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NONLINEAR STABILITY OF ELLIPTIC EQUILIBRIA IN HAMILTONIAN SYSTEMS WITH EXPONENTIAL TIME ESTIMATES

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Abstract. In the framework of nonlinear stability of elliptic equilibria in Hamiltonian systems with n degrees of freedom we provide a criterion to obtain a type of formal stability, called Lie stability. Our result generalises previous approaches, as exponential stability in the sense of Nekhoroshev (excepting a few situations) and other classical results on formal stability of equilibria. In case of Lie stable systems we bound the solutions near the equilibrium over exponentially long times. Some examples are provided to illustrate our main contributions.

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1. **Introduction.** We deal with elliptic equilibria in Hamiltonian systems with n degrees of freedom, establishing a criterion to determine their formal stability and providing asymptotic estimates on the solutions starting nearby.

The method consists in the calculation of a linear subspace of $\mathbb{R}^{2n}$ that we call S and that is contained into the orthogonal space related to the frequency vector. Then the normal-form Hamiltonian is computed up to a suitable order and we check whether the truncated Hamiltonian evaluated on S vanishes only at the origin of $\mathbb{R}^{2n}$. If it occurs we obtain a type of formal stability that is called Lie stability. The set S will be introduced in Section 2.

To our knowledge there are no examples of systems that are formally stable but not Liapunov stable, which gives an idea of the strength of formal stability in the setting of nonlinear stability of equilibria. However, very recently there have appeared examples of unstable elliptic equilibria in analytic Hamiltonian systems with three and four degrees of freedom, see [13], where the frequencies are very close to resonance, and these cases are formally stable, as they are non-resonant.

Formal stability of elliptic equilibria was started by Siegel [37], and Moser [26, 27] who established conditions on the quadratic terms of the Hamiltonians to achieve formal stability. Moser also dealt with the formal stability of systems whose terms starting at degree three in Cartesian coordinates have periodic coefficients. Glimm [15] proved formal stability provided the quartic terms in normal form when expressed in action-angle variables do not depend on the angles and are a definite function in the actions. Bryuno [3] refined earlier results getting a criterion for formal stability of Hamiltonians based on the quadratic and quartic terms.

Other members of the Russian school also contributed significantly to the research in formal stability starting in the decade of the 70s. We quote the pioneering work by Khazin [18, 19] who set up the concept of Lie stability, although he named it Birkhoff stability. More works dealing with formal stability and instability for several cases involving resonant situations in systems with at least three degrees of freedom are [23, 38, 20].

Based on Nekhoroshev theory [29] for steep functions in the context of elliptic equilibria, several authors [14, 21] established results on bounds for exponentially long times on the actual solutions near an equilibrium of an analytic Hamiltonian system. These bounds were improved later in [12, 1, 30, 32]. Recently the theory of stability has been enlarged in [36] to treat some degenerate situations where steepness is obtained from higher-order terms, and thus Nekhoroshev estimates apply. As well, paper [2] deals with very sharp estimates in case that the Diophantine condition among the frequencies ω_i is satisfied.

In addition to the above, in a series of papers Guzzo and coworkers, Niederman and Bounemoura have relaxed the hypotheses to get Nekhoroshev estimates, allowing the part of the Hamiltonian hinging only on the actions to be non-steep. More precisely, Guzzo *et al.* [17] introduced the notion of rational convexity, which roughly means that the convexity property is tested only on the subspaces of fast drift. This idea has been generalised by Niederman [31] under the name of Diophantine steepness condition, which is a weak condition of transversality that involves only the affine subspaces spanned by integer vectors. This property leads to exponential time estimates of Nekhoroshev type. Checking these conditions on a specific problem is not usually an easy task.

Surprisingly there is almost no connection in the literature between Nekhoroshev stability of elliptic equilibria and formal stability. Indeed, excepting the pioneering papers by Moser and Glimm, on the one hand the works related to formally stable systems do not consider the issue of getting estimates that measure the validity of the nonlinearly stable solutions. On the other hand the studies on elliptic equilibria from the point of view of Nekhoroshev theory have obtained very sharp bounds on the solutions but they do not deal with the existence of positive definite first integrals. However, having a closer look at the papers by Glimm [15] and Bryuno [3] it is straightforward to interpret that Glimm's criterion corresponds to Nekhoroshev stability when the elliptic equilibrium satisfies quasi-convexity whereas Bryuno's contribution is equivalent to Nekhoroshev stability under directional quasi-convexity condition. Consequently formal and Nekhoroshev stabilities are related topics. In fact, Glimm's ideas inspired Nekhoroshev to establish the concept of steepness of a function.

More recent papers on Lie stable and unstable systems are due to dos Santos and coworkers [33, 34, 35] where the authors establish several criteria dealing with Lie stable equilibria in cases of resonances. They also treat instability using suitable Chetaev functions [24]. Another related work treating a particular case of Lie stability is [25]. The instability analysis using the invariant ray technique is developed in [18, 19, 38, 6, 7]. Asymptotic estimates of exponential type for Lie stable systems where the corresponding linear subspace S is trivial are carried out in [11].

In the present work we aim at getting Lie stability with the weakest possible assumptions, thus acquiring a deeper insight in formal stability, generalising previous results on this type of formal stability. This is accomplished by exploiting the algebraic structure of the linear part of the equation as much as we can. We do not need to check whether the truncated normal-form Hamiltonian vanishes for all non-null vectors of the orthogonal space related to the frequency vector, but only for the subspace S. Thus the lower the dimension of S is, the more cases of Lie stable systems we get. This upshot allows us to get Lie stability usually with a few assumptions. In fact, this is achieved by exploiting the algebraic structure of the linear part of the equation as much as we can. The set S is constructed by means of the formal first integrals associated to the Hamiltonian in normal form.

Our second target is to provide Lie stable systems with estimates over exponentially long times. We use a result on a recent paper by Chartier *et al.* [10] where the authors determined error bounds for adiabatic invariants of Hamiltonian systems. More specifically the variation of these invariants after a process of truncation may remain bounded over exponentially long time intervals. This is due to the exponentially small character of the remainder function obtained in the normal-form calculation. In this setting we enlarge earlier approaches, as exponential stability in the sense of Nekhoroshev of elliptic equilibria, mainly because we can handle resonant terms, so this is an important novelty of our theory. Furthermore when Lie stability is decided from a Hamiltonian in normal form that does not hinge on any angle, we unravel some relations between Lie stability and Nekhoroshev-type stability, comparing the estimates obtained with both approaches and showing that our bounds are even better under certain circumstances.

Getting (formally) Lie stable systems and the asymptotic bounds on the null solution related to Hamiltonian systems, where the stability is decided from the resonant terms of the normal form, should not be underestimated. There are many examples of this type in mechanical systems with three or more degrees of freedom depending on various parameters. See for instance the various applications of Lie stable Hamiltonians in the book by Markeev [23].

The structure of the paper is the following. Section 2 is concerned with the presentation of the material we need to carry out our study as well as with the statement of the two main results. In Section 3 we provide the proof of Theorem 2.6. Section 4 is devoted to the analysis of the estimates for Lie stable systems and the main achievement is the proof of Theorem 2.7. Finally, Section 5 deals with comments relating our attainments with previous approaches. Moreover we give several examples that illustrate our theory. We also furnish some useful corollaries and remarks along Sections 3, 4 and 5, in particular Corollary 4.

We have preferred to exemplify the approach through many different examples although some of them could sound a bit artificial, but the situations displayed in Section 5 cover most cases where our theory can be applied. However we do not forget about the applications to realistic systems depending on various parameters, as it is the case of papers [8, 9] and [5], where the methods delineated in this paper allow the nonlinear analysis of quite intricate systems.

This paper is part of the Ph.D Thesis of the first author [5], where several applications to problems of celestial mechanics are studied with great detail.

2. **Statement of the Main Results.** Consider the autonomous Hamiltonian system with n degrees of freedom

$$\dot{x} = \mathcal{J}\nabla H(x), \tag{1}$$

such that the origin of the phase space is an equilibrium solution, the so called *null solution* $x = 0$. Matrix $\mathcal{J}$ is the standard $2n \times 2n$ symplectic matrix of Hamiltonian theory [24] and $H = H(x)$ is a real analytic function of $x = (q_1, \dots, q_n, p_1, \dots, p_n)$. It is assumed that the Taylor series of H in a neighborhood of the origin is

$$H = H_2 + H_3 + \cdots + H_j + \cdots , \tag{2}$$

where H_j represents a homogeneous polynomial of degree j in x, that is,

$$H_j = \sum_{|k_j|_1 + |l_j|_1 = j} h_{k_j l_j}\; q^{k_j} p^{l_j}, \tag{3}$$

with $k_j = (k_{j1}, \dots, k_{jn}) \in \mathbb{Z}^n_{\geq 0}$, $l_j = (l_{j1}, \dots, l_{jn}) \in \mathbb{Z}^n_{\geq 0}$, and $|\ |_1$ stands for the 1-norm, thus $|k_j|_1 = k_{j1} + \cdots + k_{jn}$, $|l_j|_1 = l_{j1} + \cdots + l_{jn}$, $h_{k_j l_j} = h_{k_{j1} \cdots k_{jn} l_{j1} \dots l_{jn}} \in \mathbb{R}$, $q^{k_j} = q_1^{k_{j1}} \cdots q_n^{k_{jn}}$ and $p^{l_j} = p_1^{l_{j1}} \cdots p_n^{l_{jn}}$.

The term H_2 represents the quadratic Hamiltonian

$$H_2(x) = \tfrac{1}{2} x^T B x, \tag{4}$$

with $B = B^T$ a $2n \times 2n$ real symmetric matrix. The linearised equations of motion are

$$\dot{x} = Ax, \quad A = \mathcal{J}B, \tag{5}$$

where A is a $2n \times 2n$ real Hamiltonian matrix. In the paper A is nonsingular and the linearised system is stable, i.e., all the eigenvalues of A are nonzero purely imaginary numbers (elliptic equilibrium point), say $\pm\omega_1 i, \dots, \pm\omega_n i$ and A is diagonalisable over the complex numbers. It is assumed that the non-degenerate equilibrium solution of the Hamiltonian system associated to (5) is stable in the linear approximation. We can suppose, without loss of generality (see [27, 23, 24] for more details), that a linear canonical transformation has already been constructed such that

$$H_2 = \frac{\omega_1}{2}(q_1^2 + p_1^2) + \cdots + \frac{\omega_n}{2}(q_n^2 + p_n^2). \tag{6}$$

The *normal-form Hamiltonian* of H defined in (2) up to a finite degree p is the function

$$H = H_2 + \mathcal{H}_3 + \cdots + \mathcal{H}_p + \cdots \tag{7}$$

obtained from (2) through a symplectic change of coordinates whose series expansion in x starts at degree two, such that each term $\mathcal{H}_k$ is a homogeneous polynomial of degree k in x, and satisfies $\{\mathcal{H}_k, H_2\} = 0$, $k = 2, \ldots, p$ (the operator $\{\,,\,\}$ being the classical Poisson bracket in Hamiltonian theory), see for instance [24].

Along this paper $\mathcal{H}^p$ represents the truncation of the Hamiltonian function at degree p, that is,

$$\mathcal{H}^p = H_2 + \mathcal{H}_3 + \cdots + \mathcal{H}_p, \tag{8}$$

associated with the system

$$\dot{x} = \mathcal{J}\nabla H_p(x). \tag{9}$$

We stress that the transformation to normal form can be accomplished to any finite degree.

Customarily one can introduce action-angle variables $I = (I_1, \ldots, I_n)$, $\theta = (\theta_1, \ldots, \theta_n)$ such that

$$I_j = \tfrac{1}{2}(q_j^2 + p_j^2), \quad \theta_j = \tan^{-1}\frac{p_j}{q_j},$$

where H_2 takes the form

$$H_2 = \omega_1 I_1 + \cdots + \omega_n I_n. \tag{10}$$

In [14] strong non-resonance conditions are imposed on the frequencies, which are not required here. We assume that $\omega_j \neq 0$ for all j. In addition, H_2 is generically an indefinite quadratic form in terms of x, in other words, the signs of the ω_i are not all the same, although the (trivial) case with the same signs for all the frequencies will be mentioned in Section 5.

