Manipulative agendas in four-candidate elections*

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Abstract

We consider a setting where it is known for an electorate what probability a given candidate has of beating another in a pairwise ballot. An agenda assigns candidates to the leaves of a binary tree and is called manipulative if it inverts the final winning probabilities for two candidates. We compare standard and symmetric agendas in four-candidate elections and show that in monotone environments the former are more manipulative.

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1 Introduction

There are multiple perspectives from which sequential voting procedures can be compared. One possibility is to interpret them as decision rules attaching candidates to profiles of individual preferences and to either follow an axiomatic approach by imposing properties on such rules (cf. Apesteguia et al. 2014) or to study the type of decision rules implemented by the corresponding procedures (cf. Horan 2019, Kleiner and Moldovanu 2017). Another possibility is to focus on agenda control and study how the order in which pairwise ballots take place in an election influences the sets of outcomes of the procedures (cf. Banks 1985, Miller 1995, Barberà and Gerber 2017). Yet another strand of literature is rooted in the computational social choice (cf. Vassilevska Williams 2016) where the main focus is on the computational complexity aspect of agenda control in sequential voting.

In contrast to the above mentioned works, we focus here on pairwise elimination voting procedures where only the probability of a given candidate beating another in a pairwise ballot is known (cf. Hazon et al. 2007), and compare the possible agendas in terms of their probabilistic manipulativeness. To establish ideas, consider for instance a situation where a committee has to select the host country of the next Olympic Games. The set of possible candidates consists of four countries, the procedure applied is pairwise elimination voting, and the committee chair has the power to design the structure of the pairwise ballots (the agenda) and is aware only of the pairwise winning probabilities among the candidates. One possibility for the chair is to set a *standard agenda*, i.e. the voting procedure starts with an initial ballot between two particular countries, then lets the winner participate in a second ballot against a third country, and finally determines the host country in a third ballot between the winner of the second ballot and the fourth country.[1]

However, there is another possible type of agenda. As an introduction to it, assume that two of the countries are European and the other two are Asian. In such a case, the committee chair might opt for a *symmetric agenda*, i.e. to consider two initial ballots - one European and the other Asian - and then to let the final winner be determined in a third ballot between the European

[1] Agendas in legislative amendment procedures with four alternatives (cf. Miller 1995, Rasch 2000, Rasch 2014, Barberà and Gerber 2017) are standard. As will become clear, any standard agenda in our setting is feasible. This might not be the case for amendment procedures where not all agendas make sense, given the specific nature of the alternatives (motions, amendments, status quo, etc.).

and Asian winners. We focus here on how standard and symmetric agendas can be compared in terms of their probabilistic manipulativeness.

Accordingly, we formally define agendas to assign candidates to the leaves of a binary tree and call an agenda *manipulative* if it inverts the final winning probabilities for two candidates. Of course, whether such an inversion is possible or not crucially depends on the structure of the underlying probabilistic information. We consider environments where the pairwise winning probabilities induce a weak ordering over the set of candidates and are monotone in the sense that weaker candidates are more easily beaten. We then show that in four-candidate elections standard agendas are more manipulative than symmetric agendas.

2 Basic setup

The set of candidates is $C = \{1, 2, 3, 4\}$. For $i, j \in C$, $p_{ij} \in [0, 1]$ is the probability that i will beat j in a pairwise ballot. These probabilities are collected in the matrix $p = (p_{ij})_{i,j \in C}$ with $p_{ij} + p_{ji} = 1$ for all $i, j \in C$ with $i \neq j$.

A *voting procedure* on C consists of a binary tree t with four leaves and an *agenda* a in t (i.e. a bijection labeling of the leaves of t by candidates from C). Figure 1 illustrates the two possible structures (t' and t'') of a binary tree with four leaves.

[Figure 1]

The set of all agendas in $t \in \{t', t''\}$ is denoted by $\mathbf{A}^t$. For $t \in \{t', t''\}$ and $a \in \mathbf{A}^t$, an *initial* pairwise ballot between candidates i and j is conducted if the leaves of t to which they are assigned by a have a common immediate predecessor, which is then reached by i with probability p_{ij}.

