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Boltzmann and the Statistical Multifractals

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Abstract

We extend the Boltzmann's ideas that describe the evolution to the equilibrium of many body systems to the multifractal decomposition of the unitary interval $\mathbb{I}$, in terms of sets J_α conformed by points with the same pointwise dimension, and obtain the $D(\alpha)$ singularity spectrum.

Keywords: Boltzmann's formulation, statistical multifractals, pointwise dimension, Hausdorff dimension.

1 Introduction

The multifractal formalism introduced by Halsey *et al* [1] can be understood in a simple way by applying a similar reasoning that was used by Boltzmann for obtaining the thermodynamics of an ideal gas using statistical arguments instead of the microscopic description of a system conformed by 10^{23} particles. The central idea was to introduce the relation between the entropy and the probability associated with a macrostate [2,3]. In the second section recalls briefly the Boltzmann fundamental ideas for obtaining the statistical description of an ideal gas in thermodynamic equilibrium. In the third section, we obtain the Eggleston's theorem, which relates the Hausdorff dimension with the Shannon entropy; this theorem plays a similar role in fractals like that the relation between entropy and probability in the Boltzmann treatment. The theorem is showed using a multiplicative process to decompose the unitary interval in fractals $M(\vec{\varphi})$, conformed by points with the same frequency of digits $\vec{\varphi}$; evaluate the Hausdorff dimension of $M(\vec{\varphi})$ and obtain that it is related to the Shannon entropy. In the fourth section, we introduce a Bernoulli measure with a probability vector $\vec{\mathbf{p}}$; make the multifractal decomposition in terms of sets J_α, conformed by points with the same pointwise dimension α, and show that they are conformed by an infinite number of sets $M(\vec{\varphi})$. Therefore, each J_α has a multifractal structure, to determine the Hausdorff

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dimension $D(\alpha)$, we use a basic property of the Hausdorff Dimension which plays the rule of the Boltzmann principle of maximum entropy.
In the fifth section, the Boltzmann procedure is extended to determine the distribution $\vec{\mathbf{P}}(q)$, which maximize the Eggleston's relation, given a value of α, $D(\alpha)$ is determined using $\vec{\mathbf{P}}(q)$, and $D(\alpha)$ singularity spectrum is obtained.
In the sixth section, we introduce a family of Bernoulli q-measures with a probability vector $\vec{\mathbf{P}}(q)$. We evaluate the q-measure of the sets J_α, finding that for a particular value of q the measure is concentrated in the set J_{α^*}, as a consequence of the singular behaviour of the q-measure, we obtain that $\tau(q)$ and -$D(\alpha)$ are Legendre transform of each other. The explicit values $\alpha^* = \alpha(q)$ and the Hausdorff dimension $D(\alpha(q))$ are the functions obtained in the Boltzmann procedure to determine the $D(\alpha)$ singularity spectrum.

2 Fundamental Boltzmann ideas

Boltzmann analyzed a system conformed by N particles which interact only through elastic collision. For establishing a relation between the mechanics and thermodynamics, Boltzmann introduced a probabilistic description of this system dividing in $K \lll N$ cells the phase space of a particle, and described the state of the system by the occupation number of particles of these cells:

$$\{n_1(t), n_2(t), \ldots, n_K(t)\} = \{\vec{\mathbf{n}}(t)\} \tag{1}$$

The number of microstates corresponding with this distribution is given by:

$$W\{\vec{\mathbf{n}}(t)\} = \frac{N!}{\prod_{i=1}^{K} n_i(t)!} = \left[\prod_{i=1}^{K}\left(\frac{n_i(t)}{N}\right)^{n_i(t)}\right]^{-1} = \left[\prod_{i=1}^{K}(p_i(t))^{n_i(t)}\right]^{-1} \quad with \quad p_i(t) = \frac{n_i(t)}{N} \tag{2}$$

Boltzmann postulated that $W\{\vec{\mathbf{n}}(t)\}$ is related to entropy, which is a macroscopic quantity of the system:

$$S(\vec{\mathbf{n}}(t)) = k \ln W(\vec{\mathbf{n}}(t)) \tag{3}$$

Using (2), he obtained the relation between entropy and probability:

$$\frac{S(\vec{\mathbf{n}}(t))}{N} = -k\sum_{i=1}^{K} p_i(t) \ln p_i(t) \tag{4}$$

The time evolution of the probability distribution is governed by the Boltzmann's equation. When it is introduced in (4), it can be proved that the entropy increases until it obtains its maximum value for a stationary distribution, which is the *Maxwell-Boltzmann distribution*. However, it is possible to obtain this result without invocating the Boltzmann equation, using that in the equilibrium state the entropy of the system obtains its maximum value under the restrictions:

$$\sum_{i=1}^{K} p_i(t) = 1 \quad and \quad \frac{E}{N} = \sum_{i=1}^{K} p_i(t)\varepsilon_i \tag{5}$$

Then maximizing (4) with the lateral conditions (5), it is found that the stationary distribution is given by:

$$\tilde{p}_i = \frac{1}{Z}e^{-\beta\varepsilon_i}; \quad with \quad Z = \sum_{i=1}^{K} e^{-\beta\varepsilon_i} \tag{6}$$

The value of parameter $\beta = \frac{1}{kT}$ is determined by the thermodynamic information that the internal energy of the ideal gas is given by $E = \frac{3}{2}NkT$, and (6) reduces to the Maxwell-Boltzmann distribution.

