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A fuzzy methodology for approaching fuzzy sets of the real line by fuzzy numbers

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Abstract

In this paper we introduce a novel methodology to face the problem of finding, for every fuzzy set of the real line, a fuzzy number which can be considered as an approximation of the first one in some reasonable sense. This methodology depends on a wide variety of initial parameters that each researcher may set depending on his/her own interests. The main objective of this new methodology is to ensure that many of the techniques that are currently available for fuzzy numbers can also be extended to the setting of fuzzy sets of the real line which are, in many ways, much more enriching. To do this, we carry out a study of the families of nested sets that can determine fuzzy numbers through their level sets. Next, we describe some of the main properties that this approximation methodology verifies and we show some examples to illustrate how the initial parameters influence the result of the approximation.

Keywords: Fuzzy set approximation; Fuzzy number; Level set; Nonlinear operator; Monotone operator

1. Introduction

Fuzzy sets have been successfully employed in the representation and interpretation of fuzzy information. In such contexts, the classical techniques, based on absolutely precise real numbers, lack a clear meaning and they must be replaced by procedures that take into account the intrinsically imprecise nature of the data and experimental developments. Nowadays many methodologies take into account fuzzy numbers, a remarkable subfamily of the category of all fuzzy sets of the real line under certain regularity conditions. However, fuzzy sets of the real line are much more general than fuzzy numbers and then can model information in a more varied way.

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A *fuzzy set* is a pair (X, μ_X) where X is a non-empty set and $\mu_X : X \to [0, 1]$ is a function (called the *membership function* of the fuzzy set). In the last fifty five years, several epistemic interpretations have been described about the significance of the function μ_X. Three of the most important semantics interpret it as degree of *uncertainty* (proposed by Zadeh as a key tool in *possibility theory* [42] and *approximate reasoning* [21,43]), as a degree of *similarity* or as a degree of *preference* (see [14,16]). Each of the previous view-points entails certain peculiarities depending on the context in which they use fuzzy sets. However, what is common to all of them is that fuzzy set theory has demonstrated to be a very consistent theory which have led to significant improvements in many fields of study: decision analysis [8,10,20,26,33,40], ranking of possible alternatives [1,3,7,11,22,27,32,38,39], regression theory [9,18,29,31], image [25,19,28,6], classification [23,34,35], approximation [37], Medicine [36], algebraic interrelationships [30], etc.

On the other hand, fuzzy numbers, especially trapezoidal or triangular ones, have proven to be very popular (and useful) between researchers and practitioners. Therefore they have been implemented extensively in many fuzzy methods due to their simplicity and computational efficiency. A *fuzzy number of the real line* is a fuzzy set $A : \mathbb{R} \to [0, 1]$ satisfying certain conditions: normality, fuzzy convexity and upper semicontinuity at each point (see Definition 1). There is a fuzzy arithmetic that, involving fuzzy numbers, generalize the usual arithmetic with real numbers (see [12,13,15,17]). In general, the set $\mathrm{FN}(\mathbb{R})$ of all fuzzy numbers of the real line enjoys better properties than the family $\mathrm{FS}(\mathbb{R})$ of all fuzzy sets on $\mathbb{R}$. Indeed, the set $\mathrm{FS}(\mathbb{R})$ is quite complicated so that most of techniques that can be developed with (real or) fuzzy numbers cannot be extended to the set $\mathrm{FS}(\mathbb{R})$. For instance, *ranking indices* [3,4,32] (that is, deffuzifications $\Lambda \to \mathbb{R}$ from a subset $\Lambda \subseteq \mathrm{FN}(\mathbb{R})$ into $\mathbb{R}$) are usually based on geometric arguments (centroid, center of gravity, p-sign distance, magnitude, etc.) or analytic tools (integration, differentiation, maximization, etc.) that are not available in the whole family $\mathrm{FS}(\mathbb{R})$. Other remarkable way to ranking fuzzy numbers is based on genuine fuzzy binary relations, that is, procedures that require prior knowledge of the two fuzzy numbers to be compared (see [32]). In this line, some methodologies use *degrees of dominance* of one fuzzy number over another like *possibility* or *necessity of dominance* (see chapter 10 in [15]). These procedures generate ranking processes from a pairwise comparison index that can be interpreted as how much one fuzzy number is preferable versus another one, which can help us to make a decision in the fuzzy setting.

Having in mind this open problem and its possible applications in several contexts of fuzzy setting, in this paper we introduce and study the main properties of a wide parametric family of operators

$$\Phi_{f,g,T_1,T_2} : \mathrm{FS}([0,1]) \to \mathrm{FN}([0,1])$$

that associate a unique fuzzy number to each fuzzy set on the interval [0, 1] (we will justify that the family FS([0, 1]) is rich enough to reduce the problem to this particular interval). The main advantages of the previous family of operators are the following ones.

- First of all, these operators are introduced in order to provide a novel methodology to translate to the very general family FS([0, 1]) some reasonable properties of fuzzy numbers. Taking into account that there are many procedures that can be implemented in the setting of fuzzy numbers, then the previous operators can be useful to extend such procedures to the setting of fuzzy sets of the real line (especially when they are very similar to fuzzy numbers).
- They can be useful to extend fuzzy techniques that, for the moment, are currently applied to the restricted case in which input or output data must be fuzzy numbers to a general framework in which the involved data are arbitrary fuzzy sets of the real line.
- This family provides several ways to approximate a fuzzy set of the real line by a fuzzy number. Hence, in many cases, it can be useful in approximate reasoning.
- This family depends on a wide range of functions that can be used as parameters to define a particular approximating operator. Therefore, each researcher can choose the involved parameters in order to obtain the fuzzy number that best fits to the original fuzzy set of the real line according to his/her point of view or to his/her own interests. We include an illustrative example about how such initial functions directly affect to the obtained results.

Finally, we must highlight that, although fuzzy sets can be interpreted from several semantics, throughout this manuscript we are not interested on using any of such view-points. The contents of this paper are rather algebraic. Then, we advise that we will treat fuzzy sets from an algebraic point of view assuming that a fuzzy set is only a function $A : X \to [0, 1]$ (that is, we identify the fuzzy set to its membership function). It must be the researcher who could decide how he/she can take advantage of the following contents by employing the semantic he/she is concretely interested in.

Before introducing this family of approximation operator, in Section 3, we carry out a detailed study of the behavior of the family of level sets of each fuzzy number, which will then allow us in Section 4 to define the unique fuzzy number associated to each fuzzy set of the real line by means of its corresponding level sets. Finally, in Section 5 we study the main properties of this family of approximation operators and we show, among other properties, that, as expected, normal fuzzy numbers are fixed points of certain operators of this family. We start our study with some preliminaries and basic facts.

2. Preliminaries

We include here the necessary preliminaries to understand the contents of the next sections.

2.1. Background on real functions of one real variable

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of all positive integers and let $\mathbb{R}$ stand for the family of all real numbers. A real *interval* is a set $I \subseteq \mathbb{R}$ such that $(1-\lambda)\,t + \lambda s \in I$ for all $t, s \in I$ and all $\lambda \in [0, 1]$. This definition covers the empty set, so the intervals we will consider hereinafter could be empty unless otherwise stated. Given $t, s \in \mathbb{R}$ such that $t \leq s$, there are four classes of (bounded) intervals whose *extremes* are t and s: *closed* $[t, s]$, *open* (t, s), $[t, s)$ and $(t, s]$. Last three examples are empty when $t = s$. Taking into account the great importance of the real, closed, bounded interval $[0, 1]$, for convenience, henceforth, we will denote it by $\mathbb{I}$.

From now on, we will only consider the Euclidean topology on $\mathbb{R}$ associated to the Euclidean metric $d\,(t, s) = |t - s|$ for all $t, s \in \mathbb{R}$, which determines the Euclidean measure μ such that $\mu\,([t, s]) = s - t$ for all $t, s \in \mathbb{R}$ satisfying $t \leq s$. Any subset $D \subseteq \mathbb{R}$ will be endowed with the *induced* topology. We denote by $\overline{D}$ the closure of a subset $D \subseteq \mathbb{R}$ in the Euclidean topology of $\mathbb{R}$.

Let $f : D \to \mathbb{R}$ be a function defined on a non-empty subset D of $\mathbb{R}$. Given a subset $H \subseteq D$, we say that f is *increasing on* H if $f\,(t_1) \leq f\,(t_2)$ for each $t_1, t_2 \in H$ such that $t_1 \leq t_2$ (we remark that some authors prefer the terminology *"non-decreasing"* instead of *"increasing"*); f is *strictly increasing on* H if $f\,(t_1) < f\,(t_2)$ for each $t_1, t_2 \in H$ such that $t_1 < t_2$; and f is *constant on* H if there is $c \in \mathbb{R}$ such that $f\,(t) = c$ for all $t \in H$. Similarly we can consider *decreasing* or *strictly decreasing* functions. A function is *monotone* if it is increasing or decreasing. Constant functions are increasing and decreasing at the same time. A function $f : D \to \mathbb{R}$ is *affine on an interval* $I \subseteq D$ if there are $m, n \in \mathbb{R}$ such that $f\,(t) = mt + n$ for all $t \in I$. Each affine function whose domain is an interval is continuous and monotone. If $f : D \to \mathbb{R}$ is bounded from above, we denote by $\sup\,(f) = \sup\,(\{\,f\,(t) : t \in D\,\})$ to its supremum.

2.2. Background on fuzzy sets and fuzzy numbers

In general, a *fuzzy set* [41] on a non-empty set X is a mapping $A : X \to \mathbb{I}$. A fuzzy set A is *normal* if there is $x_0 \in X$ such that $A\,(x_0) = 1$. From now on, given a fuzzy set $A : X \to \mathbb{I}$, as it is a bounded from above function, we denote by α_A the supremum of A in X, that is, $\alpha_A = \sup\,(A) = \sup\,(\{\,A\,(x) : x \in X\,\})$. We will say that A has an *absolute maximum* at $x_0 \in X$ if $A(x_0) \geq A\,(x)$ for all $x \in X$. In such a case, necessarily $\alpha_A = A(x_0)$.

Fuzzy numbers on $\mathbb{R}$ are fuzzy sets whose domain is $\mathbb{R}$ and satisfying certain constraints. There are several distinct definitions of *fuzzy number*. Throughout this manuscript, we will employ the following one.

Definition 1 *(Cf. [4,5,12,31,32]).* A *fuzzy number* $\mathcal{A}$ *on* $\mathbb{R}$ is a fuzzy set $\mathcal{A} : \mathbb{R} \to \mathbb{I}$ satisfying the following properties:

FN_1) $\mathcal{A}$ has absolute maximum on $\mathbb{R}$ (that is, there is $t_0 \in \mathbb{R}$ such that $\mathcal{A}(t_0) \geq \mathcal{A}\,(t)$ for all $t \in \mathbb{R}$).

FN_2) $\mathcal{A}$ is *fuzzy convex* (i.e., $\mathcal{A}\,(\lambda t + (1-\lambda)\,s) \geq \min\{\mathcal{A}(t), \mathcal{A}(s)\}$ for all $t, s \in \mathbb{R}$ and all $\lambda \in [0, 1]$).

FN_3) $\mathcal{A}$ is *upper semicontinuous at every* $t_0 \in \mathbb{R}$ (i.e., for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathcal{A}(t) - \mathcal{A}(t_0) < \varepsilon$, whenever $|\,t - t_0\,| < \delta$).

As a fuzzy set, a fuzzy number $\mathcal{A}$ is *normal* if there is $t_0 \in \mathbb{R}$ such that $\mathcal{A}(t_0) = 1$.

Remark 2. We highlight that, in the literature, it is very usual to assume that fuzzy numbers must satisfy the *normality condition* (that is, there is $t_0 \in \mathbb{R}$ such that $\mathcal{A}(t_0) = 1$). In fact, most of papers in this line of research consider that a *fuzzy number* is what we have just called a *normal fuzzy number*. A good reason for using such point of view is that fuzzy numbers must generalize real numbers, and the usual way to consider this extension is by identifying each real number r with the fuzzy number $\widetilde{r}_1 : \mathbb{R} \to \mathbb{I}$ that associates 1 to r and 0 to any other real number (see equation (3)). Such fuzzy number is normal by definition, so many authors consider that a fuzzy number must be normal.

However, in this study, fuzzy sets (which do not necessarily satisfy the normality condition) play as important a role as fuzzy numbers, so we consider that it may be counterproductive to force fuzzy numbers to fulfill the normality condition. As a consequence, by Definition 1, we wanted to clarify that, in this study, we will use the terminology "*fuzzy number*" to refer a fuzzy set which does not necessarily satisfy the normality condition, and the expression "*normal fuzzy number*" for fuzzy numbers that satisfy such property.

The following notions are usually associated to fuzzy numbers. Anyway, as we will use them later, we will introduce them associated to a fuzzy set.

Definition 3. Given $\alpha \in (0, 1]$, the *α-level set* (or *α-cut*) of a fuzzy set $A : \mathbb{R} \to \mathbb{I}$ is the crisp set $A_\alpha = \{ t \in \mathbb{R} : A(t) \geq \alpha \}$. The *kernel* (or *core*) *of* A is A_1. The *support of* A is the closure of the set of real points where the function A is strictly positive, that is,

$$\operatorname{supp}(A) = \overline{\{ t \in \mathbb{R} : A(t) > 0 \}} = \overline{\textstyle\bigcup_{\alpha \in (0,1]} A_\alpha} \,. \tag{1}$$

For convention, it is usual to denote $A_0 = \operatorname{supp}(A)$. Given a non-empty subset $D \subseteq \mathbb{R}$, we denote by $\mathrm{FS}(D)$ (respectively, $\mathrm{FN}(D)$) the family of all fuzzy sets (respectively, fuzzy numbers) on $\mathbb{R}$ whose supports are included on D. For simplicity, we denote $\mathrm{FS} = \mathrm{FS}(\mathbb{I})$ and $\mathrm{FN} = \mathrm{FN}(\mathbb{I})$.

Remark 4. Given a fuzzy set $A : X \to \mathbb{I}$ on a general set X, some authors call *support* to the set $\{ x \in X : A(x) > 0 \}$. This set has the advantage that it does not need to consider any topology on the set X. However, the main disadvantage of this set is that it can be closed, or open, or none of them (when X is endowed with a topology). However, for our purposes, it will be very important the property that states that the support of a fuzzy set on $\mathbb{R}$ is closed (w.r.t. the Euclidean topology). Hence, we will call *support* of the fuzzy set $A : \mathbb{R} \to \mathbb{I}$ to the closure of the set $\{ t \in \mathbb{R} : A(t) > 0 \}$.