Using action-angle variables defined above, Hamiltonian function H in (3) leads to a Poisson series (a finite Fourier series in θ whose coefficients are polynomials in $I_i^{1/2}$) with terms of the form

$$c\, I_1^{\alpha_1/2} \cdots I_n^{\alpha_n/2} \cos(\beta_1\theta_1 + \cdots + \beta_n\theta_n),$$

where c is a real constant, the α_j are non-negative integers and the β_j are integers; there is also a similar sin-term. Since the Poisson series came from a real power series the terms must have the *d'Alembert character* [24], i.e., the coefficients α_k, β_k satisfy

$$\text{for } j = 1, \ldots, n, \quad \alpha_j \geq |\beta_j| \quad \text{and} \quad \alpha_j \equiv \beta_j \mod 2. \tag{11}$$

By virtue of d'Alembert character, we write down the normal-form Hamiltonian up to degree p (when thought in rectangular coordinates) as

$$H(I,\theta) = H_2(I) + \cdots + \mathcal{H}_{2l-2}(I) + \mathcal{H}_m(I,\theta) + \cdots \tag{12}$$

with $l \geq 2$, $m = 2l-1$ or $m = 2l$ and $m \leq p$ or $m > p$. Thus m represents the lowest degree in which θ arises explicitly when transforming to action-angle coordinates. Terms of degree higher than p are not in normal form with respect to H_2. Besides we can write without loss of generality $\mathcal{H}_p(I,\theta)$, assuming that $\mathcal{H}_p$ is independent of θ when $p < m$ (then p is even), but it can hinge on the angles when $p \geq m$.

We realise that H in (12) is an analytic function of the variables $I_j^{1/2}$, θ_j and is 2π-periodic in θ_j, $j = 1, \ldots, n$ excepting at $I = 0$. To circumvent the problem at the origin of $\mathbb{R}^{2n}$ one uses d'Alembert condition. Specifically the plan is to keep track of the d'Alembert character of the Hamiltonians and related formulae in action-angle coordinates. If the d'Alembert property is maintained through the different manipulations, transforming these formulae back to rectangular coordinates, the resulting expressions are polynomials in x, thus analytic everywhere. Throughout the text all Hamiltonian functions satisfy (11).

Now we recall the notion of resonance vector.

Definition 2.1. System (1) presents a resonance relation if there exists an integer vector $k_1 = (k_{11}, \ldots, k_{1n}) \neq 0$ such that

$$k_{11}\omega_1 + \cdots + k_{1n}\omega_n = 0.$$

The number $|k_1|_1 = |k_{11}| + \cdots + |k_{1n}|$ is called the order of the resonance. On the other hand, if

$$k_{11}\omega_1 + \cdots + k_{1n}\omega_n \neq 0$$

holds for all integer vectors $k_1 = (k_{11}, \ldots, k_{1n}) \in \mathbb{Z}^n$ satisfying the equalities $|k_1|_1 = \ell$, for $\ell = 1, \ldots, l$ we say that system (1) does not present resonance relations, including order l.

The dependence of $\mathcal{H}_j$, $j \geq m$ with respect to θ occurs only through the s resonant angles generated by means of the vectors $k_1, \ldots, k_s$ with $0 \leq s \leq n-1$ where $\{k_1, \ldots, k_s\}$ is a basis of the $\mathbb{Z}$-*module* M_ω associated to H_2. Specifically we introduce,

$$M_\omega = \{k_1 = (k_{11}, \ldots, k_{1n}) \in \mathbb{Z}^n \,\big|\, k_{11}\omega_1 + \cdots + k_{1n}\omega_n = 0\}$$

and as M_ω is finitely generated we also write

$$M_\omega = k_1\,\mathbb{Z} + \cdots + k_s\,\mathbb{Z} = \{i_1 k_1 + \cdots + i_s k_s \,\big|\, i_1, \ldots, i_s \in \mathbb{Z}\}.$$

It is clear that $M_\omega = \{0\}$ is equivalent to consider $\omega_1, \ldots, \omega_n$ linearly independent over $\mathbb{Q}$, that is, $M_\omega = \{0\}$ if and only if system (1) does not possess resonances. Notice that, as the set M_ω is finitely generated, we can take a minimal set of generators, so the k_j are linearly independent. In this work we will assume that the set of generators $\{k_1, \ldots, k_s\}$ of M_ω is minimal.

We deal with the definitions of single and multiple resonances.

Definition 2.2. Assume that $M_\omega \neq \{0\}$. If M_ω is cyclic (equivalently $s = 1$) we say that system (1) possesses a single resonance, otherwise (equivalently $s > 1$) we say that the system possesses multiple resonances.

At this moment we deal with the different types of stability. We start by recalling the definition of Liapunov stability in the setting of the Hamiltonian system (1).

Definition 2.3. We say that the origin of $\mathbb{R}^{2n}$ in (1) is positively (respectively negatively) stable, if for every $\varepsilon > 0$ there is $\delta > 0$ such that if $\phi(t, \zeta)$ is the general solution of (1) then $|\phi(t, \zeta)| < \varepsilon$ for all $t \geq 0$ (respectively $t \leq 0$) whenever $|\zeta| < \delta$. The origin of $\mathbb{R}^{2n}$ is said to be Liapunov stable if it is both positively and negatively stable.

Regarding formal stability we provide the definition due to Moser [27].

Definition 2.4. We say that the equilibrium solution $x = 0$ in (1) is formally stable if there exists a real formal power series $G(x)$, which is an integral of (1) in the formal sense, and is positive definite near the origin of $\mathbb{R}^{2n}$.

Next we introduce the notion of Lie stability, which makes use of the Hamiltonian function in normal form.

Definition 2.5. We say that the equilibrium solution $x = 0$ in (1) is Lie stable if there exists $p \geq 2$ such that the truncated Hamiltonian system in normal form associated to $\mathcal{H}^q$ is stable in the sense of Liapunov for any $q \geq p$ (arbitrary).

By the truncated Hamiltonian system associated to $\mathcal{H}^q$ we consider the equation (9) where p is replaced by q.

The pioneering idea of Lie stability goes back to Khazin in [18] who dubbed it as Birkhoff stability as he dealt with equilibrium points with non-null semisimple linear part, but Lie stability is more general as it makes sense even in the non-diagonalisable case.

In the cases of stability handled in [33, 34] it is proved that Lie stability implies formal stability. Here we will prove that the same feature holds, see Corollary 1.

We need a few more ingredients in order to state our first main contribution.

The orthogonal space of M_ω is a vector subspace of $\mathbb{R}^n$ spanned by the vectors $\{a_1, \ldots, a_d\}$ with $d = n - s$ that satisfy $a_i \cdot k_j = 0$, see details in [33]. Setting $F_l = a_l \cdot I$, $l = 1, \ldots, d$, we get the independent formal first integrals of the normal-form Hamiltonian (12). The set

$$S = \{I = (I_1, \ldots, I_n) \,\big|\, I_j \geq 0,\ \ F_1(I) = \ldots = F_d(I) = 0\} \tag{13}$$

is introduced for later use, noting that $0 \leq \dim S \leq s$. It was first given in [33].

Hamiltonian $\mathcal{H}_j(I, \theta)$ is rewritten as $\mathcal{H}_j(I, \phi)$ where $\phi = (\phi_1, \ldots, \phi_s)$, $\phi_i = k_i \cdot \theta$. It is stressed that $\mathcal{H}_j$ can be independent of some angles ϕ_j or even of all of them.

We are ready to state our first main upshot on stability for the elliptic equilibria. In the following $|\ |$ stands for the Euclidean norm. Additionally Hamiltonian function H introduced in (1) is supposed to be real analytic in a neighbourhood of the origin of $\mathbb{R}^{2n}$, although we could relax it and assume that H is regular up to some order high enough.

Theorem 2.6. *Consider Hamiltonian* (12) *corresponding to the normal form up to degree p of system* (1) *with $I \in S \setminus \{0\}$, $\phi \in \mathbb{T}^s$.*

(A) Suppose there is an even integer j (with $4 \leq j \leq p$) such that $\mathcal{H}^j(I, \phi) \neq 0$ for $|I|$ small enough and all ϕ. Then the origin of $\mathbb{R}^{2n}$ is Lie stable for the Hamiltonian system (1)*.*

(B) Suppose there is an integer $i \geq 3$ such that $\mathcal{H}^i(I, \phi)$ changes sign for some I and ϕ. Then there is no index j (with $i < j \leq p$) such that $\mathcal{H}^j(I, \phi) \neq 0$ for $|I|$ sufficiently small.

The quadratic part in terms of the formal first integrals F_k is expressed as

$$H_2(I) = \sum_{k=1}^{d} \sigma_k F_k(I), \tag{14}$$

where the σ_k are linear combinations of the ω_j and $\sigma_k \neq 0$ for all k. By construction the $\sigma_1, \ldots, \sigma_d$ are rationally independent, see [11], and the vector $\sigma = (\sigma_1, \ldots, \sigma_d)$ is well defined up to the natural $GL(d; \mathbb{Z})$-action.

With the aim of getting the time estimates we need to impose a *Diophantine condition* on σ; in other words, we suppose that there are fixed constants $c > 0$ and $\nu > d - 1$ such that

$$\forall k \in \mathbb{Z}^d \setminus \{0\}, \quad |k \cdot \sigma| \geq c\, |k|_1^{-\nu}. \tag{15}$$

Next, we state our second main result on the exponential time estimates for the elliptic equilibria when Lie stability holds from Theorem 2.6.

Theorem 2.7. *If the real analytic Hamiltonian system* (1) *satisfies hypotheses (A) of Theorem 2.6 and the frequency vector σ satisfies the Diophantine condition* (15)*, then there exist $C > 0$, $K > 0$, $a > 1$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, and for all x_0 with $|x_0| < \varepsilon$ we have*

$$|x(t, x_0)| < a\,\varepsilon^{2/j} \quad \text{for all } t \text{ with } \quad 0 \le t \le T = C \exp\left(\frac{K}{\varepsilon^{1/(\nu+1)}}\right).$$

In the preceding result the constants C, K, a, ε_0 are supposed to be independent of ε. Moreover it is assumed Lie stability of the null solution, hence we obtain exponential time bounds for Lie stable systems. One of the salient points of this work is that our estimates can be applied to Hamiltonian systems that do not satisfy Nekhoroshev estimates' conditions appearing in [17, 31], see in particular the examples of Section 5. Furthermore, we can handle Hamiltonian functions that contain resonant terms in order to establish the Lie stability of a certain equilibrium point, a feature certainly not covered by Nekhoroshev theory.

3. **Proof of Lie Stability.**

Proof of Theorem 2.6. (A) We invoke Liapunov Stability Theorem, see for instance [24] Theorem 13.1.1: If there exists a function V that is positive definite with respect to ξ_0 and such that $\dot{V} \le 0$ in an open neighbourhood O of ξ_0 then the equilibrium ξ_0 is (Liapunov) stable. Consider $\xi_0 = 0$ and define V to be

$$V(x) = (F_1(x))^2 + \cdots + (F_d(x))^2 + (\mathcal{H}^p(x))^2, \tag{16}$$

with $x \in O$ and $\mathcal{H}^p$ introduced in (8). As F_k and $\mathcal{H}^p$ are first integrals of the Hamiltonian system related to $\mathcal{H}^p$ the function V is a first integral of it and that holds for every $p \ge j$ arbitrary, hence $\dot{V} = \{V, \mathcal{H}^p\} = 2(\sum_{k=1}^{d} F_k \dot{F}_k + \mathcal{H}^p \dot{\mathcal{H}}^p) = 0$. Furthermore $V = 0$ if and only if $F_1(I) = \ldots = F_d(I) = 0$ and $\mathcal{H}^p = 0$. Thus we may restrict I to the set S.

When $S = \{0\}$ we get Lie stability straightforwardly since $\mathcal{H}^p$ evaluated at $I \in S$ is trivially zero, the last term of (16) can be dropped and V is definite (no condition on the sign of $\mathcal{H}^j$ is required). In this case we set $j = 2$.