3 The geometric multifractal decomposition of the unitary interval $\mathbb{I}$

In this section we discuss the multifractal decomposition of the unitary interval $\mathbb{I}$. Any real number $x \in \mathbb{I}$, is expressed in s-base as:

$$x = \sum_{n=1}^{\infty} \frac{z_n}{s^n}; \quad z_n = 0, 1, \ldots, s-1 \tag{7}$$

Let $n_i(x,K)$ denote the number of times the digit $i \in (0, 1, \ldots, s-1)$ occurs among the first K digits of x. Then, the frequency in which this digit appears in x is given by:

$$\lim_{K\to\infty} \frac{n_i(x,K)}{K} = \lim_{K\to\infty} f_i(x,K) = \varphi_i(x), \quad i = 0, 1, \ldots, s-1 \tag{8}$$

where $0 \leq \varphi_i \leq 1$, $\sum_{i=0}^{s-1} \varphi_i = 1$; thus, x has associated a frequency vector:

$$\vec{\varphi}(x) = (\varphi_0(x), \varphi_1(x), \ldots, \varphi_{s-1}(x)) \tag{9}$$

Let $M(\vec{\varphi})$ be *the set of points in* $\mathbb{I}$ *with the same frequency vector*. For obtaining the multifractal decomposition of $\mathbb{I}$, we separate the unitary interval in the different sets $M(\vec{\varphi})$, and using the Eggleston's theorem [4], we evaluate their Hausdorff Dimension. This can be done using a *multiplicative process* that consists of dividing $\mathbb{I}$ in s-cylinders of first order:

$$C_{z_1} = \left[\frac{z_1}{s}, \frac{z_1}{s} + \frac{1}{s}\right); \quad z_1 \in (0, 1, \ldots, s-1) \tag{10}$$

Then, we divide each C_{z_1} in s-cylinders of second order:

$$C_{z_1 z_2} = \left[\frac{z_1}{s} + \frac{z_2}{s^2}, \frac{z_1}{s} + \frac{z_2}{s^2} + \frac{1}{s^2}\right); \quad z_1, z_2 \in (0, 1, \ldots, s-1) \tag{11}$$

obtaining S^2 of 2-cylinders. Repeating K times this procedure on each cylinder, we obtain S^K cylinders:

$$C_{z_1 z_2 \ldots z_K} = \left[\sum_{n=1}^{K} \frac{z_n}{s^n}, \sum_{n=1}^{K} \frac{z_n}{s^n} + \frac{1}{s^n}\right); \quad z_1, z_2, \ldots, z_K \in (0, 1, \ldots, s-1) \tag{12}$$

Each K-cylinder is characterized by the sequence $\sigma_K = z_1 z_2 \ldots z_K$, we group them by the frequency vector

$$\vec{\mathbf{f}} = (f_0, f_1, \ldots, f_{s-1})$$

where f_r is the frequency that shows that digit $r = (0, 1, \ldots, s-1)$ occurs in σ_K. The number of K-cylinders with the same frequency vector $\vec{\mathbf{f}}$ is given by

$$W(\vec{\mathbf{f}}) = \frac{K!}{n_0(K)!n_1(K)!\ldots n_{s-1}(K)!} = \left[f_0^{f_0(K)}(K) f_1^{f_1(K)}(K) \ldots f_{s-1}^{f_{s-1}(K)}(K)\right]^{-K} \tag{13}$$

On the other hand, each K-cylinder has a diameter:

$$\Lambda(C_{z_1 z_2 \ldots z_K}) = \Lambda_K = s^{-K} \tag{14}$$

due to the fact that when $K \to \infty$, each K-cylinder goes to a point x of the unitary interval with a frequency vector given by (8). The Hausdorff dimension of $M(\vec{\varphi})$ is

$$\mathrm{Dim}_H M(\vec{\varphi}) = -\lim_{K\to\infty} \frac{\ln W(\vec{\mathbf{f}})}{\ln \Lambda_K} = -\frac{1}{\ln s} \sum_{j=0}^{s-1} \varphi_j \ln \varphi_j \tag{15}$$

This relation is the *Eggleston's theorem* [4]. In the Appendix A, we show how (15), can be found using the definition of Hausdorff measure applied for dyadic intervals [6]; we find the value D for which this measure is non-singular, and it corresponds with the Hausdorff dimension of the set.

On the other hand, the relation (15) establishes a non-trivial connection between Hausdorff dimension and the Shannon entropy, which is discussed in the Appendix B.

