Generalized decomposition integral

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Abstract

In this paper we propose two different generalizations of the decomposition integral introduced by Even and Lehrer. We modify the product operator merging a given capacity and the decomposition coefficients by some more general functions F and G and compare properties of the obtained functionals with properties of the original decomposition integral. Generalized decomposition integrals corresponding to the particular decomposition systems, being generalizations of Shilkret, Choquet and concave integrals, are studied and exemplified.

Keywords: Decomposition integral; Capacity; Choquet integral; Shilkret integral

1. Introduction

Theory of non-additive integrals, i.e., integrals based on non-additive measures, proves to be useful in many fields, e.g., in multicriteria decision-making, economics, game theory, etc.

In the literature, many approaches to finding a general unifying concept of integrals can be found. One of them is the concave integral of Lehrer [11, 12] based on decompositions of integrated functions and extending the Lebesgue integral for non-additive monotone measures. In fact, the same idea was used by Wang et al. [26], considering signed measures. Considering just decompositions of integrated functions with respect to a particular decomposition system, i.e., not every decompositions being admissible, also Choquet integral, Shikret integral and PAN-integral can be expressed in a similar way. Recently, a unifying approach to such integrals appeared in the work of Even and Lehrer (see [5]), where so-called decomposition integral was introduced.

To mention another approaches based on the similar concept, we can look at the work of Greco et al. [6], where several distinguished integrals were related through the notion of superadditive and subadditive transformations of integrals, or at the work of Mesiar et al. [18] considering superdecompositions of the integrated functions (see also [15]). Various integrals were built using so-called pseudo-operations instead of classical arithmetical operations $+$ and $-$ (see [9, 13, 16, 17]). For ~~the~~ state-of-art of integral theory and generalized measures we recommend the handbook [21] and the book [25].

The aim of the present paper is to propose new integrals based on the approach similar to [5] by replacing the product operator merging a given capacity and the decomposition coefficients in the formula defining decomposition integral by some more general functions F, G. We obtain functionals $\mathcal{I}^{F}_{\mathcal{H},m}$ and $\mathcal{I}^{FG}_{\mathcal{H},m}$ covering, in both cases, the standard decomposition integrals for $F = G = \Pi$, $\Pi(u,v) = uv$.

The paper is organized as follows. Section 2 provides the basic notions and definitions needed throughout the paper. The proposal of two generalizations of the decomposition integral is confined in Section 3, wherein properties of the obtained functionals are studied and compared with properties of the standard decomposition integral. Generalized decomposition integrals with respect to (w.r.t.) decomposition systems consisting of singletons, full chains and the maximal decomposition system, respectively, are analyzed in Section 4. Finally, some concluding remarks are provided.

2. Preliminaries

Fix $n \in \mathbb{N}$ and $N = \{1, \cdots, n\}$.

Definition 2.1. *A set function $m\colon 2^N \to [0,1]$ is called a capacity if $m(C) \leq m(D)$ whenever $C \subseteq D$ and m satisfies the boundary conditions $m(\emptyset) = 0$, $m(N) = 1$.*

We denote the class of all capacities on 2^N by $\mathcal{M}_{(n)}$.

Definition 2.2. *A non-empty subset $\mathcal{D}$ of $2^N \setminus \{\emptyset\}$ is called a collection, i.e., $\emptyset \neq \mathcal{D} \subseteq 2^N \setminus \{\emptyset\}$. A non-empty family $\mathcal{H}$ of collections is called a decomposition system, i.e., $\mathcal{H} \subseteq 2^{2^N \setminus \{\emptyset\}} \setminus \{\emptyset\}$.*

Denote by $\mathbb{1}_A$ the indicator of the set $A \subseteq N$, i.e.,

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases},$$

for $x \in N$. We will use the standard bijection between the set of all functions $f\colon N \to [0,1]$ and the set of all vectors from $[0,1]^n$; $f \mapsto (f(1), \dots, f(n)) \in [0,1]^n$.

Definition 2.3 ([5]). *Let $m \in \mathcal{M}_{(n)}$ be a capacity and $\mathcal{H}$ be a decomposition system. The function $\mathcal{I}_{\mathcal{H},m} \colon [0,1]^n \to [0,\infty[$ given by*

$$\mathcal{I}_{\mathcal{H},m}(f) = \sup\left\{ \sum_{A\in\mathcal{D}} a_A m(A) \mid \sum_{A\in\mathcal{D}} a_A \mathbb{1}_A \leq f \ , \mathcal{D} \in \mathcal{H} \ , a_A \geq 0 \right\} \tag{1}$$

is called a decomposition integral w.r.t. the decomposition system $\mathcal{H}$.

Definition 2.4. *Let $f \in [0,1]^n$ be a function, $\mathcal{D} \in \mathcal{H}$ be a collection in a decomposition system $\mathcal{H}$. A finite summation $\sum_{A\in\mathcal{D}} a_A \mathbb{1}_A$ satisfying $\sum_{A\in\mathcal{D}} a_A \mathbb{1}_A \leq f$ and $a_A \geq 0$ for all $A \in \mathcal{D}$, is called a subdecomposition of the function f w.r.t. the collection $\mathcal{D}$ and will be denoted by $(a_A)_{A\in\mathcal{D}}$ when there is no fear of confusion.*

Example 2.5.

1. *Let $\mathcal{H}_1 = \{\{A\} \mid A \subseteq N \,; A \neq \emptyset\}$. Then the decomposition integral $\mathcal{I}_{\mathcal{H}_1,m}$ coincides with the Shilkret integral $\mathbf{Sh}_m$ [22].*
2. *Let $\mathcal{H}_2 = \{\mathcal{D} \mid \mathcal{D}$ is a maximal chain with respect to the inclusion$\}$. Then the decomposition integral $\mathcal{I}_{\mathcal{H}_2,m}$ coincides with the Choquet integral $\mathbf{Ch}_m$ [2]. Recall that a maximal chain in N is a subset $(A_i)_{i=1}^n \subseteq 2^N \setminus \{\emptyset\}$ such that $A_i \supsetneq A_{i+1}$ for $i = 1, \dots, i-1$.*
3. *The concave integral introduced by Lehrer [11] corresponds to $\mathcal{H}_3 = \{2^N \setminus \{\emptyset\}\}$, i.e., $\mathcal{H}_3$ is a singlenton consisting of the maximal collection $2^N \setminus \{\emptyset\}$. Note that taking the maximal decomposition system $\mathcal{H} = 2^{2^N \setminus \{\emptyset\}} \setminus \{\emptyset\}$ as a decomposition system, we obtain the same integral.*
4. *Let $\mathcal{H}_4 = \{\{A_i\}_{i\in J} \mid \{A_i\}_{i\in J}$ is a partition of $N\}$. Then the decomposition integral $\mathcal{I}_{\mathcal{H}_4,m}$ coincides with the PAN-integral [24] (based on the standard arithmetic operation $+$ and $\cdot$).*

Each decomposition integral possesses the following two properties [5]:

(i) **Positive homogeneity of degree one.**

For every $\lambda > 0$ it holds that $\mathcal{I}_{\mathcal{H},m}(\lambda f) = \lambda \mathcal{I}_{\mathcal{H},m}(f)$, for every $\mathcal{H}, m$ and every f such that $\lambda f \in [0,1]^n$.

(ii) **Monotonicity.**

For a fixed decomposition system $\mathcal{H}$ and a fixed capacity $m \in \mathcal{M}_{(n)}$ it holds $\mathcal{I}_{\mathcal{H},m}(f) \leq \mathcal{I}_{\mathcal{H},m}(g)$ whenever $f \leq g$, $f, g \in [0,1]^n$.

For a fixed decomposition system $\mathcal{H}$ and two capacitities $m, \widetilde{m} \in \mathcal{M}_{(n)}$ such that $m(A) \leq \widetilde{m}(A)$ for all $A \in \mathcal{D} \in \mathcal{H}$ it holds $\mathcal{I}_{\mathcal{H},m}(f) \leq \mathcal{I}_{\mathcal{H},\widetilde{m}}(f)$ for all $f \in [0,1]^n$.

For two decomposition systems $\mathcal{H}, \mathcal{H}^\star$ such that $\mathcal{H} \subseteq \mathcal{H}^\star$ it holds $\mathcal{I}_{\mathcal{H},m}(f) \leq \mathcal{I}_{\mathcal{H}^\star,m}(f)$ for all $f \in [0,1]^n$, $m \in \mathcal{M}_{(n)}$.

