,0\ldots,0\} \not\leq 0 = \min\{1,0\ldots,0\}$, which is a contradiction with the conjunctive property. Thus, one concludes that F is neither a general grouping function nor an n-dimensional grouping function. □

We say that an element $a \in [0,1]$ is a neutral element of $\mathcal{GG}$ if for each $x \in [0,1]$, $\mathcal{GG}(x, \underbrace{a,\ldots,a}_{(n-1)}) = x$.

Proposition 4. *Let $\mathcal{GG}\colon [0,1]^n \to [0,1]$ be a general grouping function with a neutral element $a \in [0,1]$. Then, $a = 0$ if and only if $\mathcal{GG}$ satisfies, for all $\vec{x} = (x_1,\ldots,x_n) \in [0,1]^n$, the following condition:*

($\mathcal{GG}2'$) *If $\mathcal{GG}(\vec{x}) = 0$, then $\sum_{i=1}^{n} x_i = 0$.*

Proof. ($\Rightarrow$) Suppose that (i) the neutral element of $\mathcal{GG}$ is $a = 0$ and (ii) $\mathcal{GG}(x_1,\ldots,x_n) = 0$. Then, by (i), one has that, for each $x_1 \in [0,1]$, it holds that $x_1 = \mathcal{GG}(x_1, 0\ldots,0)$. By (ii) and since $\mathcal{GG}$ is increasing, it follows that

$$x_1 = \mathcal{GG}(x_1, 0\ldots,0) \leq \mathcal{GG}(x_1,\ldots,x_n) = 0.$$

Similarly, one shows that $x_2,\ldots,x_n = 0$, that is $\sum_{i=1}^{n} x_i = 0$.
($\Leftarrow$) Suppose that $\mathcal{GG}$ satisfies ($\mathcal{GG}2'$) and that $\mathcal{GG}(x_1,\ldots,x_n) = 0$, for $(x_1,\ldots,x_n) \in [0,1]^n$. Then, by ($\mathcal{GG}2'$), it holds that $\sum_{i=1}^{n} x_i = 0$. Since a is the neutral element of $\mathcal{GG}$, one has that $\mathcal{GG}(0,a,\ldots,a) = 0$, which means that $a = 0$, by ($\mathcal{GG}2'$). □

Remark 1. Observe that the result stated by Proposition 4 does not mean that when a general general grouping function has a neutral element, then it is necessarily equal to 0. In fact, for each $a \in (0,1)$, the function $\mathcal{GG}\colon [0,1]^n \to [0,1]$, for all $\vec{x} = (x_1 \ldots, x_n) \in [0,1]^n$, defined by:

$$\mathcal{GG}(\vec{x}) = \begin{cases} \min\{\vec{x}\}, & \text{if } \max\{\vec{x}\} \leq a \\ \max\{\vec{x}\}, & \text{if } \min\{\vec{x}\} \geq a \\ \frac{\min\{\vec{x}\}+\max\{\vec{x}\}\left(1-\min\{\vec{x}\}\right)-a}{1-a}, & \text{if } \min\{\vec{x}\} < a < \max\{\vec{x}\} \end{cases}$$

is a general grouping function with a as neutral element.

Proposition 5. *If* 0 *is the neutral element of a general grouping function* $\mathcal{GG}\colon [0,1]^n \to [0,1]$ *and* $\mathcal{GG}$ *is idempotent, then* $\mathcal{GG}$ *is the maximum.*

Proof. Since $\mathcal{GG}$ is idempotent and increasing in each argument, then one has that for all $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$: **(1)** $\mathcal{GG}(x_1, \ldots, x_n) \leq \mathcal{GG}(\max(\vec{x}), \ldots, \max(\vec{x})) = \max\{\vec{x}\}$. Then we have that $x_k = \max\{\vec{x}\}$ for some $k = 1, \ldots, n$; so we have $x_k = \mathcal{GG}(0, \ldots, x_k, \ldots, 0) \leq \mathcal{GG}(x_1, \ldots, x_k, \ldots, x_n)$ and then **(2)** $\mathcal{GG}(x_1, \ldots, x_n) \geq x_k = \max\{\vec{x}\}$. So, from **(1)** and **(2)** one has that $\mathcal{GG}(x_1, \ldots, x_n) = \max\{\vec{x}\}$, for each $\vec{x} \in [0,1]^n$. □

3.1 General grouping functions on lattices

Following a similar procedure described in [20] for general overlap functions on lattices, it is possible to characterize general grouping functions. In order to do that, first we introduce some properties and notations.