4 Statistical multifractal decomposition of the unitary interval $\mathbb{I}$

When a statistical measure is assigned to each point of $\mathbb{I}$, it is decomposed in subsets J_α conformed by points with the same pointwise dimension. A simple case is when we introduce the *Bernoulli measure* μ on the unitary interval with probability vector $\vec{\mathbf{p}} = (p_0, p_1, \ldots, p_{s-1})$; assigning to each digit j belongs to x a probability p_j, it introduces a singular measure that can be characterized by the pointwise dimension of μ at x [5]:

$$\mathrm{d}_\mu(x) = \lim_{r\to 0} \frac{\ln \mu(B(x,r))}{\ln r} \tag{16}$$

where $B(x,r)$ is a ball of radius r centered in x; this quantity can be expressed in terms of the K-cylinders as:

$$\mathrm{d}_\mu(x) = \lim_{K\to\infty} \frac{\ln \mu(C_{z_1 z_2 \ldots z_K})}{\ln \Lambda(C_{z_1 z_2 \ldots z_K})} \quad \text{when} \quad x = \lim_{K\to\infty} C_{z_1 z_2 \ldots z_K} \tag{17}$$

The μ measure of the K-cylinder is given by

$$\mu(C_{z_1 z_2 \ldots z_K}) = p_{z_1} p_{z_2} \ldots p_{z_K} = \left[p_0^{f_0(K)} p_1^{f_1(K)} \ldots p_{s-1}^{f_{s-1}(K)} \right]^K \tag{18}$$

The x pointwise dimension is obtained introducing (18) and (14) into (17):

$$\mathrm{d}_\mu(x) = \lim_{K\to\infty} -\frac{1}{\ln s} \sum_{j=0}^{s-1} f_j(K) \ln p_j = -\frac{1}{\ln s} \sum_{j=0}^{s-1} \varphi_j(x) \ln p_j \tag{19}$$

We note that all the points that belong to $M(\vec{\varphi})$ have the same pointwise dimension. However, there are *an infinite number of sets $M(\vec{\varphi})$ with the same value of the pointwise dimension*, because given a particular value of $\mathrm{d}_\mu(x) = \alpha$ and the normalization condition $\sum_{j=0}^{s-1} \varphi_j(x) = 1$, we cannot determine the s components of the frequency vector.

As each $M(\vec{\varphi})$ is a fractal with the Hausdorff dimension given by Eggleston's theorem, then J_α, *the set of points with $d_\mu(x) = \alpha$, is a multifractal*:

$$J_\alpha = \left\{ x \mid \mathrm{d}_\mu(x) = \alpha \right\} = \bigcup M(\vec{\varphi}) \quad \text{such that} \quad \vec{\varphi} \cdot \ln \vec{\mathbf{p}} = \alpha \tag{20}$$

where was defined: $\vec{\boldsymbol{a}} \cdot \ln \vec{\boldsymbol{b}} = \sum_{j=0}^{s-1} a_j \ln b_j$ as the Hausdorff dimension satisfies that [6]

$$\text{if} \quad M = \bigcup M_n \quad \text{then} \quad \mathrm{Dim}_H(M) = \sup \mathrm{Dim}_H M_n \tag{21}$$

Thus, the Hausdorff dimension of J_α is given by:

$$\mathrm{D}(\alpha) = \mathrm{Dim}_H(J_\alpha) = \sup \mathrm{Dim}_H M(\vec{\varphi}) \quad \text{with} \quad \vec{\varphi} \cdot \ln \vec{\mathbf{p}} = \alpha \tag{22}$$

The statistical multifractal decomposition of $\mathbb{I}$ consists of grouping the points x in subsets with the same value of the pointwise dimension, and each subset J_α is characterized by its Hausdorff dimension $D(\alpha)$.

5 Boltzmann scheme for Multifractals

We determine $D(\alpha)$ using the Eggleston's theorem and the extremal principle given by (22); this procedure is similar to the maximum entropy principle used by Boltzmann for obtaining the stationary distribution characterizing the equilibrium state. The Hausdorff dimension $D(\alpha)$ is determined by the frequency distribution $\vec{\varphi}^*$ that maximizes:

$$\mathrm{Dim}_H M(\vec{\varphi}) = -\frac{1}{\ln s}\sum_{r=0}^{s-1}\varphi_r \ln \varphi_r$$

with lateral conditions:

$$\alpha = -\frac{1}{\ln s}\sum_{j=0}^{s-1}\varphi_j \ln p_j; \qquad \sum_{j=0}^{s-1}\varphi_j = 1 \tag{23}$$

Following the usual maximizing procedure we find that:

$$\varphi_r^* = P_r(q) = \frac{p_r^q}{Z_q} \quad \text{where} \quad Z_q = \sum_{r=0}^{s-1} p_r^q \tag{24}$$

The q parameter is determined by the equation:

$$\alpha(q) = \sum_{r=0}^{s-1} P_r(q)\left[-\frac{\ln p_r}{\ln s}\right] = \vec{\mathbf{P}}(q)\cdot\left(-\frac{\ln \vec{\mathbf{p}}}{\ln s}\right) \tag{25}$$

The Hausdorff dimension of J_α is found using (24) in (15):

$$\mathrm{D}(q) = \mathrm{D}(\alpha(q)) = -\frac{1}{\ln s}\sum_{r=0}^{s-1} P_r(q)\ln P_r(q) \tag{26}$$