When $S \ne \{0\}$ we perform a stretching of coordinates (also called dilation), say $x \to \varepsilon y$ with $\varepsilon > 0$ small, that in action-angle variables reads $I \to \varepsilon^2 J$, $\theta \to \theta$. (Notice that $|I|$ small in (A) is equivalent to consider ε small and J of order $\mathcal{O}(1)$.) To make the transformation symplectic we multiply (12) by ε^{-2} arriving at

$$H(J, \theta, \varepsilon) = H_2(J) + \cdots + \varepsilon^{2l-4}\mathcal{H}_{2l-2}(J) + \varepsilon^{m-2}\mathcal{H}_m(J, \theta) + \cdots . \tag{17}$$

Assuming hypotheses in (A) hold for $\varepsilon > 0$ sufficiently small, then $\mathcal{H}^p = 0$ if and only if $J = 0$ since there is an integer j with $4 \le j \le p$ such that $\mathcal{H}^j \ne 0$ for all $J \in S \setminus \{0\}$ and all $\phi \in \mathbb{T}^s$. Thus the addition of higher-order terms $\mathcal{H}_{j+1}$, $\mathcal{H}_{j+2}$, …, $\mathcal{H}_p$, cannot change the sign of $\mathcal{H}^j$ as a finite sum of terms is added, all factorised by powers of ε and ε is chosen as small as needed, whereby V is definite. By Liapunov Stability Theorem [24] the null solution is stable for the Hamiltonian system associated to $\mathcal{H}^p$. Considering the truncated normal form up to degree q in rectangular coordinates, i.e. $\mathcal{H}^q$ with $q > p$, and taking V as above but changing $\mathcal{H}^p$ by $\mathcal{H}^q$ it follows that V is a first integral of $\mathcal{H}^q$ which is positive definite, and the origin of $\mathbb{R}^{2n}$ is Liapunov stable for the system defined by $\mathcal{H}^q$. Since q is arbitrary and the normal-form transformation only requires a finite number of steps, the null solution of (1) is Lie stable.

(B) When there is an index $i \geq 3$ such that $\mathcal{H}^i$ changes sign for some $I \in S \setminus \{0\}$, $\phi \in \mathbb{T}^s$, then there are two possibilities: either there exists $I^* \in \mathrm{int}(S)$, $\phi^* \in \mathbb{T}^s$ with $\mathcal{H}^i(I^*, \phi^*) = 0$ or there is an integer $i' < i$ such that $\mathcal{H}^{i'}$ keeps the same sign while the change of $\mathcal{H}^i - \mathcal{H}^{i'}$ is the opposite (if this happens the change occurs at I in the boundary of S). From (17) it is clear that higher-order terms of the normal form cannot alter the sign change of $\mathcal{H}^i$, provided ε is taken small enough. Thus, one cannot find $\mathcal{H}^j$ with $j > i$ such that (A) can be applied. □

Corollary 1. *If in the function V introduced in (16), $\mathcal{H}^p$ is replaced by the normal form up to infinity, and then H is at least of class $\mathcal{C}^\infty$ in a neighbourhood of the origin, one has $H_2 + \mathcal{H}_3 + \cdots + \mathcal{H}_p + \mathcal{H}_{p+1} + \cdots$ (i.e., formally), then V becomes a formal first integral of Hamiltonian (7) which is positive definite, thus the null solution of the Hamiltonian system (1) is formally stable.*

Remark 1. Part (A) of Theorem 2.6 is essentially Theorem 1.1 of [34], but here we clarify this criterion and shows how to apply it in different situations, extending and simplifying its use. See also Remark 2 right below, the proof of Theorem 2.7 in Section 4 and the examples in Section 5.

Remark 2. When determining the sign of $\mathcal{H}^j$ for $I \in S \setminus \{0\}$ if there is an index $i < j$ such that $\mathcal{H}^i = 0$ for certain I^* in the boundary of S, $\phi^* \in \mathbb{T}$ then we have to evaluate $\mathcal{H}^{i+1}$ only at I^*, ϕ^* and proceed in this way degree by degree until reaching degree j (when the normal form is expressed in Cartesian coordinates). For an illustration see the second and the last examples in the fourth subsection of Section 5.

Remark 3. We can refine the hypotheses of the Theorem by considering s_j an integer in $[0, s]$ such that $\mathcal{H}^j$ hinges only on s_j angles – without loss of generality their first s_j angles – thus we write $\mathcal{H}^j(I, \phi_1, \ldots, \phi_{s_j})$.

Remark 4. Part (B) of the Theorem suggests that we could achieve instability of the origin of $\mathbb{R}^{2n}$ for the Hamiltonian system associated to (12). Nonetheless we would need to add extra hypotheses on some terms in normal form, $\mathcal{H}_k$ with $k > i$, in order to build a suitable Chetaev function. This analysis deserves more attention, see also Remark 11.

Remark 5. In the classical approach to Lie stability, see [23], each particular resonance was studied separately. However with the function V defined in (16) we provide a unique Liapunov function to analyse all possible situations, thus simplifying the analysis considerably. This feature is shared with the function provided in papers [33, 34] to get Lie stability.

4. Proof of the Asymptotic Estimates.

There exist only a few results dealing with estimates on formally stable equilibria. As classical achievements we report the papers by Moser [26] (dealing with the stability of periodic solutions of systems with one degree of freedom) and Glimm [15] where time estimates proportional to the reciprocal of a small parameter, and recently the paper [11] that accounts for the exponentially large estimates on time for Lie stable systems for which $S = \{0\}$.

For the cases of Lie stable equilibria provided in Theorem 2.6 we give time estimates of exponential type, similar to those of Nekhoroshev theory. Our upshot is based upon the time estimates for *adiabatic invariants*, i.e. truncated formal first integrals, established by Chartier *et al.* in [10].

Noticing that for $k = 1, \dots, d$, $F_k(I) = \varepsilon^2 F_k(J)$, Hamiltonian (17) has d independent formal first integrals given by $F_k(J)$. In practice they are adiabatic invariants as the normal-form process is pushed only to a finite order. Moreover, one can construct other adiabatic invariants whose main part, i.e. their lowest-degree terms, are $F_k(J)$. Our goal is to provide estimates on the time evolution of these adiabatic invariants. To achieve this we introduce a few ingredients before stating the attainment of Chartier *et al.* in a form suited to our needs.

Hamiltonian function (2) is rewritten in terms of the rectangular coordinates y as

$$H(y, \varepsilon) = H_2(y) + \varepsilon H_3(y) + \cdots + \varepsilon^{j-2} H_j(y) + \cdots, \tag{18}$$

here Hamiltonian $H_2(y)$ in (18) can be expressed in terms of the F_k by means of

$$H_2(y) = \sum_{k=1}^{d} \sigma_k F_k(y), \tag{19}$$

as in (14). The associated normal form of Hamiltonian (18) until degree p is

$$H = H_2 + \varepsilon \mathcal{H}_3 + \cdots + \varepsilon^{p-2} \mathcal{H}_p + \cdots, \tag{20}$$

which can be obtained directly from (18) or from (7) after doing the stretching $x = \varepsilon y$. Notice that H in (20) corresponds to the normal form (17) if one applies the standard transformation from rectangular to action-angle coordinates.

At this point we introduce some more notation. Let $\mathcal{N} = B_R$ be the open ball of radius $R > 0$ centred on 0 in $\mathbb{R}^{2n}$. Given a solution $y = y(t, y_0, \varepsilon)$ of the Hamiltonian system related to (18) with initial condition y_0 in $\mathcal{N}$ (which is equivalent to a solution $x = x(t, x_0)$ of system (1)), let $\gamma = \gamma(y_0, \varepsilon)$ be the solution's first time of escape from $\mathcal{N}$, i.e.

$$\gamma = \inf\{t > 0 \,\big|\, |y(t, y_0, \varepsilon)| \geq R\}. \tag{21}$$

Given $\gamma > 0$ and $T > 0$, we set

$$D = [0, \gamma) \cap [0, T], \tag{22}$$

thus D is the shortest of the two intervals.

In paper [10] another requirement on the well behaviour of the Fourier series related to the Hamiltonian function H in (1) is established. This is done in order to guarantee the convergence of some expansions and the existence of good bounds to achieve the main results. This requirement appears in the Hamiltonian setting under the name of Assumption B in Section 3.2 of [10]. Assuming that H in (1) is analytic, it readily satisfies Assumption B and the results about adiabatic invariants and bounds of [10] apply in our context.

Let

$$I_i^p(y, \varepsilon) = F_i(y) + \sum_{k=3}^{p} \varepsilon^{k-2} I_{i,k}(y), \quad i = 1, \dots, d,$$

be adiabatic invariants of Hamiltonian (18) truncated at degree p, where $I_{i,k}(y)$ are homogeneous polynomials in y of degree k. Then the following result appeared as Corollary 3.6 in [10].

Theorem 4.1 (Chartier, Murua and Sanz-Serna, 2015)**.** *Let the real analytic system associated to* (18) *satisfy the Diophantine condition* (15) *and let* $y_0 \in \mathcal{N}$*. Then there* *are constants $C > 0$ and $K > 0$ such that for small enough $\varepsilon > 0$, there is a positive integer p such that for arbitrary $\kappa > 0$ and for $i = 1, \dots, d$,*

$$|I_i^p(y(t, y_0, \varepsilon), \varepsilon) - I_i^p(y_0, \varepsilon)| < \kappa^2 \quad \textit{for all} \quad t \in D = [0, \gamma) \cap [0, T],$$

where

$$T = C\,\kappa^2 \exp\left(\frac{K}{\varepsilon^{1/(\nu+1)}}\right).$$

The integer p refers to the degree to which the normal form has to be carried out to get the required estimate on the time; it is the N of Theorem 3.5 and Corollary 3.6 in [10]. The parameter κ is independent of ε and is not necessarily small as it can be deduced from the proof of Theorem 3.5, while ν is the parameter related to the Diophantine condition (15). Finally it is assumed that $\varepsilon \in (0, \varepsilon_1)$, where $\varepsilon_1 > 0$ is an appropriate threshold.

Theorem 4.1 was established in the context of the averaging procedure devised by the authors (see references in [10]) to deal with vector fields – both dissipative and Hamiltonian vector fields – from the point of view of the design and analysis of numerical integrators. It is stressed that the averaging (or normal-form) transformation accomplished in [10], under the assumptions of the above theorem, are such that the corresponding remainder is exponentially small. In [11] an exponentially large time estimate based on Chartier *et al.* was established for the case $S = \{0\}$, while here we enlarge this result for the elliptic equilibria that are (formally) Lie stable using the criterion of Theorem 2.6.

Similar bounds on adiabatic invariants to those of [10] have been proved by other authors, see for instance [29, 28]. Nevertheless we have preferred to use the results by Chartier *et al.* as they are better adapted to our needs.

We deal now with Theorem 2.7.

Proof of Theorem 2.7. We prove the bounds on the system derived from the Hamiltonian in normal form, either (7) or (12), and it will imply the bounds on the system (1), as the passage to normal form involves only a finite number of steps. Therefore we work with Hamiltonian H in normal-form coordinates until the last step of the proof.

We suppose that there is an integer $j \geq 4$ such that $\mathcal{H}^j$ does not vanish for $I \in S \setminus \{0\}$, $\phi = (\phi_1, \dots, \phi_s) \in \mathbb{T}^s$. Let $p \geq j$ be the degree to which the normal-form Hamiltonian (7) has been obtained.

As a first step we prove that given a small enough $\tilde{\varepsilon}_0 > 0$ there are positive constants α, β, δ such that whenever $|x| \leq \tilde{\varepsilon}_0$ we have

$$\alpha |x|^{2j} \leq V(x), \qquad (\mathcal{H}^p(x))^2 \leq \beta |x|^4, \qquad (F_l(x))^2 \leq \delta |x|^4, \ \ l = 1, \dots, d, \tag{23}$$

where V is defined in (16).

For $|x| \leq \tilde{\varepsilon}_0 < 1$ one obtains $(\mathcal{H}^p(x))^2 = (H_2(x))^2 + \mathcal{O}(|x|^5)$, then there exists $\beta' > 0$ such that the terms of degree 5 and higher are bounded from above by $\beta' \tilde{\varepsilon}_0 |x|^4$. Setting $\beta = \beta' \tilde{\varepsilon}_0 + \beta''^2/4$ with $\beta'' = \max\{|\omega_1|, \dots, |\omega_n|\}$ we ensure that $\beta |x|^4 \geq (\mathcal{H}^p(x))^2$ for $\tilde{\varepsilon}_0$ sufficiently small.