Let us denote by $\mathfrak{G}^n$ the set of all general grouping functions. Define the ordering relation $\leq_{\mathfrak{G}^n} \in \mathfrak{G}^n \times \mathfrak{G}^n$, for all $\mathcal{GG}_1, \mathcal{GG}_2 \in \mathfrak{G}^n$ by:

$$\mathcal{GG}_1 \leq_{\mathfrak{G}^n} \mathcal{GG}_2 \Leftrightarrow \mathcal{GG}_1(\vec{x}) \leq \mathcal{GG}_2(\vec{x}), \text{ for all } \vec{x} = (x_1, \ldots, x_n) \in [0,1]^n.$$

The supremum and infimum of two arbitrary general grouping functions $\mathcal{GG}_1, \mathcal{GG}_2 \in \mathfrak{G}^n$ are, respectively, the general grouping functions $\mathcal{GG}_1 \vee \mathcal{GG}_2, \mathcal{GG}_1 \wedge \mathcal{GG}_2 \in \mathfrak{G}^n$, defined, for all $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$ by: $\mathcal{GG}_1 \vee \mathcal{GG}_2(\vec{x}) = \max\{\mathcal{GG}_1(\vec{x}), \mathcal{GG}_2(\vec{x})\}$ and $\mathcal{GG}_1 \wedge \mathcal{GG}_2(\vec{x}) = \min\{\mathcal{GG}_1(\vec{x}), \mathcal{GG}_2(\vec{x})\}$.

The following result is immediate:

Theorem 1. *The ordered set* $(\mathfrak{G}^n, \leq_{\mathfrak{G}^n})$ *is a lattice.*

Remark 2. Note that the supremum of the lattice $(\mathfrak{G}^n, \leq_{\mathfrak{G}^n})$ is given, for all $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$, by:

$$GG_{\sup}(\vec{x}) = \begin{cases} 0, & \text{if } \sum_{i=1}^{n} x_i = 0 \\ 1, & \text{otherwise.} \end{cases}$$

On the other hand, the infimum of $(\mathfrak{G}^n, \leq_{\mathfrak{G}^n})$ is given, for all $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$, by:

$$GG_{\inf}(\vec{x}) = \begin{cases} 1, & \text{if } \exists i \in \{1, \ldots, n\} : x_i = 1 \\ 0, & \text{otherwise.} \end{cases}.$$

However, neither $GG_{\sup}$ nor $GG_{\inf}$ are general grouping functions, since they are not continuous. Thus, in the lattice $(\mathfrak{G}^n, \leq_{\mathfrak{G}^n})$ there is no bottom neither top elements. Then, similarly to general overlap functions, the lattice $(\mathfrak{G}^n, \leq_{\mathfrak{G}^n})$ is not complete.

4 Characterization of General Grouping Functions and Construction Methods

In this section we provide a characterization and some constructions methods for general grouping functions.

Theorem 2. *The mapping* $\mathcal{GG}\colon [0,1]^n \to [0,1]$ *is a general grouping function if and only if*

$$\mathcal{GG}(\vec{x}) = \frac{f(\vec{x})}{f(\vec{x}) + h(\vec{x})} \tag{1}$$

for some $f, h\colon [0,1]^n \to [0,1]$ *the following properties hold, for all* $\vec{x} \in [0,1]^n$*:*

(i) f *and* h *are commutative;*
(ii) f *is increasing and* h *is decreasing.*
(iii) *If* $\sum_{i=1}^{n} x_i = 0$*, then* $f(\vec{x}) = 0$*.*
(iv) *If there exists* $i \in \{1, \ldots, n\}$ *such that* $x_i = 1$*, then* $h(\vec{x}) = 0$*.*
(v) f *and* h *are continuous.*
(vi) $f(\vec{x}) + h(\vec{x}) \neq 0$ *for any* $\vec{x} \in [0,1]^n$*.*