In [19] the authors have studied the question for which decomposition system $\mathcal{H}$ the corresponding decomposition integral gives back the capacity, i.e., $\mathcal{I}_{\mathcal{H},m}(\mathbb{1}_A) = m(A)$ for each $A \subseteq N$ and each capacity m. Recall that a decomposition system $\mathcal{H}$ is said to be complete if for each $\emptyset \neq A \subseteq N$ there is a collection $\mathcal{D} \in \mathcal{H}$ such that $A \in \mathcal{D}$. A collection $\mathcal{D}$ is said to be logically independent if it holds $\bigcap\limits_{A\in\mathcal{D}} A \neq \emptyset$.

Proposition 2.6 ([19])**.** *Let $\mathcal{H}$ be a decomposition system. Then the corresponding decomposition integral gives back the capacity if and only if $\mathcal{H}$ is a complete decomposition system and each collection $\mathcal{D} \in \mathcal{H}$ is logically independent.*

Note that all decomposition systems in Example 2.5 are complete, but $\mathcal{H}_3$ and $\mathcal{H}_4$ contain collections which are not logically independent and hence the related decomposition integrals do not give back some capacities.

3. Generalization of decomposition integral

We will generalize the decomposition integral replacing the product operator merging values of a given capacity and the coefficients of subdecomposition in the formula (1) by some more general function F. To obtain a functional with reasonable properties, we have to impose some constraints on the product-like function F. We define

$$\begin{aligned}\mathcal{F} = \{F\colon [0,1]^2 \to [0,\infty[\ \mid F \text{ is increasing in both arguments}, \\ F(0,u) = F(u,0) = 0\ \forall u \in [0,1]\}\,.\end{aligned}$$

By increasingness of F is meant that for every $x_1, x_2, y_1, y_2 \in [0,1]$, $x_1 \leq x_2, y_1 \leq y_2$ implies $F(x_1,y_1) \leq F(x_2,y_2)$.

Note that adding the conditions $F(1,u) = F(u,1) = u$, we obtain the class of pseudomultiplications $\otimes$ used for definition of universal integrals [9], i.e., ~~then~~ F is a semicopula [4].

Definition 3.1. *Let $\mathcal{H}$ be a decomposition system, $m \in \mathcal{M}_{(n)}$ be a capacity, let $F \in \mathcal{F}$. A function $\mathcal{I}^F_{\mathcal{H},m}\colon [0,1]^n \to [0,\infty[$ defined by*

$$\mathcal{I}^F_{\mathcal{H},m}(f) = \sup\left\{\sum_{A\in\mathcal{D}} F(a_A, m(A)) \mid \sum_{A\in\mathcal{D}} a_A \mathbb{1}_A \leq f\ ,\mathcal{D}\in\mathcal{H}\ ,a_A \geq 0\right\}, \quad (2)$$

for every $f \in [0,1]^n$, is called an F-decomposition integral w.r.t. decomposition system $\mathcal{H}$.

Definition 3.2. *Let $\mathcal{H}$ be a decomposition system, $m \in \mathcal{M}_{(n)}$ be a capacity, let $F, G \in \mathcal{F}$. A function $\mathcal{I}^{FG}_{\mathcal{H},m}\colon [0,1]^n \to [0,\infty[$ defined by*

$$\mathcal{I}^{FG}_{\mathcal{H},m}(f) = \sup\left\{\sum_{A\in\mathcal{D}} F(a_A, m(A)) \mid \sum_{A\in\mathcal{D}} G\left(a_A, \mathbb{1}_A(i)\right) \leq f(i)\ \ \forall i \in N, \mathcal{D}\in\mathcal{H}\ ,a_A \geq 0\right\}, \quad (3)$$

for every $f \in [0,1]^n$, is called an FG-decomposition integral w.r.t. decomposition system $\mathcal{H}$.

Remark 3.3. *Clearly, for $G \in \mathcal{F}$ satisfying $G(a,1) = a$ for all $a \in [0,1]$, both formulae (2) and (3) coincide (e.g., for all semicopulae).*

Definition 3.4. *Let $f \in [0,1]^n$ be a function and $\mathcal{I}^F_{\mathcal{H},m}(f) = \sum_{A\in\mathcal{D}} F(a_A, m(A))$ for some subdecomposition $(a_A)_{A\in\mathcal{D}}$ of the function f w.r.t. a collection $\mathcal{D} \in \mathcal{H}$ in decomposition system $\mathcal{H}$. Then the subdecomposition $(a_A)_{A\in\mathcal{D}}$ of the function f w.r.t. the collection $\mathcal{D}$ is called an optimal F-subdecomposition.*

Remark 3.5. *Although an optimal F-subdecomposition $(a_A)_{A\in\mathcal{D}}$ depends on the given decomposition system $\mathcal{H}$ and the capacity m, we will not indicate them in our notation having no fear of confusion.*

Definition 3.6. *Let $f \in [0,1]^n$ be a function, $\mathcal{D} \in \mathcal{H}$ be a collection in a decomposition system $\mathcal{H}$ and $G \in \mathcal{F}$. A set of real numbers $(a_A)_{A\in\mathcal{D}}$ satisfying $\sum_{A\in\mathcal{D}} G(a_A, \mathbb{1}_A(i)) \le f(i)$ for all $i \in N$ and $a_A \ge 0$ for all $A \in \mathcal{D}$, is called an G-subdecomposition of the function f w.r.t. the collection $\mathcal{D}$ and will be denoted by $(a_A)^G_{A\in\mathcal{D}}$ when there is no fear of confusion.*

Definition 3.7. *Let $f \in [0,1]^n$ be a function and $\mathcal{I}^{FG}_{\mathcal{H},m}(f) = \sum_{A\in\mathcal{D}} F(a_A, m(A))$ for an G-subdecomposition $(a_A)^G_{A\in\mathcal{D}}$ of the function f w.r.t. a collection $\mathcal{D} \in \mathcal{H}$ in decomposition system $\mathcal{H}$. Then the G-subdecomposition $(a_A)^G_{A\in\mathcal{D}}$ is called an optimal FG-subdecomposition (depending, like in the definition above, on the given decomposition system $\mathcal{H}$ and the capacity m) .*

Remark 3.8. *Note that due to the finiteness of the underlying space, the formula for F-decomposition integral ~~integral~~ can be seen as a linear programming task, and thus supremum in definition of $\mathcal{I}^F_{\mathcal{H},m}$ is always attained. A similar claim is valid for the FG-decomposition integral, supposing $G(.,1)$ is continuous from below (e.g., each semicopula satisfies this constraint). So, in both cases, supremum can be replaced by maximum. Nevertheless, we decided to keep the notation used in the definition of the original decomposition integral, to cover all possible situations concerning G. For G not continuous from below, supremum needs not to be attained as the following example shows.*

Let $N = \{1\}$. Considering $G(x,0) = 0, G(x,1) = x.\mathbb{1}_{[0,0.5[} + \mathbb{1}_{[0.5,1]}$, $F(x,1) = x$ and $f(1) = 0.5$, we have $\mathcal{I}^{FG}_{\mathcal{H},m}(f) = 0.5$, but there is no optimal FG-subdecomposition.

It can be easily checked that, for each $F, G \in \mathcal{F}$, the properties analogous to those of the standard decomposition integral mentioned above hold for both F-decomposition integral and FG-decomposition integral.

(i) **Positive homogeneity.**

Let $F, G \in \mathcal{F}$ be positively homogeneous in the first argument, i.e., $F(\lambda u, v) = \lambda F(u, v)$ and $G(\lambda u, v) = \lambda G(u, v)$ for every $\lambda > 0$ and every $u \in [0, 1]$ such that $\lambda u \in [0, 1]$. Then it holds

$$\mathcal{I}^{F}_{\mathcal{H},m}(\lambda f) = \lambda \mathcal{I}^{F}_{\mathcal{H},m}(f), \qquad \mathcal{I}^{FG}_{\mathcal{H},m}(\lambda f) = \lambda \mathcal{I}^{FG}_{\mathcal{H},m}(f),$$

for every $\mathcal{H}, m$ and every f such that $\lambda f \in [0, 1]^n$.