The dimension spectra for the statistical multifractal decomposition of the unitary interval is found when the q parameter is eliminated for (25) and (26). In thermodynamics, the entropy is one of the relevant functions, but there are several functions that contain the same thermodynamic information, they are the thermodynamic potentials, we show that in multifractals, a similar situation occurs. From (24) we have:

$$\ln P_r(q) = q\ln p_r(q) - \ln Z_q \tag{27}$$

Using this result in (26), we have:

$$\mathrm{D}(q) = q\alpha(q) - \tau(q) \tag{28}$$

where:

$$\tau(q) = -\frac{1}{\ln s}\ln Z_q \tag{29}$$

We proceed to show that $D(\alpha)$ and $\tau(q)$ conform a Legendre transform pair. The derivative of (29) is given by:

$$\frac{d\tau(q)}{dq} = -\frac{1}{\ln s}\frac{1}{Z_q}\sum_{r=0}^{s-1} p_r^q \ln p_r = \sum_{r=0}^{s-1} P_r(q)\left(-\frac{\ln p_r}{\ln s}\right) = \alpha(q) \tag{30}$$

Considering that $q = q(\alpha)$, the derivative of (28) with respect to α is:

$$\frac{dD}{d\alpha} = q + \alpha\frac{dq}{d\alpha} - \frac{d\tau}{dq}\frac{dq}{d\alpha} = q \tag{31}$$

This result infers that $dD = qd\alpha$, therefore $d(D - q\alpha) = -d\tau = -\alpha dq$, which implies (30) and hence $\tau(q)$ and -$D(\alpha)$ are the Legendre transform of each other.

6 Statistical q-measures of J_α

In the statistical multifractal decomposition of the unitary interval, we focused our attention to the Hausdorff dimension of the sets J_α, which is a geometrical property of these sets. However, they also have statistical properties, because they are the support of the q-measures, in such way that given a q value, this measure is supported by only one of the J_α sets. Given a probability vector $\vec{p}$, a set of probability vectors $\vec{\mathbf{P}}(q)$ can be constructed, these vectors have the ability to scan the structure of the multifractal decomposition [8]. We obtained the escort probability vector $\vec{\mathbf{P}}(q)$ in the Boltzmann scheme for multifractals, they are equivalent to the Maxwell-Boltzmann distribution in statistical physics, where the q value plays the rule of the inverse of the temperature. The statistical q-measures are Bernoulli measures in the unit interval generated by $\vec{\mathbf{P}}(q)$, which is defined by (24), i.e.

$$\vec{\mathbf{P}}(q) = \Big(\mathbf{P}_0(q), \mathbf{P}_1(q), \dots, \mathbf{P}_{s-1}(q)\Big); \quad \text{with} \quad \mathbf{P}_r(q) = \frac{p_r^q}{Z_q} \tag{32}$$

where q is any real, this vector is called the escort distribution of $\vec{p} = (p_0, p_1, \dots, p_{s-1})$ of q-order [8].

The q-measure μ_q assigns to each digit j belongs to x a probability $\mathbf{P}_j(q)$, then a K-cylinder has the q-measure:

$$\mu_q\Big(C_{z_1 z_2 \dots z_K}\Big) = \mathbf{P}_{z_1}(q)\mathbf{P}_{z_2}(q)\dots\mathbf{P}_{z_K}(q) = \frac{\mu^q\Big(C_{z_1 z_2 \dots z_K}\Big)}{Z_q^K} \tag{33}$$

We define a K_α-cylinder by the following property:

$$\lim_{K\to\infty} \frac{\ln\mu\Big(C^\alpha_{z_1 z_2 \dots z_K}\Big)}{\ln\Lambda\Big(C^\alpha_{z_1 z_2 \dots z_K}\Big)} = -\frac{1}{\ln s} \lim_{K\to\infty} \frac{\ln\mu\Big(C^\alpha_{z_1 z_2 \dots z_K}\Big)}{K} = \alpha \tag{34}$$

The set of all K_α-cylinders contains all the points of $\mathbb{I}$ with $d_\mu(x) = \alpha$, therefore this set conforms the cover $C_K(J_\alpha)$ of the set J_α. For large values of K we have that:

$$\mu\Big(C^\alpha_{z_1 z_2 \dots z_K}\Big) \approx \Big[\Lambda\Big(C^\alpha_{z_1 z_2 \dots z_K}\Big)\Big]^\alpha = s^{-\alpha K} \tag{35}$$

Using (35) and (29) in (33), we find that for large K, the q-measure of a K_α-cylinder is given by:

$$\mu_q\Big(C^\alpha_{z_1 z_2 \dots z_K}\Big) \approx s^{-K(q\alpha - \tau(q))} \tag{36}$$

On the other hand, for large K, the number of K_α-cylinders goes as:

$$N\Big(C^\alpha_{z_1 z_2 \dots z_K}\Big) \approx \Big[\Lambda_{z_1 z_2 \dots z_K}\Big]^{-D(\alpha)} = s^{KD(\alpha)} \tag{37}$$

Then, the q-measure of the cover $C_K(J_\alpha)$ for large K is given by

$$\mu_q\Big[C_K(J_\alpha)\Big] \approx \Big[s^{-K}\Big]^{q\alpha - \tau(q) - D(\alpha)} \tag{38}$$

