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Abstract homogeneous functions and consistently influenced/disturbed multi-expert decision making

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***Abstract*—In this paper we propose a new generalization for the notion of homogeneous functions. We show some properties and how it appears in some scenarios. Finally we show how this generalization can be used in order to provide a new paradigm for decision making theory called *consistent influenced/disturbed decision making*. In order to illustrate the applicability of this new paradigm, we provide a toy example.**

***Index Terms*—Homogeneity, abstract homogeneity, consistently influenced/disturbed decision making, aggregations, pre-aggregations.**

I. Introduction

HOMOGENEITY is an analytical property that has been investigated for a very long time [1], [2], [3], [4], [5], [6] and continues to be a subject under investigation.

> "Homogeneity is a certain invariance of an object (a function, a set, etc.) with respect to a class of transformations called dilations. All linear and a lot of essentially nonlinear models of mathematical physics are homogeneous (symmetric) in some sense. Homogeneous models can be utilized as local approximations of dynamical systems if, for example, linearisation is too conservative, non-informative, or simply impossible" [7, p. vii]

Homogeneity has been applied in several areas such as: image processing [29], classification [13], [30], control [7], economy [31], [32] and others. It is not difficult to see its broad application, since every linear function is homogeneous. In fusion procedures, the homogeneity of degree one implies that contracting all the inputs by the same factor λ is equivalent to contracting the output by λ – see [33]. In image processing the output image of an homogeneous operator of degree one remains proportional to the intensities of the pixels of the considered image, even if it is lightened or darkened [34].

As far as we know, the first generalization of homogeneity is due to Ebanks [8] in 1998. He has introduced the notion of quasi-homogeneity of associative functions, studying, in particular, the case of t-norms [9]. This concept was also investigated by G. Mayor et al [10], in the context of copulas [11]. Since then, homogeneity was studied in several forms. Recently, Su et al. [12] studied the characterization of all homogeneous/quasi-homogeneous binary aggregation functions in terms of single-argument functions.

In the context of overlap and grouping functions [13], [14], for example, Qiao and Hu [15] introduced the concept of pseudo-homogeneous overlap and grouping functions, which can be regarded as the generalizations of the concepts of homogeneous and quasi-homogenous overlap and grouping functions. Wang and Hu [16] studied the concept of (α, B, C)-homogeneity, (B, C)-homogeneity and B-homogeneity of overlap/grouping functions obtained by generator triples, where $B, C : L^2 \to L$ are operators on a complete lattice L and $\alpha \in L$. In fact, in the literature, one can find several works concerning the study of the homogeneity related to overlap and grouping functions, as in the works by Dimuro et al. [14], [17], [18], who studied the homogeneity property in general and consider the influence of the homogeneity for the overlap functions derived from the distortion of a positive continuous t-norm (t-conorm) by a pseudo-automorphism, in terms of their additive generator pairs.

Boczek et al. [19] studies some problems concerning the distributivity equation related to minitive and maxitive homogeneity of the upper n-Sugeno integral. Boczek and Kaluszka [20] presented the S-homogeneity property of seminormed fuzzy integral, answering to an open problem. Mesiar et al. [21] presented the generalized Choquet integral by means of fusion functions satisfying some requirements and studied the homogeneity property. Bustince et al. [22] introduced the concept of d-Choquet integral (the Choquet integral generalized by restricted dissimilarity functions) and study the homogeneity property in this context.

Lima et al. In [23] studied the pseudo-homogeneity of t-subnorms and, in [24], Lima et al. introduced the concept of h-pseudo homogeneity discussing this notion on some classes of nullnorms. Amarante [25] studied the positive homogeneity of Mm-OWA operators, proposed as generalization of OWA

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operators. In [26], Jurio et al. constructed weak homogeneity from a kind of interval homogeneity, in order to apply this concept to image segmentation.

Concerning interval-valued contexts, Lima et al. [27] introduced an interval extension of homogeneous and pseudo-homogeneous t-norms and t-conorms. Bedregal et al. [28] introduced interval-valued overlap functions and generalized interval-valued OWA operators with interval weights derived from them, studying the homogeneity property.

In this paper we propose a novel generalization of homogeneity, which differs from the works in the literature by providing more flexibility in the choice of its parameters. We show that this generalization occurs in many fields, for example in areas like fuzzy connectives and weak non-decreasing functions [35], [36]. We investigate some properties of this generalization and how it relates with known concepts. Finally, we show how it can be used to propose a new paradigm for decision making theory, called *consistently influenced/disturbed decision making*.