(ii) **Monotonicity.**

For a fixed decomposition system $\mathcal{H}$ and a fixed capacity $m \in \mathcal{M}_{(n)}$ it holds $\mathcal{I}^{F}_{\mathcal{H},m}(f) \le \mathcal{I}^{F}_{\mathcal{H},m}(g)$ and $\mathcal{I}^{FG}_{\mathcal{H},m}(f) \le \mathcal{I}^{FG}_{\mathcal{H},m}(g)$ whenever $f \le g$, $f, g \in [0, 1]^n$.

For a fixed decomposition system $\mathcal{H}$ and two capacities $m, \widetilde{m} \in \mathcal{M}_{(n)}$ such that $m(A) \le \widetilde{m}(A)$ for all $A \in \mathcal{D}$, $\mathcal{D} \in \mathcal{H}$ it holds $\mathcal{I}^{F}_{\mathcal{H},m}(f) \le \mathcal{I}^{F}_{\mathcal{H},\widetilde{m}}(f)$ and $\mathcal{I}^{FG}_{\mathcal{H},m}(f) \le \mathcal{I}^{FG}_{\mathcal{H},\widetilde{m}}(f)$ for all $f \in [0, 1]^n$.

For two decomposition systems $\mathcal{H}, \mathcal{H}^\star$ such that $\mathcal{H} \subseteq \mathcal{H}^\star$ it holds $\mathcal{I}^{F}_{\mathcal{H},m}(f) \le \mathcal{I}^{F}_{\mathcal{H}^\star,m}(f)$ and $\mathcal{I}^{FG}_{\mathcal{H},m}(f) \le \mathcal{I}^{FG}_{\mathcal{H}^\star,m}(f)$ for all $f \in [0, 1]^n$, $m \in \mathcal{M}_{(n)}$.

Proposition 3.9. *Let $\mathcal{H}, \mathcal{H}^\star$ be arbitrary decomposition systems and $F, G \in \mathcal{F}$. If for every collection $\mathcal{D} \in \mathcal{H}$ there exists a collection $\mathcal{D}^\star \in \mathcal{H}^\star$ such that $\mathcal{D} \subseteq \mathcal{D}^\star$, then $\mathcal{I}^{F}_{\mathcal{H},m} \le \mathcal{I}^{F}_{\mathcal{H}^\star,m}$ and $\mathcal{I}^{FG}_{\mathcal{H},m} \le \mathcal{I}^{FG}_{\mathcal{H}^\star,m}$, for any $m \in \mathcal{M}_{(n)}$.*

Remark 3.10. *As a consequence of the previous proposition we obtain that, for a fixed $F \in \mathcal{F}$, the greatest generalized F-decomposition integral is the generalized concave integral, i.e., integral corresponding to $\mathcal{H}_3 = \{2^N \setminus \{\emptyset\}\}$. However, there is no smallest generalized F-decomposition integral, in general.*

Moreover, defining an ordering on the class of all possible decomposition systems, i.e., on $2^{2^{2^N \setminus \{\emptyset\}}} \setminus \{\emptyset\}$, by $\mathcal{H} \prec \mathcal{H}^\star$ iff $\mathcal{I}^{F}_{\mathcal{H},m} \le \mathcal{I}^{F}_{\mathcal{H}^\star,m}$, we obtain that $\left(2^{2^{2^N \setminus \{\emptyset\}}} \setminus \{\emptyset\}, \prec\right)$ is an upper-semilattice with the top element $\mathcal{H}_3 = \{2^N \setminus \{\emptyset\}\}$.

The following property of F-decomposition integral is analogous to that of the standard decomposition integral, see [5, Theorem 1].

Proposition 3.11. *Let $\mathcal{H}$ be an arbitrary decomposition system, let $F \in \mathcal{F}$ be a function concave in the first variable, i.e., for all $\lambda \in [0, 1]$ it holds*

$$F(\lambda u_1 + (1 - \lambda) u_2, v) \ge \lambda F(u_1, v) + (1 - \lambda) F(u_2, v),$$

for all $u_1, u_2, v \in [0, 1]$ such that $u_1 + (1 - \lambda) u_2 \in [0, 1]$.

If there exists a decomposition system $\mathcal{H}'$ consisting of only one collection such that $\mathcal{I}^{F}_{\mathcal{H},m} = \mathcal{I}^{F}_{\mathcal{H}',m}$ for all capacities $m \in \mathcal{M}_{(n)}$, then $\mathcal{I}^{F}_{\mathcal{H},m}$ is concave, i.e.,

$$\mathcal{I}^{F}_{\mathcal{H},m}(\lambda f + (1 - \lambda) g) \ge \lambda \mathcal{I}^{F}_{\mathcal{H},m}(f) + (1 - \lambda) \mathcal{I}^{F}_{\mathcal{H},m}(g),$$

for all $\lambda \in [0, 1]$, $f, g \in [0, 1]^n$.

Proof: Let $\mathcal{H}' = \{\mathcal{D}\}$, $f, g \in [0,1]^n$ be two functions with respective optimal subdecompositions $(a_A)_{A\in\mathcal{D}}$ and $(b_A)_{A\in\mathcal{D}}$, i.e., $\mathcal{I}^F_{\mathcal{H},m}(f) = \sum_{A\in\mathcal{D}} F(a_A, m(A))$ and $\mathcal{I}^F_{\mathcal{H},m}(g) = \sum_{A\in\mathcal{D}} F(b_A, m(A))$. Then, $(\lambda a_A + (1-\lambda)b_A)_{A\in\mathcal{D}}$ is a subdecomposition of the function $\lambda f + (1-\lambda)g$ and due to concavity of F in the first variable we have

$$\begin{aligned}
\mathcal{I}^F_{\mathcal{H},m}(\lambda f + (1-\lambda)g) &\geq \sum_{A\in\mathcal{D}} F(\lambda a_A + (1-\lambda)b_A, m(A)) \\
&\geq \lambda \sum_{A\in\mathcal{D}} F(a_A, m(A)) + (1-\lambda) \sum_{A\in\mathcal{D}} F(b_A, m(A)) \\
&= \lambda \mathcal{I}^F_{\mathcal{H},m}(f) + (1-\lambda)\mathcal{I}^F_{\mathcal{H},m}(g),
\end{aligned}$$

for all $\lambda \in [0,1]$, $m \in \mathcal{M}_{(n)}$. □

~~Putting an assumption of convexity on a~~ function $G(.,1)$, the statement analogous to that of Proposition 3.11 can be proved for FG- decomposition integral.

Proposition 3.12. *Let $\mathcal{H}$ be an arbitrary decomposition system, let $F \in \mathcal{F}$ be a function concave in the first variable and $G \in \mathcal{F}$ be such that $G(.,1)$ is convex, i.e., for all $\lambda \in [0,1]$ it holds*

$$F(\lambda u_1 + (1-\lambda)u_2, v) \geq \lambda F(u_1, v) + (1-\lambda)F(u_2, v),$$

$$G(\lambda u_1 + (1-\lambda)u_2, 1) \leq \lambda G(u_1, 1) + (1-\lambda)G(u_2, 1),$$

for all $u_1, u_2, v \in [0,1]$ such that $u_1 + (1-\lambda)u_2 \in [0,1]$.

If there exists a decomposition system $\mathcal{H}'$ consisting of only one collection such that $\mathcal{I}^{FG}_{\mathcal{H},m} = \mathcal{I}^F G_{\mathcal{H}',m}$ for all capacities $m \in \mathcal{M}_{(n)}$, then $\mathcal{I}^{FG}_{\mathcal{H},m}$ is concave, i.e.,

$$\mathcal{I}^{FG}_{\mathcal{H},m}(\lambda f + (1-\lambda)g) \geq \lambda \mathcal{I}^{FG}_{\mathcal{H},m}(f) + (1-\lambda)\mathcal{I}^{FG}_{\mathcal{H},m}(g),$$

for all $\lambda \in [0,1]$, $f, g \in [0,1]^n$.