For the sake of clarity, we highlight that, in what follows, we will only consider both "*fuzzy sets*" and "*fuzzy numbers*" of the real line (that is, $X = \mathbb{R}$). In this setting, the α-cuts completely characterize the fuzzy sets.

Proposition 5. *Two fuzzy sets $A, B \in \mathrm{FS}(\mathbb{R})$ are equal if, and only if, their level sets are equal.*

Each α-cut A_α can be bounded or not. To our study, we are only interested on fuzzy numbers whose supports are included on $\mathbb{I}$, that is, the family FS. The following result justifies that this case covers all possibilities in which we are interested.

Lemma 6.

(a) *If $a, b \in \mathbb{R}$ are such that $a < b$ and $\phi : \mathbb{I} \to [a, b]$ is defined by $\phi(t) = (1-t)a + tb$ for all $t \in \mathbb{I}$, then the mapping*

$$\psi_1 : \mathrm{FS}([a, b]) \to \mathrm{FS}, \quad \psi_1(A)(t) = \begin{cases} A(\phi(t)), & \text{if } t \in \mathbb{I}, \\ 0, & \text{in any other case,} \end{cases}$$

is a bijection.

(b) *If $\varphi:(0,1)\to\mathbb{R}$ is bijective,*[1] *then the mapping*

$$\psi_2:\mathrm{FS}(\mathbb{R})\to\mathrm{FS},\qquad \psi_2(A)(t)=\begin{cases}A(\varphi(t)), & \text{if } t\in(0,1),\\ 0, & \text{in any other case,}\end{cases}$$

is injective.

Proof. Item (a). Clearly, $\phi:[0,1]\to[a,b]$, defined by $\phi(t)=(1-t)a+tb$ for all $t\in[0,1]$, is a bijection, and its inverse function is given by

$$\phi^{-1}:[a,b]\to[0,1],\qquad \phi^{-1}(s)=\frac{s-a}{b-a}\quad\text{for all } s\in[a,b].$$

To prove that ψ_1 is injective, let $A,B\in\mathrm{FS}([a,b])$ be two fuzzy sets such that $\psi_1(A)=\psi_1(B)$. Then $A,B:[a,b]\to[0,1]$ satisfy

$$A(\phi(t))=B(\phi(t))\quad\text{for all } t\in[0,1].$$

As ϕ is a bijection, then

$$A(s)=B(s)\quad\text{for all } s\in[a,b].$$

Then $A=B$ and ψ_1 is injective. To prove that ψ_1 is surjective, let $A'\in\mathrm{FS}$ be arbitrary. Then $A':[0,1]\to[0,1]$. Let $A=A'\circ\phi^{-1}:[a,b]\to[0,1]$. Then $A\in\mathrm{FS}([a,b])$ is a fuzzy set on $[a,b]$ such that, for all $s\in[a,b]$,

$$\psi_1(A)(s)=A(\phi(s))=\left(A'\circ\phi^{-1}\right)(\phi(s))=A'\left(\phi^{-1}(\phi(s))\right)=A'(s).$$

Hence $\psi_1(A)=A'$ and ψ_1 is surjective.

The proof of the second item is similar. □

In the previous result, since $\psi_2(A)(0)=\psi_2(A)(1)=0$, the mapping ψ_2 is not surjective because it cannot model a fuzzy set $B\in\mathrm{FS}$ such that $B(0)>0$ or $B(1)>0$. Notice that fuzzy sets $A:\mathbb{R}\to\mathbb{I}$ belonging to FS can be seen as functions $A:\mathbb{I}\to\mathbb{I}$ whose domains and codomains are the interval $\mathbb{I}$ because $A(t)=0$ for all $t\in\mathbb{R}\diagdown\mathbb{I}$. Hence, for the sake of clarity, we restrict our study to fuzzy sets on FS and fuzzy numbers on FN. Obviously $\mathrm{FN}\subset\mathrm{FS}$.

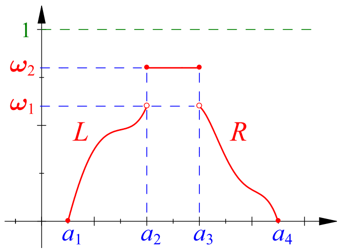
2.3. Some notable families of fuzzy numbers

In the setting of fuzzy numbers with compact support, adapting this notion from [31], a *generalized left-right fuzzy number* (for short, an *LR fuzzy number*) is a fuzzy number $\mathcal{A}=(a_1/a_2/a_3/a_4;\omega_1,\omega_2)_{LR}$, where $a_1,a_2,a_3,a_4\in\mathbb{R}$ (called the *corners* of $\mathcal{A}$) satisfy $a_1\le a_2\le a_3\le a_4$, $\omega_1,\omega_2\in(0,1]$, $\omega_1\le\omega_2$, defined by:

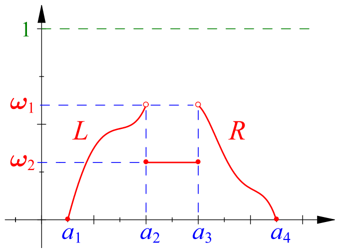
$$\mathcal{A}(t)=\begin{cases}L(t), & \text{if } a_1<t<a_2,\\ \omega_2, & \text{if } a_2\le t\le a_3,\\ R(t), & \text{if } a_3<t<a_4,\\ 0, & \text{in any other case,}\end{cases}\tag{2}$$

where $L:[a_1,a_2]\to[0,\omega_1]$ is a continuous, strictly increasing function such that $L(a_1)=0$ and $L(a_2)=\omega_1$, and $R:[a_3,a_4]\to[0,\omega_1]$ is a continuous, strictly decreasing function such that $R(a_3)=\omega_1$ and $L(a_4)=0$. Notice that if $a_1=a_2$, then the function L does not play a role in (2), and if $a_3=a_4$, the function R does not appear in (2). In general, the support of $\mathcal{A}=(a_1/a_2/a_3/a_4;\omega_1,\omega_2)_{LR}$ is $[a_1,a_4]$ and its core is $[a_2,a_3]$, if $\omega_2=1$, and it is empty, if $\omega_2<1$. The supremum of $\mathcal{A}=(a_1/a_2/a_3/a_4;\omega_1,\omega_2)_{LR}$ is $\alpha_{\mathcal{A}}=\max\{\omega_1,\omega_2\}$. It is an absolute maximum when $\omega_1\le\omega_2$ (in such a case, $\alpha_{\mathcal{A}}=\omega_2$).

[1] The reader can check that the function $\varphi:(0,1)\to\mathbb{R}$, defined by $\varphi(t)=(t-0.5)/(t(1-t))$ for all $t\in(0,1)$, can be employed here.



Fuzzy number ($\omega_1 \leq \omega_2$).



Fuzzy set ($\omega_1 > \omega_2$).

Functions given by (2) describe fuzzy numbers only when $\omega_1 \leq \omega_2$. In general, if $\omega_2 < \omega_1$, then it is a fuzzy set (it does not reach absolute maximum and it is not upper semi-continuous).

If $\omega_1 \leq \omega_2$, the LR-fuzzy number $(a_1/a_2/a_3/a_4;\omega_1,\omega_2)_{LR}$ can only be discontinuous at $t = a_2$ (when $a_1 = a_2$) or at $t = a_3$ (when $a_3 = a_4$). Furthermore, if $a_1 < a_2$ and $a_3 < a_4$, taking into account that $L^{-1} : [0,\omega_1] \to [a_1,a_2]$ is continuous and strictly increasing, $R^{-1} : [0,\omega_1] \to [a_3,a_4]$ is continuous and strictly decreasing, $L^{-1}(0) = a_1$, $L^{-1}(\omega_1) = a_2$, $R^{-1}(\omega_1) = a_3$ and $R^{-1}(0) = a_4$, the level sets of the LR-fuzzy number are given, for each $\alpha \in (0,1]$, by:

$$[(a_1/a_2/a_3/a_4;\omega_1,\omega_2)_{LR}]_\alpha = \begin{cases} \varnothing, & \text{if } \alpha \in (\omega_2, 1], \\ [\,a_2,\, a_3\,], & \text{if } \alpha \in [\omega_1,\omega_2], \\ \left[\,L^{-1}(\alpha),\ R^{-1}(\alpha)\,\right], & \text{if } \alpha \in (0,\omega_1). \end{cases}$$

If

$$L(t) = \omega_1 \cdot \frac{t - a_1}{a_2 - a_1} \text{ for all } t \in [a_1,a_2], \qquad R(t) = \omega_1 \cdot \frac{a_4 - t}{a_4 - a_3} \text{ for all } t \in [a_3,a_4],$$

then $\mathcal{A} = (a_1/a_2/a_3/a_4;\omega_1,\omega_1)$ is called a *trapezoidal fuzzy number*. When $a_1 < a_2$ and $a_3 < a_4$, its graphic representation corresponds to the trapezoid whose height is ω_1, whose long base is $[a_1,a_4] \times \{0\}$ and whose short base is $[a_2,a_3] \times \{\omega_1\}$. Notice that if $a_1 = a_2$ or $a_3 = a_4$, it corresponds to a right trapezoid. Particular cases of trapezoidal fuzzy numbers are: *triangular fuzzy numbers* (when $a_2 = a_3$), *rectangular fuzzy numbers* (when $a_1 = a_2$ and $a_3 = a_4$) and *crisp fuzzy numbers* (when $a_1 = a_2 = a_3 = a_4$). Normal crisp fuzzy numbers can be seen as real numbers.

For convenience, we introduce the following notation. Given $r,\omega \in \mathbb{I}$, we denote by $\mathbf{r}$ and $\widetilde{r}_\omega$ to the following self-mappings on $\mathbb{I}$:

$$\mathbf{r}(t) = r \quad \text{for all } t \in \mathbb{I}; \qquad \widetilde{r}_\omega(t) = \begin{cases} \omega, & \text{if } t = r, \\ 0, & \text{if } t \neq r. \end{cases} \tag{3}$$

On the one hand, although it is not normal, the fuzzy number $\mathbf{0}$ is a very special fuzzy number: for instance, it is the unique fuzzy number whose support is empty. Furthermore, any coherent fuzzy ranking among fuzzy numbers in FN should consider that $\mathbf{0}$ is its absolute minimum. On the other hand, crisp fuzzy numbers $\{\widetilde{r}_1 : r \in \mathbb{I}\}$ correspond to real scalars of the interval $\mathbb{I}$. From our point of view, the crisp fuzzy number $\widetilde{1}_1$, which corresponds to the real number 1, should be considered as the absolute maximum of any ranking index on FN.

3. Fuzzy sets generated by families of nested sets

In this section, inspired by the family of level sets associated to each fuzzy set, we study its main properties and we describe a procedure to consider a fuzzy set starting from a family of nested subsets of $\mathbb{I}$. Later, we take advantage of this study in order to introduce the lateral limits of the extremes of such families.

3.1. Some properties of the family of level sets associated to a fuzzy set

In our study, first we show some basic properties.

Proposition 7. *Given $A \in$ FS, the following properties hold.*

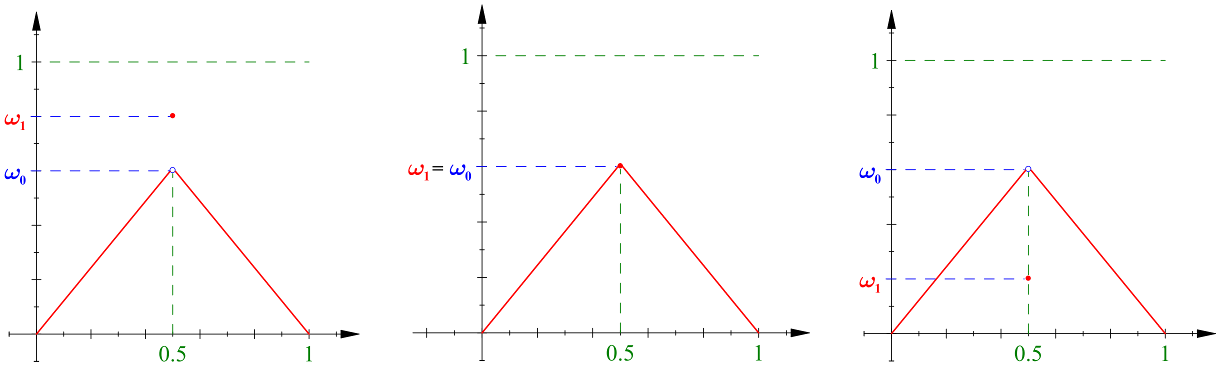


Fig. 1. Plot of A_{ω_0,ω_1} corresponding to $\omega_0 < \omega_1$, $\omega_0 = \omega_1$ and $\omega_0 > \omega_1$.

1. $\alpha_A \in \mathbb{I}$.
2. *If* $\alpha, \beta \in \mathbb{I}$ *are such that* $\alpha \le \beta$, *then* $A_1 \subseteq A_\beta \subseteq A_\alpha \subseteq A_0 \subseteq \mathbb{I}$.
3. $\alpha_A = 0$ *if, and only if,* $A = \mathbf{0}$.
4. *If* $\alpha \in [0, \alpha_A)$, *then* $A_\alpha \neq \varnothing$, *and if* $\alpha \in (\alpha_A, 1]$, *then* $A_\alpha = \varnothing$.

Let us introduce a multiparametric family of fuzzy sets on $\mathbb{I}$.

Example 8. Let $\omega_0, \omega_1 \in \mathbb{I}$ be such that $\omega_0 > 0$. Let $A_{\omega_0,\omega_1} \in \mathrm{FS}$ be the fuzzy set defined, for all $t \in \mathbb{I}$, by:

$$A_{\omega_0,\omega_1}(t) = \begin{cases} \omega_1, & \text{if } t = \dfrac{1}{2}, \\ 2\omega_0 t, & \text{if } t \in \left[0, \dfrac{1}{2}\right), \\ 2\omega_0(1-t), & \text{if } t \in \left(\dfrac{1}{2}, 1\right]. \end{cases}$$

The graphic representation of A_{ω_0,ω_1} corresponds to an isosceles triangle whose bases measure 1 and whose heights measure ω_0. However, the highest vertex of the triangle is placed to a height of ω_1. Fig. 1 shows three distinct cases corresponding to $\omega_0 < \omega_1$, $\omega_0 = \omega_1$ and $\omega_0 > \omega_1$. Notice that the unique fuzzy sets of the type A_{ω_0,ω_1} that are fuzzy numbers correspond to cases in which $\omega_0 \le \omega_1$, and they are normal fuzzy numbers when $\omega_0 \le \omega_1 = 1$. Furthermore, A_{ω_0,ω_1} is continuous if, and only if, $\omega_0 = \omega_1$.