Proof. It follows that:
($\Rightarrow$) Suppose that $\mathcal{GG}$ is a general grouping function, and take $f(\vec{x}) = \mathcal{GG}(\vec{x})$ and $h(\vec{x}) = 1 - f(\vec{x})$. Then one always have $f(\vec{x}) + h(\vec{x}) \neq 0$, and so Equation (1) is well defined. Also, conditions **(i)**-**(v)** trivially hold.
($\Leftarrow$) Consider $f, h\colon [0,1]^n \to [0,1]$ satisfying conditions **(i)**-**(v)**. We will show that $\mathcal{GG}$ defined according to Equation (1) is a general grouping function. It is immediate that $\mathcal{GG}$ is commutative ($\mathcal{GG}1$) and continuous ($\mathcal{GG}5$). To prove ($\mathcal{GG}2$), notice that if $\sum_{i=1}^{n} x_i = 0$ then $f(\vec{x}) = 0$ and thus $\mathcal{GG}(\vec{x}) = 0$. Now, let us prove that ($\mathcal{GG}3$) holds. For that, observe that if there exists $i \in \{1, \ldots, n\}$ such that $x_i = 1$, then $h(\vec{x}) = 0$, and, thus, it is immediate that $\mathcal{GG}(\vec{x}) = 1$. The proof of ($\mathcal{GG}4$) is similar to [20, Theorem 3]. □

Example 2. Observe that Theorem 2 provides a method for constructing general grouping functions. For example, take the maximum powered by p, defined by:

$$\max^p(\vec{x}) = \max_{1 \leq i \leq n}\{x_i^p\},$$

with $p > 0$. So, if we take the function $T\max_\alpha^p : [0,1]^n \to [0,1]$, called α-truncated maximum powered by p, given, for all $\vec{x} \in [0,1]^n$ and $\alpha \in (0,1)$, by:

$$T\max_\alpha^p(\vec{x}) = \begin{cases} 0, & \text{if } \max^p(\vec{x}) \leq \alpha \\ \max^p(\vec{x}), & \text{if } \max^p(\vec{x}) > \alpha \end{cases} \tag{2}$$

then it is clear that $T\max_{\alpha}^{p}$ is not continuous. However, one can consider the function $CT\max_{\alpha,\epsilon}^{p}\colon [0,1]^n \to [0,1]$, called the continuous truncated maximum powered by p, for all $\vec{x} \in [0,1]^n$, $\alpha \in [0,1]$ and $\epsilon \in (0,\alpha]$, which is defined by:

$$CT\max_{\alpha,\epsilon}^{p}(\vec{x}) = \begin{cases} 0, & \text{if } \max^p(\vec{x}) \leq \alpha - \epsilon \\ \frac{\alpha}{\epsilon}\left(\max^p(\vec{x}) - (\alpha - \epsilon)\right), & \text{if } \alpha - \epsilon < \max^p(\vec{x}) < \alpha \\ \max^p(\vec{x}), & \text{if } \max^p(\vec{x}) \geq \alpha. \end{cases} \quad (3)$$

Observe that taking $f = CT\max_{\alpha,\epsilon}^{p}$, then f satisfies conditions **(i)-(iii)** and **(v)** in Theorem 2. Now, take $h(\vec{x}) = \min_{1\leq i\leq n}\{1 - x_i\}$, which satisfies conditions **(i)-(ii)** and **(iv)-(v)** required in Theorem 2. Thus, this assures that

$$\mathcal{GG}(\vec{x}) = \frac{CT\max_{\alpha,\epsilon}^{p}(\vec{x})}{CT\max_{\alpha,\epsilon}^{p}(\vec{x}) + \min_{1\leq i\leq n}\{1 - x_i\}}$$

is a general grouping function.

Remark 3. Observe that the maximum powered by p is an n-dimensional grouping function [15] and that $CT\max_{\alpha,\epsilon}^{p}$ is a general grouping function. However, $CT\max_{\alpha,\epsilon}^{p}$ is not an n-dimensional grouping function, for $\alpha - \epsilon > 0$, since $CT\max_{\alpha,\epsilon}^{p}(\alpha - \epsilon, \ldots, \alpha - \epsilon) = 0$.