Proof: The proof is similar to that of Proposition 3.11. It is based on the fact that for optimal G-subdecompositions $(a_A)^G_{A\in\mathcal{D}}$ and $(b_A)^G_{A\in\mathcal{D}}$ of f and g, respectively, $(\lambda a_A + (1-\lambda)b_A)_{A\in\mathcal{D}}$ is a G-subdecomposition of the function $\lambda f + (1-\lambda)g$ due to constraints on G. □

Remark 3.13. *A statement analogous to the previous proposition does not hold for FF-decomposition integrals, as the following example shows, since due to concavity of F we have*

$$\sum_{A\in\mathcal{D}} F(\lambda a_A + (1-\lambda)b_A, \mathbb{1}_A(i)) \geq \lambda \sum_{A\in\mathcal{D}} F(a_A, \mathbb{1}_A(i)) + (1-\lambda) \sum_{A\in\mathcal{D}} F(b_A, \mathbb{1}_A(i)),$$

for all $i \in N$, thus $(\lambda a_A + (1-\lambda)b_A)^F_{A\in\mathcal{D}}$ need not to be an F-subdecomposition of the function $\lambda f + (1-\lambda)g$.

Example 3.14. *Let us take $n = 2$, $\mathcal{D} = \{\{1,2\},\{1\},\{2\}\}$, $\mathcal{H} = \{\mathcal{D}\}$. Denote $\varphi(u) = F(u,1)$ and $\varphi^{(-1)}$ the so-called pseudo-inverse of φ defined as $\varphi^{(-1)}(y) = \sup\{x \in [0,1] \mid \varphi(x) \le y\}$). Then, according to (5) below, we have*

$$\mathcal{I}^{FF}_{\mathcal{H},m}(x,y) = \sup_{p\in[0,x\wedge y]} \{p + F(\varphi^{(-1)}(x-p), a) + F(\varphi^{(-1)}(y-p), b)\},$$

where $a = m(\{1\})$, $b = m(\{2\})$ ($x \wedge y = \min\{x,y\}$).

Considering $F(u,v) = \sin(uv)$, $f = (0.5, 0.8)$, $g = (0.9, 0.6)$, $a = 0.4$, $b = 0.7$ we obtain

$$\mathcal{I}^{FF}_{\mathcal{H},m}(0.5, 0.8) = 0.812387, \qquad \mathcal{I}^{FF}_{\mathcal{H},m}(0.9, 0.6) = 0.868452.$$

However, for $\lambda = 0.5$ we get $\lambda f + (1-\lambda)g = (0.7, 0.7)$ and

$$\mathcal{I}^{FF}_{\mathcal{H},m}(0.7, 0.7) = 0.821727 < 0.8404195 = \lambda\mathcal{I}^{FF}_{\mathcal{H},m}(f) + (1-\lambda)\mathcal{I}^{FF}_{\mathcal{H},m}(g),$$

hence $\mathcal{I}^{FF}_{\mathcal{H},m}$ is not concave, although F is concave in the first variable and $\mathcal{H}$ consists of the only one collection.

Definition 3.15. *Let $f \in [0,1]^n$, $\mathcal{D}$ be a collection. A subdecomposition $(a_A)_{A\in\mathcal{D}}$ of the function f w.r.t. $\mathcal{D}$ is defined to be a maximal subdecomposition of f w.r.t. $\mathcal{D}$ if for all subdecompositions $(b_A)_{A\in\mathcal{D}}$ of the function f w.r.t. $\mathcal{D}$ such that $b_A \ge a_A$ for all $A \in \mathcal{D}$, it holds $b_A = a_A$ for all $A \in \mathcal{D}$.*

A~~n~~ G-subdecomposition ${(a_A)^G}_{A\in\mathcal{D}}$ of the function f w.r.t. $\mathcal{D}$ is defined to be a maximal G-subdecomposition of f w.r.t. $\mathcal{D}$ if for all G-subdecompositions ${(b_A)^G}_{A\in\mathcal{D}}$ of the function f w.r.t. $\mathcal{D}$ such that $b_A \ge a_A$ for all $A \in \mathcal{D}$, it holds $b_A = a_A$ for all $A \in \mathcal{D}$.

Due to the increasingness of F in the first argument, when determining the value of F-decomposition integral and FG-decomposition integral, it is enough to take supremum (actually, maximum in that case) in formula (2) and (3), respectively, over all maximal subdecompositions of f and maximal G-subdecomposition of f w.r.t. $\mathcal{D} \in \mathcal{H}$, respectively.

Lemma 3.16. *Let $\mathcal{H}$ be a decomposition system, $m \in \mathcal{M}_{(n)}$ be a capacity, $F \in \mathcal{F}$. Then for all $f \in [0,1]^n$ it holds*

$$\mathcal{I}^{F}_{\mathcal{H},m}(f) = \max\{\sum_{A\in\mathcal{D}} F(a_A, m(A)) \mid (a_A)_{A\in\mathcal{D}} \text{ is a maximal subdecomposition of } f \text{ w.r.t. } \mathcal{D};\, \mathcal{D}\in\mathcal{H}\}$$

and

$$\mathcal{I}^{FG}_{\mathcal{H},m}(f) = \max\{\sum_{A\in\mathcal{D}} F(a_A, m(A)) \mid {(a_A)^G}_{A\in\mathcal{D}} \text{ is a maximal } G\text{-subdecomposition of } f \text{ w.r.t. } \mathcal{D};\, \mathcal{D}\in\mathcal{H}\}\,.$$

Focusing on the question of giving back the capacity for F-decomposition integral, we find out that, similarly as for the standard decomposition integral, the completeness of the decomposition system ensures one of desired inequalities (compare with [19, Lemma 3.3] and Proposition 2.6 above).

Proposition 3.17. *Let $F \in \mathcal{F}$ be a function satisfying $F(1,1) \neq 0$. A decomposition system $\mathcal{H}$ is complete if and only if for all capacities m and all $A \subseteq N$ it holds $\mathcal{I}^F_{\mathcal{H},m}(\mathbb{1}_A) \geq F(1, m(A))$.*

Proof: Let $\mathcal{H}$ be a complete decomposition system. Then, for $A \subseteq N$ there is a collection $\mathcal{D} \in \mathcal{H}$ such that $A \in \mathcal{D}$. Taking the decomposition $(a_B)_{B\in\mathcal{D}}$ of the $\mathbb{1}_A$ w.r.t. $\mathcal{D}$ defined as $a_B = 1$ if $B = A$ and $a_B = 0$ otherwise, we obtain $\mathcal{I}^F_{\mathcal{H},m}(\mathbb{1}_A) \geq F(1, m(A))$.

To show the necessity, suppose that there is $A \subseteq N$ such that for each collection $\mathcal{D} \in \mathcal{H}$ it holds $A \notin \mathcal{D}$. Then for each $\mathcal{D} \in \mathcal{H}$ and each decomposition $(a_B)_{B\in\mathcal{D}}$ of the $\mathbb{1}_A$ w.r.t. $\mathcal{D} \in \mathcal{H}$ it holds either $a_B = 0$ or $B \subsetneq A$. Taking the capacity δ_A defined as $\delta_A(B) = 1$ if $A \subseteq B$ and vanishing otherwise, we have

$$\mathcal{I}^F_{\mathcal{H},\delta_A}(\mathbb{1}_A) = \sum_{B\subsetneq A} F(a_B, 0) + \sum_{B, B\setminus A\neq\emptyset} F(0, m(B)) = 0 \lneq F(1,1) = F(1, \delta_A(A)),$$

and the necessity follows. $\square$

For the standard decomposition integral the other inequality, i.e., $\mathcal{I}^F_{\mathcal{H},m}(\mathbb{1}_A) \leq F(1, m(A))$, is ensured by the logical independency of each collection in the decomposition system (compare with [19, Lemma 3.1] and Proposition 2.6 above). However, this is not the case of general F-decomposition integral as the following example shows.

Example 3.18. *Let $\mathcal{H}_2 = \{\mathcal{D} \mid \mathcal{D}$ is a maximal chain with respect to the inclusion$\}$, $n = 2$, $F(u,v) = \sqrt{u}v$, $m \in \mathcal{M}_{(2)}$ be a capacity with $m(\{1\}) = a \neq 0$, $m(\{2\}) = b \neq 0$. Then each collection in $\mathcal{H}_2$ is logically independent and, according to Example 4.3.2 below (see also Figure 2 (left)), we have*

$$\mathcal{I}^F_{\mathcal{H},m}(\mathbb{1}_N) = \mathcal{I}^F_{\mathcal{H},m}(1,1) = \begin{cases} \sqrt{a^2+1} & \text{if } a \leq b \\ \sqrt{b^2+1} & \text{otherwise} \end{cases} \gneq 1 = F(1, m(N)).$$

In particular case of $F(u,v) = G(u,v) = \varphi(u)\psi(v)$, ~~there is~~ the following connection between the FG-decomposition integral and the standard decomposition integral corresponding to the distorted capacity.