The structure of this paper is as follows. In section II we recall some basic concepts and results that will be of interest for the remainder of this paper. Section III is devoted to review the notion of homogeneity. Our definition of *abstract homogeneity* and the verification of some properties are provided in sections IV and V. In section VI we investigate the relation of abstract homogeneity with aggregation functions. In section VII we propose the use of abstract homogeneous functions in multi-expert decision making as the basis to formalize the notion of *consistently influenced/disturbed multi-expert decision making*. We finish the paper with some final remarks and a list of references.

II. Notation and Preliminaries

In this section we review some basic concepts and notations that are used in this paper. Sometimes we use the following vector notations: $\vec{x}$ for $(x_1, \ldots, x_n)$, $\vec{0}$ for $(0, \ldots, 0)$ and $\vec{1}$ for $(1, \ldots, 1)$.

A. Automorphisms and Fuzzy negations

Definition 1 *[37], [38] A function $\varphi\colon [0,1] \to [0,1]$ is said to be an automorphism on $[0,1]$ whenever it is continuous, strictly increasing, $\varphi(0) = 0$ and $\varphi(1) = 1$. Given functions $f, g : [0,1]^n \to [0,1]$, g is the conjugated of f if there is an automorphism φ such that $g = f^{\varphi}$ and $f^{\varphi}(x_1, \ldots, x_n) = \varphi^{-1}(f(\varphi(x_1), \ldots, \varphi(x_n)))$.*

Definition 2 *[39], [40] A function $N\colon [0,1] \to [0,1]$ is called fuzzy negation if : **(N1)** N is decreasing and **(N2)** $N(0) = 1$ and $N(1) = 0$. If $N(N(x)) = x$, then N is called strong. Given a function f, the application: $f_N(x_1, \ldots, x_n) = N(f(N(x_1), \ldots, N(x_n)))$ is called the N-dual of f.*

The function $N_Z(x) = 1 - x$ is generally called standard negation or Zadeh negation. It is a strong negation.

Theorem 3 *[41] A function $N : [0,1] \to [0,1]$ is a strong negation if and only if there exists an automorphism φ such that $N(x) = \varphi^{-1}(N_Z(\varphi(x)))$.*

B. Aggregation and Pre-Aggregation Functions

We recall the notion of aggregation functions [42], [43], [44], [45]; t-norms/t-conorms [9]; overlap/grouping functions [13], [14], [17], [18], [29], [34], [46], [47], [48]; weak [35], [36] and directional [49] monotonicity; and pre-aggregation functions [50], [51].

Remark 4 *In what follows we assume: (1) $\mathbb{N}^+ = \mathbb{N}\setminus\{0\}$ and (2) for any function $f : A \to B$ and $S \subseteq A$, the restriction of f to S is the function $f \restriction S : S \to B$, such that for all $x \in S$, $\big(f \restriction S\big)(x) = f(x)$.*

Definition 5 *An increasing n-ary function $A : [0,1]^n \to [0,1]$, $n \geq 1$, is called aggregation if $A(\vec{0}) = 0$ and $A(\vec{1}) = 1$. It is averaging (or a mean) if for every $\vec{x} \in [0,1]^n$, $\min(\vec{x}) \leq A(\vec{x}) \leq \max(\vec{x})$. We denote by $\mathcal{A}_n$ the set of all n-ary aggregation functions. An extended aggregation is a function $A : \bigcup\limits_{n\in\mathbb{N}^+} [0,1]^n \to [0,1]$ such that for every $n \geq 1$, the restriction $A^{(n)} = \big(A \restriction [0,1]^n\big)$ is also an aggregation, with the convention $A(x) = x$ for $n = 1$.*

Example 6 *The arithmetic mean: $M(\vec{x}) = \frac{1}{n}\sum\limits_{i=1}^{n} x_i$ and the geometric mean: $G_1(\vec{x}) = (\Pi_{i=1}^{n} x_i)^{\frac{1}{n}}$ are averaging aggregations.*

Note that every averaging aggregation function is idempotent.