It is straightforward to notice that for $l = 1, \dots, d$, one has $|F_l(x)| \leq \sqrt{\delta_l} |x|^2$ for some $\delta_l > 0$, thus δ is chosen as $\max\{\delta_1, \dots, \delta_d\}$.

When $x = 0$ the first inequality of (23) holds trivially as an equality, so we consider $x \neq 0$. Setting $W(x) = V(x) - \alpha |x|^{2j}$ we get $W(x) = \sum_{l=1}^d (F_l(x))^2 + (H_2(x))^2 + \mathcal{O}(|x|^5)$ provided $|x| \leq \tilde{\varepsilon}_0 < 1$. When x is in correspondence with an action $I \notin S$ it is clear that $W(x) > 0$ as its terms of degree 4 are strictly positive and higher-order terms do not change it when $|x|$ is small enough. Then we focus on x such that its corresponding $I \in S \setminus \{0\}$, ending up with

$$\begin{aligned} W(x) &= (\mathcal{H}^p(x))^2 - \alpha|x|^{2j} = (\mathcal{H}^j(x))^2 - \alpha|x|^{2j} + \mathcal{O}(|x|^{2j+1}) \\ &= (\mathcal{H}^j(x) - \sqrt{\alpha}|x|^j)(\mathcal{H}^j(x) + \sqrt{\alpha}|x|^j) + \mathcal{O}(|x|^{2j+1}). \end{aligned}$$

Since hypotheses (A) of Theorem 2.6 hold, without loss of generality we assume $\mathcal{H}^j(x) > 0$, thus it suffices to prove that $U(x) = \mathcal{H}^j(x) - \sqrt{\alpha}|x|^j$ is non-negative for x small enough, i.e. $0 < |x| \le \tilde{\varepsilon}_0$, and an adequate choice of α. We introduce the small parameter ε through the stretching $x \to \varepsilon y$ and apply it to U, arriving at

$$\begin{aligned} U^*(y,\varepsilon) &= \frac{1}{\varepsilon^3}\left(\varepsilon^2 \mathcal{H}^j(y,\varepsilon) - \varepsilon^j \sqrt{\alpha}|y|^j\right) \\ &= \mathcal{H}_3(y) + \varepsilon\mathcal{H}_4(y) + \cdots + \varepsilon^{j-3}\left(\mathcal{H}_j(y) - \sqrt{\alpha}|y|^j\right), \end{aligned} \tag{24}$$

where $U^*(y,\varepsilon) = \varepsilon^{-3}U(\varepsilon y)$, $\mathcal{H}^j(y,\varepsilon) = H_2(y) + \varepsilon\mathcal{H}_3(y) + \cdots + \varepsilon^{j-2}\mathcal{H}_j(y)$ and we have taken into account that $\mathcal{H}^j(x) = \varepsilon^2\mathcal{H}^j(y,\varepsilon)$ and $H_2(y) = 0$ for y associated to $J \in S$ with $J = \varepsilon^{-2}I$. According to part (B) of Theorem 2.6, we note that given y related to $J \in S \setminus \{0\}$, for all k between 3 and $j-1$, each $\mathcal{H}_k(y)$ in (24) cannot change sign, henceforth $\mathcal{H}_k(y) \ge 0$. Moreover, if for some $k' = 3, \ldots, j-1$, $\mathcal{H}_{k'}$ is not identically zero but vanishes at some y^*, these values are necessarily associated to $J^* \neq 0$ being in the boundary of S. Otherwise we would obtain a term $\mathcal{H}_{k'}$ such that it changes sign for values of y related to J in the interior of S, contradicting the hypotheses of part (A) in Theorem 2.6. Following Remark 2, every time we get $\mathcal{H}_{k'}(y^*) = 0$, we continue checking the sign of $\mathcal{H}_{k'+1}$ (provided it is not null), but restricting it to the values that annihilate $\mathcal{H}_{k'}$. Proceeding in ascending degree we arrive at degree j and then $\mathcal{H}_j$ is analysed only in the subset of the ball $|y| \le \varepsilon^{-1}\tilde{\varepsilon}_0$ where all previous $\mathcal{H}_k$, $k = 3, \ldots, j-1$ vanish and in addition y is in correspondence with $J \in S \setminus \{0\}$. We call this subset $B^*_{\tilde{\varepsilon}_0,\varepsilon}$. So, for $y \in B^*_{\tilde{\varepsilon}_0,\varepsilon}$ it necessarily follows that $\mathcal{H}^j(y,\varepsilon) = \varepsilon^{j-2}\mathcal{H}_j(y) > 0$. The positiveness of $\mathcal{H}_j$, a homogeneous polynomial in y of degree j, restricted to $B^*_{\tilde{\varepsilon}_0,\varepsilon}$ ensures the existence of $\alpha > 0$ such that $\mathcal{H}_j(y) \ge \sqrt{\alpha}|y|^j$ for $y \in B^*_{\tilde{\varepsilon}_0,\varepsilon}$. Therefore $U^*(y,\varepsilon) \ge 0$ for all y related to $J \in S \setminus \{0\}$. In consequence $V(x) \ge \alpha|x|^{2j}$ where $|x| \le \tilde{\varepsilon}_0$, $\tilde{\varepsilon}_0$ is small enough and x is in correspondence with $I \in S \setminus \{0\}$. As before terms of order $\mathcal{O}(|x|^{2j+1})$ in $W(x)$ cannot change its sign when $|x| \le \tilde{\varepsilon}_0$.

From (23) we conclude that for all x with $|x| \le \tilde{\varepsilon}_0$ the following inequalities hold:

$$\alpha|x|^{2j} \le V(x) \le (\beta + d\delta)|x|^4, \tag{25}$$

where α is chosen smaller than $\beta + d\delta$.

The second step consists in proving that when $t \in D$ with D defined in (22), we get

$$|\mathcal{H}^p(x(t))| < Q\varepsilon^2 \quad \text{and} \quad |F_l(x(t))| < Q'\varepsilon^2, \tag{26}$$

with certain positive constants Q, Q' independent of ε that will be specified later. Notice that $\mathcal{H}^p(x) = \varepsilon^2\mathcal{H}^p(y,\varepsilon)$ with $\mathcal{H}^p(y,\varepsilon) = H_2(y) + \sum_{k=3}^{p}\varepsilon^{k-2}\mathcal{H}_k(y)$, $F_l(x) = \varepsilon^2 F_l(y)$. Then we use the fact that $\mathcal{H}^p$ and F_l are adiabatic invariants – i.e. formal first integrals truncated at degree p – of the normal-form Hamiltonian H in (20) and that the Poisson brackets between H and these adiabatic invariants are made exponentially small (see the different bounds in Theorem 3.5 and additional Lemmas and Propositions of [10]). Thus, we can apply the estimate given in Theorem 4.1

to $\mathcal{H}^p$, F_1, …, F_d, from where it is readily deduced that for $t \in D$ and for arbitrary $\kappa > 0$:

$$|\mathcal{H}^p(y(t), \varepsilon) - \mathcal{H}^p(y_0, \varepsilon)| < \kappa^2,$$

where $y_0 = \varepsilon^{-1} x_0$. Thus

$$|\mathcal{H}^p(x(t)) - \mathcal{H}^p(x_0)| < \kappa^2 \varepsilon^2.$$

Applying the second inequality in (23) we find that

$$|\mathcal{H}^p(x(t))| < |\mathcal{H}^p(x_0)| + \kappa^2\varepsilon^2 \leq \sqrt{\beta}|x_0|^2 + \kappa^2\varepsilon^2 < Q\varepsilon^2 \quad \text{with} \quad Q = \kappa^2 + \sqrt{\beta},$$

when $t \in D$.

As $|F_l(y(t)) - F_l(y_0)| < \kappa^2$ one has $|F_l(x(t)) - F_l(x_0)| < \kappa^2\varepsilon^2$, $l = 1, \ldots, d$, therefore $|F_l(x(t))| < |F_l(x_0)| + \kappa^2\varepsilon^2$, but we know that $|F_l(x_0)| \leq \sqrt{\delta}|x_0|^2 < \sqrt{\delta}\varepsilon^2$, hence when t is in D, $|F_l(x(t))| < Q'\varepsilon^2$ with $Q' = \kappa^2 + \sqrt{\delta}$.

The third step consists in providing an upper small bound for $x(t)$. Combining the previous inequalities with (23) and setting $Q'' = dQ'^2 + Q^2$ we end up with

$$\alpha|x(t)|^{2j} \leq V(x(t)) = \sum_{l=1}^{d}(F_l(x(t)))^2 + (\mathcal{H}^p(x(t)))^2 < dQ'^2\varepsilon^4 + Q^2\varepsilon^4 = Q''\varepsilon^4 \quad (27)$$

for $t \in D$, and then

$$|x(t)| < a'\,\varepsilon^{2/j} \quad \text{where} \quad a' = \left(\frac{Q''}{\alpha}\right)^{1/(2j)}, \quad (28)$$

stressing that $a' > 1$ because $\alpha < \beta + d\delta < Q''$. It is remarkable that both inequalities in (27) apply when $|x(t)| \leq \tilde{\varepsilon}_0$, thus imposing $a'\,\varepsilon^{2/j} \leq \tilde{\varepsilon}_0$ we get the bound $\varepsilon \leq (\tilde{\varepsilon}_0/a')^{j/2}$ and choose $\varepsilon_0 = \min\{(\tilde{\varepsilon}_0/a')^{j/2}, \varepsilon_1\}$, where $\varepsilon_1 > 0$ is the threshold guaranteed by Theorem 4.1.

Since $\kappa > 0$ is arbitrary we set it equal to one converting T of Theorem 4.1 into

$$T = C \exp\left(\frac{K}{\varepsilon^{1/(\nu+1)}}\right).$$

As a four step we want to show that γ given in (21) is bigger than T implying that $D = [0, T]$. Assume the contrary, that is, take $\gamma \leq T$ so that $D = [0, \gamma)$ and consider $\varepsilon < \min\{\varepsilon_0, (R/(2a'))^{j/(2-j)}\}$. Now by assumption $|y(t, y_0, \varepsilon)| \nearrow R$ as $t \nearrow \gamma$. Applying Theorem 4.1 and estimate (28) given above, we arrive at $|y(t, y_0, \varepsilon)| < a'\,\varepsilon^{2/j-1} < R/2$ for all $t \in [0, \gamma)$, which is a contradiction. It follows then that $\gamma > T$, so $D = [0, T]$ as desired.

Hitherto we have obtained the estimates in the normal-form coordinates. The last step consists in putting inequality (28) in terms of the coordinates prior to the normal-form transformation. We rename x in (28) as x^{nf}, using x for the variables related to the Hamiltonian system (1). We know that $|x(t) - x^{nf}(t)|$ is of order ε, choosing the same initial condition for the untransformed and transformed coordinates, in other words, $x_0 = x_0^{nf}$, see for instance [24]. This remains true for all $t \in D$ because the transformation to normal form is performed up to a finite order, thus the process is convergent. Hence, there exists a constant $b > 0$ such that $|x(t) - x^{nf}(t)| < b\,\varepsilon$ from where we get $|x(t)| < |x^{nf}(t)| + b\,\varepsilon$. Applying (28) to $x^{nf}(t)$ we arrive at $|x(t)| < a'\,\varepsilon^{2/j} + b\,\varepsilon = (a' + b\,\varepsilon^{1-2/j})\,\varepsilon^{2/j} \leq a'(1 + a'^{-j/2}\,b\,\tilde{\varepsilon}_0^{j/2-1})\,\varepsilon^{2/j}$, hence

$$|x(t, x_0)| < a\,\varepsilon^{2/j} \quad \text{where} \quad a = a'(1 + a'^{-j/2}\,b\,\tilde{\varepsilon}_0^{j/2-1}) > 1 \quad \forall\, t \in [0, T]. \quad (29)$$

□

Remark 6. Theorem 2.7 generalises Theorem 5.1 of [11], because when $S = \{0\}$, in the proof of Theorem 2.7 we can drop $\mathcal{H}^p$ in V, set $j = 2$, $\kappa = 1$, improving the time estimate given in [11]. In this case the bounds on $x(t)$ are obtained working essentially with the functions F_l.