Proposition 3.19. *Let $\mathcal{H}$ be a decomposition system, $m \in \mathcal{M}_{(n)}$ and $F(u,v) = \varphi(u)\psi(v)$, where both $\varphi, \psi\colon [0,1] \to [0,1]$ are increasing functions and $\varphi(0) = \psi(0) = 0$, $\varphi(1) = \psi(1) = 1$. Then it holds $\mathcal{I}^{FF}_{\mathcal{H},m} = \mathcal{I}_{\mathcal{H},\psi(m)}$.*

Proof: Let $f \in [0,1]^n$ and $(a_A)^F_{A\in\mathcal{D}}$ be an F-subdecomposition of the function f w.r.t. a collection $\mathcal{D} \in \mathcal{H}$, i.e., $\sum_{A\in\mathcal{D}} \varphi(a_A)\,\psi(\mathbb{1}_A(i)) \leq f(i)$ for all $i \in N$. Since $\psi(\mathbb{1}_A(i)) = \mathbb{1}_A(i)$ for all $i \in N$, it holds $\sum_{A\in\mathcal{D}} \varphi(a_A)\,\mathbb{1}_A(i) \leq f(i)$ for all $i \in N$, i.e., $\sum_{A\in\mathcal{D}} \varphi(a_A)\, \mathbb{1}_A \le f$. Thus, denoting $b_A = \varphi(a_A)$, we obtain that $(b_A)_{A\in\mathcal{D}}$ is a subdecomposition of the function f w.r.t. a collection $\mathcal{D} \in \mathcal{H}$. Hence, we get

$$\begin{aligned}
& \mathcal{I}^{FF}_{\mathcal{H},m}(f) \\
= \; & \sup\left\{ \sum_{A\in\mathcal{D}} \varphi(a_A)\, \psi(m(A)) \mid \sum_{A\in\mathcal{D}} \varphi(a_A)\, \psi(\mathbb{1}_A(i)) \le f(i) \;\; \forall i \in N, \mathcal{D}\in\mathcal{H}\,, a_A \ge 0 \right\} \\
= \; & \sup\left\{ \sum_{A\in\mathcal{D}} b_A \psi(m(A)) \mid \sum_{A\in\mathcal{D}} b_A \mathbb{1}_A \le f, \mathcal{D}\in\mathcal{H}\,, a_A \ge 0 \right\} \\
\le \; & \mathcal{I}_{\mathcal{H},\psi(m)}(f).
\end{aligned}$$

On the other hand, for each subdecomposition $(b_A)_{A\in\mathcal{D}}$ of the function f w.r.t. a collection $\mathcal{D} \in \mathcal{H}$ there is the F-subdecomposition $(a_A)^F_{A\in\mathcal{D}}$ of the function f w.r.t. a collection $\mathcal{D} \in \mathcal{H}$, where $a_A = \varphi^{(-1)}(b_A)$. Thus, similarly as above, we get $\mathcal{I}^{FF}_{\mathcal{H},m}(f) \ge \mathcal{I}_{\mathcal{H},\psi(m)}(f)$ and the claim follows. □

4. *F*-decomposition integrals and *FG*-decomposition integrals w.r.t. particular decompositions systems

In this section we focus on F-decomposition integrals and FG-decomposition integrals w.r.t. particular ~~decompositions~~ decomposition systems mentioned in Example 2.5 and corresponding in case of the standard decomposition integral to Shilkret, Choquet and concave integrals, respectively. As such, related F-decomposition integrals and FG-decomposition integrals can be regarded as their generalizations. In particular, we study binary F-decomposition integrals w.r.t. the respective particular decompositions systems.

4.1. Singletons - generalization of the Shilkret integral

Let $\mathcal{H}_1 = \{\{A\} \mid A \subseteq N,\ A \ne \emptyset\}$, $F \in \mathcal{F}$. Then, due to Lemma 3.16, for $m \in \mathcal{M}_{(n)}$ we have

$$\begin{aligned}
\mathcal{I}^{F}_{\mathcal{H}_1,m}(f) &= \sup\{F(a_A, m(A)) \mid a_A \mathbb{1}_A \le f\,, A \subseteq N\,, a_A \ge 0\} \\
&= \max\left\{ F(\min_{i\in A} f(i), m(A)) \mid A \subseteq N \right\},
\end{aligned}$$

for $f \in [0,1]^n$.

Denote $F(u,1) = \varphi(u)$. Then we have

$$\begin{aligned}
\mathcal{I}^{FF}_{\mathcal{H}_1,m}(f) &= \sup\{F(a_A, m(A)) \mid F(a_A, \mathbb{1}_A(i)) \le f \;\forall i \in N,\ A \subseteq N,\ a_A \ge 0\} \\
&= \max\left\{ F(\min_{i\in A} \varphi^{(-1)}(f(i)), m(A)) \mid A \subseteq N \right\} \\
&= \varphi^{(-1)}\left(\mathcal{I}^{F}_{\mathcal{H}_1,m}(f)\right),
\end{aligned}$$

for $f \in [0,1]^n$.

Note that both F-decomposition integral and FF-decomposition integral for $F(u,v) = uv$ coincide with the Shilkret integral (observe that then $\varphi = \mathrm{id}_{[0,1]}$).

Remark 4.1. *Let $m \in \mathcal{M}_{(n)}$, $f \in [0,1]^n$. Recall that the Sugeno integral of f w.r.t. m can be expressed as*

$$\mathbf{Su}_m(f) = \sup_{A \subseteq N} \left\{ \min\{m(A), \min_{i \in A} f(i)\} \right\}$$

(see [23]). Thus, taking $F(u,v) = \min\{u,v\}$, we have $\mathcal{I}^F_{\mathcal{H}_1,m} = \mathbf{Su}_m$ for all $m \in \mathcal{M}_{(n)}$. Consequently, F-decomposition integral, for $F(u,v) = \min\{u,v\}$, can be regarded as a generalization of the Sugeno integral (unlike the standard decomposition integral of Lehrer and Even). For another generalizations of the Sugeno integral see, e.g., [1, 3, 7, 8].

Moreover, the well-known fact that Choquet and Sugeno integrals coincide for $\{0,1\}$-valued capacities turns into the following, even stronger, form:

For ~~an~~ arbitrary semicopulae S and $\tilde{S}$, a decomposition system $\mathcal{H}$ and a $\{0,1\}$-valued capacity m it holds

$$\mathcal{I}^S_{\mathcal{H},m} = \mathcal{I}^{S\tilde{S}}_{\mathcal{H},m} = \mathcal{I}_{\mathcal{H},m}.$$

For instance, taking the maximal capacity m^*, defined as $m^*(A) = 1$ for $A \neq \emptyset$ and $m^*(\emptyset) = 0$, we obtain

$$\begin{aligned}
\mathcal{I}^S_{\mathcal{H}_1,m^*}(f) = \mathcal{I}^{S\tilde{S}}_{\mathcal{H}_1,m^*}(f) = \mathcal{I}_{\mathcal{H}_1,m^*}(f) &= \max_{i \in N} f(i), \\
\mathcal{I}^S_{\mathcal{H}_2,m^*}(f) = \mathcal{I}^{S\tilde{S}}_{\mathcal{H}_2,m^*}(f) = \mathcal{I}_{\mathcal{H}_2,m^*}(f) &= \max_{i \in N} f(i), \\
\mathcal{I}^S_{\mathcal{H}_3,m^*}(f) = \mathcal{I}^{S\tilde{S}}_{\mathcal{H}_3,m^*}(f) = \mathcal{I}_{\mathcal{H}_3,m^*}(f) &= \sum_{i \in N} f(i), \\
\mathcal{I}^S_{\mathcal{H}_4,m^*}(f) = \mathcal{I}^{S\tilde{S}}_{\mathcal{H}_4,m^*}(f) = \mathcal{I}_{\mathcal{H}_4,m^*}(f) &= \sum_{i \in N} f(i),
\end{aligned}$$

for every $f \in [0,1]^n$. Note that these are the maximal values which can be attained by the decomposition integrals $\mathcal{I}^S_{\mathcal{H}_i,m}$, $\mathcal{I}^{S\tilde{S}}_{\mathcal{H}_i,m}$, $\mathcal{I}_{\mathcal{H}_i,m}$, $i = 1, \ldots, 4$, for a given f, an arbitrary semicopula S and an arbitrary capacity m.