As $0 \le \mu_q\Big[C_K(J_\alpha)\Big] \le 1$ and $s^{-K} < 1$, for all values of α is satisfied the inequality:

$$\tau(q) \le q\alpha - D(\alpha) \tag{39}$$

The q-measure of the set J_α is given by:

$$\mu_q(J_\alpha) = \lim_{K\to\infty} \Big[s^{-K}\Big]^{q\alpha - \tau(q) - D(\alpha)} = \delta(\alpha - \alpha^*) \tag{40}$$

Thus, the q-measure of J_α is null for all values of $\alpha \neq \alpha^*$ and is one for $\alpha = \alpha^*$, which is defined by the relation:

$$\tau(q) = q\alpha^* - D(\alpha^*) \tag{41}$$

Then, the set J_{α^*} is the support of the q-measure. From (39) and (41), we conclude that:

$$\tau(q) = \inf\Big[q\alpha - D(\alpha)\Big] \tag{42}$$

where the infimum is taken with respect to α; thus, (42) defines $\tau(q)$ as the Legendre transform of $-D(\alpha)$. We note that this result is obtained from (40), which is the generalization of the random weighted curdling proposed by Mandelbrot [9] and [10].

The relations (41) and (42) imply that α^*, satisfies the following conditions:

$$\frac{\partial}{\partial\alpha}\Big[q\alpha - D(\alpha)\Big]\Bigg|_{\alpha^*} = q - \frac{dD(\alpha)}{d\alpha}\Bigg|_{\alpha^*} = 0 \Rightarrow \frac{dD(\alpha)}{d\alpha}\Bigg|_{\alpha^*} = q \tag{43}$$

$$\frac{d^2D(\alpha)}{d\alpha^2}\Bigg|_{\alpha^*} < 0 \tag{44}$$

Taking the derivative of (41) and using (43) and (29), we obtain that

$$\alpha^* = \frac{d\tau(q)}{dq} = -\frac{1}{\ln s}\frac{d\ln Z_q}{dq} = \alpha(q) \tag{45}$$

Therefore, the q-measure is concentrated in the set $J_{\alpha(q)}$ with $\alpha(q)$, which is given by (45), and it can be rewritten as the following average on $\vec{\mathbf{P}}(q)$:

$$\alpha(q) = \sum_{i=0}^{s-1}\mathbf{P}_i(q)\Big(-\frac{\ln p_i}{\ln s}\Big) = \Big\langle -\frac{\ln p_i}{\ln s}\Big\rangle_q = -\frac{\vec{\mathbf{P}}(q)\cdot\ln\vec{\mathbf{p}}}{\ln s} \tag{46}$$

The Hausdorff dimension of $J_{\alpha(q)}$ is obtained by using (46) into (41):

$$D(\alpha(q)) = q\alpha(q) - \tau(q) = -\frac{1}{\ln s}\vec{\mathbf{P}}(q)\cdot\ln\vec{\mathbf{P}}(q) \tag{47}$$

7 Conclusions

When a Bernoulli statistical measure characterized by a probability vector $\vec{\mathbf{p}}$ is introduced in a fractal, it is decomposed into sets J_α, which are multifractal. The determination of their Hausdorff dimension $D(\alpha)$ requires to use an extremal property of the Hausdoff dimension, similar to the maximum entropy principle. $D(\alpha)$ is determined in terms of a probability distribution $\vec{\mathbf{P}}(q)$, we find that each set J_α is an statistical attractor set where the q-measure defined in terms of $\vec{\mathbf{P}}(q)$ is concentrated.
As a consequence of the singular behaviour of the q-measure on the sets J_α, given by (40), we obtain the following:

(1) $\tau(q)$ and $-D(\alpha)$ are Legendre transform of each other and,

(2) the information on the set J_α where the q-measure is supported, and its Hausdorff dimension, are given by (25) and (26), respectively.

An important case of (40) is when $q = 1$. The 1-measure is generated by the probability vector $\vec{\mathbf{p}}$, the statistical attractor or the curdling set is conformed by the points with their pointwise dimension is identical with the Hausdorff dimension of the set, and it is given by Shannon entropy of $\vec{\mathbf{p}}$, i.e.

$$\alpha(q=1) = D(q=1) = -\frac{1}{\ln s}\sum_{j=0}^{s-1} p_j \ln p_j = -\frac{\vec{\mathbf{p}}\cdot\ln\vec{\mathbf{p}}}{\ln s} \tag{48}$$

Then, $D(q=1)$ is the Hausdorff dimension of the measure theoretical support of $\vec{\mathbf{p}}$, which was found by Billingsley [6] in his work about the Hausdorff dimension in probability theory. Chhabra and Jensen [11] applied this result to $\vec{\mathbf{P}}(q)$ and found (47), after using heuristic arguments introduces (46), and with these expressions they found an alternative method for obtaining the singularity spectrum. On the other hand, Mandelbrot [10] introduced the curdling set for explaining the energy dissipation in fully developed turbulence using a multiplicative cascade process and identified this set with the Besicovitch fractal [9] extended the Mandelbrot suggestions, Feder [7] obtained and showed that (48) characterizes the set where the 1-measure is concentrated. The result (40) can be showed for a singular measure, and obtain an unified description of the multifractal decomposition can be obtained, which relates the Halsey *et al* [2] and Chhabra and Jensen [11] methods for obtaining the spectral singularity (see sections 5 to 7 of reference [12]).