Definition 7 *An associative and commutative bivariate aggregation function $A : [0,1]^2 \to [0,1]$ is called a t-norm whenever $A(x,1) = x$. On the other hand, it is called a t-conorm whenever $A(x,0) = x$.*

The minimum, the product and the Łukasiewicz conjunction defined, respectively, by: $T_M(x,y) = \min\{x,y\}, T_P(x,y) = x \cdot y$, and $T_{Ł}(x,y) = \max\{x + y - 1, 0\}$ are examples of t-norms. Examples of t-conorms are: the maximum, the probabilistic sum and the Łukasiewicz disjunction, defined by: $S_M(x,y) = \max\{x,y\}, S_P(x,y) = x + y - xy$, and $S_{Ł}(x,y) = \min\{x + y, 1\}$, respectively.

Definition 8 *[52], [53], [54], [55] A fuzzy implication is a bivariate function $I\colon [0,1]^2 \to [0,1]$ such that: **(I1)** if $x \leq y$, then $I(y,z) \leq I(x,z)$; **(I2)** if $y \leq z$, then $I(x,y) \leq I(x,z)$; **(I3)** $I(0,0) = 1$; **(I4)** $I(1,1) = 1$; and **(I5)** $I(1,0) = 0$.*

Example 9

1) $I_{Ł}(x,y) = \min(1, 1 - x + y)$
2) $$I_G(x,y) = \begin{cases} 1 & \text{, if } x \leq y \\ \frac{y}{x} & \text{, otherwise} \end{cases}$$

Definition 10 *[13], [48], [56] An overlap function is a bivariate function $O\colon [0,1]^2 \to [0,1]$, such that for all $x, y \in [0,1]$: **(O1)** $O(x,y) = O(y,x)$; **(O2)** $O(x,y) = 0$ if and only if $x \cdot y = 0$; **(O3)** $O(x,y) = 1$ if and only if $x = y = 1$; **(O4)** O is increasing; and **(O5)** O is continuous.*

Example 11 $O(x, y) = x^p \cdot y^p$, *for* $p > 0$.

Definition 12 *A grouping is a bivariate function* $G\colon [0,1]^2 \to [0,1]$ *such that for all* $x, y \in [0,1]$*:* ***(G1)*** $G(x,y) = G(y,x)$*;* ***(G2)*** $G(x,y) = 0$ *if and only if* $x = y = 0$*;* ***(G3)*** $G(x,y) = 1$ *if and only if* $x = 1$ *or* $y = 1$*;* ***(G4)*** G *is increasing; and* ***(G5)*** G *is continuous.*

Example 13 $G(x,y) = 1 - \sqrt{(1-x)(1-y)}$.

Definition 14 *Given a grouping function G (res. an overlap O) and a pair of continuous negations* N_1 *and* N_2*, s.t.* $N_i(x) = 0$ *iff* $x = 1$ *and dually* $N_i(x) = 1$ *iff* $x = 0$*. The function* $\overline{G}_{N_1,N_2} = N_1(G(N_2(x), N_2(y)))$ *is called the dual grouping (res. overlap) with respect to* N_1 *and* N_2*.*

Example 15 *Let* $O(x,y) = \sqrt{x \cdot y}$*, then* $G(x,y) = \overline{O}_{N_Z,N_Z}(x,y) = N_Z(O(N_Z(x), N_Z(y))) = 1 - \sqrt{(1-x)(1-y)}$.

Definition 16 *[35], [36] A function* $F : [0,1]^n \to [0,1]$ *is weakly increasing if for all points* $(x_1, \ldots, x_n) \in [0,1]^n$ *and for all* $c > 0$ *such that* $(x_1 + c, \ldots, x_n + c) \in [0,1]^n$*,*

$$F(x_1 + c, \ldots, x_n + c) \geq F(x_1, \ldots, x_n).$$

Dually we define weakly decreasing functions.

Definition 17 *[49] Let* $\vec{r} = (r_1, \ldots, r_n)$ *be a real n-dimensional vector* $\vec{r} \neq \vec{0}$*. A function* $F : [0,1]^n \to [0,1]$ *is* $\vec{r}$*-increasing if for all points* $(x_1, \ldots, x_n) \in [0,1]^n$ *and for all* $c > 0$ *such that* $(x_1 + cr_1, \ldots, x_n + cr_n) \in [0,1]^n$ *it holds*

$$F(x_1 + cr_1, \ldots, x_n + cr_n) \geq F(x_1, \ldots, x_n).$$

Dually, we define $\vec{r}$*-decreasing functions.*

Definition 18 *A function* $PA : [0,1]^n \to [0,1]$ *is said to be a n-ary pre-aggregation function [50], [51] if the following conditions hold:* ***(PA1)*** PA *is* $\vec{r}$*-increasing and* ***(PA2)*** $PA(\vec{0}) = 0$ *and* $PA(\vec{1}) = 1$*. If* F *is a pre-aggregation function and* $\vec{r}$*-increasing, then* F *is also called a* $\vec{r}$*-pre-aggregation function.* PA *is an internal pre-aggregation function [57] whenever for all* $\vec{x} \in [0,1]^n$*,* $PA(\vec{x}) = x_j$ *for some* $j \in \{1, \ldots, n\}$*.*