Corollary 1. *Consider the functions $f, h\colon [0,1]^n \to [0,1]$ and let $\mathcal{GG}\colon [0,1]^n \to [0,1]$ be a general grouping function constructed according to Theorem 2, and taking into account functions f and h. Then $\mathcal{GG}$ is idempotent if and only if, for all $x \in [0,1)$, it holds that:*

$$f(x,\ldots,x) = \frac{x}{1-x} h(x,\ldots,x).$$

Proof. ($\Rightarrow$) If $\mathcal{GG}$ is idempotent, then by Theorem 2 it holds that:

$$\mathcal{GG}(x,\ldots,x) = \frac{f(x,\ldots,x)}{f(x,\ldots,x) + h(x,\ldots,x)} = x.$$

It follows that: $f(x,\ldots,x) = x(f(x,\ldots,x) + h(x,\ldots,x))$

$$(1-x)f(x,\ldots,x) = x\, h(x,\ldots,x)$$
$$f(x,\ldots,x) = \frac{x}{1-x}\, h(x,\ldots,x).$$

($\Leftarrow$) It is immediate. □

Example 3. Take the function $\alpha\beta$-truncated maximum powered by p, $T\max_{\alpha\beta}^{p}\colon [0,1]^n \to [0,1]$, for all $\vec{x} \in [0,1]^n$; $\alpha, \beta \in (0,1)$ and $\alpha < \beta$, defined by:

$$T\max_{\alpha\beta}^{p}(\vec{x}) = \begin{cases} 0, & \max^p(\vec{x}) \leq \alpha \\ \max^p(\vec{x}), & \alpha < \max^p(\vec{x}) < \beta \\ 1, & \max^p(\vec{x}) \geq \beta \end{cases}$$

It is clear that $T\max^p_{\alpha\beta}$ is not continuous. However, we can define its continuous version, $CT\max^p_{\alpha\beta,\epsilon\delta}\colon [0,1]^n \to [0,1]$, for all $\vec{x} \in [0,1]^n$; $\alpha \in [0,1)$; $\beta, \epsilon, \delta \in (0,1]$; $\alpha+\epsilon, \beta-\delta \in (0,1)$ and $\alpha+\epsilon < \beta-\delta$, as follows:

$$CT\max^p_{\alpha\beta,\epsilon\delta}(\vec{x}) = \begin{cases} 0, & \max^p(\vec{x}) \leq \alpha \\ \frac{1-(\alpha+\epsilon)}{\epsilon}(\alpha - \max^p(\vec{x})), & \alpha < \max^p(\vec{x}) < \alpha+\epsilon \\ 1-\max^p(\vec{x}), & \alpha+\epsilon \leq \max^p(\vec{x}) \leq \beta-\delta \\ 1-(\beta-\delta) - \frac{\beta-\delta}{\delta}(\beta-\delta-\max^p(\vec{x})), & \beta-\delta < \max^p(\vec{x}) < \beta \\ 1, & \max^p(\vec{x}) \geq \beta \end{cases}$$

Observe that $CT\max^p_{\alpha\beta,\epsilon\delta}$ satisfies conditions $(\mathcal{GG}1)$-$(\mathcal{GG}5)$ from Def. 9, and then it is a general grouping function. But, whenever $\alpha \neq 0$ or $\beta \neq 1$, then $CT\max^p_{\alpha\beta,\epsilon\delta}$ is not an n-dimensional grouping function, once $CT\max^p_{\alpha\beta,\epsilon\delta}(\alpha-\epsilon,\ldots,\alpha-\epsilon) = 0$, for $\alpha-\epsilon > 0$, because $\max^p(\alpha-\epsilon,\ldots,\alpha-\epsilon) = \alpha-\epsilon < \alpha$.

The following Theorem generalizes Example 3 providing a construction method for general grouping functions from truncated n-dimensional grouping functions.

Theorem 3. *Consider $\alpha \in [0,1)$; $\beta, \epsilon, \delta \in (0,1]$; $\alpha+\epsilon, \beta-\delta \in (0,1)$ and $\alpha < \beta$, $\alpha+\epsilon < \beta-\delta$. Let $\mathcal{G}$ be an n-dimensional grouping function, whose $\alpha\beta$-truncated version is defined, for all $\vec{x} = (x_1,\ldots,x_n) \in [0,1]^n$, by:*

$$T\mathcal{G}_{\alpha\beta}(\vec{x}) = \begin{cases} 0, & \mathcal{G}(\vec{x}) \leq \alpha \\ \mathcal{G}(\vec{x}), & \alpha < \mathcal{G}(\vec{x}) < \beta \\ 1, & \mathcal{G}(\vec{x}) \geq \beta \end{cases}$$