Appendix A

In this work, was used the alternative definition of Hausdorff dimension given by Billingsley [13] was used where the covering to sets M on the unitary interval is conformed by s-adic intervals:

$$v_i = \left[\frac{j}{s^K}, \frac{j+1}{s^K}\right), K = 1,2,\ldots, j = 0,1,\ldots,s-1 \tag{A-1}$$

instead of arbitrary intervals s_K of length $|s_K|$, Billingsley considers the measure of M in the unitary interval as:

$$\Lambda_\alpha(M,\rho) = \inf\sum_K |C_K|^\alpha \tag{A-2}$$

where $|C_K|$ denotes the length of the K-cylinder, the infimum extends only over coverings of M by cylinders of length less than ρ, then $\Lambda(M,\rho)$ differs from the Hausdorff measure $H_\alpha(M,\rho)$ but not in any way that is significant for the computation of dimensions. Defining $\Lambda_\alpha(M) = \lim_{\rho\to 0}\Lambda_\alpha(M,\rho)$ is possible redefine $\dim_H M$ as that D such that $\Lambda_\alpha(M) = \infty$ if $\alpha < D$ and $\Lambda_\alpha(M) = 0$ if $\alpha > D$. This definition is useful for problems involving s-dyadic expansions.

We use the Billingsley definition of the Hausdorff dimension to obtain the Eggleston's theorem.
Let $M(f_0, f_1, \ldots, f_{s-1})$ be the set of numbers in s-base, that belongs to the unitary interval and that in their first K digits have the same frequency $f_i = \frac{N_i}{K}$ of the digits $i = 0,1,\ldots,s-1$. This set covers $W(\vec{\mathbf{f}})$ cylinders with length s^{-K} which is given by (13), then

$$\Lambda_\alpha(M,\rho = s^{-K}) = W(\vec{\mathbf{f}})s^{-K\alpha} = \left[f_0^{f_0(K)}(K)f_1^{f_1(K)}(K)\ldots f_{s-1}^{f_{s-1}(K)}(K)s^\alpha\right]^{-K} \tag{A-3}$$

Taking the limit when $K\to\infty$ we obtain the function $\Lambda_\alpha(M(\vec{\varphi}))$:

$$\Lambda_\alpha(M(\vec{\varphi})) = \lim_{K\to\infty}\left[\varphi_0^{\varphi_0}\varphi_1^{\varphi_1}\ldots\varphi_{s-1}^{\varphi_{s-1}}s^\alpha\right]^{-K} \tag{A-4}$$

where

$$\varphi_i = \lim_{K\to\infty}\frac{N_i(K)}{K} = \lim_{K\to\infty} f_i(K) \tag{A-5}$$

This function is different for zero or infinite, only for the value $\alpha = D$, which satisfies:

$$\varphi_0^{\varphi_0}\varphi_1^{\varphi_1}\dots\varphi_{s-1}^{\varphi_{s-1}}s^D = 1 \tag{A-6}$$

Therefore

$$\mathrm{Dim}_H M(\vec{\varphi}) = -\frac{1}{\ln s}\sum_{j=0}^{s-1}\varphi_j \ln\varphi_j \tag{A-7}$$

This result is the Eggleston's theorem.

Appendix B

The Eggleston's theorem was originally shown and used in number theory, for obtaining a measure of how to compare different non-normal set numbers; which have a null value of Lebesgue measure. The solution proposed by Eggleston was used to evaluate the Hausdorff dimension of these sets. The original version of Eggleston's theorem [4] is the following:

Theorem. For any real number x $(0 \le x \le 1)$ is expressed as a decimal in the scale s (i.e. involving digits $0, 1, 2, \dots, s-1$), let $N_i(K)$ denote the number of times the digit i occurs amongst the first K digits of this decimal.

The set $M(\varphi_0, \varphi_1, \dots, \varphi_{s-1})$ of these x $(0 \le x \le 1)$ for which

$$\lim_{K\to\infty}\frac{N_i(K)}{K} = \varphi_i \tag{B-1}$$

where $0 \le \varphi_i \le 1$, $\sum_{i=0}^{s-1}\varphi_i = 1$ has a fractional dimension α, given by

$$s^{-\alpha} = \prod_{i=0}^{s-1}\varphi_i^{\varphi_i} \tag{B-2}$$

Eggleston used this theorem to characterize the "length" of non-normal sets of numbers. However, when it is rewritten in the form:

$$\mathrm{Dim}_H M(\varphi_0, \varphi_1, \dots, \varphi_{s-1}) = -\frac{1}{\ln s}\sum_{j=0}^{s-1}\varphi_j \ln\varphi_j \tag{B-3}$$

This result suggests a connection with probability theory, because the r.h.s recalls the entropy of a stochastic process, but here $\{\varphi_i\}$ is a frequency vector instead of a probability vector.