Definition 19 *Let be a tuple* $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$*,* $k(i, \vec{x})$ *be the number of occurences of* x_i *in* $\vec{x}$ *– i.e.* $k(i, \vec{x}) = \#\{j : x_i = x_j, 1 \leq j \leq n\}$*; where* $\#$ *denotes the cardinality of a set – and* $m = \max\{k(i, \vec{x}) : 1 \leq i \leq n\}$*, the multimode of* $\vec{x}$ *is the set of all modes of* $\vec{x}$*, i.e.* $mmode(\vec{x}) = \{x_i : k(i, \vec{x}) = m\}$*.*

Example 20 *For* $\vec{x} = (0.2, 0.3, 0.5, 0.7, 0.3, 0.9, 0.7)$*,* $k(2, \vec{x}) = \#\{2, 5\} = 2$*,* $m = \max\{1, 2\} = 2$ *and* $mmode(\vec{x}) = \{0.3, 0.7\}$*.*

Example 21 *Let* $\mathcal{P}^{fin}([0,1])$ *be the set of all non-empty finite subsets of* $[0,1]$ *and* $ch : \mathcal{P}^{fin}([0,1]) \to [0,1]$ *a choice function (i.e.,* $ch(\{x_1, \ldots x_k\}) \in \{x_1, \ldots x_k\}$*). If* $\{x_1, \ldots x_n\} \in \mathcal{P}^{fin}([0,1])$ *and* $k \leq 1 - \max(x_1, \ldots x_n)$*,* $ch(\{x_1, \ldots, x_n\}) + k = ch(\{x_1 + k, \ldots, x_n + k\})$*, then the composed function* $(ch \circ mmode)$ *is an internal pre-aggregation.*

III. Homogeneity

Definition 22 *Consider* $\gamma \in [0, +\infty[$*. A function* $F : [0,1]^n \to [0,1]$ *is said to be homogeneous of order* γ *whenever for every* $\lambda, x_1, \ldots, x_n \in [0,1]$*,*

$$F(\lambda x_1, \ldots, \lambda x_n) = \lambda^\gamma F(x_1, \ldots, x_n)$$

We consider $0^0 = 0$*.*

Example 23

1) *A constant function is homogeneous of order* 0*.*
2) *The maximum and the minimum are* 1*-homogeneous functions.*
3) *The* n*-dimensional product* $\Pi_n(x_1, \ldots, x_n) = \Pi_{i=1}^n x_i$ *is homogeneous of order* n*.*
4) *Given* $\gamma > 0$*, the function* $G_\gamma : [0,1]^n \to [0,1]$ *given by* $G_\gamma(\vec{x}) = (\Pi_{i=1}^n x_i)^{\frac{\gamma}{n}}$ *is homogeneous of order* γ*.*

A. Homogeneity and Aggregations

Let $\mathcal{H}_\gamma^n$ be the family of all n-ary γ-homogeneous functions and $\mathcal{AH}_n^\gamma$ the family of all n-ary γ-homogeneous aggregation functions.

Remark 24 *On the usual definition of homogeneous functions either one considers* $\lambda > 0$ *or the point* $\vec{0}$ *is discarded from the domain. However, since we are also interested in homogeneous aggregation functions, we consider both* $\lambda = 0$ *and* $\vec{0}$*. Observe that whenever an aggregation function* A *is homogeneous of order* γ*, we have that*

$$A(\vec{0}) = \begin{cases} A(0 \cdot x_1, \ldots, 0 \cdot x_n) = 0^\gamma \cdot A(\vec{x}) = 0 & \text{if } \gamma > 0 \\ A(0 \cdot x_1, \ldots, 0 \cdot x_n) = 0^0 \cdot A(\vec{x}) = 0 & \text{if } \gamma = 0, \end{cases}$$

which is one of the boundary conditions ***(A2)****. Hence, we do not lose any generality.*