Then, the continuous version of $T\mathcal{G}_{\alpha\beta}$, for all $\vec{x} = (x_1,\ldots,x_n) \in [0,1]^n$, is given by:

$$CT\mathcal{G}_{\alpha\beta,\epsilon\delta}(\vec{x}) = \begin{cases} 0, & \mathcal{G}(\vec{x}) \leq \alpha \\ \frac{1-(\alpha+\epsilon)}{\epsilon}(\alpha - \mathcal{G}(\vec{x})), & \alpha < \mathcal{G}(\vec{x}) < \alpha+\epsilon \\ 1-\mathcal{G}(\vec{x}), & \alpha+\epsilon \leq \mathcal{G}(\vec{x}) \leq \beta-\delta \\ 1-(\beta-\delta) - \frac{\beta-\delta}{\delta}(\beta-\delta-\mathcal{G}(\vec{x})), & \beta-\delta < \mathcal{G}(\vec{x}) < \beta \\ 1, & \mathcal{G}(\vec{x}) \geq \beta \end{cases}$$

and it is a general grouping function. Besides that, whenever $\alpha = 0$ and $\beta = 1$, then $CT\mathcal{G}_{\alpha\beta,\epsilon\delta}$ is an n-dimensional grouping function.

The following proposition shows a construction method of general grouping functions that generalizes Example 1(4).

Proposition 6. *Let $\mathcal{G}\colon [0,1]^n \to [0,1]$ be an n-dimensional grouping function and let $F\colon [0,1]^n \to [0,1]$ be a commutative and continuous aggregation function such that, for all $\vec{x} = (x_1,\ldots,x_n) \in [0,1]^n$, if there exists $i \in \{1,\ldots,n\}$ such that $x_i = 1$, then $F(\vec{x}) = 1$. Then $\mathcal{GG}_{\mathcal{G}F}(\vec{x}) = \mathcal{G}(\vec{x})F(\vec{x})$ is a general grouping function.*

Proof. It is immediate that $\mathcal{GG}_{\mathcal{G}F}$ is well defined, ($\mathcal{GG}1$) commutative, ($\mathcal{GG}4$) increasing and ($\mathcal{GG}5$) continuous, since $\mathcal{G}$, F and the product operation are commutative, increasing and continuous. To prove ($\mathcal{GG}2$), whenever $\sum_{i=1}^{n} x_i = 0$, then by ($\mathcal{G}2$), it holds that $\mathcal{G}(\vec{x}) = 0$, and, thus, $\mathcal{GG}_{\mathcal{G}F}(\vec{x}) = \mathcal{G}(\vec{x})F(\vec{x}) = 0$. For ($\mathcal{GG}3$), whenever there exists $i \in \{1, \ldots, n\}$ such that $x_i = 1$, then, by ($\mathcal{G}3$), one has that $\mathcal{G}(\vec{x}) = 1$, and, by the property of F, it holds that $F(\vec{x}) = 1$. It follows that: $\mathcal{GG}_{\mathcal{G}F}(\vec{x}) = \mathcal{G}(\vec{x})F(\vec{x}) = 1$. □

The following result is immediate.

Corollary 2. *Let $\mathcal{GH}\colon [0,1]^n \to [0,1]$ be a general grouping function and let $F\colon [0,1] \to [0,1]$ be a commutative and continuous aggregation function such that, for all $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$, if there exists $i \in \{1, \ldots, n\}$ such that $x_i = 1$, then $F(\vec{x}) = 1$. Then $\mathcal{GG}_{\mathcal{GH},F}(\vec{x}) = \mathcal{GH}(\vec{x})F(\vec{x})$ is a general grouping function.*

Note that $\mathfrak{G}^n$ is closed with respect to some aggregation functions, as stated by the following results, which provide a construction methods of general grouping functions.

Theorem 4. *Consider $M\colon\ [0,1]^2 \to [0,1]$. For any $\mathcal{GG}_1, \mathcal{GG}_2 \in \mathfrak{G}^n$, define the mapping $M_{\mathcal{GG}_1,\mathcal{GG}_2}\colon [0,1]^n \to [0,1]$, for all $\vec{x} \in [0,1]^n$, by:*

$$M_{\mathcal{GG}_1,\mathcal{GG}_2}(\vec{x}) = M(\mathcal{GG}_1(\vec{x}), \mathcal{GG}_2(\vec{x})).$$

Then, $M_{\mathcal{GG}_1,\mathcal{GG}_2} \in \mathfrak{G}^n$ if and only if M is a continuous aggregation function.