Remark 7. The estimate (29) gets worse for the confinement of the solution $x(t)$ as j grows, indicating the fact that the more terms one needs to conclude Lie stability the poorer the bounds on the solutions are. However the exponential estimates on the time T do not hinge on j.

Remark 8. As stated in [10], when $d = 1$ it is possible to set $\nu = 0$ in (15) because small divisors cannot arise and the Diophantine condition is dropped. In this case the time T can be very large, moreover the constants C and K are better. In particular this happens in fully resonant Hamiltonians, where there is only an adiabatic invariant (or formal first integral).

Corollary 2. *Suppose hypotheses (A) of Theorem 2.6 are valid with $j = 4$ and the corresponding frequency vector σ is Diophantine. Then for all $\epsilon \in (0, \epsilon_0)$ (with ϵ_0 of order ε_0^2) and for all initial conditions $I(0)$ with $|I(0)|_1 < \epsilon$ we have*

$$|I(t)|_1 < \tilde{a}\, \epsilon^{1/2} \quad \textit{for all} \;\; t \;\; \textit{with} \quad 0 \le t \le T = C \exp\left(\frac{K}{\epsilon^{1/(2(\nu+1))}} \right),$$

where $\tilde{a} > 1$ is independent of ϵ.

Definition 4.2. Suppose that in Hamiltonian (12) the inequalities $4 \le j < m$ hold, then $\mathcal{H}^j(I)$ is convex (C.) at $I = 0$ if the quadratic form $\mathcal{H}_4(I)$ is definite; it is quasi-convex (Q.C.) at $I = 0$ if $H_2(I) = \mathcal{H}_4(I) = 0$ imply $I = 0$; it is directionally quasi-convex (D.Q.C.) at $I = 0$ if H_2 and $\mathcal{H}_4$ vanish simultaneously for $I_i \ge 0$ only at $I = 0$.

Remark 9. If in Corollary 2, $\mathcal{H}^4$ depends only on the actions I, i.e., $m > 4$, our estimates are usually better than those of Nekhoroshev type obtained in [1] in the case of directional quasi-convexity, provided σ in (15) is Diophantine. In fact when D.Q.C. holds the estimates given in [1] are of the form

$$|I(t)|_1 < \epsilon^{1/n} \quad \text{for all} \;\; t \;\; \text{with} \quad 0 \le t \le T = \exp\left(\epsilon^{-1/n}\right),$$

or,

$$|I(t)|_1 < \epsilon^{1/2} \quad \text{for all} \;\; t \;\; \text{with} \quad 0 \le t \le T = \exp\left(\epsilon^{-1/(2n)}\right).$$

In the bounds given above no Diophantine condition is required. Taking into account that C, K and $\tilde{a}$ in Corollary 2 can be considered of order $\mathcal{O}(1)$ we realise that in the worst situation $d = n$, implying that $\nu + 1 > n$ and then our estimate of Corollary 2 is comparable to the second of the estimates of [1] (given in their Theorem 1). Nevertheless, when $d < n$ our result enhances the estimates of [1], especially in the case of fully resonant Hamiltonians. Besides, as mentioned before, in Corollary 2 we can incorporate resonant terms of any arbitrary form, starting at degree 3.

Corollary 3. *Suppose hypotheses (A) of Theorem 2.6 hold with $j = 6$ and the corresponding frequency vector σ is Diophantine. Then for all $\epsilon \in (0, \epsilon_0)$ and for* *all initial conditions* $I(0)$ *with* $|I(0)|_1 < \epsilon$ *we have*

$$|I(t)|_1 < \tilde{a}\, \epsilon^{1/3} \quad \textit{for all} \;\; t \;\; \textit{with} \quad 0 \le t \le T = C \exp\left(\frac{K}{\epsilon^{1/(2(\nu+1))}}\right),$$

where $\tilde{a} > 1$ *is independent of* ϵ.

Remark 10. If in Corollary 3 Hamiltonians $\mathcal{H}^4$ and $\mathcal{H}^6$ hinge only on the actions I, our estimates are different from those of Nekhoroshev type mentioned in [1] for the case of 3-jet non-degenerate systems (these Hamiltonians are steep functions). Indeed, when $n = 3$ the bounds claimed in [1] read

$$|I(t)|_1 < \epsilon \quad \text{for all} \;\; t \;\; \text{with} \quad 0 \le t \le T = \exp\left(\epsilon^{-b}\right),$$

where $b = \min\{\frac{p-7}{20}, \frac{p+1}{36}\}$, $p \ge 8$ stands for the degree reached in the normal-form transformation and such that no resonant terms are encountered in $\mathcal{H}^p$. In a recent study on steep Hamiltonian systems Guzzo *et al.* [16] have obtained sharp bounds for exponentially large time in terms of the steepness indices of the Hamiltonian function independent of angles. Nevertheless the application to elliptic equilibria still seems far from being fully achieved. Indeed for $n > 3$ no explicit estimate based on Nekhoroshev theory is available yet for 3-jet non-degenerate systems. Note however that the estimates of Corollary 3 apply regardless of the presence of resonant terms.

Remark 11. It might happen that a certain Hamiltonian is a 3-jet non-degenerate function, thence Nekhoroshev stability holds and even the estimates given in Remark 10 apply if $n = 3$. Nonetheless, if Hamiltonian $\mathcal{H}_4(I)$ changes sign in the interior of S (then at least $\dim S > 1$), and if moreover $\mathcal{H}_3 = \mathcal{H}_5 = 0$, the application of part (B) in Theorem 2.6 prevents us of getting Lie stability from $\mathcal{H}^6(I)$. In this situation we could not obtain a definite function V as (16), therefore we could not apply Theorem 2.7 to find asymptotic estimates. So, we content ourselves with the estimates of Nekhoroshev type, as we have done in [8].

Remark 12. When convexity holds better time bounds, compared to the ones of D.Q.C. written in Remark 9, are obtained whether the normalisation can be executed to an adequate high order without encountering resonant terms, see for instance [30, 32]. If the order reached by the normal-form Hamiltonian is high enough, these bounds are also better than the estimates obtained in Corollary 2, or the estimates of Theorem 2.7.

On our part we can improve the confinement of the solution with a slight change in hypotheses as in Theorem 2.6, part (A) and Theorem 2.7, as we show in the following result.

Corollary 4. *Suppose there are two even integers* j', j *such that* $4 \le j' \le j \le p$, $\mathcal{H}_k(x) = 0$ *for* $k = 3, \ldots, j'-1$ *and* $\sum_{k=j'}^{j} \mathcal{H}_k(x) \ne 0$ *for all* x *associated to* $I \in S \setminus \{0\}$. *Suppose in addition that the frequency vector* σ *is Diophantine. Then with the same notation as in Theorem 2.7, the estimates are of the form*

$$|x(t, x_0)| < a\, \varepsilon^{j'/j} \quad \textit{for all} \;\; t \;\; \textit{with} \quad 0 \le t \le T = C \exp\left(\frac{K}{\varepsilon^{1/(\nu+1)}}\right).$$

Proof. For an integer $r \geq j$ we define $\mathcal{H}^{*,r} = \mathcal{H}^r - H_2 = \mathcal{H}_{j'} + \cdots + \mathcal{H}_j + \mathcal{H}_{j+1} + \cdots + \mathcal{H}_r$, and choose

$$\begin{aligned} V(x) &= F_1^{j'}(x) + \cdots + F_d^{j'}(x) + (\mathcal{H}^{*,p}(x))^2 \\ &= F_1^{j'}(x) + \cdots + F_d^{j'}(x) + (\mathcal{H}^{j'}(x) + \cdots + \mathcal{H}^j(x) + \cdots + \mathcal{H}^p(x))^2. \end{aligned}$$

We observe that j' cannot be odd. If this occurs then $\mathcal{H}_{j'}$ would be composed of terms of the form $c\, I_1^{\alpha_1/2} \cdots I_n^{\alpha_n/2} \cos(\beta_1\theta_1 + \cdots + \beta_n\theta_n)$ (as well as sin-terms) with α_k, β_k satisfying d'Alembert character. This in turns leads to $\mathcal{H}_{j'}(x)$ would change sign in the interior of S, then preventing $\mathcal{H}^{*,j}$ to be definite in S.

As $\{\mathcal{H}_k, H_2\} = 0$, $k \in \{j', \ldots, p\}$ and $\{F_l, H_2\} = 0$, $l \in \{1, \ldots, d\}$, V is a first integral of $\mathcal{H}^p$. Notice in addition that since j' is even, then $V(x) \geq 0$ for all x. Besides for $|x| \leq \tilde{\varepsilon}_0$ small enough, $V(x) = 0$ if and only if $x = 0$. The reasoning is almost the same as the one done in the proof of Theorem 2.6, part (A). Applying Liapunov Stability Theorem the null solution is stable for the Hamiltonian system associated to $\mathcal{H}^p$. The same holds if we consider the normal form to any degree $q > p$ thus, as in the proof of Theorem 2.6, Lie stability is established.

Next, we claim that there are positive constants α, β and δ independent of ε such that $\beta + d\delta > \alpha > 0$ and

$$\alpha|x|^{2j} \leq V(x) \leq (\beta + d\delta)|x|^{2j'}, \tag{30}$$

for all $|x| \leq \tilde{\varepsilon}_0$.

To prove that $V(x) - \alpha|x|^{2j} \geq 0$ notice that for $x = 0$ we get a trivial identity, so we distinguish between x related to $I \notin S$ and $I \in S \setminus \{0\}$. Choose x such that its corresponding I is not in S, the lowest-degree terms of V are of degree $2j'$, concretely, $F_1^{j'}(x) + \cdots + F_d^{j'}(x) + (\mathcal{H}_{j'}(x))^2$ and it is a homogeneous polynomial in x strictly positive. Then, higher-order terms cannot change the sign of $V(x) - \alpha|x|^{2j}$ for x selected in such a way that $|x| \leq \tilde{\varepsilon}_0$. Next, take x in correspondence with $I \in S \setminus \{0\}$, set $V(x) - \alpha|x|^{2j} = (\mathcal{H}^{*,j}(x))^2 - \alpha|x|^{2j} + \mathcal{O}(|x|^{2j+1})$ and assume that $\mathcal{H}^{*,j}$ is positive definite for $|x|$ small. Then use the same argument as in the proof of Theorem 2.7 where we showed that $U(x) \geq 0$, proving that $\mathcal{H}^{*,j}(x)) - \sqrt{\alpha}|x|$ remains non-negative for $|x|$ small and an appropriate choice of $\alpha > 0$ small enough. As terms of order $\mathcal{O}(|x|^{2j+1})$ cannot turn $\mathcal{H}^{*,j}(x) - \sqrt{\alpha}|x|^j$ into a negative expression, the inequality holds.

That $(\beta + d\delta)|x|^{2j'} \geq V(x)$ follows first from the bounds $F_l^{j'}(x) \leq \delta|x|^{2j'}$, for an adequate choice of δ. Besides, we need to take into account that $(\mathcal{H}^{*,p}(x))^2$ can be rewritten as $(\mathcal{H}_{j'}(x))^2 + \mathcal{O}(|x|^{2j'+1})$, observing that $(\mathcal{H}_{j'})^2$ is a homogeneous polynomial in x of degree $2j'$. So, for $|x| \leq \tilde{\varepsilon}_0$ there is a positive constant, say β, such that $\beta|x|^{2j'} - (\mathcal{H}_{j'}(x))^2 + \mathcal{O}(|x|^{2j'+1}) \geq 0$ and β can be selected satisfying $\beta + d\delta > \alpha > 0$.