Theorem 25 *Consider* $A_1, \ldots, A_m \in \mathcal{AH}_\gamma^n$ *for some* $\gamma \geq 0$*. Then, for every* $A \in \mathcal{AH}_\eta^m$ *(with* $\eta \geq 0$*), it holds that* $A(A_1, \ldots, A_m) \in \mathcal{AH}_{\gamma\eta}^n$*, where:* $A(A_1, \ldots, A_m)(\vec{x}) = A(A_1(\vec{x}), \ldots, A_m(\vec{x}))$*. In particular, if we take* $\eta = 1$*, it is immediate that* $A(A_1, \ldots, A_m) \in \mathcal{AH}_\gamma^n$*.*

Proof: It follows from a straightforward calculation. □

For all $A_1, A_2 \in \mathcal{AH}_\gamma^n$ let be the functions $A_1 \vee A_2, A_1 \wedge A_2 : [0,1]^n \to [0,1]$ s.t. $A_1 \vee A_2(\vec{x}) = \max\{A_1(\vec{x}), A_2(\vec{x})\}$ and $A_1 \wedge A_2(\vec{x}) = \min\{A_1(\vec{x}), A_2(\vec{x})\}$.

Corollary 26 *If* $A_1, A_2 \in \mathcal{AH}_\gamma^n$ *and* $\gamma \geq 0$*, then* $A_1 \vee A_2, A_1 \wedge A_2 \in \mathcal{AH}_\gamma^n$*.*

Corollary 27 *For all* $A \in \mathcal{AH}_\gamma^n$ *and some* $\gamma > 0$*,* $\min(\vec{x})^\gamma \leq A(\vec{x}) \leq \max(\vec{x})^\gamma$*.*

As a consequence we have the following:

Theorem 28 *For $\gamma > 0$, $(\mathcal{AH}^n_\gamma, \leq)$ is a bounded lattice, with top and bottom elements given, respectively, by the functions $A_\top, A_\perp : [0,1]^n \to [0,1]$ defined by: $A_\top(\vec{x}) = \max\{\vec{x}\}^\gamma$ and $A_\perp(\vec{x}) = \min\{\vec{x}\}^\gamma$.*

Proof: Firstly, we show that, for all $A_1, A_2 \in \mathcal{AH}^n_\gamma$, it holds that $\sup\{A_1, A_2\} = A_1 \vee A_2$. From corollary 26, for all $A_1, A_2 \in \mathcal{AH}^n_\gamma$, one has that $A_1 \vee A_2 \in \mathcal{AH}^n_\gamma$, and it is immediate that $A_1, A_2 \leq A_1 \vee A_2$, since for all $\vec{x} \in [0,1]^n$, one has that $A_1(\vec{x}) \leq \max\{A_1(\vec{x}), A_2(\vec{x})\} = A_1 \vee A_2(\vec{x})$, and similarly for A_2. Now, consider that there exists $A_3 \in \mathcal{AH}^n_\gamma$ such that $A_1, A_2 \leq A_3$ and $A_1 \vee A_2 \not\leq A_3$. Then there exists $\vec{x} \in [0,1]^n$ such that: $A_3(\vec{x}) < A_1 \vee A_2(\vec{x}) = \max\{A_1(\vec{x}), A_2(\vec{x})\}$.

Now, suppose that $\max\{A_1(\vec{x}), A_2(\vec{x})\} = A_1(\vec{x})$. It follows that $A_3(\vec{x}) < A_1(\vec{x})$, which is a contradiction with the fact that $A_1 \leq A_3$. A similar contradiction is obtained whenever one considers that $\max\{A_1(\vec{x}), A_2(\vec{x})\} = A_2(\vec{x})$. So, $A_1 \vee A_2 \leq A_3$ and $\sup\{A_1, A_2\} = A_1 \vee A_2$. Analogously one proves that $\inf\{A_1, A_2\} = A_1 \wedge A_2$. This proves that $(\mathcal{AH}^n_\gamma, \leq)$ is a lattice. Finally, for $\lambda \in [0,1]$ and $\vec{\lambda x} = (\lambda x_1, \ldots, \lambda x_n)$, one has that: $A_\top(\vec{\lambda x}) = \max\{\vec{\lambda x}\}^\gamma = \lambda^\gamma \max\{\vec{x}\}^\gamma = \lambda^\gamma A_\top(\vec{x})$, and, thus, $A_\top \in \mathcal{AH}^n_\gamma$. Similarly, one proves that $A_\perp \in \mathcal{AH}^n_\gamma$. By corollary 27, one has that $A_\perp \leq A \leq A_\top$, for all $A \in \mathcal{AH}^n_\gamma$. □

Proposition 29 *Let $A : [0,1]^n \to [0,1]$ be a γ-homogeneous aggregation function. Then $A(x, \ldots, x) = x^\gamma$ for every $x \in [0,1]$. Here we assume $0^0 = 0$.*

Proof: Straight from the homogeneity. □

IV. Abstract homogeneity

In this section we propose a generalization for homogeneity called **abstract homogeneity**. In a nutshell we replace the operation of multiplication by a general function and investigate the consequences of this abstraction. We focus on the case of homogeneous functions of order 1.