A *standard agenda* is an agenda in t'; for instance, $a' = (1, 3, 2, 4) \in \mathbf{A}^{t'}$ indicates that the winner in the ballot between candidates 1 and 3 faces candidate 2 in the second ballot with the final winner being determined in the third ballot between the winner of the second and candidate 4. Clearly, the number of pairwise ballots for standard agendas is three.

A *symmetric agenda* is an agenda in t''; for example, $a'' = (13, 24) \in \mathbf{A}^{t''}$ means that there are two initial pairwise ballots (between 1 and 3, and 2 and 4, respectively) with the final winner being determined in a third ballot between the winners of the initial ballots. The number of pairwise ballots for symmetric agendas is also three.

Clearly, standard agendas can be made manipulative by prioritizing certain candidates, while symmetric agendas can be made so by partitioning the set of candidates in a particular way. Below, we confront these two modalities of manipulativeness.

2.1 Manipulative agendas

For a binary tree $t \in \{t', t''\}$, agenda $a \in \mathbf{A}^t$, and probability matrix p, denote by $\varphi_k(t, a, p)$ the probability that candidate $k \in C$ will be the final winner (i.e. he/she reaches t's root) in the election. For a set $\mathbf{P}$ of probability matrices, $a \in \mathbf{A}^t$ is said to be *manipulative at* $p \in \mathbf{P}$ if there are candidates, $i, j \in C$ such that $p_{ij} > 0.5$ and $\varphi_i(t, a, p) < \varphi_j(t, a, p)$ holds. The *agenda collection* $\mathbf{A}^t$ is

- *manipulative at* $p \in \mathbf{P}$, if there exists $a \in \mathbf{A}^t$ which is manipulative at p.

The idea contained in the definition of a manipulative agenda is explained as follows. Assume that there are two candidates i and j with respect to whom the agenda setter knows that i is likely to beat j in a pairwise ballot ($p_{ij} > p_{ji}$). Now assume that the agenda setter prefers candidate j over candidate i. In such a case he/she would rather look for an agenda $a \in \mathbf{A}^t$ that guarantees that the probability of i being the final winner lower than that of j. Such an agenda is then called manipulative at the probability matrix p.

In our setting and in four-candidate elections, the agenda setter has the possibility of opting for either standard or symmetric agendas and of selecting a particular agenda of the corresponding type. In what follows, we assume that the agenda setter compares the two agenda types based on how manipulative they are. For two binary trees t^* and t^{**}, we adopt the basic idea from Pathak and Sönmez (2013) as to compare the agenda collections $\mathbf{A}^{t^*}$ and $\mathbf{A}^{t^{**}}$. Given a set $\mathbf{P}$ of probability matrices, we say that

- $\mathbf{A}^{t^*}$ is *at least as manipulative as* $\mathbf{A}^{t^{**}}$ *on* $\mathbf{P}$ if for each $p \in \mathbf{P}$ it results that $\mathbf{A}^{t^{**}}$ being manipulative at p implies that $\mathbf{A}^{t^*}$ is manipulative at p;

- $\mathbf{A}^{t^*}$ is *more manipulative than* $\mathbf{A}^{t^{**}}$ *on* $\mathbf{P}$ if
  - $\mathbf{A}^{t^*}$ is at least as manipulative as $\mathbf{A}^{t^{**}}$ on $\mathbf{P}$, and
  - there exist $p \in \mathbf{P}$ such that $\mathbf{A}^{t^*}$ is manipulative at p but $\mathbf{A}^{t^{**}}$ is not manipulative at p.

2.2 Monotone environments

We focus on environments where the agenda setter has an a priori ranking of the candidates in terms of their "strengths". Such a ranking could be the result of his/her own subjective perceptions or be induced by polls or historical data. We formalize this idea as follows.

Let R be a complete binary relation defined over the set of candidates. For $i, j \in C$, we interpret iRj as expressing the fact that candidate i is weakly stronger than candidate j. We further say that a probabilistic matrix p and a binary relation R are *compatible* if for all $i, j \in C$, iRj if and only if $p_{ij} \geq 0.5$. Additionally, the following *monotonicity* condition imposed on p guarantees that R is transitive when it is compatible with p:

- for all $i, j \in C$, $p_{ij} \geq 0.5$ implies $p_{ik} \geq p_{jk}$ for each $k \in C \setminus \{i, j\}$.