Now perform the stretching $x = \varepsilon y$ noting that $V(x) = \varepsilon^{j'} V(y, \varepsilon)$ and assume $|x_0| < \varepsilon$ and $y_0 = \varepsilon^{-1} x_0$. As V is composed by first integrals of $\mathcal{H}^p$, namely, F_l and $\mathcal{H}^{*,p}$, we can apply the bounds provided in Theorem 4.1. Following the same approach as in the proof of Theorem 2.7, and applying (30), we arrive at $V(x(t)) < Q\varepsilon^{2j'}$ for t in the interval D introduced in (22), where the constant $Q > \alpha$ is independent of ε. Then $\alpha|x(t)|^{2j} < \delta\, \varepsilon^{2j'}$ from where we obtain

$$|x(t)| < a'\, \varepsilon^{j'/j},$$

and $a' = (Q/\alpha)^{1/(2j)} > 1$ depends only upon α, β, δ and d. The bound for $x(t)$ is true for all $t \in [0, T]$ with the same T as in Theorem 2.7. The rest of the proof follows similar steps to those given in the course of the proof of Theorem 2.7. □

Remark 13. Notice that the optimal situation is $j' = j$ as then the confinement of $|x(t)|$ is of order ε, similar to that obtained in [30, 32], although in our case j' can be any even integer greater than 3. Besides in V we could retain resonant terms, thus enlarging considerably the approach of the previous contributions handling convex situations and mentioned in Remark 12. Suppose for example a Hamiltonian system with 4 degrees of freedom where $\mathcal{H}^{*,8}$ is $\mathcal{H}_6 + \mathcal{H}_7 + \mathcal{H}_8 = I_1^3 + 2I_2^3 + 5I_1 I_4^{5/2} \sin(2\theta_1 + 5\theta_4) + 4I_3^4 - 30I_2^4$ while $\mathcal{H}_3 = \mathcal{H}_4 = \mathcal{H}_5 = 0$. It is easy to infer that $\mathcal{H}^8$ is positive for x related to any $I \in S$ and $0 < |x| \leq \tilde{\varepsilon}_0$ with $\tilde{\varepsilon}_0$ small enough. (We also assume $\dim S = 3$.) Then the origin of $\mathbb{R}^8$ is Lie stable for a Hamiltonian system where the terms in normal form up to degree 8 are given above while H_2 could be taken as any indefinite form such that $\{H_2, \mathcal{H}^8\} = 0$ and $\dim S = 3$. Then one takes $j' = 6, j = 8$ and $x(t)$ satisfies $|x(t)| < a\,\varepsilon^{3/4}$ for an exponentially large time.

Remark 14. In case that Lie stability is not accomplished (or even instability is obtained) one can still deduce asymptotic bounds for some action coordinates as follows. Since we have d adiabatic invariants F_l satisfying Theorem 4.1, there always exists a linear change of coordinates from I, θ to $\tilde{I}, \tilde{\theta}$ such that $\tilde{I}_l = F_l$ for $l = 1, \ldots, d$ and $\tilde{\theta}_m = k_m \cdot \theta$ for $m = d+1, \ldots, n$. Therefore, these actions satisfy Chartier *et al.*'s estimates and from the proof of Theorem 2.7 one gets bounds of the form $|\tilde{I}_l(x(t, x_0)) - \tilde{I}_l(x_0)| < \varepsilon^2$ for exponentially large time.

Remark 15. When $d > 1$ it would be desirable to lessen the Diophantine condition stated above, replacing it by another weaker non-resonance condition, but at present the Diophanticity of the vector σ is required. Perhaps applying the techniques of Lochak's method of averaging [21] by analysing the neighbourhoods of periodic solutions of the unperturbed system and approximating all other initial positions by periodic ones, see also [30, 32], would help to relax the Diophantine hypothesis on the vector σ. Additionally it is a well-known fact that for a fixed ν the Lebesgue measure of the set of vectors $\sigma \in \mathbb{R}^d$ that does not satisfy the Diophantine condition for any $c > 0$ is null. On the other hand although in the contributions [21, 12, 30, 32] no Diophantine hypothesis is needed to get Nekhoroshev stability, a non-resonance condition affecting the frequencies ω_i is required, usually involving resonances of orders 3 and 4.

5. Implications and Examples.

5.1. The case $n = 2$.

For two degrees of freedom, our result is the same as the stability part in Cabral-Meyer's Theorem (Theorem 4.1 of [4]) that includes Arnold's Theorem of stability as well as other results of Alfriend (for references, see [4]) and Markeev [22, 23]. Therefore Lie stability becomes Liapunov stability. The reason of this is that the function $\mathcal{H}^j$ of Theorem 2.6 agrees with the function Ψ in Theorem 4.1 of [4].

5.2. The case $S = \{0\}$.

On this occasion one always achieve Lie stability, see also details in [34]. More specifically, part (A) of Theorem 2.6 applies trivially with $V = \sum_{k=1}^{d} F_k^2$ and no normal-form computation has to be carried out.

When H_2 is definite Dirichlet Theorem [24] applies and Liapunov stability is fulfilled.

Corollary 5. *Assuming that H_2 is definite the null solution of (1) is Lie stable.*

Proof. It is clear that the formal first integrals F_k are written as linear combinations of the form $\sum_j \alpha_{j,k} I_j$ and without loss of generality we assume $\alpha_{j,k} > 0$. Hence for every $k = 1, \ldots, d$, $F_k = 0$ if and only if $I_j = 0$ for all j appearing in F_k, therefore $S = \{0\}$. □

Corollary 6. *In the absence of resonances among the ω_i the null solution is Lie stable.*

Proof. In fact, since $\omega_1, \ldots, \omega_n$ are linearly independent over $\mathbb{Q}$, the formal first integrals are $F_j = I_j$ with $j = 1, \ldots, n$, then $d = n$, $s = 0$ and the set S is null. □

Remark 16. A particular situation of Corollary 6 occurs when $(\omega_1, \ldots, \omega_n)$ is Diophantine. Then, under an additional condition on the coefficients of the normal form which is of full Lebesgue measure, the bounds on time have been largely sharpened, becoming super-exponentially long. See for instance [2] and references therein. Thus our bounds of Theorem 2.7 are considerably improved.

Remark 17. Normal stability introduced in [25] is a particular case of Lie stability with $S = \{0\}$ as it was concluded in [34]. The estimates obtained in [11] when $S = \{0\}$ are comparable to those provided by Theorem 2.7, see Remark 6.

An example of a Hamiltonian function with $S = \{0\}$ is the following:

$$H = (1 - \sqrt{2}) I_1 - \sqrt{2} I_2 + (2 - \sqrt{2}) I_3 - \sqrt{2} I_4 + \cdots ,$$

where $\cdots$ refers to higher-order terms in normal form starting at degree three, has resonance vectors $k_1 = (2, 0, -1, -1)$, $k_2 = (0, 2, 0, -2)$ and two formal first integrals, namely, $F_1 = I_1 + 2I_2 + 2I_4$ and $F_2 = I_1 + I_2 + I_3 + I_4$. Hence it is easily deduced that S is null, concluding Lie stability. As $H_2 = \sigma_1 F_1 + \sigma_2 F_2$ with $(\sigma_1, \sigma_2) = (-1, 2 - \sqrt{2})$, which is a Diophantine vector, the estimates of Theorem 2.7 apply with $j = 2$ and $\nu > 1$.

We stress that when $S = \{0\}$ Lie stability can be achieved even for Hamiltonians whose first nonlinear term $\mathcal{H}_3$ is non-zero. If this occurs then $\mathcal{H}_3$ in terms of action-angle coordinates hinges on angles because d'Alembert character (11) is satisfied.

For instance, the equations of motion corresponding with Hamiltonian

$$\begin{aligned} H \;=\; & 5(\sqrt{5} - 1) I_1 + 2(\sqrt{5} - 1) I_2 + (\sqrt{5} - 1) I_3 - 18 I_4 + 18(1 + \sqrt{5}) I_5 \\ & + 3\sqrt{I_2} I_3 \sin(\theta_2 - 2\theta_3) + \cdots , \end{aligned}$$

has the null solution as Lie stable because $S = \{0\}$. Note that the corresponding term $\mathcal{H}_3$ is given by $3\sqrt{I_2} I_3 \sin(\theta_2 - 2\theta_3)$ and satisfies d'Alembert character so the perturbation starts with a polynomial of degree three in x. This Hamiltonian has two other independent resonances of orders 6 and 21. The estimates are the same as in the preceding example, but more details can be looked at [11].

5.3. **Lie stability decided from terms that do not depend on angles.** Our theory extends previous results in the sense that we can get Lie stability for Hamiltonian systems that even do not satisfy the conditions needed in Nekhoroshev theory, obtaining Lie stable systems under rather weak conditions. More specifically we assume in this subsection that hypotheses (A) hold for some $j \geq 4$ and such that $\mathcal{H}^j$ does not hinge on angles, so $j < m$ in (12).

Corollary 7. *Directional quasi-convexity of elliptic equilibria is a particular case of Lie stability.*

Proof. When $j = 4$ and $\mathcal{H}^4(I) = H_2(I) + \mathcal{H}_4(I)$ is directionally quasi-convex, then $H_2(I) = \mathcal{H}_4(I) = 0$ with $I_i \geq 0$ imply $I = 0$. Then exponential estimates of Nekhoroshev type apply [1]. Choosing $I \in S \setminus \{0\}$ we get $H_2(I) = 0$ and as D.Q.C. holds, $\mathcal{H}_4(I) \neq 0$ so $\mathcal{H}^4(I) \neq 0$, entailing the application of Theorem 2.6. □

We observe that Corollary 7 applies regardless of the number and type of resonances the Hamiltonian function has, provided $m > 4$. In addition we notice that Theorem 2.6 particularises to D.Q.C. not only when $j = 4$, $\mathcal{H}_3 = 0$, and also $d = 1$, $\dim S = n - 1$. In contrast, if $\dim S < n - 1$, the requirement $\mathcal{H}^4(I) \neq 0$ is checked in a space of lower dimension.

Remark 18. It is well known that 3-jet non-degenerate functions [29, 36] are a particular case of steep functions, hence they are Nekhoroshev stable. The non-degeneracy is concluded whether from the system $H_2(I) = \mathcal{H}_4(I) = \mathcal{H}_6(I) = 0$ the only solution with $I_i \geq 0$ is $I = 0$. If $\mathcal{H}^4(I)$ changes sign at some $I \in S \setminus \{0\}$, then we cannot decide on the Lie stability of the system whereas if, on the contrary, $\mathcal{H}^4(I)$ does not change sign for any $I \in S \setminus \{0\}$, then the system is Lie stable. So, there are steep systems for which we cannot decide on their Lie stability. When higher-order jets are needed, according to Schirinzi and Guzzo [36], extra hypotheses are required in order to guarantee steepness. Although these assumptions have been established for 4-jets when $n = 3, 4$, for degrees higher than 4 this property is hard to analyse. Nevertheless we emphasise that steepness does not imply nonlinear stability of an elliptic equilibrium since instability could occur through a slow diffusion mechanism.

Remark 19. As mentioned in the introduction, exponential estimates of Nekhoroshev type have been obtained recently by several authors relaxing steepness conditions, see the papers [17, 31]. The hypotheses that one has to examine are not straightforward, but they essentially involve to check whether some Hessian matrices obtained in suitable affine subspaces of $\mathbb{R}^{2n}$ are non-degenerate. Translated to the setting of elliptic equilibria it implies a significant restriction in the terms $\mathcal{H}_4(I)$, $\mathcal{H}_6(I)$, …, $\mathcal{H}_j(I)$. However we can handle rather simply examples of Lie stable systems with exponential bounds that are very degenerate, some of non-steep nature, as we see below.

Next example illustrates that we can have Lie stability from $\mathcal{H}^4(I)$ under very weak conditions. Consider the Hamiltonian with three degrees of freedom

$$H = H_2 + \mathcal{H}_4 + \cdots, \tag{31}$$

with

$$\begin{aligned} H_2 &= -\tfrac{1}{10}(6 + \sqrt{6})I_1 + \tfrac{1}{10}(-2 + 3\sqrt{6})I_2 + I_3, \\ \mathcal{H}_4 &= I_1^2 + \alpha I_2^2 + I_3^2 + I_1 I_2 + I_1 I_3 + I_2 I_3, \end{aligned}$$

and α a real parameter. Hamiltonian H is supposed to be in normal form up to a certain order. The $\mathbb{Z}$-module M_ω is spanned by $k_1 = (3, 1, 2)$ and the functions $F_1 = -2I_1 + 3I_3$, $F_2 = -I_1 + 3I_2$ are the formal first integrals of the system whose Hamiltonian is H, so $d = 2$. The corresponding set S related to the quadratic terms of (31) is given by $\{(3I_3, I_3, 2I_3) \mid I_3 \geq 0\}$, thus $\dim S = 1$. As $\mathcal{H}^4(I) = H_2(I) + \mathcal{H}_4(I) = (24 + \alpha)I_3^2$ for $I \in S$, one gets $\mathcal{H}^4 \neq 0$ when $I_3 \neq 0$, $\alpha \neq -24$. Thence, applying Theorem 2.6 the null solution of the Hamiltonian system associated to (31) is Lie stable, provided $\alpha \neq -24$.