Remark 4.2. *Let $F(u,v) = \phi(u)\psi(v)$, where both φ, $\psi \colon [0,1] \to [0,1]$ are increasing functions and $\varphi(0) = \psi(0) = 0$, $\varphi(1) = \psi(1) = 1$. Then we get*

$$\begin{aligned}
\mathcal{I}^F_{\mathcal{H}_1,m}(f) &= \sup \left\{ \varphi \left(\min_{i \in A} f(i) \right) \psi\left(m(A)\right) \mid A \subseteq N \right\} \\
&= \sup \left\{ \min_{i \in A} \varphi(f(i))\, \psi\left(m(A)\right) \mid A \subseteq N \right\}
\end{aligned}$$

for $f \in [0,1]^n$, which is the Shilkret integral of $\varphi \circ f$ w.r.t. the transformed measure $\psi(m)$ (compare Proposition 3.19).

Binary case. Let $n = 2$, $F \in \mathcal{F}$. Let $m \in \mathcal{M}_{(2)}$ be a capacity with $m(\{1\}) = a$, $m(\{2\}) = b$. We have

$$\begin{aligned}
\mathcal{I}^F_{\mathcal{H}_1,m}(x,y) &= \max\{F(x,a), F(y,b), F(x \wedge y, 1)\} \\
&= \begin{cases} \max\{F(x,a), F(y,b), F(x,1)\} & \text{if } x \le y \\ \max\{F(x,a), F(y,b), F(y,1)\} & \text{otherwise} \end{cases} \\
&= \begin{cases} \max\{F(y,b), F(x,1)\} & \text{if } x \le y \\ \max\{F(x,a), F(y,1)\} & \text{otherwise,} \end{cases}
\end{aligned}$$

see Figure 1.

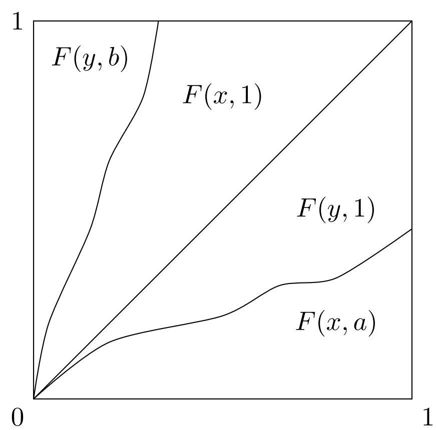

Figure 1: Binary generalization of the Shilkret integral.

Denoting again $\varphi(u) = G(u,1)$, for FG-decomposition integral, we obtain

$$\mathcal{I}^{FG}_{\mathcal{H}_1,m}(x,y) = \max\{F(\varphi^{(-1)}(x), a), F(\varphi^{(-1)}(y), b), F(\varphi^{(-1)}(x \wedge y), 1)\}$$

In particular, for FF-decomposition integral, we have

$$\mathcal{I}^{FF}_{\mathcal{H}_1,m}(x,y) = \max\{F(\varphi^{(-1)}(x), a), F(\varphi^{(-1)}(y), b), x \wedge y\}.$$

4.2. Maximal chains - generalization of the Choquet integral

Let $\mathcal{H}_2 = \{\mathcal{D} \mid \mathcal{D} \text{ is a maximal chain}\}$. Then, for $F, G \in \mathcal{F}$, the functionals $\mathcal{I}^F_{\mathcal{H}_2,m}$ and $\mathcal{I}^{FG}_{\mathcal{H}_2,m}$ can be regarded as generalizations of the Choquet integral.

Binary case. Let $n = 2$. Let $m \in \mathcal{M}_{(2)}$ be a capacity with $m(\{1\}) = a$, $m(\{2\}) = b$. We have two collections in $\mathcal{H}_2$, $\mathcal{D}_1 = \{\{1,2\},\{1\}\}$ and $\mathcal{D}_2 = \{\{1,2\},\{2\}\}$.

The families of the maximal subdecompositions of $f = (x, y) \in [0, 1]^2$ w.r.t. $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively, are of the following respective forms:

$$\left\{p\mathbb{1}_{\{1,2\}} + (x-p)\mathbb{1}_{\{1\}} \mid p \in [0, x \wedge y]\right\}$$
$$\left\{p\mathbb{1}_{\{1,2\}} + (y-p)\mathbb{1}_{\{2\}} \mid p \in [0, x \wedge y]\right\}.$$

Thus, using Lemma 3.16, we get

$$\mathcal{I}^F_{\mathcal{H}_2,m}(x, y) = \sup_{p\in[0,x\wedge y]} \{F(p, 1)+F(x-p, a)\} \vee \sup_{p\in[0,x\wedge y]} \{F(p, 1)+F(y-p, b)\} \quad (4)$$

Looking for the maximal G-subdecompositions of $f = (x, y) \in [0, 1]^2$ w.r.t. $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively, we have to consider the following inequalities (here again $\varphi(u) = G(u, 1)$):

$$\varphi(a_{\{1,2\}}) + \varphi(a_{\{1\}}) \leq x,$$
$$\varphi(a_{\{1,2\}}) + \varphi(a_{\{2\}}) \leq y.$$

Thus, denoting $\varphi(a_{\{1,2\}}) = p$ we obtain

$$\begin{aligned}\mathcal{I}^{FG}_{\mathcal{H}_2,m}(x, y) \quad = \quad & \sup_{p\in[0,x\wedge y]} \{F(\varphi^{(-1)}(p), 1) + F(\varphi^{(-1)}(x-p), a)\} \\ & \vee \sup_{p\in[0,x\wedge y]} \{F(\varphi^{(-1)}(p), 1) + F(\varphi^{(-1)}(y-p), b)\}.\end{aligned}$$

In particular, for $F = G$, we have

$$\mathcal{I}^{FF}_{\mathcal{H}_2,m}(x, y) = \sup_{p\in[0,x\wedge y]} \{p + F(\varphi^{(-1)}(x-p), a)\} \vee \sup_{p\in[0,x\wedge y]} \{p + F(\varphi^{(-1)}(y-p), b)\}.$$

Example 4.3.

1. *Let $F(u, v) = u \wedge v$. Let $x \in [0, 1]$ and $a \geq 0$. If $x - a \geq 0$, then we have*

   $$p \wedge 1 + (x-p) \wedge a = \begin{cases} p + a \leq x & \text{if } 0 \leq p \leq x - a \\ p + (x-p) = x & x - a \leq p \leq x \wedge y. \end{cases}$$

   If $x - a < 0$, then $x - p < a$ and $p \wedge 1 + (x-p) \wedge a = x$.
   Thus, for fixed $x \in [0, 1]$ and $a \geq 0$, $p\wedge 1+(x-p)\wedge a$ is always nondecreasing w.r.t. p. Similarly, nondecreasingness of $p\wedge 1+(y-p)\wedge b$ for fixed $y \in [0, 1]$ and $b \geq 0$ can be shown.
   Thus, both $p\wedge 1+(x-p)\wedge a$ and $p\wedge 1+(y-p)\wedge b$ attain their maximum in $x \wedge y$. Using (4) and properties of F, we obtain

   $$\mathcal{I}^F_{\mathcal{H}_2,m}(x, y) = \begin{cases} y \wedge 1 + (x-y) \wedge a = x \wedge (a + y) & \text{if } x \geq y \\ x \wedge 1 + (y-x) \wedge b = y \wedge (b + x) & \text{otherwise,} \end{cases}$$

   see Figure 2 (left).
   Note that according to Remark 3.3 in this case it holds $\mathcal{I}^F_{\mathcal{H},m} = \mathcal{I}^{FF}_{\mathcal{H},m}$.