Given a fuzzy set $A \in \mathrm{FS}$, the family $\mathcal{F}^A = \{A_\alpha : \alpha \in (0,1]\}$ of all its α-cuts satisfies a concrete property that is directly related to the employment of the inequality $\ge$ in the definition of $A_\alpha = \{x \in \mathbb{R} : A(x) \ge \alpha\}$.

Proposition 9. *If* $A \in \mathrm{FS}$, *then*

$$A_\alpha = \bigcap_{\beta \in (0,\alpha)} A_\beta \quad \textit{for all } \alpha \in (0,1]. \tag{4}$$

Proof. It is straightforward. □

Similarly, it can be proved the following fact, that also directly depends on the usage of $\ge$ in A_α.

Proposition 10. *If a fuzzy set* $A \in \mathrm{FS}$ *is continuous as function* $A : \mathbb{I} \to \mathbb{I}$, *then its* α*-cut* A_α *is closed on* $\mathbb{I}$ *(and also closed on* $\mathbb{R}$*) for all* $\alpha \in \mathbb{I}$.

Proof. It follows from the fact that $A_\alpha = A^{-1}([\alpha, 1]) = A^{-1}([\alpha, \infty) \cap \mathbb{I})$ for each $\alpha \in (0,1]$, and $A_0 = \mathrm{supp}(A)$ is closed by definition. □

3.2. Fuzzy sets and fuzzy numbers

In this subsection we show some properties associated to the level sets of fuzzy numbers. First of all, we highlight the following result in which the involved function satisfies, on a closed set, the condition (FN_3) in Definition 1, that is, it is upper semicontinuous on an appropriate closed set.

Theorem 11. *Given a fuzzy set $A : \mathbb{R} \to \mathbb{I}$, let $\{t_n\} \subset \mathbb{R}$ be a sequence such that $\{A(t_n)\} \to \sup(A)$. If $\{t_n\}$ is bounded on $\mathbb{R}$ and there is $n_0 \in \mathbb{N}$ such that A is upper semicontinuous at every point in a closed subset of $\mathbb{R}$ containing $\{t_n : n \geq n_0\}$, then A has absolute maximum.*

Proof. If $A = \mathbf{0}$, then A has absolute maximum on each $t \in \mathbb{R}$. Suppose that $A \neq \mathbf{0}$, that is, $\alpha_A = \sup(A) > 0$. Suppose that this supremum is not an absolute maximum (and we will get a contradiction). As $\{t_n\}$ is bounded, it has a partial convergent subsequence $\{t_{\sigma(n)}\}$. Let $s_0 \in \mathbb{R}$ be such that $\{t_{\sigma(n)}\} \to s_0$. As s_0 is not an absolute maximum for A, then $A(s_0) < \alpha_A$. Let define $\varepsilon = (\alpha_A - A(s_0))/2 > 0$ and $\beta_0 = A(s_0) + \varepsilon$. Then

$$\beta_0 = A(s_0) + \varepsilon = A(s_0) + \frac{\alpha_A - A(s_0)}{2} = \frac{A(s_0) + \alpha_A}{2}.$$

Therefore β_0 is a constant such that $\beta_0 < \alpha_A$. Since A is upper semicontinuous at every point in a closed subset Ω of $\mathbb{R}$ containing $\{t_n : n \geq n_0\}$ (for some $n_0 \in \mathbb{N}$), then A is upper semicontinuous at s_0 (because $s_0 \in \overline{\{t_n : n \geq n_0\}} \subseteq \overline{\Omega} = \Omega$). As a consequence, there exists $\delta > 0$ such that $A(t) - A(s_0) < \varepsilon$ for all $t \in (s_0 - \delta, s_0 + \delta)$. Hence $A(t) < A(s_0) + \varepsilon = \beta_0$ for all $t \in (s_0 - \delta, s_0 + \delta)$. Since $\{t_{\sigma(n)}\} \to s_0$, there is $n_1 \in \mathbb{N}$, $n_1 > n_0$, such that $t_{\sigma(n)} \in (s_0 - \delta, s_0 + \delta)$ for all $n \geq n_1$. Therefore $A(t_{\sigma(n)}) < \beta_0 < \alpha_A$ for all $n \geq n_1$, which contradicts the fact that $\{A(t_n)\} \to \alpha_A$. This contradiction guarantees that A has absolute maximum. □

Corollary 12. *If the support of a fuzzy set $A : \mathbb{R} \to \mathbb{I}$ is bounded on $\mathbb{R}$ and A is upper semicontinuous at every $s_0 \in \operatorname{supp}(A)$, then A has absolute maximum.*

Proof. It follows from the fact that if $\{A(t_n)\} \to \sup(A) > 0$, then there is $n_0 \in \mathbb{N}$ such that $A(t_n) > 0$ for all $n \in \mathbb{N}$. Then $\{t_n\}_{n \geq n_0} \subseteq \operatorname{supp}(A)$ and Theorem 11 is applicable to this subsequence. □

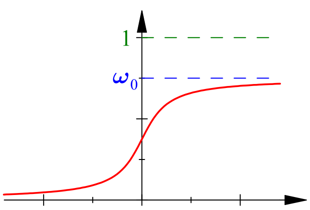
Corollary 13. *If a fuzzy set $A \in \mathrm{FS}$ is an upper semicontinuous function at every point of $\mathbb{I}$, then A has absolute maximum.*

Remark 14. Theorem 11 means that

$$(FN_3) + \text{bounded support} \Rightarrow (FN_1).$$

However, this property is false if we do not assume that the support of A is bounded on $\mathbb{R}$, even if the fuzzy set A satisfies (FN_2) and (FN_3). To show it, given $\omega_0 \in (0, 1]$, let $A : \mathbb{R} \to \mathbb{I}$ be defined by:

$$A(t) = \omega_0 \left(\frac{\arctan(t)}{\pi} + \frac{1}{2} \right) \quad \text{for all } t \in \mathbb{R}.$$



Then A is an strictly increasing, continuous bijection from $\mathbb{R}$ onto $(0, \omega_0)$. Hence, it satisfies the assumptions (FN_2) and (FN_3). However, it does not reach an absolute maximum on $\mathbb{R}$. Notice that its kernel is empty ($A_1 = \varnothing$). In fact, the rest of level sets are the non-bounded closed intervals:

$$A_\alpha = \begin{cases} \left[A^{-1}(\alpha), \infty\right), & \text{if } \alpha \in (0, \omega_0), \\ \varnothing, & \text{if } \alpha \in [\omega_0, 1]. \end{cases}$$

If $\mathcal{A} : \mathbb{R} \to \mathbb{I}$ is a fuzzy number, condition (FN_2) is equivalent to say that each α-cut $\{\mathcal{A}_\alpha : \alpha \in (0, 1]\}$ is a subinterval of $\mathbb{R}$ (including the possibility in which it is empty), and assumption (FN_3) means (with an obvious interpretation) that each interval $\mathcal{A}_\alpha$ is closed on $\mathbb{R}$ (it may be bounded or not). In this case, its support $\operatorname{supp}(\mathcal{A})$ is also a closed subinterval of $\mathbb{R}$. Furthermore, condition (FN_1) can be interpreted by saying that the α-cuts $\{\mathcal{A}_\alpha\}$ are non-empty for each α in a maximal closed interval $[0, \alpha_0] \subseteq \mathbb{I}$. This proves the following statement.

Proposition 15. *A fuzzy set $A : \mathbb{R} \to \mathbb{I}$ is a fuzzy number if, and only if, it has absolute maximum and each α-cut A_α, with $\alpha \in (0, 1]$, is a closed subinterval of $\mathbb{R}$ (including the possibility in which it is empty). In addition to this, A is normal if, and only if, its kernel A_1 is a non-empty set.*

Notice that the previous result is false if we do not assume that A has absolute maximum (recall the example in Remark 14). However, if its support is bounded, we don't need to assume this condition. If this is the case, we deduce the following result in which we appreciate how the extremes of the level sets associated to a fuzzy number completely characterize it. In fact, the functions that determine such level sets must satisfy certain conditions. The reader can compare this result with those included in [17,24].

Theorem 16. *If $A : \mathbb{R} \to \mathbb{I}$ is a fuzzy set with bounded support and $A \neq \mathbf{0}$, then the following conditions are equivalent.*

(a) *A is a fuzzy number.*
(b) *There are two left-continuous functions $A_L, A_U : (0, \alpha_A] \to \mathbb{R}$ such that A_L is increasing, A_L is decreasing and*

$$A_\alpha = \begin{cases} [\, A_L(\alpha)\,,\ A_U(\alpha)\,], & \text{if } \alpha \in (0, \alpha_A]\,, \\ \varnothing, & \text{if } \alpha \in (\alpha_A, 1]\,. \end{cases}$$

In such a case, the following properties hold.

1. *There is $t_0 \in \mathbb{R}$ such that A reaches absolute maximum on $\mathbb{R}$ at t_0, and $A(t_0) = \alpha_A > 0$.*
2. *The restriction $A|_{(-\infty, t_0]}$ is an increasing and right-continuous function.*
3. *The restriction $A|_{[t_0, \infty)}$ is a decreasing and left-continuous function.*
4. *$A_L((0, 1]) \subseteq (-\infty, t_0]$ and $A_U((0, 1]) \subseteq [t_0, \infty)$, so we can consider the functions A_L and A_U defined as:*

$$A_L : (0, 1] \to (-\infty, t_0] \quad \text{and} \quad A_U : (0, 1] \to [t_0, \infty)\,.$$

Proof. $(b) \Rightarrow (a)$. As each α-cut A_α is an interval, then A is fuzzy convex, that is, it satisfies (FN_2). To prove (FN_3), suppose that A is not upper semicontinuous at some $s_0 \in \mathbb{R}$. Then there is $\varepsilon_0 > 0$ and a sequence $\{t_n\} \to s_0$ such that $A(t_n) \geq A(s_0) + \varepsilon_0$. Define $\beta_0 = A(s_0) + \varepsilon_0$. Since $A(s_0) < \beta_0$, then $s_0 \notin A_{\beta_0}$. However, as $A(t_n) \geq \beta_0$, then $t_n \in A_{\beta_0}$ for all $n \in \mathbb{N}$. By hypothesis (b), as $A_{\beta_0} \neq \varnothing$, then $A_{\beta_0} = [\, A_L(\alpha)\,,\ A_U(\alpha)\,]$ is a closed, bounded subinterval of $\mathbb{R}$. Then $s_0 \in \overline{\{\, t_n : n \in \mathbb{N}\,\}} \subseteq \overline{A_{\beta_0}} = A_{\beta_0}$, which contradicts the fact that $s_0 \notin A_{\beta_0}$. This contradiction means that A is upper semicontinuous at each point $s_0 \in \mathbb{R}$, so A satisfies (FN_3). As we suppose that A has compact support, Corollary 12 guarantees that A has absolute maximum, so it also verifies condition (FN_1).

$(a) \Rightarrow (b)$. It is obvious by choosing t_0 as the absolute maximum of A, and A_L and A_U as the extremes of each level set (a proof in the case of normal fuzzy numbers can be found in [17,24], but its arguments are valid in the non-normal case). $\square$

3.3. The fuzzy set associated to a family of nested subsets of $\mathbb{I}$

In this subsection we describe how any family of nested subsets of $\mathbb{I}$ generates a fuzzy set whose levels sets are very similar to the subsets of the family. For the sake of clarity, we declare that we will use the term *nested* for a family $\mathcal{F} = \{\, \Omega_\alpha : \alpha \in (0, 1]\,\}$ of subsets of $\mathbb{R}$ such that $\Omega_\alpha \subseteq \Omega_\beta$ for all $\alpha, \beta \in (0, 1]$ satisfying $\beta \leq \alpha$. We highlight that this notion includes the case in which some subsets Ω_α are empty (for instance, for $\alpha \in (\omega_0, 1]$), or even all of them are empty. Notice that in the following results the possible set Ω_0 does not play any role because the support of a fuzzy set is always defined as in (1).

Property (4) is not satisfied by all families of nested subsets, even if they are non-empty, closed intervals, as in the following example.

Example 17. For each $\alpha \in (0, 1]$, let

$$\Omega_\alpha = \begin{cases} \{0.5\}, & \text{if } \alpha \in [0.5, 1]\,, \\ [0.25, 0.75]\,, & \text{if } \alpha \in (0, 0.5)\,. \end{cases} \tag{5}$$

Then $\mathcal{F} = \{\, \Omega_\alpha : \alpha \in (0, 1] \,\}$ is a family of non-empty, nested, closed intervals (such that $\Omega_\alpha \subseteq \Omega_\beta \subseteq \mathbb{I}$ for all $\alpha, \beta \in \mathbb{I}$ satisfying $\beta \leq \alpha$). However, if $\alpha = 0.5$:

$$\Omega_{0.5} = \{0.5\} \quad \subsetneqq \quad \bigcap_{\beta \in (0,0.5)} \Omega_\beta = [0.25, 0.75]\,.$$

Anyway, a family like the previous one permits us to consider an associated fuzzy set by using the following procedure.

Theorem 18. *Let $\mathcal{F} = \{\, \Omega_\alpha : \alpha \in (0, 1] \,\}$ be a family of nested subsets of $\mathbb{I}$ (maybe empty some of them). Let define $A^{\mathcal{F}} : \mathbb{I} \to \mathbb{I}$, for each $t \in \mathbb{I}$, by:*

$$A^{\mathcal{F}}(t) = \begin{cases} 0, & \text{if } t \in \mathbb{I} \setminus \left(\cup_{\alpha \in (0,1]} \Omega_\alpha \right), \\ \sup\left(\{\, \beta : t \in \Omega_\beta \,\} \right), & \text{if there is } \beta_0 \in (0, 1] \text{ such that } t \in \Omega_{\beta_0}. \end{cases} \tag{6}$$

Then $A^{\mathcal{F}} \in \mathrm{FS}$ is a fuzzy set satisfying the following properties.