This condition simply expresses the fact that in a pairwise ballot any candidate is likelier to beat a weaker candidate than a stronger one. It also implies that if two candidates are equally strong ($p_{ij} = 0.5$) then in a pairwise ballot they should be equally likely to defeat any third candidate. The above condition (together with $p_{ij} + p_{ji} = 1$ for all $i, j \in C$ with $i \neq j$) is equivalent to what is sometimes referred as "strong stochastic transitivity" of the representing probability matrix (cf. David 1963). Below, w.l.o.g., we set the weak ordering R to be such that $1R2R3R4$ and denote by $\mathbf{P}_R$ the set of all probability matrices compatible with R.

3 A comparison in monotone environments

Our setup allows for the following comparison of standard and symmetric agendas.

Theorem 1 *In four-candidate elections and monotone environments, standard agendas are more manipulative than symmetric agendas.*

Proof. Recall that $\mathbf{A}^{t'}$ and $\mathbf{A}^{t''}$ stand for the collections of standard and symmetric agendas, respectively. We first show that $\mathbf{A}^{t'}$ is at least as manipulative as $\mathbf{A}^{t''}$ on $\mathbf{P}_R$ and then proceed to show that $\mathbf{A}^{t'}$ is more manipulative than $\mathbf{A}^{t''}$ on $\mathbf{P}_R$.

Part 1 ($\mathbf{A}^{t'}$ *is at least as manipulative as* $\mathbf{A}^{t''}$ *on* $\mathbf{P}_R$) We have to show that if $\mathbf{A}^{t''}$ is manipulative at $p \in \mathbf{P}_R$, then also $\mathbf{A}^{t'}$ is manipulative at p. So, assume that $a'' \in \mathbf{A}^{t''}$ is manipulative at p and let us show first that $a'' \neq (14, 23)$ should hold.[2]

For $a'' = (14, 23)$ we have

$$\begin{aligned}
\varphi_1(t'', a'', p) &= p_{14} \cdot p_{12} \cdot p_{23} + p_{14} \cdot p_{13} \cdot p_{32}, \\
\varphi_2(t'', a'', p) &= p_{23} \cdot p_{21} \cdot p_{14} + p_{23} \cdot p_{24} \cdot p_{41}, \\
\varphi_3(t'', a'', p) &= p_{32} \cdot p_{31} \cdot p_{14} + p_{32} \cdot p_{34} \cdot p_{41}, \\
\varphi_4(t'', a'', p) &= p_{41} \cdot p_{42} \cdot p_{23} + p_{41} \cdot p_{43} \cdot p_{32}.
\end{aligned}$$

By $p \in \mathbf{P}_R$, it emerges that $p_{12} \geq 0.5$, $p_{14} \geq p_{24}$, $p_{13} \geq p_{23}$, and $p_{32} \geq p_{41}$ (otherwise, $p_{23} > p_{14}$ and $p_{14} \geq p_{24}$ would imply $p_{23} > p_{24}$, a contradiction to p being monotone). Hence, $\varphi_1(t'', a'', p) \geq \varphi_2(t'', a'', p)$ follows. Similarly, $p_{23} \geq 0.5$, $p_{21} \geq p_{31}$, and $p_{24} \geq p_{34}$ results in $\varphi_2(t'', a'', p) \geq \varphi_3(t'', a'', p)$. Finally, $p_{34} \geq 0.5$ implies $p_{31} \geq p_{41}$ and $p_{32} \geq p_{42}$, and hence, $\varphi_3(t'', a'', p) \geq \varphi_4(t'', a'', p)$ holds. In other words, it is impossible for $a'' = (14, 23)$ to be manipulative at $p \in \mathbf{P}_R$. That leaves the following two possibilities (corresponding to Cases 1.1 and 1.2) for a manipulative agenda $a'' \in \mathbf{A}^{t''}$.