Proof. It follows that:
($\Rightarrow$) Suppose that $M_{\mathcal{GG}_1,\mathcal{GG}_2} \in \mathfrak{G}^n$. Then it is immediate that M is continuous and increasing (A2). Now consider $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$ and suppose that $\sum_{i=1}^{n} x_i = 0$. Then, by ($\mathcal{GG}2$), one has that: $M_{\mathcal{GG}_1,\mathcal{GG}_2}(\vec{x}) = M(\mathcal{GG}_1(\vec{x}), \mathcal{GG}_2(\vec{x})) = 0$ and $\mathcal{GG}_1(\vec{x}) = \mathcal{GG}_2(\vec{x}) = 0$. Thus, it holds that $M(0,0) = 0$. Now, consider $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$, such that there exists $i \in \{1, \ldots, n\}$ such that $x_i = 1$. Then, by ($\mathcal{GG}3$), one has that: $M_{\mathcal{GG}_1,\mathcal{GG}_2}(\vec{x}) = M(\mathcal{GG}_1(\vec{x}), \mathcal{GG}_2(\vec{x})) = 1$ and $\mathcal{GG}_1(\vec{x}) = \mathcal{GG}_2(\vec{x}) = 1$. Therefore, it holds that $M(1,1) = 1$. This proves that M also satisfies (A1), and, thus, M is a continuous aggregation function.
($\Leftarrow$) Suppose that M is a continuous aggregation function. Then it is immediate that $M_{\mathcal{GG}_1,\mathcal{GG}_2}$ is ($\mathcal{GG}1$) commutative, ($\mathcal{GG}4$) increasing and ($\mathcal{GG}5$) continuous. For ($\mathcal{GG}2$), consider $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$ such that $\sum_{i=1}^{n} x_i = 0$. Then, by ($\mathcal{GG}2$), one has that $\mathcal{GG}_1(\vec{x}) = \mathcal{GG}_2(\vec{x}) = 0$. It follows that: $M_{\mathcal{GG}_1,\mathcal{GG}_2}(\vec{x}) = M(\mathcal{GG}_1(\vec{x}), \mathcal{GG}_2(\vec{x}))= M(0,0) = 0$, by (A1), since M is an aggregation function. Finally, for ($\mathcal{GG}3$) consider that there exists $i \in \{1, \ldots, n\}$ such that $x_i = 1$ for some $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$. Then, it holds that $\mathcal{GG}_1(\vec{x}) = \mathcal{GG}_2(\vec{x}) = 1$. It follows that: $M_{\mathcal{GG}_1,\mathcal{GG}_2}(\vec{x})= M(\mathcal{GG}_1(\vec{x}), \mathcal{GG}_2(\vec{x})) = M(1,1) = 1$, by (A1), since M is an aggregation function. This proves that $M_{\mathcal{GG}_1,\mathcal{GG}_2} \in \mathfrak{G}^n$. □

Example 4. In the sense of Theorem 4, $\mathfrak{G}^n$ is closed under any bidimensional overlap functions, grouping functions and continuous t-norms and t-conorms [18].

Corollary 3. *Consider $M\colon [0,1]^2 \to [0,1]$. For any n-dimensional grouping functions $\mathcal{G}_1, \mathcal{G}_2\colon [0,1]^n \to [0,1]$, define the mapping $M_{\mathcal{G}_1,\mathcal{G}_2}\colon [0,1]^n \to [0,1]$, for all $\vec{x} = (x_1,\ldots,x_n) \in [0,1]^n$, by:*

$$M_{\mathcal{G}_1,\mathcal{G}_2}(\vec{x}) = M(\mathcal{G}_1(\vec{x}), \mathcal{G}_2(\vec{x})).$$

Then, $M_{\mathcal{G}_1,\mathcal{G}_2} \in \mathfrak{G}^n$ if and only if M is a continuous aggregation function.