2. *Let* $F(u,v) = \sqrt{uv}$*. Let* $(x,y) \in [0,1]^2$ *be fixed. For a fixed* $(x,y) \in [0,1]^2$*, we define functions* $K, L\colon [0, x \wedge y] \to [0, \infty[$ *as*

$$K(p) = \sqrt{p} + a\sqrt{x-p}, \qquad L(p) = \sqrt{p} + b\sqrt{y-p}.$$

Then, using (4), we have

$$\mathcal{I}^F_{\mathcal{H}_2,m}(x,y) = \sup_{p \in [0, x\wedge y]} K(p) \vee \sup_{p \in [0, x\wedge y]} L(p).$$

Taking into account concavity of K *and* L *on* $[0, x \wedge y]$ *and the fact that* $p = \frac{x}{a^2+1}$ *and* $p = \frac{y}{b^2+1}$ *are the stationary points of* K *and* L*, respectively, we get*

$$\mathcal{I}^F_{\mathcal{H}_2,m}(x,y) = \begin{cases} \sqrt{y} + a\sqrt{x-y} & \text{if } \frac{x}{a^2+1} \geq y \\ \sqrt{x} + b\sqrt{y-x} & \text{if } \frac{y}{b^2+1} \geq x \\ (\sqrt{a^2+1}\sqrt{x}) \vee (\sqrt{b^2+1}\sqrt{y}) & \text{otherwise,} \end{cases}$$

see Figure 2 (right).

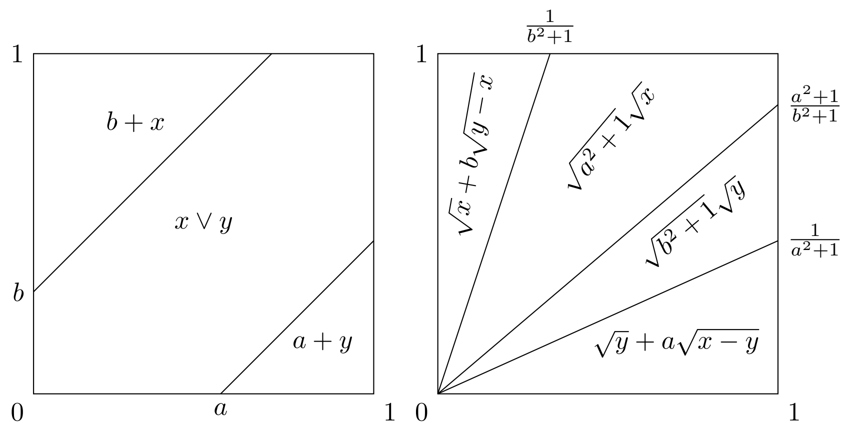

Figure 2: $\mathcal{I}^F_{\mathcal{H}_2,m}$ *from Example 4.3.1.(left) and* $\mathcal{I}^F_{\mathcal{H}_2,m}$ *from Example 4.3.2. for* $a \leq b$*(right)*

3. *Let* $F(u,v) = u^2 v$*. Similarly as in the previous example, we denote*

$$K(p) = p^2 + a(x-p)^2, \qquad L(p) = p^2 + b(y-p)^2.$$

Then, we have

$$\mathcal{I}^F_{\mathcal{H}_2,m}(x,y) = \sup_{p \in [0, x\wedge y]} K(p) \vee \sup_{p \in [0, x\wedge y]} L(p).$$

Both K *and* L *are convex, therefore they attain their maximum either in* 0 *or in* $x \wedge y$*. Thus,*

$$\sup_{p\in[0,x\wedge y]} K(p) = \begin{cases} ax^2 \vee x^2 = x^2 & \textit{if } x \le y \\ ax^2 \vee (y^2 + a(x-y)^2) & \textit{otherwise.} \end{cases}$$

and

$$\sup_{p\in[0,x\wedge y]} L(p) = \begin{cases} (x^2 + b(y-x)^2) \vee by^2 & \textit{if } x \le y \\ y^2 \vee by^2 = y^2 & \textit{otherwise.} \end{cases}$$

Consequently,

$$\mathcal{I}^F_{\mathcal{H}_2,m}(x,y) = \begin{cases} (x^2 + b(y-x)^2) \vee by^2 & \textit{if } x \le y \\ ax^2 \vee (y^2 + a(x-y)^2) & \textit{otherwise.} \end{cases}$$

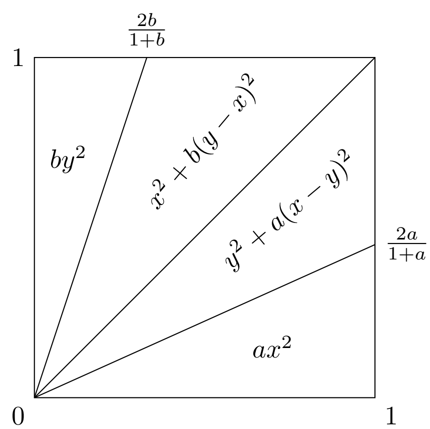

Figure 3: $\mathcal{I}^F_{\mathcal{H}_2,m}$ from Example 4.3.3.

The observation made in Example 4.3.3. can easily be generalized in the following way.

Proposition 4.4. *Let $F \in \mathcal{F}$ be a function convex in the first argument. Then, for every $m \in \mathcal{M}_{(2)}$, it holds*

$$\mathcal{I}^F_{\mathcal{H}_2,m}(x,y) = \begin{cases} F(y,b) \vee (F(x,1) + F(y-x,b)) & \textit{if } x \le y \\ F(x,a) \vee (F(y,1) + F(x-y,a)) & \textit{otherwise,} \end{cases}$$

for every $(x,y) \in [0,1]^2$.

4.3. $\mathcal{H}_3 = \{2^N \setminus \{\emptyset\}\}$ - generalization of the concave integral

Let $\mathcal{H}_3 = \{2^N \setminus \{\emptyset\}\}$. Then, for $F, G \in \mathcal{F}$, the functionals $\mathcal{I}^F_{\mathcal{H}_2,m}$ and $\mathcal{I}^{FG}_{\mathcal{H}_2,m}$ can be regarded as generalizations of the concave integral.

Binary case. Let $n = 2$. Let $m \in \mathcal{M}_{(2)}$ be a capacity with $m(\{1\}) = a$, $m(\{2\}) = b$. We have $\mathcal{H}_3 = \{\mathcal{D}\}$; $\mathcal{D} = \{\{1,2\}, \{1\}, \{2\}\}$.

The family of the maximal subdecompositions of $f = (x, y) \in [0,1]^2$ w.r.t. $\mathcal{D}$ takes the following form:

$$\left\{p\mathbb{1}_{\{1,2\}} + (x-p)\mathbb{1}_{\{1\}} + (y-p)\mathbb{1}_{\{2\}} \mid p \in [0, x \wedge y]\right\}.$$

Thus, we get

$$\mathcal{I}^F_{\mathcal{H}_3,m}(x,y) \quad = \quad \sup_{p\in[0,x\wedge y]} \{F(p,1) + F(x-p,a) + F(y-p,b)\}.$$

For getting the maximal G-decompositions we have to consider following inequalities (here again $\varphi(u) = G(u,1)$):

$$\varphi(a_{\{1,2\}}) + \varphi(a_{\{1\}}) \le x,$$
$$\varphi(a_{\{1,2\}}) + \varphi(a_{\{2\}}) \le y.$$

Thus, similarly as in the case of the generalized chain integral, denoting $\varphi(a_{\{1,2\}}) = p$, we obtain

$$\mathcal{I}^{FG}_{\mathcal{H}_3,m}(x,y) = \sup_{p\in[0,x\wedge y]} \{F(\varphi^{(-1)}(p),1) + F(\varphi^{(-1)}(x-p),a) + F(\varphi^{(-1)}(y-p),b)\} \tag{5}$$

Example 4.5.

1. *Taking* $F(u,v) = uv$, *we get*

   $$F(p,1)+F(x-p,a)+F(y-p,b) = p+(x-p)a+(y-p)b = ax+by+p(1-a-b),$$

   which is an increasing function w.r.t. p *iff* $a + b \le 1$, *i.e., only for a superadditive measure* $m \in \mathcal{M}_{(2)}$. *For such a measure we have*

   $$\mathcal{I}^F_{\mathcal{H}_3,m}(x,y) = ax + by + (x\wedge y)(1-a-b) = \mathcal{I}_{\mathcal{H}_3,m}(x,y).$$

   On the other hand, for a subadditive measure $m \in \mathcal{M}_{(2)}$ *we have*

   $$\mathcal{I}^F_{\mathcal{H}_3,m}(x,y) = ax + by = \mathcal{I}_{\mathcal{H}_3,m}(x,y).$$

   It confirms the well-known fact that the standard concave integral coincide with the Choquet integral iff the underlying measure is supermodular (in binary case it means superadditivity of the measure).
2. *Let* $F(u,v) = u \wedge v$. *Denoting* $K(p) = p\wedge 1 + (x-p)\wedge a + (y-p)\wedge b$ *for a fixed* $(x,y) \in [0,1]^2$, *we have* $\mathcal{I}^F_{\mathcal{H}_3,m}(x,y) = \sup_{p\in[0,x\wedge y]} K(p)$ *and consequently we obtain*

   $$\mathcal{I}^F_{\mathcal{H}_3,m}(x,y) = \begin{cases} x+y & \text{if } x \le a,\ y \le b \\ y+a & \text{if } x \ge a,\ y \le x+b-a \\ x+b & \text{otherwise,} \end{cases}$$

   see Figure 4 (left).