1. *$A^{\mathcal{F}} = \mathbf{0}$ if, and only if, Ω_α is empty for all $\alpha \in (0, 1]$.*
2. *The fuzzy set $A^{\mathcal{F}}$ can be equivalently described as:*

$$A^{\mathcal{F}}(t) = \begin{cases} 1, & \text{if } t \in \Omega_\alpha \text{ for all } \alpha \in (0, 1)\,, \\ \inf\left(\{\, \beta : t \notin \Omega_\beta \,\} \right), & \text{if there is } \beta_0 \in (0, 1) \text{ such that } t \notin \Omega_{\beta_0}. \end{cases} \tag{7}$$

3. *For all $\alpha \in (0, 1]$,*

$$\Omega_\alpha \subseteq (A^{\mathcal{F}})_\alpha = \bigcap_{\beta \in (0,\alpha)} \Omega_\beta. \tag{8}$$

4. *The supremum of $A^{\mathcal{F}}$ is:*

$$\begin{aligned} \sup(A^{\mathcal{F}}) &= \left\{ \begin{array}{ll} 0, & \text{if } \Omega_\alpha = \varnothing \text{ for all } \alpha \in (0, 1)\,, \\ \sup\left(\{\, \beta : \Omega_\beta \neq \varnothing \,\} \right), & \text{if there is } \beta_0 \in (0, 1) \text{ such that } \Omega_{\beta_0} \neq \varnothing \end{array} \right\} \\ &= \begin{cases} 1, & \text{if } \Omega_\alpha \neq \varnothing \text{ for all } \alpha \in (0, 1)\,, \\ \inf\left(\{\, \beta : \Omega_\beta = \varnothing \,\} \right), & \text{if there is } \beta_0 \in (0, 1) \text{ such that } \Omega_{\beta_0} = \varnothing. \end{cases} \end{aligned} \tag{9}$$

Proof. Clearly $A^{\mathcal{F}}$ is well-defined and $A^{\mathcal{F}} \in \mathrm{FS}$. Throughout this proof, we denote by $A^{\mathcal{F}}_\alpha$ the α-cut of $A^{\mathcal{F}}$, that is, $A^{\mathcal{F}}_\alpha = (A^{\mathcal{F}})_\alpha$ for all $\alpha \in (0, 1]$.

1.- If $\Omega_\alpha = \varnothing$ for all $\alpha \in (0, 1]$, then definition (6) leads to $A^{\mathcal{F}}(t) = 0$ for all $t \in \mathbb{I}$. Hence $A^{\mathcal{F}} = \mathbf{0}$. If there is $\alpha_0 \in (0, 1]$ such that $\Omega_{\alpha_0} \neq \varnothing$, then there exists some $t_0 \in \Omega_{\alpha_0}$. Since $A^{\mathcal{F}}(t_0) = \sup\left(\{\, \beta : t_0 \in \Omega_\beta \,\} \right) \geq \alpha_0 > 0$, then the fuzzy number $A^{\mathcal{F}}$ is distinct than $\mathbf{0}$ (because $A^{\mathcal{F}}(t_0) > 0$).

2.- Let $t \in \mathbb{I}$ be arbitrary. Definitions (6) and (7) coincide when $A^{\mathcal{F}}(t) = 0$ or $A^{\mathcal{F}}(t) = 1$. Suppose that the numbers $s_0 = \sup\left(\{\, \beta : t \in \Omega_\beta \,\} \right)$ and $i_0 = \inf\left(\{\, \beta : t \notin \Omega_\beta \,\} \right)$ exist (that is, $t \notin \Omega_1$ and there is $\beta \in (0, 1)$ such that $t \in \Omega_\beta$), and we are going to prove that $s_0 = i_0$ by contradiction.

Suppose that $s_0 < i_0$. Let $\beta_0 \in (s_0, i_0)$, that is, $s_0 < \beta_0 < i_0$. If $t \in \Omega_{\beta_0}$, then s_0 is not the supremum of the set $\{\, \beta : t \in \Omega_\beta \,\}$ (because β_0 is greater than s_0). However, if $t \notin \Omega_{\beta_0}$, then i_0 is not the infimum of the set $\{\, \beta : t \notin \Omega_\beta \,\}$ (because β_0 is less than i_0). In any case we get a contradiction.

Next suppose that $s_0 > i_0$. Since $s_0 = \sup\left(\left\{\beta : t \in \Omega_\beta\right\}\right)$ is a supremum and $i_0 < s_0$, then there is $\beta_1 \in (i_0, s_0]$ such that $t \in \Omega_{\beta_1}$. Hence, $i_0 < \beta_1 \leq s_0$. Similarly, since $i_0 = \inf\left(\left\{\beta : t \notin \Omega_\beta\right\}\right)$ is an infimum and $i_0 < \beta_1$, then there is $\beta_2 \in [i_0, \beta_1)$ such that $t \notin \Omega_{\beta_2}$. Hence, $i_0 \leq \beta_2 < \beta_1 \leq s_0$, which means that $\Omega_{\beta_1} \subseteq \Omega_{\beta_2}$. However, this is a contradiction because $t \in \Omega_{\beta_1} \subseteq \Omega_{\beta_2}$ but $t \notin \Omega_{\beta_2}$. This proves that $i_0 = s_0$, that is, both descriptions of $A^{\mathcal{F}}$ lead to the same function.

3.- Let $\alpha \in (0, 1]$ be arbitrary. If $t \in \Omega_\alpha$, then $A^{\mathcal{F}}(t) = \sup\left(\left\{\beta : t \in \Omega_\beta\right\}\right) \geq \alpha$, so $t \in A^{\mathcal{F}}_\alpha$. This proves that $\Omega_\alpha \subseteq A^{\mathcal{F}}_\alpha$ for all $\alpha \in (0, 1]$. Next we prove the equality in (8). Let $t \in A^{\mathcal{F}}_\alpha$ for some given $\alpha \in (0, 1]$. Then $A^{\mathcal{F}}(t) \geq \alpha > 0$. Let $\beta \in (0, \alpha)$ be arbitrary and we are going to prove that $t \in \Omega_\beta$ (this will mean that $t \in \cap_{\beta \in (0,\alpha)} \Omega_\beta$). Since $A^{\mathcal{F}}(t) = \sup\left(\left\{\gamma : t \in \Omega_\gamma\right\}\right) \geq \alpha > \beta$, then there is $\gamma_0 \in \left(\beta, A^{\mathcal{F}}(t)\right]$ such that $t \in \Omega_{\gamma_0}$. Since $\beta < \gamma_0$, then $t \in \Omega_{\gamma_0} \subseteq \Omega_\beta$. Conversely, let $t \in \cap_{\beta \in (0,\alpha)} \Omega_\beta$ and we have to prove that $t \in A^{\mathcal{F}}_\alpha$. Let $\{\beta_n\} \subset (0, \alpha)$ be an strictly increasing sequence such that $\{\beta_n\} \to \alpha$. Since $t \in \Omega_{\beta_n} \subseteq A^{\mathcal{F}}_{\beta_n}$, then $A^{\mathcal{F}}(t) \geq \beta_n$ for all $n \in \mathbb{N}$. And as $\{\beta_n\} \to \alpha$, then $A^{\mathcal{F}}(t) \geq \alpha$, so $t \in A^{\mathcal{F}}_\alpha$.

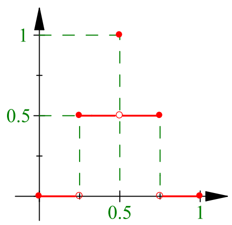
4.- We check that (9) holds. If $\Omega_\alpha = \varnothing$ for all $\alpha \in (0, 1)$, then $A^{\mathcal{F}} = \mathbf{0}$ and $\sup(A^{\mathcal{F}}) = 0$. Suppose that there is $\beta \in (0, 1)$ such that $\Omega_\beta \neq \varnothing$ and let $s_0 = \sup\left(\left\{\beta : \Omega_\beta \neq \varnothing\right\}\right)$. Clearly $s_0 > 0$. Let $\varepsilon \in (0, s_0/2)$ be arbitrary. Then there is $\beta_\varepsilon \in (s_0 - \varepsilon, s_0]$ such that $\Omega_{\beta_\varepsilon} \neq \varnothing$. Let $t_\varepsilon \in \Omega_{\beta_\varepsilon}$. Since $\Omega_{\beta_\varepsilon} \subseteq A^{\mathcal{F}}_{\beta_\varepsilon}$, then $A^{\mathcal{F}}(t_\varepsilon) \geq \beta_\varepsilon > s_0 - \varepsilon$. This proves that $\sup(A^{\mathcal{F}}) \geq A^{\mathcal{F}}(t_\varepsilon) > s_0 - \varepsilon$ for all $\varepsilon \in (0, s_0/2)$. Therefore $\sup(A^{\mathcal{F}}) \geq s_0$. To prove the equality, suppose, by contradiction, that $s_0 < \sup(A^{\mathcal{F}})$. Then there is $t_0 \in \mathbb{I}$ such that $s_0 < A^{\mathcal{F}}(t_0) \leq \sup(A^{\mathcal{F}})$. If we call $\alpha_0 = A^{\mathcal{F}}(t_0)$, then $s_0 < \alpha_0 \leq \sup(A^{\mathcal{F}})$ and, by item (3),

$$t_0 \in A^{\mathcal{F}}_{\alpha_0} = \bigcap_{\beta \in (0,\alpha_0)} \Omega_\beta.$$

If we take any $\alpha_1 \in (s_0, \alpha_0)$, then $s_0 < \alpha_1 < \alpha_0$ and $t_0 \in \Omega_{\alpha_1}$. Then $\Omega_{\alpha_1} \neq \varnothing$, which contradicts the fact that $s_0 = \sup\left(\left\{\beta : \Omega_\beta \neq \varnothing\right\}\right)$ and $s_0 < \alpha_1$.

To prove that both descriptions of $\sup(A^{\mathcal{F}})$ are the same, it is only necessary to repeat the arguments given in the proof of item 2. $\square$

Example 19. Let $\mathcal{F} = \{\Omega_\alpha : \alpha \in (0, 1]\}$ be the family described by (5) in Example 17. Recall that $\mathcal{F}$ does not satisfy the property $A_\alpha = \cap_{\beta \in (0,\alpha)} A_\beta$ for all $\alpha \in (0, 1]$. However, if we compute the associated fuzzy set $A^{\mathcal{F}}$ defined in Theorem 18, then we obtain the following function.



$$A^{\mathcal{F}}(t) = \begin{cases} 1, & \text{if } t = 0.5, \\ 0.5, & \text{if } t \in [0.25, 0.75] \diagdown \{0.5\}, \\ 0, & \text{if } t \in [0, 0.25) \cup (0.75, 1]. \end{cases}$$

Notice that, for all $\alpha \in (0, 1]$,

$$(A^{\mathcal{F}})_\alpha = \begin{cases} \{0.5\}, & \text{if } \alpha \in (0.5, 1], \\ [0.25, 0.75], & \text{if } \alpha \in (0, 0.5], \end{cases}$$

so $\Omega_\alpha \subseteq (A^{\mathcal{F}})_\alpha$ for all $\alpha \in (0, 1]$. Also notice that $A^{\mathcal{F}}$ is, in fact, a finite fuzzy number (see [2]).

Corollary 20. *Let $\mathcal{F} = \{\Omega_\alpha : \alpha \in (0, 1]\}$ be a family of nested subsets of $\mathbb{I}$ (maybe empty some of them). Let $A^{\mathcal{F}}$ be the fuzzy set defined in Theorem 18. Then the following properties are equivalent.*

a) $(A^{\mathcal{F}})_\alpha = \Omega_\alpha$ *for all* $\alpha \in (0, 1]$.
b) $\Omega_\alpha = \cap_{\beta \in (0,\alpha)} \Omega_\beta$ *for all* $\alpha \in (0, 1]$.

Proof. $(a) \Rightarrow (b)$. Since $A^{\mathcal{F}}$ is a fuzzy set, its α-cuts, that in this case are the sets Ω_α of the family $\mathcal{F}$, satisfy condition (b) by Proposition 9.

$(b) \Rightarrow (a)$. Item 3 of Theorem 18 guarantees that property (a) holds. $\square$

Corollary 21. *Given a fuzzy set $A \in \mathrm{FS}$, let $\mathcal{F}^A = \{A_\alpha : \alpha \in (0,1]\}$ be the family of all its α-cuts and let $A^{\mathcal{F}^A} \in \mathrm{FS}$ be the fuzzy set associated to the family $\mathcal{F}^A$ as in Theorem 18. Then $A^{\mathcal{F}^A} = A$.*

Proof. It follows from Proposition 5 taking into account that both fuzzy sets have the same α-cuts, that is, $(A^{\mathcal{F}^A})_\alpha = \cap_{\beta\in(0,\alpha)} A_\beta = A_\alpha$ for all $\alpha \in (0,1]$ (recall Proposition 9 and Corollary 20). □

Remark 22. Notice that the corresponding process by using a family of nested subsets does not lead to the same result. For instance, if $\mathcal{F} = \{\Omega_\alpha : \alpha \in (0,1]\}$ is the family described in Example 17, and $A^{\mathcal{F}}$ is the fuzzy set associated to $\mathcal{F}$ as in Theorem 18, then the family of α-cuts $\mathcal{F}^{A^{\mathcal{F}}} = \{(A^{\mathcal{F}})_\alpha : \alpha \in (0,1]\}$ is distinct to $\mathcal{F}$. In fact, $\Omega_\alpha \subseteq (A^{\mathcal{F}})_\alpha$ for all $\alpha \in (0,1]$, but $\Omega_{0.5} \neq (A^{\mathcal{F}})_{0.5}$ (recall Example 19).

As a consequence of Corollary 20, we derive the following consequence by employing closed subintervals of $\mathbb{I}$.

Corollary 23. *Let $\mathcal{F} = \{I_\alpha : \alpha \in (0,1]\}$ be a family of nested closed subintervals of $\mathbb{I}$ (maybe empty some of them). Let define $A^{\mathcal{F}} : \mathbb{I} \to \mathbb{I}$, for each $t \in \mathbb{I}$, by:*

$$A^{\mathcal{F}}(t) = \begin{cases} 0, & \text{if } t \in \mathbb{I} \setminus \left(\cup_{\alpha\in(0,1]} I_\alpha\right), \\ \sup\left(\{\beta : t \in I_\beta\}\right), & \text{if there is } \beta_0 \in (0,1] \text{ such that } t \in I_{\beta_0}. \end{cases}$$

Then $A^{\mathcal{F}}$ is a fuzzy set satisfying the following property:

$$\left[\, (A^{\mathcal{F}})_\alpha = I_\alpha \ \text{ for all } \alpha \in (0,1] \,\right] \Leftrightarrow \left[\, I_\alpha = \cap_{\beta\in(0,\alpha)} I_\beta \ \text{ for all } \alpha \in (0,1] \,\right]. \tag{10}$$

If this is the case, then $A^{\mathcal{F}}$ is a fuzzy number on $\mathbb{I}$, and $A^{\mathcal{F}}$ is a normal fuzzy number if, and only if, I_1 is non-empty.