Proof. It follows from Theorem 4, since any n-dimensional grouping function is a general grouping function. □

Theorem 4 can be easily extended for n-ary functions $M^n\colon [0,1]^n \to [0,1]$:

Theorem 5. *Consider $M^n\colon [0,1]^n \to [0,1]$. For any $\mathcal{GG}_1,\ldots,\mathcal{GG}_n \in \mathfrak{G}^n$, define the mapping $M_{\mathcal{GG}_1,\ldots,\mathcal{GG}_n}\colon [0,1]^n \to [0,1]$, for all $\vec{x} \in [0,1]^n$, by:*

$$M_{\mathcal{GG}_1,\ldots,\mathcal{GG}_n}(\vec{x}) = M^n(\mathcal{GG}_1(\vec{x}),\ldots,\mathcal{GG}_n(\vec{x})).$$

Then, $M_{\mathcal{GG}_1,\ldots,\mathcal{GG}_n} \in \mathfrak{G}^n$ if and only if $M^n : [0,1]^n \to [0,1]$ is a continuous n-ary aggregation function.

Proof. Analogous to the proof of Theorem 4. □

This result can be extended for n-dimensional grouping functions.

Corollary 4. *Consider $M^n\colon [0,1]^n \to [0,1]$ and fr any n-dimensional grouping functions $\mathcal{G}_1,\ldots,\mathcal{G}_n$ define the mapping $M_{\mathcal{G}_1,\ldots,\mathcal{G}_n}\colon [0,1]^n \to [0,1]$, for all $\vec{x} = (x_1,\ldots,x_n) \in [0,1]^n$, by:*

$$M_{\mathcal{G}_1,\ldots,\mathcal{G}_n}(\vec{x}) = M^n(\mathcal{G}_1(\vec{x}),\ldots,\mathcal{G}_n(\vec{x})).$$

Then, $M_{\mathcal{G}_1,\ldots,\mathcal{G}_n}$ is a general grouping function if and only if $M^n\colon [0,1]^n \to [0,1]$ is a continuous n-ary aggregation function.

Corollary 5. *Let $\mathcal{GG}_1,\ldots,\mathcal{GG}_m\colon [0,1]^n \to [0,1]$ be general grouping functions and consider weights $w_1,\ldots,w_m \in [0,1]$ such that $\sum\limits_{i=1}^{m} w_i = 1$. Then the convex sum $\mathcal{GG} = \sum\limits_{i=1}^{m} w_i \mathcal{GG}_i$ is also a general grouping function.*

Proof. Since the weighted sum is a continuous commutative n-ary aggregation function, the result follows from Theorem 5. □

It is possible to obtain general grouping functions from the generalized composition of general grouping functions and aggregation functions satisfying especial conditions:

Theorem 6. *Let $\mathcal{GG}_2\colon [0,1]^n \to [0,1]$ be a general grouping function and let the n-ary aggregation functions $A_1,\ldots,A_n\colon [0,1]^n \to [0,1]$ be continuous, commutative and disjunctive. Then, the function $\mathcal{GG}_1\colon [0,1]^n \to [0,1]$ defined, for all $\vec{x} = (x_1,\ldots,x_n) \in [0,1]^n$, by: $\mathcal{GG}_1(\vec{x}) = \mathcal{GG}_2(A_1(\vec{x}),\ldots,A_n(\vec{x}))$ is a general grouping function.*

Proof. Since $\mathcal{GG}_2$, $A_1, \ldots, A_n$ are commutative, increasing and continuous functions, then $\mathcal{GG}_1$ satisfies conditions ($\mathcal{GG}1$), ($\mathcal{GG}4$) and ($\mathcal{GG}5$). So, it remains to prove:
($\mathcal{GG}2$) Let $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$ be such that $\sum_{i=1}^{n} x_i = 0$. Then, since A_1 is disjunctive, we have that $A_1(\vec{x}) \geq \max(\vec{x}) = 0$, that is $A_1(\vec{x}) = 0$. Equivalently, one obtains $A_2(\vec{x}), \ldots, A_n(\vec{x}) = 0$. Thus, since $\mathcal{GG}_2$ is a general grouping function, one has that $\mathcal{GG}_1(\vec{x}) = \mathcal{GG}_2(A_1(\vec{x}), \ldots, A_n(\vec{x})) = \mathcal{GG}_2(0, \ldots, 0) = 0$.
($\mathcal{GG}3$) Suppose that, for some $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$, there exists $i \in \{1, \ldots, n\}$ such that $x_i = 1$. So, since A_1 is disjunctive then $A_1(\vec{x}) \geq \max(\vec{x}) = 1$, that is $A_1(\vec{x}) = 1$. Since $\mathcal{GG}_2$ is a general grouping function, it follows that $\mathcal{GG}_1(\vec{x}) = \mathcal{GG}_2(A_1(\vec{x}), \ldots, A_n(\vec{x})) = \mathcal{GG}_2(1, A_2(\vec{x}), \ldots, A_n(\vec{x})) = 1$. □