Case 1.1 ($a'' = (12, 34)$) Notice first that $p_{12} \geq 0.5$ and $p \in \mathbf{P}_R$ implies $p_{13} \geq p_{23}$ and $p_{14} \geq p_{24}$. Thus,

$$\begin{aligned}
\varphi_1(t'', a'', p) &= p_{12} \cdot p_{13} \cdot p_{34} + p_{12} \cdot p_{14} \cdot p_{43} \\
&\geq p_{21} \cdot p_{23} \cdot p_{34} + p_{21} \cdot p_{24} \cdot p_{43} \\
&= \varphi_2(t'', a'', p)
\end{aligned}$$

follows. From $p \in \mathbf{P}_R$ it further results that $p_{34} \geq 0.5$, $p_{31} \geq p_{41}$, and $p_{32} \geq p_{42}$. Hence,

$$\begin{aligned}
\varphi_3(t'', a'', p) &= p_{34} \cdot p_{31} \cdot p_{12} + p_{34} \cdot p_{32} \cdot p_{21} \\
&\geq p_{43} \cdot p_{41} \cdot p_{12} + p_{43} \cdot p_{42} \cdot p_{21} \\
&= \varphi_4(t'', a'', p)
\end{aligned}$$

[2]See Arlegi and Dimitrov (2020) for a generalization of this fact and a characterization of non-manipulativeness situations in a context of sports competitions.

also follows.

It can thus be concluded from a'' being manipulative at p and from the above two inequalities that there are $i \in \{1,2\}$ and $j \in \{3,4\}$ with $p_{ij} > 0.5$ and $\varphi_j(t'', a'', p) > \varphi_i(t'', a'', p)$. Let $i' \in \{1,2\} \setminus \{i\}$, $j' \in \{3,4\} \setminus \{j\}$ (i.e., $a'' = (ii', jj')$), and consider the agenda $a' = (i, i', j, j') \in \mathbf{A}^{t'}$. The following emerges:

$$\begin{aligned}
\varphi_j(t', a', p) &= \varphi_j(t'', a'', p) = p_{ii'} \cdot p_{ji} \cdot p_{jj'} + p_{i'i} \cdot p_{ji'} \cdot p_{jj'} \\
&> \varphi_i(t'', a'', p) = p_{ii'} \cdot p_{ij} \cdot p_{jj'} + p_{ii'} \cdot p_{ij'} \cdot p_{j'j} \\
&\geq p_{ii'} \cdot (\min\{p_{ij}, p_{ij'}\} \cdot p_{jj'} + \min\{p_{ij}, p_{ij'}\} \cdot p_{j'j}) \\
&= p_{ii'} \cdot \min\{p_{ij}, p_{ij'}\} \\
&\geq p_{ii'} \cdot p_{ij} \cdot p_{ij'} = \varphi_i(t', a', p),
\end{aligned}$$

where the first inequality follows by assumption and the last one from $1 \geq \min\{p_{ij}, p_{ij'}\} \in \{p_{ij}, p_{ij'}\}$. Thus, a' is manipulative at p.

Case 1.2 ($a'' = (13, 24)$) It results from $p_{13} \geq 0.5$ and $p \in \mathbf{P}_R$ that $p_{14} \geq p_{34}$ holds and thus,

$$\begin{aligned}
\varphi_1(t'', a'', p) &= p_{13} \cdot p_{12} \cdot p_{24} + p_{13} \cdot p_{14} \cdot p_{42} \\
&\geq p_{31} \cdot p_{32} \cdot p_{24} + p_{31} \cdot p_{34} \cdot p_{42} \\
&= \varphi_3(t'', a'', p)
\end{aligned}$$

follows. From $p \in \mathbf{P}_R$ we further have $p_{24} \geq 0.5$ and $p_{21} \geq p_{41}$, and thus,

$$\begin{aligned}
\varphi_2(t'', a'', p) &= p_{24} \cdot p_{21} \cdot p_{13} + p_{24} \cdot p_{23} \cdot p_{31} \\
&\geq p_{42} \cdot p_{41} \cdot p_{13} + p_{42} \cdot p_{43} \cdot p_{31} \\
&= \varphi_4(t'', a'', p)
\end{aligned}$$

also follows. Finally, from $p \in \mathbf{P}_R$ it results that $p_{13}, p_{23}, p_{24} \geq 0.5$, $p_{21} \geq p_{31}$, $p_{24} \geq p_{34}$, and hence,

$$\begin{aligned}
\varphi_2(t'', a'', p) &= p_{24} \cdot p_{21} \cdot p_{13} + p_{24} \cdot p_{23} \cdot p_{31} \\
&\geq p_{31} \cdot p_{32} \cdot p_{24} + p_{31} \cdot p_{34} \cdot p_{42} \\
&= \varphi_3(t'', a'', p)
\end{aligned}$$

also holds.