Definition 30 *Let be the functions $g : [0,1]^2 \to [0,1]$ and $F : [0,1]^n \to [0,1]$ and an automorphism $\varphi : [0,1] \to [0,1]$. A partial function F is said to be abstract homogeneous with respect to g and φ or just (g,φ)-homogeneous if for every $\lambda, x_1, \ldots, x_n \in [0,1]$, s.t. $\Big(g(\lambda, x_1), \ldots, g(\lambda, x_n)\Big) \in [0,1]^n$,*

$$F\big(g(\lambda, x_1), \ldots, g(\lambda, x_n)\big) = g\big(\varphi(\lambda), F(x_1, \ldots, x_n)\big),$$

if φ is the identity function, then g is called g-homogeneous instead of (g,φ)-homogeneous.

Note that this is a generalization of Def. 22.

Proposition 31 *Let $F : [0,1]^n \to [0,1]$ be a homogeneous function of order $\gamma \in [0, +\infty[$. Then it is (g,φ)-homogeneous for $g(x,y) = x \cdot y$ and $\varphi(x) = x^\gamma$.*

Proof: Straightforward. □

The next examples assume the identity automorphism.

Example 32

1) *Consider the arithmetic mean:*
$$M(x_1, \ldots, x_n) = \frac{x_1 + \cdots + x_n}{n} \ .$$
If $g(x,y) = \frac{x+y}{2}$, then
$$g\big(\lambda, M(x_1, \ldots, x_n)\big) = M\big(g(\lambda, x_1), \ldots, g(\lambda, x_n)\big)$$
for every $\lambda \in [0,1]$. So, M is g-homogeneous.
2) *If $g(x,y) = \sqrt{xy}$, then for $\lambda \in [0,1]$,*
$$\max(\sqrt{\lambda x_1}, \ldots, \sqrt{\lambda x_n}) = \sqrt{\lambda \max(x_1, \ldots, x_n)}$$
and
$$\min(\sqrt{\lambda x_1}, \ldots, \sqrt{\lambda x_n}) = \sqrt{\lambda \min(x_1, \ldots, x_n)} \ .$$
So both max *and* min *are g-homogeneous.*

The next example will be used in our toy algorithm at the end of this paper.

Example 33 *Consider the multimode function of Def. 19, the choice function* max *and the weighted average function $g_a(x,y) = a \cdot x + (1-a) \cdot y$, for $0 \leq a \leq 1$. Then $\max \circ mmode$ is g_a-homogeneous for any $a \in [0,1]$.*

In fact, for any $\vec{x} \in [0,1]^n$, $\lambda \in [0,1]$ and $x_1, \ldots, x_n \in [0,1]$, let be: $\lambda^n = \overbrace{(\lambda, \ldots, \lambda)}^{n}$, $\lambda \cdot \{x_1, \ldots, x_n\} = \{\lambda \cdot x_1, \ldots, \lambda \cdot x_n\}$, and $\lambda + \{x_1, \ldots, x_n\} = \{\lambda + x_1, \ldots, \lambda + x_n\}$. Then $mmode(\lambda \cdot \vec{x}) = \lambda \cdot mmode(\vec{x})$ and $mmode(\lambda^n + \vec{x}) = \lambda + mmode(\vec{x})$. Hence, $\lambda \cdot a + (1-a) \cdot mmode(\vec{x}) = mmode((\lambda \cdot a)^n + (1-a) \cdot \vec{x}) = mmode(\lambda \cdot a + (1-a) \cdot x_1, \ldots, \lambda \cdot a + (1-a) \cdot x_n) = mmode(g_a(\lambda, x_1), \ldots, g_a(\lambda, x_n))$. Therefore $\max(mmode(g_a(\lambda, x_1), \ldots, g_a(\lambda, x_n))) = \max(\lambda \cdot a + (1-a) \cdot mmode(\vec{x})) = \lambda \cdot a + \max((1-a) \cdot mmode(\vec{x})) = \lambda \cdot a + (1-a) \cdot \max(mmode(\vec{x})) = g_a(\lambda, \max(mmode(\vec{x})))$.