Next proposition uses the n-duality property.

Proposition 7. *Consider a continuous fuzzy negation* $N\colon [0,1] \to [0,1]$ *and a general overlap function* $\mathcal{GO}\colon [0,1]^n \to [0,1]$*, then for all* $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$*:*

$$\mathcal{GG}(\vec{x}) = N(\mathcal{GO}(N(x_1), \ldots, N(x_n))) \tag{4}$$

is a general grouping function. Reciprocally, if $\mathcal{GG}\colon [0,1]^n \to [0,1]$ *is a general grouping function, then for all* $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$*:*

$$\mathcal{GO}(\vec{x}) = N(\mathcal{GG}(N(x_1), \ldots, N(x_n))) \tag{5}$$

is a general overlap function.

Proof. Since we have a continuous fuzzy negation and bearing in mind that general overlap functions and general grouping functions are commutative, increasing and continuous functions according to Def. 8 and Def. 9, respectively, then $\mathcal{GO}$ and $\mathcal{GG}$ satisfy conditions ($\mathcal{GO}1$), ($\mathcal{GG}1$); ($\mathcal{GO}4$), ($\mathcal{GG}4$) and ($\mathcal{GO}5$), ($\mathcal{GG}5$). So, it remains to prove:
($\mathcal{GG}2$) For Eq. (4), take $x_i = 0$, for all $i \in \{1, \ldots, n\}$. Therefore,

$$\mathcal{GG}(\vec{x}) = N(\mathcal{GO}(N(0), \ldots, N(0))) \overset{N2}{=} N(\mathcal{GO}(1, \ldots, 1)) \overset{\mathcal{GO}3}{=} N(1) \overset{N2}{=} 0.$$

($\mathcal{GG}3$) If there exists a $x_i = 1$, for some $i \in \{1, \ldots, n\}$, then

$$\begin{aligned} \mathcal{GG}(\vec{x}) &= N(\mathcal{GO}(N(x_1), \ldots, N(1), \ldots, N(x_n))) \\ &\overset{N2}{=} N(\mathcal{GO}(N(x_1), \ldots, 0, \ldots, N(x_n))) \\ &\overset{\mathcal{GO}2}{=} N(0) \overset{N2}{=} 1. \end{aligned}$$

($\mathcal{GO}2$) Similarly, for Eq. (5), take a $x_i = 0$ for some $i \in \{1, \ldots, n\}$. So,

$$\begin{aligned} \mathcal{GO}(\vec{x}) &= N(\mathcal{GG}(N(x_1), \ldots, N(0), \ldots, N(x_n))) \\ &\overset{N2}{=} N(\mathcal{GG}(N(x_1), \ldots, 1, \ldots, N(x_n))) \\ &\overset{\mathcal{GG}3}{=} N(1) \overset{N2}{=} 0. \end{aligned}$$

($\mathcal{GO}3$) Now, consider that $x_i = 1$, for all $i \in \{1, \ldots, n\}$. Then,

$$\mathcal{GO}(\vec{x}) = N(\mathcal{GG}(N(1), \ldots, N(1))) \overset{N2}{=} N(\mathcal{GG}(0, \ldots, 0)) \overset{\mathcal{GG}2}{=} N(0) \overset{N2}{=} 1.$$

□

5 Conclusions

In this paper, we first introduced the concept of general grouping functions and studied some of their properties. Then we provide a characterization of general grouping functions and some construction methods.

The theoretical developments presented here allow for a more flexible approach when dealing with decision making problems with multiple alternatives. Immediate future work is concerned with the development of an application in multi-criteria decision making based on n-ary fuzzy heterogeneous, incomplete preference relations.

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