Nonetheless, Hamiltonian function $\mathcal{H}^4$ in (31) is convex at $I = 0$ only if $\alpha > 1/3$. This condition can be somewhat relaxed, assuming directional quasi-convexity as introduced in [1]. After some straightforward computations we conclude that $\mathcal{H}_4$ is directionally quasi-convex at $I = 0$ for $\alpha > 2(\sqrt{6} - 4)/3$, which is the bound for α in order to get Nekhoroshev type of stability from $\mathcal{H}_4$. Nevertheless, the previous bound is extended applying the notion of rational convexity of Guzzo *et al.* [17], where convexity is tested only in the affine planes of fast drift, which are subspaces formed by integer vectors of dimensions up to $n - 1$. In this case it is enough to take into account the one-dimensional subspace spanned by k_1. By proceeding as in [17] one arrives at exponential stability when $\alpha \neq -24$, i.e. the same restriction we found to achieve Lie stability, suggesting that when Lie stability is established from Hamiltonian functions using only the two first terms of the form $H_2(I) + \mathcal{H}_4(I)$ is equivalent to rational convexity of Guzzo and coworkers.

In case the origin of $\mathbb{R}^6$ is Lie stable we note that $H_2 = \sigma_2 F_1 + \sigma_2 F_2$ with $\sigma_1 = 1/3$, $\sigma_2 = (3\sqrt{6} - 2)/30$, thus (σ_1, σ_2) is Diophantine and the estimates of Theorem 2.7 hold. Notice in addition that ν can be taken above one, thence applying Corollary 2, $T = C \exp(K\epsilon^{-1/4})$, and we get a better estimate than the one corresponding to the application of Nekhoroshev theory (choosing $a = 1/2$ and $b = 1/(2n)$). The confinement is in general of order $\epsilon^{1/2}$ but applying Corollary 4 it becomes of order ϵ.

For the following example we take

$$H = H_2 + \mathcal{H}_{10} + \cdots = 3I_1 - 2I_2 + 6I_3 - I_2^5 + \cdots . \tag{32}$$

The resonance vectors are $k_1 = (2, 0, -1)$, $k_2 = (0, 3, 1)$ and

$$S = \{(2(I_2 - 3I_3), 3I_2, 3I_3) \mid I_2 \geq 3I_3 \geq 0\},$$

the only formal first integral is $F_1 = 3I_1 - 2I_2 + 6I_3$, hence $d = 1$. Considering $I \in S$ it is clear that $\mathcal{H}^{10}(I) = H_2(I) + \mathcal{H}_{10}(I) = -I_2^5 = 0$ if and only if $I_2 = 0$, but then $I_1 = I_3 = 0$, thus $\mathcal{H}^{10} < 0$ for $I \in S \setminus \{0\}$ and Lie stability holds. However the system is too degenerate to obtain stability from Nekhoroshev theory. In fact as $\mathcal{H}_4 = \mathcal{H}_6 = 0$ steepness condition fails and the more relaxed conditions of rational convexity [17] and Diophantine steepness [31] break down as well since we can select a two-dimensional affine subspace of $\mathbb{R}^3$ so that the corresponding Hessian matrix is degenerate. Regarding the estimates on $x(t)$ and the time T they can be straightforwardly applied as $d = 1$, thus no Diophantine condition is required in this case, see Remark 8. Theorem 2.7 is applied with $\nu = 0$ and $j = 10$. Notice that Corollary 4 applies, then the bound on $x(t)$ is of order ε.

Let us consider now $H = H_2 + \mathcal{H}_6 + \cdots$ where H_2 is as in the preceding example and $\mathcal{H}_6 = 40I_3^3 - I_2^3/2$. Then $\mathcal{H}_6(I^*) = 0$ for points of the form $I^* = (I_1^*, I_2^*, I_3^*) = ((2/3 - 10^{-1/3})I_2^*, I_2^*, I_2^*/(2 \cdot 10^{1/3}))$, which are in the interior of S for $I_2^* > 0$. So, we cannot build a positive definite first integral V as defined in (16) and then, applying part (B) of Theorem 2.6, Lie stability for the null solution of the system related to H cannot be achieved by adding higher-order terms to $\mathcal{H}^6$. In Fig. 1 we compare the effect of taking two different $\mathcal{H}_6$ with the same H_2, leading to different behaviours. Theorem 4.1 can be applied to the formal first integral F_1 getting an exponential time estimate for it, see Remark 14.

For the next example we choose a Hamiltonian (7) with n degrees of freedom, $\mathcal{H}_3 = 0$ and such that $\mathcal{H}_4$ is independent of the angles and a directionally quasi-convex function of I, so Nekhoroshev stability of the origin of $\mathbb{R}^{2n}$ holds. Furthermore it is assumed that there is only one resonant angle, thus $s = 1$ and one can take the integer vector $k_1 = (k_{11}, \ldots, k_{1n})$ as the resonance vector. We notice

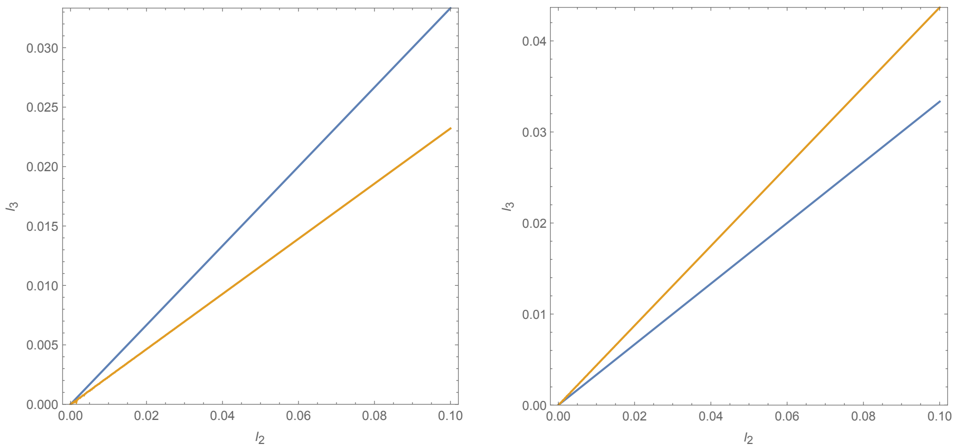

FIGURE 1. On the left we plot the curves $I_2 = 3I_3$ (blue) and $40I_3^3 = I_2^3/2$ (orange) showing that $\mathcal{H}_6$ changes sign in S, hence Lie stability cannot be accomplished. On the right we consider $\mathcal{H}_6 = 4I_3^3 - I_2^3/3$ and plot the curves $I_2 = 3I_3$ (blue) and $4I_3^3 = I_2^3/3$ (orange) showing that the origin of $\mathbb{R}^6$ is Lie stable for the Hamiltonian $H_2 + \mathcal{H}_6 + \cdots$.

that $d = n-1$ and the corresponding formal first integrals F_j are obtained as $F_j = k_{11}I_j - k_{1j}I_1$ with $j = 2, \ldots, n$ where without loss of generality we can take $k_{11} \neq 0$. If $I \in S$ and $\dim S = 1$, we get $I_j = (k_{1j}/k_{11})I_1$ for $j = 2, \ldots, n$. Thus, Hamiltonian (7) evaluated at $I \in S$ takes the form

$$H = \frac{1}{k_{11}^2}\mathcal{H}_4(k_{11}, k_{12}, \ldots, k_{1n})I_1^2 + \cdots . \tag{33}$$

Since $\mathcal{H}_4$ is directionally quasi-convex, $\mathcal{H}^4 \neq 0$ for $I_1 \neq 0$ and the null solution is Lie stable for the Hamiltonian system associated to H. This should be expected as Nekhoroshev stability of elliptic equilibria implies Lie stability in this case, see Corollary 7. Regarding the estimates issue one can apply Nekhoroshev's bounds provided in [1] (or the ones provided in Remark 9). Alternatively it is possible to use the estimates of Corollary 2 with $T = C\exp(K\epsilon^{-1/(2n-2)})$, slightly improving the bound given by Nekhoroshev theory. Finally for the action confinement, when $\dim S = 0$ then $j = 2$ and one gets $|I(t)|_1 < \tilde{a}\,\epsilon$, $\tilde{a} > 1$. Applying Corollary 4 the same bound on the actions is achieved regardless of the dimension of S.

In [8], see also [5], we dealt with the Lie stability of the equilibrium points L_4/L_5 in the spatial circular restricted three-body problem when both points are elliptic. Specifically we show that there is an interval depending on the parameter of the problem, say μ, where Lie stability can be accomplished from $\mathcal{H}^4(I) = H_2(I) + \mathcal{H}_4(I)$ since it is possible that $\mathcal{H}^4(I)$ does not change sign for $I \in S \setminus \{0\}$. In these cases we have $\dim S = 0$ or 1. (Note that when $S = \{0\}$ no normal form is needed.) Nonetheless directional quasi-convexity does not hold in this interval and $\mathcal{H}_6(I)$ has to be computed to achieve 3-jet non-degeneracy, hence steepness, excluding only a few values of μ, see [1].

5.4. **Lie stability decided from terms that depend on angles.** As the Hamiltonian $\mathcal{H}^j$ of Theorem 2.6 can hinge on ϕ_k, our result generalises Theorem 3.1 of

[33] and Theorem 1.1 of [34], dealing with the situations of single and multiple resonances, respectively. Furthermore, our upshot remains valid for any $\mathcal{H}^j$ satisfying the hypotheses in (A) regardless of the dependence of intermediate Hamiltonians $\mathcal{H}_i$ with $i < j$ with respect to some angles ϕ_k. This makes the approach of Theorem 2.6 of broader application than any other result we know on Lie stability of elliptic equilibria.

An example of a Hamiltonian system with three degrees of freedom that has two resonances, one of order 4 and the other of order 5 and is derived from the Hamiltonian

$$H = H_2 + \mathcal{H}_4 + \mathcal{H}_5 + \cdots, \tag{34}$$

where

$$\begin{aligned} H_2 &= \left(\sqrt{2} - 7\right) I_1 + 3\left(7 - \sqrt{2}\right) I_2 + \tfrac{1}{2}\left(5\sqrt{2} - 35\right) I_3, \\ \mathcal{H}_4 &= I_1^{3/2}\sqrt{I_2}\cos\phi_1 + I_1^2 + I_2^2 + I_3^2 + I_1 I_2 + I_1 I_3 + I_2 I_3, \\ \mathcal{H}_5 &= \sqrt{I_1} I_2 I_3 \cos\phi_2, \end{aligned}$$

and $\phi_1 = 3\theta_1 + \theta_2$, $\phi_2 = \theta_1 + 2\theta_2 + 2\theta_3$. Notice that H satisfies d'Alembert character. The resonance vectors are $k_1 = (3, 1, 0)$, $k_2 = (1, 2, 2)$; $F_1 = 2I_1 - 6I_2 + 5I_3$ is the corresponding formal first integral and the set S is given by $\{(6I_2 - 5I_3, 2I_2, 2I_3) \mid 0 \leq 5I_3 \leq 6I_2\}$ with $\dim S = 2$. Taking $I \in S$ we get

$$\mathcal{H}_4(I, \phi_1) = 52I_2^2 + 19I_3^2 - 54I_2I_3 + \sqrt{2I_2}(6I_2 - 5I_3)^{3/2}\cos\phi_1.$$

At this point we notice that when $I_3 \in (0, 6I_2/5)$ then $52I_2^2 + 19I_3^2 - 54I_2I_3 > |\sqrt{2I_2}(6I_2 - 5I_3)^{3/2}|$ from where it is easily deduced that $\mathcal{H}^4(I, \phi_1)$ is positive for $I \in S \setminus \{0\}$ and any $\phi_1 \in \mathbb{T}$. Thus, the null solution of the Hamiltonian system associated to H defined in (34) is Lie stable. We observe that for $\mathcal{H}_5$ we could have chosen any Hamiltonian in normal form in terms of I and ϕ_2 provided it satisfies (11). As in this case we only have a first integral, F_1, the estimates of Theorem 2.7 on $x(t)$ and T apply with $j = 4$ and $d = 1$, noticing that no Diophantine hypothesis is required for the estimates. Specifically the time estimate is $T = C\exp(K\varepsilon^{-1})$. Moreover, according to Corollary 4 the bound for $x(t)$ is of order ε.