Billingsley [13] was the first to show that the Eggleston's theorem can be interpreted as a relation between dimension and entropy for the stochastic process; after he showed an important theorem [14] where he found the relationship between the Hausdorff dimensions taking with the statistical measures μ and ν of a special set M, $\mathrm{Dim}_\mu M$ and $\mathrm{Dim}_\nu M$. He selected one of them as Λ Lebesgue measure, and like other as a special stochastic measure. He generalized the Eggleston's theorem, in such a way that he showed the relation between Hausdorff dimension and entropy.

In the next sections, we present the original arguments written by Billingsley in reference [14] for obtaining a generalization of Eggleston's theorem, and how considering a particular case, he found the Eggleston's theorem. A more detailed discussion can be found in [6] and [13].

Generalization of the Hausdorff dimension to statistical measure

Let $\{x_1, x_2, \dots\}$ be a stochastic process, with a finite state space σ, defined on a probabilistic measure space (Ω, B, μ), a Hausdorff dimension with μ-measure $\mathrm{Dim}_\mu M$ is defined for each set $M \subset \Omega$ in the following way:

A cylinder of rank K is defined to be a set of the form $\{\omega \mid x_k(\omega) = a_k, k = 1,2,\ldots,K\}$ where $a_k \in \sigma$ If $M \subset \Omega$ and $\rho > 0$, a $\mu - \rho$ covering of M is a finite collection $\{\mathrm{v}_i\}$ of cylinders such that $M \subset \bigcup \mathrm{v}_i$ and $\mu(\mathrm{v}_i) < \rho$ for each i. If $\rho, \alpha > 0$, put $L_\mu(M,\alpha,\rho) = \inf \sum_i \mu(\mathrm{v}_i)^\alpha$, where the infimum extends over all $\mu - \rho$ coverings $\{\mathrm{v}_i\}$ of M, and let $L_\mu(M,\alpha) = \lim_{\rho \to 0} L_\mu(M,\alpha,\rho)$. If $L_\mu(M,\alpha) < \infty$, then $L_\mu(M,\alpha+\varepsilon) = 0$ for all $\varepsilon > 0$; hence, we can define

$$\mathrm{Dim}_\mu M = \sup\{\alpha : L_\mu(M,\alpha) = \infty\} = \inf\{\alpha : L_\mu(M,\alpha) = 0\} \tag{B-4}$$

It was shown in [13] that if Ω is the unit interval $(0,1]$, if μ is a Lebesgue measure, and if $\sum_{n=1}^{\infty} x_n(\omega)s^{-n}$ is for each ω, the nonterminating base s expansion of ω, then this definition reduces to the classical one due to Hausdorff. The dimension of M depends both on the measure μ and the process $\{x_n\}$. Note that if M is in the Borel field, generating by $\{x_n\}$ then $L_\mu(M,\alpha = 1) \geq \mu(M)$, so that $\mu(M) > 0$ implies $\mathrm{Dim}\, M = 1$ [13].

On an especial Billinsley's theorem

In reference [14] Billingsley investigates how $\mathrm{Dim}_\mu M$ varies as μ varies. For $\omega \in \Omega$ and $K = 1,2,\ldots$, put

$$C_K(\omega) = \left\{\omega' \mid x_i(\omega') = x_i(\omega), i = 1,2,\ldots,K\right\} \tag{B-5}$$

In other words, $C_K(\omega)$ is that cylinder of range K which contains ω. In section 2 of the reference [14], Billingsley proved that if μ and ν are probability measures on B and if

$$M \subset \left\{\omega \;\middle|\; \lim_{K\to\infty} \frac{\ln \nu\left(C_K(\omega)\right)}{\ln \mu\left(C_K(\omega)\right)} = \delta\right\} \tag{B-6}$$

then

$$\mathrm{Dim}_\mu M = \delta\, \mathrm{Dim}_\nu M \tag{B-7}$$

The essential Billingsley's idea is to compute $\mathrm{Dim}_\mu M$ for certain sets M by constructing a measure ν such that (B-6) holds and such that $\mathrm{Dim}_\nu M = 1$. It then follows from (B-7) that $\mathrm{Dim}_\mu M = \delta$. Applying this result Billingsley obtained the Eggleston's theorem.

Heuristic proof of Billingsley's theorem

Billingsley gave a heuristic proof of the fact that (B-6) implies (B-7). He assumes that for each $\omega \in M$, not only $\frac{\ln \nu(C_K(\omega))}{\ln \mu(C_K(\omega))}$ approaches δ, but is also equal to δ for all K. If $\{\mathrm{v}_i\}$ is any covering of M, each element of which intersects M, then any element v of the covering has the form $\mathrm{v} = C_K(\omega)$ with $\omega \in M$, so that $\nu(\mathrm{v}) = \mu(\mathrm{v})^\delta$. Then

$$\sum \nu(\mathrm{v}_i)^\alpha = \sum \mu(\mathrm{v}_i)^{\alpha\delta}$$

for any covering $\{\mathrm{v}_i\}$ of M. It follows that $L_\nu(M,\alpha) = L_\mu(M,\alpha\delta)$; using (B-4), we obtained (B-7):

$$\mathrm{Dim}_\mu M = \delta\, \mathrm{Dim}_\nu M$$

In the third section of [14], Billlingsley applies his theorem for obtaining the generalization of Eggleston's theorem that is used in the present work.