Proof. It follows from Corollary 20. In such a case, $A^{\mathcal{F}}$ is a fuzzy number on $\mathbb{I}$ because its α-cuts are closed subintervals of $\mathbb{I}$. Then Theorem 18 is applicable, and condition (10) means that the functions A_L and A_U are left-continuous. □

3.4. The lateral limits of the extremes of level sets of a fuzzy set

It follows from item 4 of Proposition 7 that, given a fuzzy set $A \in \mathrm{FS}$, the α-cut A_α is non-empty when α is less than α_A and it is empty when α is greater than α_A. Next, we study what is the case for $\alpha = \alpha_A$. It will depend on whether the fuzzy set A has an absolute maximum or not.

Lemma 24. *Given $A \in \mathrm{FS} \setminus \{\mathbf{0}\}$, the following properties hold.*

1. *For all $\alpha \in (0, \alpha_A]$, the following limits exist:*

$$\lim_{\beta\to\alpha^-} \inf A_\beta, \quad \lim_{\beta\to\alpha^-} \sup A_\beta \in \mathbb{I},$$

and they satisfy:

$$\lim_{\beta\to\alpha^-} \inf A_\beta \le \inf A_\alpha \le \sup A_\alpha \le \lim_{\beta\to\alpha^-} \sup A_\beta \qquad \text{for all } \alpha \in (0, \alpha_A)$$

(this property also holds for $\alpha = \alpha_A$ if A_{α_A} is non-empty).

2. *If $\alpha_1, \alpha_2 \in (0, \alpha_A]$ are such that $\alpha_2 \le \alpha_1$, then*

$$\lim_{\beta\to\alpha_2^-} \inf A_\beta \le \lim_{\beta\to\alpha_1^-} \inf A_\beta \le \lim_{\beta\to\alpha_1^-} \sup A_\beta \le \lim_{\beta\to\alpha_2^-} \sup A_\beta.$$

3. *The following limits*

$$\ell_L^A = \lim_{\alpha\to\alpha_A^-} \inf A_\alpha \quad \text{and} \quad \ell_U^A = \lim_{\alpha\to\alpha_A^-} \sup A_\alpha$$

exist and they satisfy

$$0 \le \inf A_\alpha \le \ell_L^A \le \ell_U^A \le \sup A_\alpha \le 1 \quad \textit{for all } \alpha \in [0, \alpha_A).$$

4. *The level set A_{α_A} is non-empty if, and only if, A has absolute maximum on $\mathbb{I}$ (that is, there is $t_0 \in \mathbb{I}$ such that $A(t_0) = \alpha_A$).*
5. *If $A_{\alpha_A} \neq \varnothing$ (A has absolute maximum), then*

$$\ell_L^A \le \inf A_{\alpha_A} \le \sup A_{\alpha_A} \le \ell_U^A. \tag{11}$$

Proof. Since $A \neq \mathbf{0}$, then $\alpha_A > 0$. Item 4 of Proposition 7 guarantees that $A_\beta \neq \varnothing$ for all $\beta \in (0, \alpha_A)$.

1, 2, 3.- If $\alpha_1, \alpha_2 \in [0, \alpha_A)$ are such that $\alpha_2 \le \alpha_1$, then $\varnothing \neq A_{\alpha_1} \subseteq A_{\alpha_2}$. As a consequence:

$$\inf A_{\alpha_2} \le \inf A_{\alpha_1} \le \sup A_{\alpha_1} \le \sup A_{\alpha_2}. \tag{12}$$

Given $\alpha \in (0, \alpha_A]$, the function $\phi_\alpha : (0, \alpha) \to \mathbb{I}$ given by $\phi_\alpha(\beta) = \inf A_\beta$ is well defined on the interval $(0, \alpha)$. Taking into account that it is bounded and increasing, then the limit $\lim_{\beta \to \alpha^-} \inf A_\beta$ exists and it belongs to $\mathbb{I}$ (because $\operatorname{supp} A \subseteq \mathbb{I}$). Similarly we can deduce that $\lim_{\beta \to \alpha^-} \sup A_\beta$ exists. The other items follow from the properties of functions $\{\phi_\alpha : \alpha \in (0, \alpha_A]\}$.

4.- If $A_{\alpha_A} \neq \varnothing$, then there is $t_1 \in A_{\alpha_A}$. This point satisfies $A(t_1) \ge \alpha_A = \sup(A) \ge A(t)$ for all $t \in \mathbb{I}$. Hence t_1 is an absolute maximum of A on $\mathbb{I}$. Conversely, suppose that A has an absolute maximum on $\mathbb{I}$, that is, there is $t_0 \in \mathbb{I}$ such that $A(t_0) \ge A(t)$ for all $t \in \mathbb{I}$. Therefore $\alpha_A = \sup(A) = A(t_0)$, so $t_0 \in A_{\alpha_A}$ and this set is non-empty.

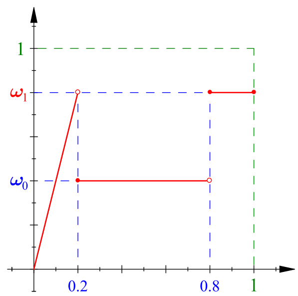
5.- Suppose that α_A is the maximum of A on $\mathbb{I}$. The previous item shows that $A_{\alpha_A} \neq \varnothing$, and the reasoning given in (12) guarantees that

$$\inf A_\alpha \le \inf A_{\alpha_A} \le \sup A_{\alpha_A} \le \sup A_\alpha \quad \text{for all } \alpha \in [0, \alpha_A].$$

Letting $\alpha \to \alpha_A^-$ we deduce that $\ell_L^A \le \inf A_{\alpha_A} \le \sup A_{\alpha_A} \le \ell_U^A$. □

Example 25. Property (4) that states that, given a fuzzy set $A \in \mathrm{FS}$, $A_\alpha = \cap_{\beta \in (0,\alpha)} A_\beta$ for all $\alpha \in (0, 1]$, may lead us to believe that, when A has absolute maximum, the equalities hold in (11), that is, $\ell_L^A = \inf A_{\alpha_A}$ and $\ell_U^A = \sup A_{\alpha_A}$. However, this is false, as we next show. Let $\omega_0, \omega_1 \in \mathbb{I}$ be such that $0 < \omega_0 < \omega_1 \le 1$ and let define $A : \mathbb{I} \to \mathbb{I}$ by:

$$A(t) = \begin{cases} \dfrac{\omega_1 t}{0.2}, & \text{if } 0 \le t < 0.2, \\ \omega_0, & \text{if } 0.2 \le t < 0.8, \\ \omega_1, & \text{if } 0.8 \le t \le 1. \end{cases}$$



The α-cuts of A are:

$$A_\alpha = \begin{cases} \varnothing, & \text{if } \alpha \in (\omega_1, 1], \\ [0.8, 1], & \text{if } \alpha = \omega_1, \\ \left[\dfrac{0.2\alpha}{\omega_1}, 0.2\right) \cup [0.8, 1], & \text{if } \alpha \in (\omega_0, \omega_1), \\ \left[\dfrac{0.2\alpha}{\omega_1}, 1\right], & \text{if } \alpha \in [0, \omega_0]. \end{cases}$$

Notice that A has absolute maximum, which is $\alpha_A = \omega_1 > 0$. Furthermore,

$$A_{\omega_1} = [0.8, 1] \quad \text{and} \quad A_\alpha = \left[\dfrac{0.2\alpha}{\omega_1}, 0.2\right) \cup [0.8, 1] \quad \text{for all } \alpha \in (\omega_0, \omega_1).$$

Then $\inf A_{\alpha_A} = \min A_{\omega_1} = 0.8$ but

$$\inf A_\alpha = \frac{0.2\alpha}{\omega_1} \quad \text{for all } \alpha \in (\omega_0, \omega_1).$$

Therefore

$$\ell_L^A = \lim_{\alpha \to \alpha_A^-} \inf A_\alpha = \lim_{\alpha \to \omega_1^-} \frac{0.2\alpha}{\omega_1} = 0.2 < 0.8 = \min A_{\alpha_A}.$$

4. Approximating fuzzy sets by fuzzy numbers

In this subsection we introduce a procedure in order to associate a unique normal fuzzy number $\mathcal{A} \in \mathrm{FN}$ to each fuzzy set $A \in \mathrm{FS}$. This process can be interpreted as a way to approximate each fuzzy set $A \in \mathrm{FS}$ by a unique normal fuzzy number $\Phi(A) = \mathcal{A} \in \mathrm{FN}$, that is, we introduce an approximation operator $\Phi : \mathrm{FS} \to \mathrm{FN}$. Next we study the main properties of this operator.

The following methodology will depend on:

- two increasing, continuous functions $f, g : \mathbb{I} \to \mathbb{I}$ such that $f \le g$ (that is, $f(t) \le g(t)$ for all $t \in \mathbb{I}$); and on
- two functions $T_1, T_2 : \triangle \to \mathbb{I}$ such that $f(t) \le T_1(t,s) \le T_2(t,s) \le g(s)$ for all $(t,s) \in \triangle$, where $\triangle$ denotes the triangle on the plane whose vertices are $(0,0)$, $(0,1)$ and $(1,1)$ (that is, $\triangle = \{(t,s) \in \mathbb{I} \times \mathbb{I} : t \le s\}$).

In the next result we introduce the association methodology. If $A = \mathbf{0}$, then $\Phi(A) = \mathbf{0}$, so we reduce our study to fuzzy sets such that $A \neq \mathbf{0}$.

Theorem 26. *Let $f, g : \mathbb{I} \to \mathbb{I}$ be two increasing, continuous functions and let $T_1, T_2 : \triangle \to \mathbb{I}$ be two functions such that $f(t) \le T_1(t,s) \le T_2(t,s) \le g(s)$ for all $(t,s) \in \triangle$.*

Given a fuzzy set $A \in \mathrm{FS} \setminus \{\mathbf{0}\}$, there is a unique normal fuzzy number $\mathcal{A} \in \mathrm{FN}$ whose α-level sets $\{\mathcal{A}_\alpha = [A_L(\alpha), A_U(\alpha)] : \alpha \in (0,1]\}$ are given by the following extremes:

$$A_L(\alpha) = \begin{cases} f\left(\lim_{\beta \to \alpha^-} \inf A_\beta\right), & \text{if } \alpha \in (0, \alpha_A], \\ T_1\left(\ell_L^A, \ell_U^A\right), & \text{if } \alpha \in (\alpha_A, 1]; \end{cases} \tag{13}$$

$$A_U(\alpha) = \begin{cases} g\left(\lim_{\beta \to \alpha^-} \sup A_\beta\right), & \text{if } \alpha \in (0, \alpha_A], \\ T_2\left(\ell_L^A, \ell_U^A\right), & \text{if } \alpha \in (\alpha_A, 1]. \end{cases} \tag{14}$$

Proof. Since $A \neq \mathbf{0}$, then $\alpha_A \in (0,1]$. Item 1 of Lemma 24 guarantees that the limits $\lim_{\beta \to \alpha^-} \inf A_\beta$ and $\lim_{\beta \to \alpha^-} \sup A_\beta$ exist for all $\alpha \in (0, \alpha_A]$. Therefore, for all $\alpha \in (0, \alpha_A]$,

$$A_L(\alpha) = f\left(\lim_{\beta \to \alpha^-} \inf A_\beta\right) \le f\left(\lim_{\beta \to \alpha^-} \sup A_\beta\right) \le g\left(\lim_{\beta \to \alpha^-} \sup A_\beta\right) = A_U(\alpha).$$

Furthermore, if $\alpha \in (\alpha_A, 1]$, then $A_L(\alpha) = T_1\left(\ell_L^A, \ell_U^A\right) \le T_2\left(\ell_L^A, \ell_U^A\right) = A_U(\alpha)$. Hence $\mathcal{A}_\alpha = [A_L(\alpha), A_U(\alpha)]$ is a non-empty closed subinterval of $\mathbb{I}$ for each $\alpha \in (0,1]$. We claim that A_L is increasing. To prove it, let $\alpha_1, \alpha_2 \in (0,1]$ be such that $\alpha_1 \le \alpha_2$. If $\alpha_1, \alpha_2 \in (\alpha_A, 1]$, then $A_L(\alpha_1) = T_1\left(\ell_L^A, \ell_U^A\right) = A_L(\alpha_2)$. If $\alpha_1, \alpha_2 \in (0, \alpha_A]$, then item 2 of Lemma 24 shows that $\lim_{\beta \to \alpha_1^-} \inf A_\beta \le \lim_{\beta \to \alpha_2^-} \inf A_\beta$, and as f is increasing, then

$$A_L(\alpha_1) = f\left(\lim_{\beta \to \alpha_1^-} \inf A_\beta\right) \le f\left(\lim_{\beta \to \alpha_2^-} \inf A_\beta\right) = A_L(\alpha_2).$$

Finally, if $\alpha_1 \in (0, \alpha_A]$ and $\alpha_2 \in (\alpha_A, 1]$, then

$$A_L(\alpha_1) = f\left(\lim_{\beta \to \alpha_1^-} \inf A_\beta\right) \le f\left(\lim_{\beta \to \alpha_A^-} \inf A_\beta\right) = f\left(\ell_L^A\right) \le T_1\left(\ell_L^A, \ell_U^A\right) = A_L(\alpha_2).$$

In any case, we deduce that $A_L(\alpha_1) \leq A_L(\alpha_2)$ for all $\alpha_1, \alpha_2 \in (0,1]$ such that $\alpha_1 \leq \alpha_2$, so A_L is increasing. Similarly it can be checked that A_U is a decreasing function.

Now we prove that A_L is a left-continuous function. Notice that A_L is constant on $(\alpha_A, 1]$, so it is continuous on this interval. Let $\alpha \in (0, \alpha_A]$ be arbitrary and let $\{\alpha_n\} \subset (0,\alpha)$ be an strictly increasing sequence such that $\{\alpha_n\} \to \alpha$. Since the function $\phi_\alpha : (0,\alpha) \to \mathbb{I}$ given by $\phi_\alpha(\beta) = \inf\left(A_\beta\right)$ is well defined and increasing on the interval $(0,\alpha)$, and the function f is continuous, then

$$\lim_{n\to\infty} A_L(\alpha_n) = \lim_{n\to\infty} f\left(\lim_{\beta\to\alpha_n^-} \inf A_\beta\right) = f\left(\lim_{n\to\infty}\lim_{\beta\to\alpha_n^-} \inf A_\beta\right) = f\left(\lim_{\beta\to\alpha^-} \inf A_\beta\right) = A_L(\alpha).$$

Therefore, A_L is left-continuous on α, which proves that A_L is left-continuous on $(0,1]$. Similarly, it can be proved that A_U is left-continuous on $(0,1]$.