3. *Let* $F(u,v) = u^2 v$. *For a fixed* $(x,y) \in [0,1]^2$, *we have* $\mathcal{I}^F_{\mathcal{H},m}(x,y) = \sup_{p \in [0, x \wedge y]} K(p)$, *where* $K(p) = p^2 + a(x-p)^2 + b(y-p)^2$. *Since* K *is convex, it attains its maximum either in* 0 *or in* $x \wedge y$. *Thus*

$$\mathcal{I}^F_{\mathcal{H}_3,m}(x,y) = \begin{cases} (x^2 + b(y-x)^2) \vee (ax^2 + by^2) & \textit{if } x \leq y \\ (ax^2 + by^2) \vee (y^2 + a(x-y)^2) & \textit{otherwise.} \end{cases}$$

For a measure satisfying $1 \leq a+b$, *i.e., for a subadditive measure, it holds*

$$x^2 + b(y-x)^2 \leq ax^2 + by^2, \quad y^2 + a(x-y)^2 \leq ax^2 + by^2,$$

therefore

$$\mathcal{I}^F_{\mathcal{H}_3,m}(x,y) = ax^2 + by^2,$$

for all $(x,y) \in [0,1]^2$. *For a superadditive measure,* $\mathcal{I}^F_{\mathcal{H}_3,m}$ *is illustrated in Figure 4 (right).*

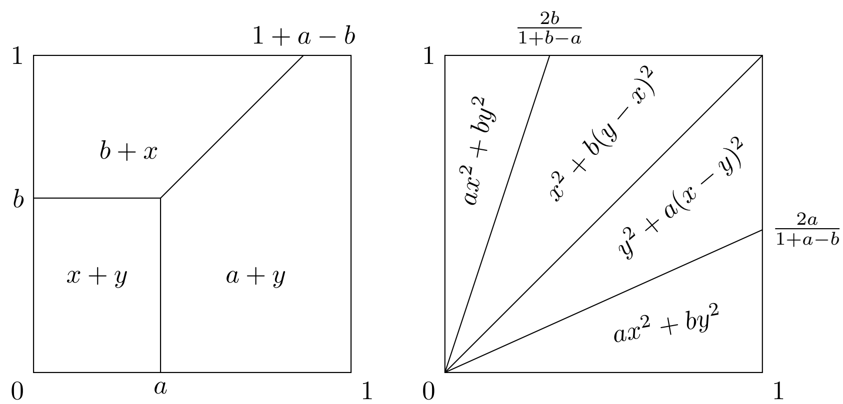

Figure 4: $\mathcal{I}^F_{\mathcal{H}_3,m}$ *for* $a \leq b$ *from Example 4.5.2.(left) and* $\mathcal{I}^F_{\mathcal{H}_3,m}$ *for a superadditive measure from Example 4.5.3.(right)*

5. Conclusion

We have introduced two new classes of non-additive integrals generalizing the decomposition integrals. We have studied their properties and compared them with corresponding properties of the standard decomposition integral. We have shown that the desired property of being monotone (w.r.t. integrated functions, underlying measures and decomposition systems, respectively) is possessed by both our generalizations, while the property of being positively homogeneous is satisfied under some particular constraints on F and G. However, as we have shown, supposing completeness and logical independency of the considered decomposition system, they need not to give back the underlying measure, unlike the standard decomposition integral.

Noteworthy, although well-known Choquet, Shilkret and concave integrals are covered by the standard decomposition integrals, this is not true for the case of the Sugeno integral, as well as for the case of minimal semicopula-based universal integrals introduced in [9] when the considered semicopula differs from the standard product. However, our approach to F-decomposition integrals covers all above mentioned integrals.

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References

[1] J. Borzová-Molnárová, L. Halčinová, O. Hutník, The smallest semicopula-based universal integrals I: Properties and characterizations, Fuzzy Sets and Systems 271 (2015) 1–17.

[2] G. Choquet, Theory of capacities, Annales de l'Institut Fourier 5 (1953–1954) 131–295.

[3] D. Dubois, H. Prade, A. Rico, B. Teheux, Generalized Qualitative Sugeno Integrals, Information Sciences 415-416 (2017) 429–445.

[4] F. Durante, C. Sempi, Semicopulae, Kybernetika (Prague) 41(3) (2005) 315–328.

[5] Y. Even, T. Lehrer, Decomposition-integral: unifying Choquet and the concave integrals, Economic Theory 56 (2014) 33–58.

[6] S. Greco, R. Mesiar, F. Rindone, L. Sipeky, Superadditive and subadditive transformations of integrals and aggregation functions, Fuzzy Sets and Systems 291 (2016) 40–53.

[7] Ľ. Horanská, A. Šipošová, A generalization of the discrete Choquet and Sugeno integrals based on a fusion function, Information Sciences 451 (2018) 83–99.

[8] M. Kaluszka, A. Okolewski, M. Boczek, On Chebyshev type inequalities for generalized Sugeno integrals, Fuzzy Sets and Systems 244 (2014) 51–62.

[9] E. P. Klement, R. Mesiar, E.Pap, Universal integral as common frame for Choquet and Sugeno integral, IEEE Transactions on Fuzzy Systems 18 (2010) 178–187.

[10] E. P. Klement, F. Spizzichino, R. Mesiar, A. Stupňanová, Universal integrals based on copulas, Fuzzy Optimization and Decision Making 13 (2014) 273–286.

[11] E. Lehrer, A new integral for capacities, Economic Theory 39 (2009) 157–176.

[12] E. Lehrer, R. Teper, The concave integral over large spaces, Fuzzy Sets and Systems 159 (16)(2008) 2130–2144.

[13] R. Mesiar, Choquet-like integrals, Journal of Mathematical Analysis and Applications 194 (1995) 477–488.

[14] R. Mesiar, A. Kolesárová, H. Bustince, G. Pereira Dimuro, B. Bedregal, Fusion functions based discrete Choquet-like integrals. European Journal of Operational Research 252(2) (2016) 601–609.

[15] R. Mesiar, J. Li, E. Pap, Discrete pseudo-integrals, International Journal of Approximate Reasoning 54(3), (2013) 357–364.

[16] R. Mesiar, J. Li, E. Pap, Generalized concave integral, 34th Seminar on Fuzzy Set Theory, (2013) 100–102.

[17] R. Mesiar, J. Li, E. Pap, Pseudo-concave integrals, Nonlinear Mathematics for Uncertainty and its Applications. Springer, Berlin, Heidelberg (2011) 43–49.

[18] R. Mesiar, J. Li, E. Pap, Superdecomposition integrals, Fuzzy Sets and Systems 259 (2015) 3–11.

[19] R. Mesiar, A. Stupňanová, Decomposition integrals, International Journal of Approximate Reasoning 54 (2013) 1252–1259.

[20] T. Murofushi, M. Sugeno, M. Machida, Non-monotonic fuzzy measures and the Choquet integral, Fuzzy Sets and Systems 64 (1) (1994) 73–86.

[21] E. Pap (Ed.), Handbook of Measure Theory, Elsevier, Amsterdam, 2002.

[22] N. Shilkret, Maxitive measure and integration, Indagationes Mathematicae 33 (1971) 109–116.

[23] M. Sugeno, Theory of fuzzy integrals and its applications, Ph.D. thesis, Tokyo Institute of Technology (1974).

[24] Q. Yang, The Pan-integral on the Fuzzy Measure Space, Fuzzy Mathematics 3 (1985) 107–114.

[25] Z. Wang, G. Klir, Generalized Measure Theory, IFSR International Series on Systems Science and Engineering 25, Springer, 2009.

[26] Z. Wang, K. Leung, M. Wong, J. Fang, A new type of nonlinear integrals and the computational algorithm, Fuzzy Sets and Systems 112 (2000) 223–231.