The generalization of Eggleston' theorem

Suppose that the state space σ is finite, say $\sigma = \{0,1,\ldots,s-1\}$. Suppose further that under μ the process $\{x_n\}$ is independent of

$$\mu\{\omega : x_n(\omega) = i\} = p_i > 0, \quad i = 0,1,\ldots,s-1 \tag{B-8}$$

Suppose finally that if $\vec{\varphi} = (\varphi_0, \varphi_1, \ldots, \varphi_{s-1})$ is any set of nonnegative numbers, which sums to 1, then there is a measure $\nu = \nu_\varphi$ such that $\{x_n\}$ is independent under ν and $\nu\{\omega : x_n(\omega) = i\} = \varphi_i$. This last assumption holds, if Ω is the unit interval and $\sum_{n=1}^{\infty} \frac{x_n(\omega)}{s^n}$ is the base s expansion of ω.

Let A be the set of vectors $\vec{\varphi} = (\varphi_0, \varphi_1, \ldots, \varphi_{s-1})$ of s-space such that $\varphi_i \geq 0$ and $\sum_{i=0}^{s-1} \varphi_i = 1$. For each $\omega \in \Omega$, $n \geq 1$, and $i = 0, \ldots, s-1$, let $f_i(\omega, K) = \frac{N_i(\omega,K)}{K}$ be where $N_i(\omega, K)$ denotes the number of times the digit i occurs amongst the first K digits of ω, and let be $\vec{\mathbf{f}}(\omega, K) = (f_0(\omega, K), f_1(\omega, K), \ldots, f_{s-1}(\omega, K))$ be a vector that belongs to A.
We are interested in evaluating $\mathrm{Dim}_\mu M(\vec{\varphi})$ where $M(\vec{\varphi}) = \{\omega \mid \lim_{K\to\infty} \vec{\mathbf{f}}(\omega, K) = \vec{\varphi}\}$. To do this, let ν be that measure on B under which $\{x_n\}$ is independent of $\nu\{\omega : x_n(\omega) = i\} = \varphi_i$. Since

$$\begin{aligned} \mu(C_K(\omega)) &= p_0^{N_0(\omega,K)} p_1^{N_1(\omega,K)} \ldots p_{s-1}^{N_{s-1}(\omega,K)} \\ \nu(C_K(\omega)) &= \varphi_0^{N_0(\omega,K)} \varphi_1^{N_1(\omega,K)} \ldots \varphi_{s-1}^{N_{s-1}(\omega,K)} \end{aligned} \tag{B-9}$$

Thus, we have:

$$\frac{\ln \nu(C_K(\omega))}{\ln \mu(C_K(\omega))} = \frac{\sum_{i=0}^{s-1} f_i(\omega, K) \ln \varphi_i}{\sum_{i=0}^{s-1} f_i(\omega, K) \ln p_i}$$

and therefore,

$$\lim_{K\to\infty} \frac{\ln \nu(C_K(\omega))}{\ln \mu(C_K(\omega))} = \frac{\sum_{i=0}^{s-1} \varphi_i \ln \varphi_i}{\sum_{i=0}^{s-1} \varphi_i \ln p_i} = H(\vec{\varphi}, \vec{\mathbf{p}}) \tag{B-10}$$

Using (B-7), we have:

$$\mathrm{Dim}_\mu M(\vec{\varphi}) = H(\vec{\varphi}, \vec{\mathbf{p}})\, \mathrm{Dim}_\nu M(\vec{\varphi}) \tag{B-11}$$

However, since $\nu(M(\vec{\varphi})) = 1$ by the strong law of large numbers, then $\mathrm{Dim}_\nu M(\vec{\varphi}) = 1$, and therefore,

$$\mathrm{Dim}_\mu M(\vec{\varphi}) = H(\vec{\varphi}, \vec{\mathbf{p}}) \tag{B-12}$$

The Eggleston's theorem
We consider the special case when under μ the process $\{x_n\}$ is independent with $\mu\{\omega : x_n(\omega) = i\} = \frac{1}{s}$, so the function $H(\vec{\varphi}, \vec{\mathbf{p}})$ takes the form:

$$H(\vec{\varphi}, \vec{\mathbf{p}}) = -\frac{1}{\ln s} \sum_{i=0}^{s-1} \varphi_i \ln \varphi_i \tag{B-13}$$

which implies that

$$\mathrm{Dim}_\mu M(\vec{\varphi}) = -\frac{1}{\ln s} \sum_{i=0}^{s-1} \varphi_i \ln \varphi_i \tag{B-14}$$

as $\mu(C_K(\omega)) = \left(\frac{1}{s}\right)^K$, the μ measure of the K-cylinder is identical with its Lebesgue measure $\Lambda(C_K)$; therefore $\mathrm{Dim}_\mu M(\vec{\varphi}) = \mathrm{Dim}_H M(\vec{\varphi})$. Thus, in the r.h.s. of (B-14) we have the entropy of the stochastic process ν_φ, and it is precisely the Eggleston's theorem used in (15).

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