Next let $\mathcal{A} \in \mathrm{FS}$ be the fuzzy set associated to the family $\mathcal{F} = \{[A_L(\alpha), A_U(\alpha)] : \alpha \in (0,1]\}$ by Theorem 18, that is, $\mathcal{A}$ is defined by

$$\mathcal{A}(t) = \begin{cases} 0, & \text{if } t \in \mathbb{I} \setminus \left(\cup_{\alpha\in(0,1]}[A_L(\alpha), A_U(\alpha)]\right), \\ \sup\left(\{\beta : t \in [A_L(\beta), A_U(\beta)]\}\right), & \text{if there is } \beta_0 \in (0,1] \text{ such that } t \in [A_L(\beta_0), A_U(\beta_0)]. \end{cases}$$

Item 3 of Theorem 18 shows that

$$[A_L(\alpha), A_U(\alpha)] \subseteq \mathcal{A}_\alpha = \bigcap_{\beta\in(0,\alpha)} [A_L(\beta), A_U(\beta)].$$

But as A_L and A_U are left-continuous on $(0,1]$, then

$$\mathcal{A}_\alpha = \bigcap_{\beta\in(0,\alpha)} [A_L(\beta), A_U(\beta)] = [A_L(\alpha), A_U(\alpha)] \quad \text{for all } \alpha \in (0,1].$$

Notice that $\mathcal{A}_1 = [A_L(1), A_U(1)] = \left[T_1\left(\ell_L^A, \ell_U^A\right), T_2\left(\ell_L^A, \ell_U^A\right)\right]$ is a non-empty interval, so $\mathcal{A}$ is a normal fuzzy set. As a consequence, Theorem 16 (applied with $\omega_0 = 1$) guarantees that $\mathcal{A}$ is a fuzzy number. Furthermore, $\mathcal{A}$ is the unique fuzzy number whose level sets are given by (13)-(14) because the level sets characterize the fuzzy set (recall Proposition 5). □

The previous theorem let us to introduce an approximation operator

$$\Phi = \Phi_{f,g,T_1,T_2} : \mathrm{FS} \to \mathrm{FN}$$

defined, for each $A \in \mathrm{FS}$ by:

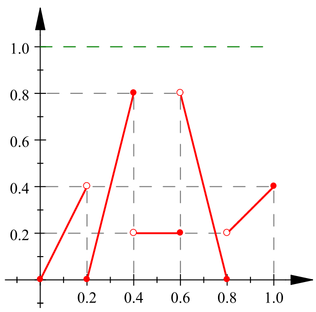
$$\Phi(A) = \begin{cases} \mathbf{0}, & \text{if } A = \mathbf{0}, \\ \mathcal{A}, & \text{if } A \neq \mathbf{0}, \end{cases}$$

where $\mathcal{A} \in \mathrm{FN}$ is the unique normal fuzzy number whose α-cuts $\{\mathcal{A}_\alpha = [A_L(\alpha), A_U(\alpha)] : \alpha \in (0,1]\}$ are given by the equalities (13)-(14). Notice that this operator directly depends on the functions f, g, T_1 and T_2.

We are mainly interested in the following case, that we will call *standard choice*: f and g are the identity mapping on $\mathbb{I}$, $T_1 = \min$ and $T_2 = \max$. In such a case, we denote by $\Phi_0 : \mathrm{FS} \to \mathrm{FN}$ the above mentioned operator under the standard choice for f, g, T_1 and T_2.

Example 27. Let $A \in \mathrm{FS}$ be the fuzzy set defined as:

$$A(t) = \begin{cases} 2t, & \text{if } 0 \leq t < 0.2, \\ 4t - 0.8, & \text{if } 0.2 \leq t \leq 0.4, \\ 0.2, & \text{if } 0.4 < t \leq 0.6, \\ 3.2 - 4t, & \text{if } 0.6 < t \leq 0.8, \\ t - 0.6, & \text{if } 0.8 < t \leq 1. \end{cases} \tag{15}$$



Then, for all $\alpha \in (0, 1]$,

$$A_\alpha = \begin{cases} \varnothing, & \text{if } \alpha \in (0.8, 1]\,, \\ \left[\dfrac{\alpha+0.8}{4}, 0.4\right] \cup \left(0.6, \dfrac{3.2-\alpha}{4}\right], & \text{if } \alpha \in (0.4, 0.8]\,, \\ \left[\dfrac{\alpha}{2}, 0.2\right) \cup \left[\dfrac{\alpha+0.8}{4}, 0.4\right] \cup \left(0.6, \dfrac{3.2-\alpha}{4}\right] \cup [\alpha+0.6, 1]\,, & \text{if } \alpha \in (0.2, 0.4]\,, \\ \left[\dfrac{\alpha}{2}, 0.2\right) \cup \left[\dfrac{\alpha+0.8}{4}, \dfrac{3.2-\alpha}{4}\right] \cup [\alpha+0.6, 1]\,, & \text{if } \alpha \in (0, 0.2]\,, \\ [0, 1]\,, & \text{if } \alpha = 0. \end{cases}$$

Therefore, since $\alpha_A = 0.8$, we have that for all $\alpha \in (0, 0.8]$,

$$\inf A_\alpha = \begin{cases} \dfrac{\alpha+0.8}{4}, & \text{if } \alpha \in (0.4, 0.8]\,, \\ \dfrac{\alpha}{2}, & \text{if } \alpha \in (0, 0.4]\,; \end{cases} \qquad \sup A_\alpha = \begin{cases} \dfrac{3.2-\alpha}{4}, & \text{if } \alpha \in (0.4, 0.8]\,, \\ 1, & \text{if } \alpha \in (0, 0.4]\,. \end{cases}$$

Notice that the functions $\alpha \in (0, 0.8] \longmapsto \inf A_\alpha$ and $\alpha \in (0, 0.8] \longmapsto \sup A_\alpha$ are left-continuous, but they are not continuous at $\alpha = 0.4$. Also notice that

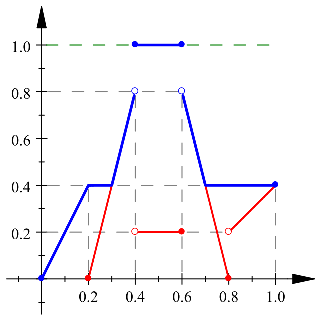
$$\ell_L^A = \lim_{\beta \to 0.8^-} \inf A_\beta = 0.4 \quad \text{and} \quad \ell_U^A = \lim_{\beta \to 0.8^-} \sup A_\beta = 0.6.$$

Under the standard choice, $T_1\left(\ell_L^A, \ell_U^A\right) = \min\{0.4, 0.6\} = 0.4$ and $T_2\left(\ell_L^A, \ell_U^A\right) = \max\{0.4, 0.6\} = 0.6$. Therefore

$$\Phi_0(A)_\alpha = \begin{cases} [0.4, 0.6]\,, & \text{if } \alpha \in (0.8, 1]\,, \\ \left[\dfrac{\alpha+0.8}{4}, \dfrac{3.2-\alpha}{4}\right], & \text{if } \alpha \in (0.4, 0.8]\,, \\ \left[\dfrac{\alpha}{2}, 1\right], & \text{if } \alpha \in (0, 0.4]\,. \end{cases}$$

Thus, under the standard choice, its associated normal fuzzy number $\Phi_0(A)$ is:

$$\Phi_0(A)(t) = \begin{cases} 2t, & \text{if } 0 \le t < 0.2, \\ 0.4, & \text{if } 0.2 \le t \le 0.3, \\ 4t - 0.8, & \text{if } 0.3 < t < 0.4, \\ 1, & \text{if } 0.4 \le t \le 0.6, \\ 3.2 - 4t, & \text{if } 0.6 < t < 0.7, \\ 0.4, & \text{if } 0.7 \le t \le 1. \end{cases}$$

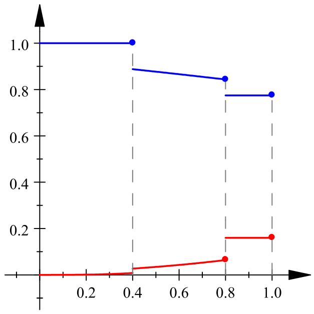


Example 28. Suppose that $f(t) = t^3$, $T_1(t, s) = (\min\{t, s\})^2$, $T_2(t, s) = \sqrt{\max\{t, s\}}$ and $g(t) = \sqrt[3]{t}$ for all $t, s \in \mathbb{I}$ such that $t \le s$. If we take the same fuzzy number $A \in \mathrm{FS}$ defined by (15) in Example 27, then

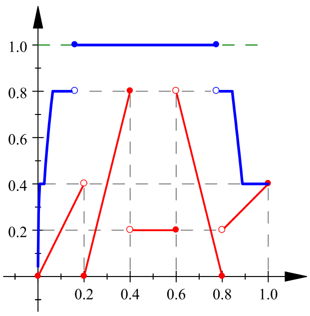
$$A_L(\alpha) = \left\{ \begin{array}{ll} T_1\left(\ell_L^A, \ell_U^A\right), & \text{if } \alpha \in (0.8, 1]\,, \\ f\left(\lim\limits_{\beta \to \alpha^-} \inf A_\beta\right), & \text{if } \alpha \in (0, 0.8] \end{array} \right\} = \begin{cases} 0.4^2, & \text{if } \alpha \in (0.8, 1]\,, \\ \left(\dfrac{\alpha+0.8}{4}\right)^3, & \text{if } \alpha \in (0.4, 0.8]\,, \\ \left(\dfrac{\alpha}{2}\right)^3, & \text{if } \alpha \in (0, 0.4]\,; \end{cases}$$

$$A_U(\alpha) = \left\{ \begin{array}{ll} T_2\left(\ell_L^A, \ell_U^A\right), & \text{if } \alpha \in (0.8, 1], \\ g\left(\lim_{\beta \to \alpha^-} \sup A_\beta\right), & \text{if } \alpha \in (0, 0.8] \end{array} \right\} = \begin{cases} \sqrt{0.6}, & \text{if } \alpha \in (0.8, 1], \\ \sqrt[3]{\dfrac{3.2 - \alpha}{4}}, & \text{if } \alpha \in (0.4, 0.8], \\ 1, & \text{if } \alpha \in (0, 0.4]. \end{cases}$$

If we plot these functions, we obtain the following graphic:



Then, the unique fuzzy number whose extremes of its α-cuts are given by the previous functions is the following one:



$$\Phi_{f,g,T_1,T_2}(A)(t) = \begin{cases} \sqrt[3]{8t}, & \text{if } 0 \leq t \leq 0.008, \\ 0.4, & \text{if } 0.008 < t \leq 0.027, \\ 4\sqrt[3]{t} - 0.8, & \text{if } 0.027 < t \leq 0.064, \\ 0.8, & \text{if } 0.064 < t < 0.16, \\ 1, & \text{if } 0.16 \leq t \leq \sqrt{0.6}, \\ 0.8, & \text{if } \sqrt{0.6} < t \leq \sqrt[3]{0.6}, \\ 3.2 - 4t^3, & \text{if } \sqrt[3]{0.6} < t \leq \sqrt[3]{0.7}, \\ 0.4, & \text{if } \sqrt[3]{0.7} < t \leq 1. \end{cases}$$

5. Some properties of the approximation operator

We start this section by showing a key property of the above mentioned operator: under certain conditions, the normal fuzzy number associated to each normal fuzzy number is itself, that is, normal fuzzy numbers are approximated by themselves.

Theorem 29. *If f and g are chosen as the identity mapping on $\mathbb{I}$, then each normal fuzzy number on $\mathbb{I}$ is a fixed point of the approximation operator $\Phi = \Phi_{f,g,T_1,T_2}$. In this case, the operator Φ is surjective onto the family of all normal fuzzy numbers.*

Proof. The fuzzy number $\mathbf{0}$ is a fixed point of Φ by definition. Let $\mathcal{B} \in \mathrm{FN}$ be a normal fuzzy number such that $\mathcal{B} \neq \mathbf{0}$. Then $\alpha_{\mathcal{B}} = 1$ and its level sets are given by $\mathcal{B}_\alpha = [B_L(\alpha), B_U(\alpha)]$ for all $\alpha \in (0, 1]$, where the function $B_L : (0, 1] \to \mathbb{R}$ is left-continuous (recall Theorem 16). Let $\mathcal{A} = \Phi(\mathcal{B}) \in \mathrm{FN}$ be the unique normal fuzzy number associated to $\mathcal{B}$ following the procedure described in Theorem 26. As f is the identity mapping on $\mathbb{I}$, then, for all $\alpha \in (0, 1]$,

$$A_L(\alpha) = f\left(\lim_{\beta \to \alpha^-} \inf \mathcal{B}_\beta\right) = \lim_{\beta \to \alpha^-} \inf \mathcal{B}_\beta = \lim_{\beta \to \alpha^-} B_L(\beta) = B_L(\alpha),$$

where the last equality occurs because B_L is left-continuous. Similarly it can be proved that $A_U(\alpha) = B_U(\alpha)$ for all $\alpha \in (0, 1]$. We conclude by Proposition 5 that $\mathcal{A} = \mathcal{B}$ because they have the same level sets. Hence $\mathcal{B} = \mathcal{A} = \Phi(\mathcal{B})$ is a fixed point of Φ. □

Fuzzy sets are generalizations of crisp sets to a setting in which some uncertainty must be considered. The most important binary relation among crisp sets is the *inclusion* $\subseteq$. In fact, it is a partial order. In the context of fuzzy sets, the inclusion is generalized by the following binary relation: given two fuzzy sets $A, B : \mathbb{R} \to \mathbb{I}$, the fuzzy set A is *included* in the fuzzy set B if $A \leq B$, that is, $A(t) \leq B(t)$ for all $t \in \mathbb{R}$.

Lemma 30. *Given two fuzzy sets $A, B : \mathbb{R} \to \mathbb{I}$, A is included on B ($A \leq B$) if, and only if, $A_\alpha \subseteq B_\alpha$ for all $\alpha \in (0, 1]$.*

Proof. Suppose that $A \leq B$. Given $\alpha \in (0, 1]$, let $t \in A_\alpha$. Since $B(t) \geq A(t) \geq \alpha$, then $t \in B_\alpha$, so $A_\alpha \subseteq B_\alpha$. Conversely, suppose that $A_\alpha \subseteq B_\alpha$ for all $\alpha \in (0, 1]$. Let $t \in \mathbb{R}$ be arbitrary and let $\alpha = A(t)$. If $\alpha = 0$, then $A(t) = 0 \leq B(t)$. Next, suppose that $\alpha > 0$. Then $t \in A_\alpha \subseteq B_\alpha$, so $B(t) \geq \alpha = A(t)$. In any case, $A \leq B$. □

A first result considering the inclusion of fuzzy sets is the following one.