Proposition 34 *Let be a bijection $\rho : [0,1] \to [0,1]$ and a vector $\overrightarrow{g^\rho(\lambda, x)} = (g^\rho(\lambda, x_1), \ldots, g^\rho(\lambda, x_n))$. If $F : [0,1]^n \to [0,1]$ is g-homogeneous, then F^ρ is g^ρ-homogeneous, where $g^\rho(x,y) = \rho^{-1}(g(\rho(x), \rho(y)))$.*

Proof:

$$\begin{aligned}
F^\rho(\overrightarrow{g^\rho(\lambda, x)}) &= \rho^{-1}(F(\rho(g^\rho(\lambda, x_1)), \ldots, \rho(g^\rho(\lambda, x_n)))) \\
&= \rho^{-1}(F(\rho(\rho^{-1}(g(\rho(\lambda), \rho(x_1)))), \ldots, \\
&\qquad \rho(\rho^{-1}(g(\rho(\lambda), \rho(x_n)))) \ )) \\
&= \rho^{-1}(F(g(\rho(\lambda), \rho(x_1)), \ldots, g(\rho(\lambda), \rho(x_n)))) \\
&= \rho^{-1}(g(\rho(\lambda), F(\rho(x_1), \ldots, \rho(x_n)))) - g\text{-homogeneity} \\
&= \rho^{-1}(g(\rho(\lambda), \rho(\rho^{-1}(F(\rho(x_1), \ldots, \rho(x_n)))))) \\
&= \rho^{-1}(g(\rho(\lambda), \rho(F^\rho(x_1, \ldots, x_n)))) \\
&= g^\rho(\lambda, F^\rho(x_1, \ldots, x_n))
\end{aligned}$$

□

Lemma 35 *Let be a function $g : [0,1]^2 \to [0,1]$ and a bijective function $\rho : [0,1] \to [0,1]$ s.t.*

$$\rho(g(x,y)) = g(\rho(x), \rho(y)), \tag{1}$$

then $\rho^{-1}(g(x,y)) = g(\rho^{-1}(x), \rho^{-1}(y))$.

Proof: Observe that $g(\rho^{-1}(x), \rho^{-1}(y)) = \rho^{-1}(\rho(g(\rho^{-1}(x), \rho^{-1}(y))))$. By hypothesis it is equal to $\rho^{-1}(g(\rho(\rho^{-1}(x)), \rho(\rho^{-1}(y)))) = \rho^{-1}(g(x,y))$. □

Proposition 36 *For every bijection ρ, if F is g-homogeneous and $\rho(g(x,y)) = g(\rho(x), \rho(y))$, then F^ρ is also g-homogeneous.*

Proof: Given a bijection ρ, suppose that F is g-homogeneous and g satisfies (1), then for $\overrightarrow{g(\lambda, x)} = (g(\lambda, x_1), \ldots, g(\lambda, x_n))$,

$$\begin{aligned} F^\rho(\overrightarrow{g(\lambda, x)}) &= \rho^{-1}(F(\rho(g(\lambda, x_1)), \ldots, \rho(g(\lambda, x_n)))) \\ &= \rho^{-1}(F(g(\rho(\lambda), \rho(x_1)), \ldots, g(\rho(\lambda), \rho(x_n))))\text{by hypth} \\ &= \rho^{-1}(g(\rho(\lambda), F(\rho(x_1), \ldots \rho(x_n))))F \text{ is } g\text{-homog.} \\ &= g(\rho^{-1}(\rho(\lambda)), \rho^{-1}(F(\rho(_1), \ldots, \rho(x_n))))\text{by lemma 35} \\ &= g(\lambda, F^\rho(x_1, \ldots, x_n)) \end{aligned}$$

□

Example 37 *Consider $F(x,y) = x \cdot y$ and, $\rho_1(x) = x^k$ or $\rho_2(x) = x^{\frac{1}{k}}$, for $k \geq 2$.*