A slight variation of the previous example consists in a Hamiltonian function with the same H_2 and $\mathcal{H}_5$ as before, but where we modify $\mathcal{H}_4$ taking it as

$$\mathcal{H}_4 = \tfrac{1}{4} I_1^{3/2}\sqrt{I_2}\cos\phi_1 + I_1^2 + I_2^2 - I_3^2 - I_1I_2 + 2I_1I_3 + \tfrac{11}{30} I_2 I_3,$$

and also we add $\mathcal{H}_6 = I_3^3$. Then we calculate $\mathcal{H}_4(I, \phi_1)$ for $I \in S$, obtaining

$$\mathcal{H}_4(I, \phi_1) = \tfrac{1}{60}(6I_2 - 5I_3)\left(280I_2 - 12I_3 + 15\sqrt{2I_2(6I_2 - 5I_3)}\cos\phi_1\right),$$

which is non-negative and vanishes for $I_2 = I_3 = 0$ and for $I_3 = 6I_2/5$, thus $\mathcal{H}^4 = 0$ for $I^* = (0, 2I_2^*, 12I_2^*/5)$, with $I_2^* \geq 0$. Then, we proceed to check $\mathcal{H}^5$ for I^*, ϕ_1, ϕ_2, getting $\mathcal{H}^5(I^*, \phi_1, \phi_2) = 0$. Next we consider $\mathcal{H}^6(I^*, \phi_1, \phi_2)$ arriving at $\mathcal{H}^6(I^*, \phi_1, \phi_2) = 13824{I_2^*}^3/125 > 0$ if $I_2^* > 0$. This is enough in order to conclude Lie stability, a feature which is not possible to achieve applying the theory of [34]. The application of Corollary 4 in terms of the actions gives $|I(t)|_1 < \tilde{a}\,\epsilon^{2/3}$ with time estimate $C\exp(K\epsilon^{-1/2})$. It is remarkable that this example does not contradict part (B) of Theorem 2.6 because $\mathcal{H}^4$, $\mathcal{H}^5$ vanish only on the boundary of S for small enough $|I|$.

Our next example represents a Hamiltonian with multiple resonances of orders 3 and 4 for which the Hamiltonian function is

$$H = H_2 + \mathcal{H}_3 + \mathcal{H}_4 + \cdots, \tag{35}$$

where

$$\begin{aligned} H_2 &= \tfrac{1}{20}\left(2-3\sqrt{6}\right) I_1 + \tfrac{1}{10}\left(3\sqrt{6}-2\right) I_2 - \tfrac{3}{20}\left(3\sqrt{6}-2\right) I_3, \\ \mathcal{H}_3 &= I_1\sqrt{I_2}\cos\phi_1, \\ \mathcal{H}_4 &= \sqrt{I_1 I_3} I_2 \cos\phi_2 + 2I_1^2 - 5I_1I_2 - I_1I_3 + 3I_2I_3, \end{aligned}$$

with $\phi_1 = 2\theta_1 + \theta_2$, $\phi_2 = \theta_1 + 2\theta_2 + \theta_3$. D'Alembert property is satisfied, thus H is regular at any point. The resonance vectors are $k_1 = (2,1,0)$, $k_2 = (1,2,1)$, $F_1 = I_1 - 2I_2 + 3I_3$ is a formal first integral and the set S is given by $\{(2I_2 - 3I_3, I_2, I_3) \mid 0 \le 3I_3 \le 2I_2\}$ with $\dim S = 2$. Taking a point in the interior of S, say I^* with $0 < I_3^* < 2I_2^*/3$, we build the function $\mathcal{H}_3(I^*, \phi_1) = (2I_2^* - 3I_3^*)\sqrt{I_2^*}\cos\phi_1$ that has a simple zero at $\phi_1^* = \pi/2$. Then part (B) of Theorem 2.6 applies and we cannot deduce stability of the null solution of $\mathbb{R}^6$. In fact it is likely that it is unstable for the Hamiltonian system associated to (35), but currently none of the known theorems on instability applies. Even when stability does not likely hold, Theorem 4.1 applies on the (transformed) action given as the first integral F_1 and the exponential time estimate of Chartier *et al.* is true for it. Note that $\mathcal{H}_4$ plays no role in the analysis performed.

We present a case of a Hamiltonian system with three degrees of freedom that has multiple resonances of order 4 and that is given by the function

$$H = H_2 + \mathcal{H}_4 + \cdots, \tag{36}$$

where

$$\begin{aligned} H_2 &= \tfrac{1}{20}\left(2-3\sqrt{6}\right) I_1 + \tfrac{3}{20}\left(3\sqrt{6}-2\right) I_2 + \tfrac{7}{20}\left(3\sqrt{6}-2\right) I_3, \\ \mathcal{H}_4 &= I_1^2 + I_2^2 - I_3^2 + I_1I_2 + I_1I_3 + I_2I_3 + I_1^{3/2}\sqrt{I_2}\cos\phi_1 + \sqrt{I_1I_3}I_2\cos\phi_2, \end{aligned}$$

and $\phi_1 = 3\theta_1 + \theta_2$, $\phi_2 = \theta_1 - 2\theta_2 + \theta_3$. We emphasise that d'Alembert character (11) fulfills. The resonance vectors are $k_1 = (3,1,0)$, $k_2 = (1,-2,1)$. The corresponding formal first integral reads as $F_1 = -I_1 + 3I_2 + 7I_3$ whereas the set S is given by $\{(3I_1, I_1 - 7I_3, 3I_3) \mid I_1 \ge 7I_3 \ge 0\}$, so $\dim S = 2$. Taking $I \in S$ we get

$$\begin{aligned} \mathcal{H}_4(I,\phi_1,\phi_2) \;\; = \;\; & 13I_1^2 + 19I_3^2 - 23I_1I_3 + 3\sqrt{3}I_1^{3/2}\sqrt{I_1 - 7I_3}\cos\phi_1 \\ & +3\sqrt{I_1I_3}(I_1 - 7I_3)\cos\phi_2, \end{aligned}$$

and assuming that $I_3 \in [0, I_1/7]$, $I_1 > 0$, we know that $13I_1^2 + 19I_3^2 - 23I_1I_3 > 0$ and moreover

$$13I_1^2 + 19I_3^2 - 23I_1I_3 > |3\sqrt{3}I_1^{3/2}\sqrt{I_1 - 7I_3}| + |3\sqrt{I_1I_3}(I_1 - 7I_3)|,$$

concluding that $\mathcal{H}_4(I,\phi_1,\phi_2)$ is positive for $I \in S \setminus \{0\}$ and any $\phi_1, \phi_2 \in \mathbb{T}$. Thus, the null solution of the equations of motion related to H in (36) is Lie stable. Since $d = 1$ no Diophantine condition is needed and we can apply the bounds obtained in Theorem 2.7 with $j = 4$ and $\nu = 0$. By virtue of Corollary 4 the bound on the confinement of $x(t)$ is of order ε

For the last example we consider a Hamiltonian which is in normal form up to terms of degree 6 and is given by

$$H = H_2 + \mathcal{H}_4 + \mathcal{H}_6 + \cdots \tag{37}$$

with

$$\begin{aligned} H_2 &= 2\sqrt{2}I_1 - 2I_2 + 4I_3 - 3\sqrt{2}I_4 + 4I_5, \\ \mathcal{H}_4 &= 3I_1^2 + 4I_5^2, \\ \mathcal{H}_6 &= 2I_3^3 + I_4^3 + 5I_5(I_3 + I_5)^2 - 2I_2^2 I_5 \sin(4\theta_2 + 2\theta_5). \end{aligned}$$

As in the preceding cases H is regular at any point since d'Alembert property holds. Analysing H_2 it is straightforward to deduce that M_ω is spanned by three vectors, specifically $k_1 = (0,2,0,0,1)$, $k_2 = (0,0,1,0,-1)$, $k_3 = (3,0,0,2,0)$. From the orthogonal space of the linear subspace of $\mathbb{R}^5$ spanned by k_1, k_2, k_3 we build the two formal first integrals, namely $F_1 = -I_2 + 2I_3 + 2I_5$ and $F_2 = -2I_1 + 3I_4$. Next the set S is obtained from F_k, yielding the three-dimensional subspace of $\mathbb{R}^5$ given by $\{(3I_4/2, 2(I_3+I_5), I_3, I_4, I_5) \mid I_3, I_4, I_5 \geq 0\}$.

Considering $\mathcal{H}^4 = H_2 + \mathcal{H}_4$ we realise that it becomes zero for $I \in S \setminus \{0\}$, in particular for $I^* = (0, 2I_3^*, I_3^*, 0, 0)$ with any $I_3^* > 0$, and besides $\mathcal{H}_4 \geq 0$. Thus we need to take into account the next non-null term and consider $\mathcal{H}^6 = H_2 + \mathcal{H}_4 + \mathcal{H}_6$. When $I \in S$ we arrive at

$$\mathcal{H}^6(I,\phi_1) = \tfrac{27}{4} I_4^2 + 4I_5^2 + 2I_3^3 + I_4^3 + I_5(I_3+I_5)^2\left(5 - 8\sin(2\phi_1)\right),$$

where $\phi_1 = 2\theta_2 + \theta_5$. To check whether $\mathcal{H}^6$ changes sign when I is in $S \setminus \{0\}$ and close to the origin, it is enough to prove it doing $I_4 = I_5 = 0$. We get $\mathcal{H}_6 = 2I_3^3 > 0$ for $I = (0, 2I_3, I_3, 0, 0)$ and $I_3 > 0$, which is a term in $S \setminus \{0\}$. Then, for $|I|$ small enough, $\mathcal{H}^6 \geq 0$ and $\mathcal{H}^6 = 0$ if and only if $I_j = 0$. As a consequence the origin of $\mathbb{R}^{10}$ is Lie stable for the Hamiltonian system associated to H in (37). Note however that Lie stability cannot be concluded applying other known results on Lie stability. From the identity $H_2 = 2F_1 - \sqrt{2}F_2$ the frequency vector is $(\sigma_1, \sigma_2) = (2, -\sqrt{2})$, which is Diophantine, and Theorem 2.7 applies with $j = 6$, see also Corollary 3. The time T is of the form $C\exp(K\epsilon^{-1/4})$ with an action confinement of order $\epsilon^{1/3}$, which is improved to be of order $\epsilon^{2/3}$, applying Corollary 4. As $\mathcal{H}^4$ vanishes only on the boundary of S, part (B) of Theorem 2.6 does not apply.

As said in the introduction there are many interesting cases where Lie stability is obtained from a given Hamiltonian $\mathcal{H}^j$ depends on one or several resonant angles. One example corresponds to the motion in the three-dimensional space of a satellite with respect to its centre of mass where the orbit followed by the centre of mass is circular and the satellite has unequal principal central moments of inertia. This problem can be written as an autonomous Hamiltonian system with three degrees of freedom and some of its equilibria are of elliptic type. The problem hinges basically on two external parameters and this leads to many different possibilities of getting Lie stability for the elliptic equilibria, either obtained from non-resonant or resonant normal-form terms. A deep analysis of this problem has been made in the monograph [5], see also [9] and there are several (formally) Lie stable situations where the stability is deduced from resonant normal forms of degrees 3, 4 or 5. In case of stability, exponential estimates are provided for these cases.

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