Theorem 31. *If f and g satisfy $f(t) \leq t \leq g(t)$ for all $t \in \mathbb{I}$, then $A \leq \Phi(A)$ for all $A \in \mathrm{FS}$.*

Proof. If $A = \mathbf{0}$, then $\Phi(A) = A$. Suppose that $A \neq \mathbf{0}$, that is, $\alpha_A > 0$. Let $t_0 \in \mathbb{I}$ be arbitrary and let $\alpha_0 = A(t_0)$. If $\alpha_0 = 0$, then $A(t_0) = 0 \leq \Phi(A)(t_0)$. Next, suppose that $\alpha_0 = A(t_0) > 0$. Clearly, $t_0 \in A_{\alpha_0}$, so A_{α_0} is non-empty. Item 1 of Lemma 24 guarantees that

$$\lim_{\beta\to\alpha_0^-} \inf A_\beta \leq \inf A_{\alpha_0} \leq \sup A_{\alpha_0} \leq \lim_{\beta\to\alpha_0^-} \sup A_\beta .$$

Therefore:

$$A_L(\alpha_0) = f\left(\lim_{\beta\to\alpha_0^-} \inf A_\beta\right) \leq \lim_{\beta\to\alpha_0^-} \inf A_\beta \leq \inf A_{\alpha_0} \qquad \text{and}$$

$$A_U(\alpha_0) = g\left(\lim_{\beta\to\alpha_0^-} \sup A_\beta\right) \geq \lim_{\beta\to\alpha_0^-} \sup A_\beta \geq \sup A_{\alpha_0}.$$

As a consequence,

$$t_0 \in A_{\alpha_0} \subseteq \left[\inf A_{\alpha_0}, \sup A_{\alpha_0}\right] \subseteq [A_L(\alpha_0), A_U(\alpha_0)] = \Phi(A)_{\alpha_0}.$$

This means that $\Phi(A)(t_0) \geq \alpha_0 = A(t_0)$, which completes the proof. □

The following result shows that Φ satisfies a minimizing property.

Theorem 32. *Let $A \in \mathrm{FS}$ be a fuzzy set such that $\alpha_A = 1$ and let $\mathcal{B} \in \mathrm{FN}$ be a fuzzy number such that $A \leq \mathcal{B}$. If f and g are the identity mapping on $\mathbb{I}$, then $\Phi(A) \leq \mathcal{B}$.*

Proof. Let $\mathcal{A} = \Phi(A) \in \mathrm{FN}$. Since $A \leq \mathcal{B}$, then $A_\alpha \subseteq \mathcal{B}_\alpha = [B_L(\alpha), B_U(\alpha)]$ for all $\alpha \in (0, 1]$ (see Lemma 30). Since $\alpha_A = 1$, then $A_\beta \neq \varnothing$ for all $\beta \in (0, 1)$. In particular, $B_L(\beta) \leq \inf A_\beta \leq \sup A_\beta \leq B_U(\beta)$ for all $\beta \in (0, 1)$. Hence, for all $\alpha \in (0, 1]$,

$$A_L(\alpha) = f\left(\lim_{\beta\to\alpha^-} \inf A_\beta\right) = \lim_{\beta\to\alpha^-} \inf A_\beta \geq \lim_{\beta\to\alpha^-} B_L(\beta) = B_L(\alpha),$$

$$A_U(\alpha) = g\left(\lim_{\beta\to\alpha^-} \sup A_\beta\right) = \lim_{\beta\to\alpha^-} \sup A_\beta \leq \lim_{\beta\to\alpha^-} B_U(\beta) = B_U(\alpha).$$

As a consequence, $\mathcal{A}_\alpha = [A_L(\alpha), A_U(\alpha)] \subseteq [B_L(\alpha), B_U(\alpha)] = \mathcal{B}_\alpha$ for all $\alpha \in (0, 1]$. Lemma 30 guarantees that $\Phi(A) = \mathcal{A} \leq \mathcal{B}$. □

Next we study how the approximation operator works on two very similar fuzzy sets. Before that, we need the following technical result.

Proposition 33. *If a fuzzy set $A \in \mathrm{FS}$ has not absolute maximum, then there is a sequence $\{t_n\} \subseteq \mathbb{I}$ such that $\{t_n\} \to s_0 \in \mathbb{I}$, $\{A(t_n)\}$ is strictly increasing and $\{A(t_n)\} \to \alpha_A$.*

Proof. Since A has not absolute maximum, $A(t) < \alpha_A$ for all $t \in \mathbb{I}$. As α_A is a supremum, then there is a sequence $\{s_n\} \subset \mathbb{I}$ such that $\{A(s_n)\} \to \alpha_A$. Since $A(s_n) < \alpha_A$ for all $n \in \mathbb{N}$, the sequence $\{s_n\}$ has a partial subsequence $\{s_{\sigma(n)}\}$ such that $\{A(s_{\sigma(n)})\}$ is strictly increasing (and also convergent to α_A). Finally, taking into account that $\{s_{\sigma(n)}\}$ is bounded, then it has a convergent partial subsequence $\{t_n\}$ that satisfies all requirements. □

Theorem 34. *Given a fuzzy set $A \in \mathrm{FS}$ with no absolute maximum, let $\{t_n\} \subseteq \mathbb{I}$ be a sequence such that $\{t_n\} \to s_0 \in \mathbb{I}$, $\{A(t_n)\}$ is strictly increasing and $\{A(t_n)\} \to \alpha_A$. Let define*

$$B(t) = \begin{cases} \alpha_A, & \text{if } t = s_0, \\ A(t), & \text{if } t \neq s_0. \end{cases}$$

Then $B \in \mathrm{FS}$ is a fuzzy set, B has absolute maximum (which is α_A) and $\Phi(A) = \Phi(B)$.

Proof. Let $\gamma_0 = A(s_0)$. Since A has not absolute maximum, then $\gamma_0 = A(s_0) < \alpha_A$. Clearly B is a fuzzy set having $\alpha_B = \alpha_A$ as absolute maximum. Furthermore,

$$B_\alpha = \begin{cases} \varnothing, & \text{if } \alpha \in (\alpha_A, 1], \\ A_\alpha \cup \{s_0\}, & \text{if } \alpha \in (\gamma_0, \alpha_A], \\ A_\alpha, & \text{if } \alpha \in (0, \gamma_0]. \end{cases}$$

Item 4 of Proposition 7 guarantees that $A_\alpha \neq \varnothing$ for all $\alpha \in (0, \alpha_A)$ and $A_\alpha = \varnothing$ for all $\alpha \in (\alpha_A, 1]$. As A has not absolute maximum, then $A_\alpha = \varnothing$ for all $\alpha \in [\alpha_A, 1]$. In particular, $B_{\alpha_A} = \{s_0\}$ and $s_0 \in B_\alpha \neq \varnothing$ for all $\alpha \in (0, \alpha_A]$. Since $A_\alpha \subseteq B_\alpha$ for all $\alpha \in (0, \alpha_A)$, then $\inf B_\alpha \leq \inf A_\alpha$ for all $\alpha \in (0, \alpha_A)$. Let define $\phi_A, \phi_B : (0, \alpha_A) \to \mathbb{R}$ by

$$\phi_A(\alpha) = \inf A_\alpha \quad \text{and} \quad \phi_B(\alpha) = \inf B_\alpha \quad \text{for all } \alpha \in (0, \alpha_A).$$

Both functions ϕ_A and ϕ_B are increasing and bounded from above (recall $\operatorname{supp} A \subseteq \mathbb{I}$). We divide the rest of the proof into seven steps.

Step 1. We claim that $\inf B_\beta \leq s_0$ *for all* $\beta \in (0, \alpha_A]$ *and* $\ell_L^B \leq s_0$. Since $s_0 \in B_\beta$ for all $\beta \in (0, \alpha_A]$, then $\inf B_\beta \leq s_0$ for all $\beta \in (0, \alpha_A]$. In particular, $\ell_L^B = \lim_{\beta \to \alpha_A^-} \inf B_\beta \leq s_0$.

Step 2. We claim that $\ell_L^A \leq s_0$. Define $\beta_n = A(t_n)$ for all $n \in \mathbb{N}$. Since $\{A(t_n)\}$ is strictly increasing but A has not absolute maximum, then $\beta_n < \beta_{n+1} < \alpha_A$ for all $n \in \mathbb{N}$ and $\{\beta_n\} \to \alpha_A$. As $t_n \in A_{\beta_n}$, then $\inf A_{\beta_n} \leq t_n$ for all $n \in \mathbb{N}$. Since $\{t_n\} \to s_0$ and ϕ_A is increasing and bounded from above, then

$$\ell_L^A = \lim_{\beta \to \alpha_A^-} \inf A_\beta = \lim_{\beta \to \alpha_A^-} \phi_A(\beta) = \lim_{n \to \infty} \phi_A(\beta_n) = \lim_{n \to \infty} \inf A_{\beta_n} \leq \lim_{n \to \infty} t_n = s_0.$$

Step 3. We claim that $\inf A_\alpha \leq s_0$ *for all* $\alpha \in (0, \alpha_A)$. Let $\alpha \in (0, \alpha_A)$ be arbitrary. If $\beta \in (\alpha, \alpha_A)$, then $\alpha < \beta < \alpha_A$. Since ϕ_A is increasing, then $\phi_A(\alpha) \leq \phi_A(\beta)$. Therefore:

$$\inf A_\alpha = \phi_A(\alpha) \leq \lim_{\beta \to \alpha_A^-} \phi_A(\beta) = \ell_L^A \leq s_0.$$

Step 4. We claim that $\inf A_\alpha = \inf B_\alpha$ *for all* $\alpha \in (0, \alpha_A)$. If $\alpha \in (0, \gamma_0]$, then $B_\alpha = A_\alpha$, so $\inf A_\alpha = \inf B_\alpha$ for all $\alpha \in (0, \gamma_0]$. Suppose that $\alpha \in (\gamma_0, \alpha_A)$. In this case, A_α is a non-empty subset of $\mathbb{R}$ bounded from below and $s_0 \notin A$. Since $\inf A_\alpha \leq s_0$ by Step 3, then $\inf A_\alpha = \inf(A_\alpha \cup \{s_0\}) = \inf B_\alpha$.

Step 5. We claim that $\ell_L^A = \ell_L^B$. By Step 4,

$$\ell_L^B = \lim_{\beta \to \alpha_A^-} \inf B_\beta = \lim_{\beta \to \alpha_A^-} \inf A_\beta = \ell_L^A.$$

Repeating Steps 1-5, it can similarly be proved that $\ell_U^A = \ell_U^B$.

Step 6. We claim that $\lim_{\beta \to \alpha^-} \inf A_\beta = \lim_{\beta \to \alpha^-} \inf B_\beta$ *for all* $\alpha \in (0, \alpha_A]$. It directly follows from Step 4.

Step 7. $A_L(\alpha) = B_L(\alpha)$ *for all* $\alpha \in (0,1]$. Taking into account (13) and Step 6, if $\alpha \in (0, \alpha_A]$, then:

$$A_L(\alpha) = f\left(\lim_{\beta\to\alpha^-} \inf A_\beta\right) = f\left(\lim_{\beta\to\alpha^-} \inf B_\beta\right) = B_L(\alpha),$$

and if $\alpha \in (\alpha_A, 1]$,

$$A_L(\alpha) = T_1\left(\ell_L^A, \ell_U^A\right) = T_1\left(\ell_L^B, \ell_U^B\right) = B_L(\alpha).$$

In any case, $A_L(\alpha) = B_L(\alpha)$ for all $\alpha \in (0,1]$. In a similar way it can be proved that $A_U(\alpha) = B_U(\alpha)$ for all $\alpha \in (0,1]$. Hence $\Phi(A)_\alpha = \Phi(B)_\alpha$ for all $\alpha \in (0,1]$, so $\Phi(A) = \Phi(B)$. □

Theorem 34 implies that the approximation operator Φ is never injective. Furthermore, it means that we can reduce the computation of Φ to fuzzy sets in FS having absolute maximum.

Theorem 35. *Under the standard choice, the approximation operator* Φ_0 *is increasing in* FS *w.r.t. the binary relation* $\preceq$. *Furthermore, it satisfies* $\Phi_0(\mathbf{0}) = \mathbf{0}$ *and* $\Phi_0(\widetilde{1}_1) = \widetilde{1}_1$.

Proof. Let $A, B \in$ FS be such that $A \preceq B$. By Theorem 31, $B \preceq \Phi_0(B)$, so $A \preceq \Phi_0(B)$. Since $\Phi_0(B) \in$ FN, Theorem 32 guarantees that $\Phi_0(A) \preceq \Phi_0(B)$. Thus Φ_0 is increasing in FS w.r.t. the binary relation $\preceq$. □

5.1. LR-*fuzzy numbers*

In this subsection we describe how the approximation operator Φ acts on the family of LR-fuzzy sets under the standard choice. In the following result, we use the LR-fuzzy set $A = (a_1/a_2/a_3/a_4; \omega_1, \omega_2)_{LR}$ described in (2), where $a_1, a_2, a_3, a_4 \in \mathbb{I}$ satisfy $0 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq 1$, $\omega_1, \omega_2 \in (0,1]$ (recall that A is a fuzzy number if, and only if, $\omega_1 \leq \omega_2$), $L : [a_1, a_2] \to [0, \omega_1]$ is a continuous, strictly increasing function such that $L(a_1) = 0$ and $L(a_2) = \omega_1$, and $R : [a_3, a_4] \to [0, \omega_1]$ is a continuous, strictly decreasing function such that $R(a_3) = \omega_1$ and $L(a_4) = 0$.

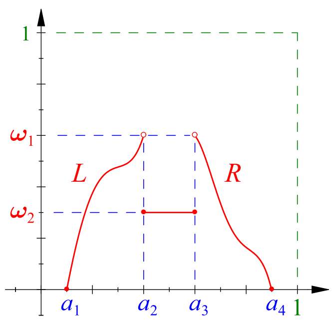
Lemma 36. *Under the standard choice, if* $A = (a_1/a_2/a_3/a_4; \omega_1, \omega_2)_{LR}$ *is an* LR-*fuzzy set, then* $\Phi_0(A) = (a_1/a_2/a_3/a_4; \omega_1, 1)_{LR}$.