We thus conclude from a'' being manipulative at p and from the above three inequalities that there are $i \in \{1,3\}$ and $j \in \{2,4\}$ with $p_{ij} > 0.5$ and

$\varphi_j(t'', a'', p) > \varphi_i(t'', a'', p)$. Let $i' \in \{1,3\} \setminus \{i\}$ and $j' \in \{2,4\} \setminus \{j\}$ (i.e., $a'' = (ii', jj')$). It is then possible to proceed as in Case 1.1 and show that the agenda $a' = (i, i', j, j') \in \mathbf{A}^{t'}$ is manipulative at p.

Part 2 (*$\mathbf{A}^{t'}$ is more manipulative than $\mathbf{A}^{t''}$ on $\mathbf{P}_R$*) Take p to be such that $p_{ij} = 0.6$ for all $i, j \in C$ with $j > i$, and notice that $p \in \mathbf{P}_R$. We first show that for each $a'' \in \mathbf{A}^{t''}$, a'' is not manipulative at p. As already checked at the beginning of Part 1 of the proof, $a'' = (14, 23)$ is not manipulative at any $p \in \mathbf{P}_R$ and thus, the following two possibilities are left (corresponding to Cases 2.1 and 2.2) for what $a'' \in \mathbf{A}^{t''}$ might look like.

Case 2.1 ($a'' = (13, 24)$):

$$\begin{aligned}
\varphi_1(t'', a'', p) &= p_{13} \cdot p_{12} \cdot p_{24} + p_{13} \cdot p_{14} \cdot p_{42} = 0.36, \\
\varphi_2(t'', a'', p) &= p_{24} \cdot p_{21} \cdot p_{13} + p_{24} \cdot p_{23} \cdot p_{31} = 0.288, \\
\varphi_3(t'', a'', p) &= p_{31} \cdot p_{32} \cdot p_{24} + p_{31} \cdot p_{34} \cdot p_{42} = 0.192, \\
\varphi_4(t'', a'', p) &= p_{42} \cdot p_{41} \cdot p_{13} + p_{42} \cdot p_{43} \cdot p_{31} = 0.16,
\end{aligned}$$

and thus, a'' is not manipulative at p.

Case 2.2 ($a'' = (12, 34)$):

$$\begin{aligned}
\varphi_1(t'', a'', p) &= p_{12} \cdot p_{13} \cdot p_{34} + p_{12} \cdot p_{14} \cdot p_{43} = 0.36, \\
\varphi_2(t'', a'', p) &= p_{21} \cdot p_{23} \cdot p_{34} + p_{21} \cdot p_{24} \cdot p_{43} = 0.24, \\
\varphi_3(t'', a'', p) &= p_{34} \cdot p_{31} \cdot p_{12} + p_{34} \cdot p_{32} \cdot p_{21} = 0.24, \\
\varphi_4(t'', a'', p) &= p_{43} \cdot p_{41} \cdot p_{12} + p_{43} \cdot p_{42} \cdot p_{21} = 0.16,
\end{aligned}$$

i.e., a'' is not manipulative at p. We conclude that the agenda collection $\mathbf{A}^{t''}$ is not manipulative at p.

It remains to be shown that there exists $a' \in \mathbf{A}^{t'}$ which is manipulative at the above p. For this, take $a' = (1, 2, 3, 4)$. It follows that

$$\begin{aligned}
\varphi_1(t', a', p) &= p_{12} \cdot p_{13} \cdot p_{14} = 0.216, \\
\varphi_3(t', a', p) &= p_{12} \cdot p_{31} \cdot p_{34} + p_{21} \cdot p_{32} \cdot p_{34} = 0.24,
\end{aligned}$$

and thus a' is manipulative at p since $p_{13} > 0.5$ and $\varphi_1(t', a', p) < \varphi_3(t', a', p)$. We conclude that $\mathbf{A}^{t'}$ is more manipulative than $\mathbf{A}^{t''}$ on $\mathbf{P}_R$. ■

4 Conclusion

Our analysis of probabilistic manipulativeness in pairwise elimination procedures with four candidates shows that, when candidates can be ordered by strength, standard agendas are more manipulative than symmetric agendas. To the best of our knowledge, the latter agenda type is extensively used in sports competitions but not in voting situations. Thus, our result provides a strong motivation for the use of symmetric agendas in the voting environment described.

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