Proof. Let $\mathcal{B}$ be the fuzzy number $(a_1/a_2/a_3/a_4; \omega_1, 1)_{LR}$. Its α-cuts are given by:

$$\mathcal{B}_\alpha = \begin{cases} \left[L^{-1}(\alpha), R^{-1}(\alpha)\right], & \text{if } \alpha \in (0, \omega_1], \\ [a_2, a_3], & \text{if } \alpha \in (\omega_1, 1]. \end{cases}$$

Let $\mathcal{A} = \Phi_0(A)$ and denote by $\{\mathcal{A}_\alpha = [A_L(\alpha), A_U(\alpha)] : \alpha \in (0,1]\}$ to its α-cuts. We study the case $\omega_2 < \omega_1$ because it is more difficult than the contrary case. Also assume that $a_1 < a_2$ and $a_3 < a_4$, so $L^{-1} : [0, \omega_1] \to [a_1, a_2]$ is strictly increasing, $R^{-1} : [0, \omega_1] \to [a_3, a_4]$ is strictly decreasing, $L^{-1}(0) = a_1$, $L^{-1}(\omega_1) = a_2$, $R^{-1}(\omega_1) = a_3$ and $R^{-1}(0) = a_4$. Both functions L^{-1} and R^{-1} are also continuous. In this case:

$$A_\alpha = \begin{cases} \varnothing, & \text{if } \alpha \in (\omega_1, 1], \\ \left[L^{-1}(\alpha), a_2\right) \cup \left(a_3, R^{-1}(\alpha)\right], & \text{if } \alpha \in (\omega_2, \omega_1), \\ \left[L^{-1}(\alpha), R^{-1}(\alpha)\right], & \text{if } \alpha \in (0, \omega_2]. \end{cases}$$



Hence, if $\beta \in (0, \omega_1) = (0, \alpha_A)$, then $\inf A_\beta = L^{-1}(\beta) \in [a_1, a_2]$ and $\sup A_\beta = R^{-1}(\beta) \in [a_3, a_4]$. As a consequence, for all $\alpha \in (0, \alpha_A]$,

$$A_L(\alpha) = f\left(\lim_{\beta\to\alpha^-} \inf A_\beta\right) = \lim_{\beta\to\alpha^-} \inf A_\beta = \lim_{\beta\to\alpha^-} L^{-1}(\beta) = L^{-1}(\alpha),$$

$$A_U(\alpha) = g\left(\lim_{\beta\to\alpha^-} \sup A_\beta\right) = \lim_{\beta\to\alpha^-} \sup A_\beta = \lim_{\beta\to\alpha^-} R^{-1}(\beta) = R^{-1}(\alpha).$$

Notice that

$$\ell_L^A = \lim_{\alpha\to\alpha_A^-} \inf A_\alpha = \lim_{\alpha\to\omega_1^-} L^{-1}(\alpha) = L^{-1}(\omega_1) = a_2 \quad \text{and}$$

$$\ell_U^A = \lim_{\alpha\to\alpha_A^-} \sup A_\alpha = \lim_{\alpha\to\omega_1^-} R^{-1}(\alpha) = R^{-1}(\omega_1) = a_3.$$

Therefore $T_1\left(\ell_L^A, \ell_U^A\right) = \min\{a_2, a_3\} = a_2$ and $T_2\left(\ell_L^A, \ell_U^A\right) = \max\{a_2, a_3\} = a_3$. This means that:

$$A_L(\alpha) = \begin{cases} L^{-1}(\alpha), & \text{if } \alpha \in (0, \omega_1], \\ a_2, & \text{if } \alpha \in (\omega_1, 1]; \end{cases} \qquad A_U(\alpha) = \begin{cases} R^{-1}(\alpha), & \text{if } \alpha \in (0, \omega_1], \\ a_3, & \text{if } \alpha \in (\omega_1, 1]. \end{cases}$$

Thus, for all $\alpha \in (0, 1]$,

$$\mathcal{A}_\alpha = [A_L(\alpha), A_U(\alpha)] = \left\{ \begin{array}{ll} \left[L^{-1}(\alpha), R^{-1}(\alpha)\right], & \text{if } \alpha \in (0, \omega_1], \\ [a_2, a_3], & \text{if } \alpha \in (\omega_1, 1] \end{array} \right\} = \mathcal{B}_\alpha$$

These are exactly the α-cuts of the fuzzy number $\mathcal{B} = (a_1/a_2/a_3/a_4; \omega_1, 1)_{LR}$ so, by Proposition 5, we conclude that $\Phi_0(A) = \mathcal{A} = \mathcal{B} = (a_1/a_2/a_3/a_4; \omega_1, 1)_{LR}$. □

Corollary 37. *For all $r \in \mathbb{I}$ and $\omega \in (0, 1]$, $\Phi_0(\widetilde{r}_\omega) = \widetilde{r}_1 \equiv r$.*

6. Conclusions

In this paper we have introduced a wide family of operators $\Phi_{f,g,T_1,T_2} : \mathrm{FS}([0, 1]) \to \mathrm{FN}([0, 1])$, depending on a great range of initial functions, and we have studied their main properties. This family can be interpreted as a first stage in order to translate the best properties of the set FN([0, 1]) to the very general family FS([0, 1]). In this line of research, as they are able to approximate fuzzy sets by fuzzy numbers, it could be useful, for instance, in approximate reasoning when the initial input data are near to be fuzzy numbers.

In prospect works, we will analyze the potential of this approach in real-life contexts (for instance, in fuzzy ranking, fuzzy decision making, fuzzy regression, etc.) where fuzzy sets (or even type-2 fuzzy sets) are a key tool to represent fuzzy information without considering crisp (non-fuzzy) data in the solution of the problem.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] S. Abbasbandy, T. Hajjari, A new approach for ranking of trapezoidal fuzzy numbers, Comput. Math. Appl. 57 (3) (2009) 413–419.

[2] G. Alfonso, A.F. Roldán López de Hierro, C. Roldán, A fuzzy regression model based on finite fuzzy numbers and its application to real-world financial data, J. Comput. Appl. Math. 318 (2017) 47–58.

[3] B. Asady, A. Zendehnam, Ranking fuzzy numbers by distance minimization, Appl. Math. Model. 31 (2007) 2589–2598.

[4] A.I. Ban, L. Coroianu, Simplifying the search for effective ranking of fuzzy numbers, IEEE Trans. Fuzzy Syst. 23 (2) (2015) 327–339.
[5] H. Bustince, E. Barrenechea, M. Pagola, J. Fernández, Z. Xu, B. Bedregal, J. Montero, H. Hagras, F. Herrera, B. de Baets, A historical account of types of fuzzy sets and their relationships, IEEE Trans. Fuzzy Syst. 24 (2016) 179–194.
[6] H. Bustince, M. Pagola, E. Barrenechea, J. Fernández, P. Melo-Pinto, P. Couto, H.R. Tizhoosh, J. Montero, Ignorance functions. An application to the calculation of the threshold in prostate ultrasound images, Fuzzy Sets Syst. 161 (1) (2010) 20–36.
[7] P.-T. Chang, E.S. Lee, Ranking of fuzzy sets based on the concept of existence, Comput. Math. Appl. 27 (1994) 1–21.
[8] S.M. Chen, J.H. Chen, Fuzzy risk analysis based on ranking generalized fuzzy numbers with different heights and different spreads, Expert Syst. Appl. 36 (3) (2009) 6833–6842.
[9] L.H. Chen, C.-C. Hsueh, Fuzzy regression models using the least-squares method based on the concept of distance, IEEE Trans. Fuzzy Syst. 17 (6) (2009) 1259–1272.
[10] S.-M. Chen, K. Sanguansat, Analyzing fuzzy risk based on a new fuzzy ranking method between generalized fuzzy numbers, Expert Syst. Appl. 38 (2011) 2163–2171.
[11] T.C. Chu, C.T. Tsao, Ranking fuzzy numbers with an area between the centroid point and original point, Comput. Math. Appl. 43 (1–2) (2002) 111–117.
[12] D. Dubois, H. Prade, Operations on fuzzy numbers, Int. J. Syst. Sci. 9 (3) (1978) 613–626.
[13] D. Dubois, H. Prade, Fuzzy Sets and Systems: Theory and Applications, Academic Press, 1980.
[14] D. Dubois, H. Prade, The three semantics of fuzzy sets, Fuzzy Sets Syst. 90 (1997) 141–150.
[15] D. Dubois, H. Prade (Eds.), Fundamentals of Fuzzy Sets, Kluwer Academic Press, 2000.
[16] D. Dubois, H. Prade, Gradualness, uncertainty and bipolarity: making sense of fuzzy sets, Fuzzy Sets Syst. 192 (2012) 3–24.
[17] R. Goetschel, W. Voxman, Elementary fuzzy calculus, Fuzzy Sets Syst. 18 (1986) 31–43.
[18] G. Hesamian, M.G. Akbari, A fuzzy additive regression model with exact predictors and fuzzy responses, Appl. Soft Comput. 95 (106507) (2020).
[19] L.G. Jaime, E.E. Kerre, M. Nachtegael, H. Bustince, Consensus image method for unknown noise removal, Knowl.-Based Syst. 70 (2014) 64–77.
[20] R. Jain, Decision-making in the presence of fuzzy variables, IEEE Trans. Syst. Man Cybern. 6 (1976) 698–703.
[21] L.-W. Lee, S.-M. Chen, Fuzzy interpolative reasoning for sparse fuzzy rule-based systems based on the ranking values of fuzzy sets, Expert Syst. Appl. 35 (2008) 850–864.
[22] T.-S. Liou, M.-J. Wang, Ranking fuzzy numbers with integral value, Fuzzy Sets Syst. 50 (1992) 247–255.
[23] G. Lucca, G.P. Dimuro, J. Fernández, H. Bustince, B. Bedregal, J.A. Sanz, Improving the performance of fuzzy rule-based classification systems based on a nonaveraging generalization of CC-integrals named $C_{F_1F_2}$ integrals, IEEE Trans. Fuzzy Syst. 27 (1) (2018) 124–134.
[24] M. Ma, M. Friedman, A. Kandel, A new fuzzy arithmetic, Fuzzy Sets Syst. 108 (1999) 83–90.
[25] C. Marco-Detchart, H. Bustince, J. Fernández, R. Mesiar, J. Lafuente, E. Barrenechea, J.M. Pintor, Ordered directional monotonicity in the construction of edge detectors, Fuzzy Sets Syst. 421 (2021) 111–132, https://doi.org/10.1016/j.fss.2020.07.002.
[26] S. Murakami, S. Maeda, S. Imamura, Fuzzy decision analysis on the development of centralized regional energy control system, in: Proceedings of the IFAC Symposium on Fuzzy Information, Knowledge Representation and Decision Analysis, 1983, pp. 363–368.
[27] A.M. Nejad, M. Mashinchi, Ranking fuzzy numbers based on the areas on the left and the right sides of fuzzy number, Comput. Math. Appl. 61 (2011) 431–442.
[28] M. Pagola, C. López-Molina, J. Fernández, E. Barrenechea, H. Bustince, Interval type-2 fuzzy sets constructed from several membership functions: application to the fuzzy thresholding algorithm, IEEE Trans. Fuzzy Syst. 21 (2) (2012) 230–244.
[29] C. Roldán, A. Roldán, J. Martínez-Moreno, A fuzzy regression model based on distances and random variables with crisp input and fuzzy output data: a case study in biomass production, Soft Comput. 16 (5) (2012) 785–795.
[30] A. Roldán, J. Martínez-Moreno, C. Roldán, On interrelationships between fuzzy metric structures, Iran. J. Fuzzy Syst. 10 (2) (2013) 133–150.
[31] A.F. Roldán López de Hierro, C. Roldán López de Hierro, J. Martínez-Moreno, C. Aguilar Peña, Estimation of a fuzzy regression model using fuzzy distances, IEEE Trans. Fuzzy Syst. 24 (2016) 344–359.
[32] A.F. Roldán López de Hierro, C. Roldán, F. Herrera, On a new methodology for ranking fuzzy numbers and its application to real economic data, Fuzzy Sets Syst. 353 (2018) 86–110.
[33] A.F. Roldán López de Hierro, M. Sánchez, C. Roldán, Multi-criteria decision making involving uncertain information via fuzzy ranking and fuzzy aggregation functions, J. Comput. Appl. Math. (2020) 113138.
[34] J.A. Sanz, D. Bernardo, F. Herrera, H. Bustince, H. Hagras, A compact evolutionary interval-valued fuzzy rule-based classification system for the modeling and prediction of real-world financial applications with imbalanced data, IEEE Trans. Fuzzy Syst. 23 (4) (2014) 973–990.
[35] J.A. Sanz, A. Fernández, H. Bustince, F. Herrera, IVTURS: a linguistic fuzzy rule-based classification system based on a new interval-valued fuzzy reasoning method with tuning and rule selection, IEEE Trans. Fuzzy Syst. 21 (3) (2013) 399–411.
[36] J.A. Sanz, M. Galar, A. Jurio, A. Brugos, M. Pagola, H. Bustince, Medical diagnosis of cardiovascular diseases using an interval-valued fuzzy rule-based classification system, Appl. Soft Comput. 20 (2014) 103–111.
[37] Y. Shen, W. Pedrycz, X. Wang, Approximation of fuzzy sets by interval type-2 trapezoidal fuzzy sets, IEEE Trans. Cybern. 50 (11) (2020) 4722–4734.
[38] X. Wang, E.E. Kerre, Reasonable properties for the ordering of fuzzy quantities (I), Fuzzy Sets Syst. 118 (3) (2001) 375–385.
[39] Y.-J. Wang, H.-S. Lee, The revised method of ranking fuzzy numbers with an area between the centroid and original points, Comput. Math. Appl. 55 (9) (2008) 2033–2042.
[40] B. Yatsalo, L. Martínez, Fuzzy rank acceptability analysis: a confidence measure of ranking fuzzy numbers, IEEE Trans. Fuzzy Syst. 26 (6) (2018) 3579–3593.
[41] L.A. Zadeh, Fuzzy set, Inf. Control 8 (1965) 338–353.

[42] L.A. Zadeh, Fuzzy sets as a basis for a theory of possibility, Fuzzy Sets Syst. 1 (1978) 3–28.

[43] L.A. Zadeh, A theory of approximate reasoning, in: J.E. Hayes, D. Michie, L.I. Mikulich (Eds.), Machine Intelligence, vol. 9, Elsevier, New York, 1979, pp. 149–194.