A. Abstract homogeneity, Shift-invariance, weak monotonicity and pre-aggregations

Proposition 38 *A function $F : [0,1]^n \to [0,1]$ is shift-invariant, – i.e. $F(\lambda + x_1, \ldots, \lambda + x_n) = F(x_1, \ldots, x_n) + \lambda \in [0,1]$ whenever $\lambda, x_1, \ldots, x_n, \max(x_1, \ldots, x_n) + \lambda \in [0,1]$ – if and only if it is abstract homogeneous with respect to the Łukasiewicz T-conorm $S_Ł(x,y) = \min(y + x, 1)$.*

Definition 39 *Let $g : [0,1]^2 \to [0,1]$ be a function. A partial function $F : [0,1]^n \to [0,1]$ is g-weak increasing if $F(g(\lambda, x_1), \ldots, g(\lambda, x_n)) \geq F(x_1, \ldots, x_n)$, for $(g(\lambda, x_1), \ldots, g(\lambda, x_n)) \in [0,1]^n$ and $\lambda > 0$.*

Theorem 40 *Let $g : [0,1]^2 \to [0,1]$ be a function such that $g(x,y) \geq y$. If $F : [0,1]^n \to [0,1]$ is g-homogeneous, then it is g-weak increasing. Moreover, for any bijection ρ satisfying (1), F^ρ is also g-weak increasing.*

Proof: Indeed, $F(g(\lambda, x_1), \ldots, g(\lambda, x_n)) = g(\lambda, F(x_1, \ldots, x_n)) \geq F(x_1, \ldots, x_n)$. Moreover, by proposition 36, F^ρ is also g-weak increasing. □

Corollary 41

1) *For any T-conorm S, if F is S-homogeneous, then it is S-weak increasing.*
2) *If F is $S_Ł$-homogeneous, then it is weak increasing.*
3) *For every T-conorm generated by Łukasiewicz T-conorm and an automorphism φ, $S_Ł^\varphi(x,y) = \varphi^{-1}(S_Ł(\varphi(x), \varphi(y)))$ – if φ satisfies equation (1) and F is $S_Ł^\varphi$-homogeneous, then F is also weak increasing.*
4) *Let $\vec{r} = (r, \ldots, r) \in]0, +\infty[^n$ be a real n-dimensional vector, then for any automorphism φ that satisfies equation (1), if a function $F : [0,1]^n \to [0,1]$ is $S_Ł^\varphi$-homogeneous, then F is $\vec{r}$-increasing.*

Proof:

1) Observe that $x, y \leq \max(x,y) \leq S(x,y)$ and apply Theorem 40.
2) Given $(x_1, \ldots, x_n) \in [0,1]^n$ and $\lambda > 0$, such that $(x_1 + \lambda, \ldots, x_n + \lambda) \in [0,1]^n$, since $S_Ł(x,y) = \min(x + y, 1)$, then $F(x_1 + \lambda, \ldots, x_n + \lambda) = F(S_Ł(\lambda, x_1), \ldots, S_Ł(\lambda, x_n)) = S_Ł(\lambda, F(x_1, \ldots, x_n)) \geq F(x_1, \ldots, x_n)$.
3) Apply the previous result plus proposition 40.
4) It follows from item 3), considering $\lambda = cr$ and $c > 0$.

□

Proposition 42 *Let $\vec{r} = (r, \ldots, r) \in]0, +\infty[^n$ be a real n-dimensional vector and an automorphism φ that satisfies equation (1). If a function $F : [0,1]^n \to [0,1]$ is $S_Ł^\varphi$-homogeneous and $F(x, \ldots, x) = 0$ for some $x \in [0,1]$, then F is a pre-aggregation function.*

Proof: By item 4) in corollary 41, F is $\vec{r}$-increasing. In addition, $F(1, \ldots, 1) = F(S_Ł^\varphi(1,0), \ldots, S_Ł^\varphi(1,0)) = S_Ł^\varphi(1, F(0, \ldots, 0)) = 1$.

Finally, suppose F is $S_Ł^\varphi$-homogeneous and $F(x, \ldots, x) = 0$, for some $x \in [0,1]$, then $S_Ł^\varphi(x, F(0, \ldots, 0)) = F(S_Ł^\varphi(x,0), \ldots, S_Ł^\varphi(x,0)) = F(0 + x, \ldots, 0 + x) = F(x, \ldots, x) \overset{\text{hip}}{=} 0$. Therefore, $x = 0$ and $F(0, \ldots, 0) = 0$. □

B. Abstract Homogeneity in Fuzzy Logic

.

Abstract homogeneity appears in Fuzzy Logic under the well-known name of distributivity. For example, if H is a T-norm/T-conorm and I is an implication which distributes over H:

$$I(x, H(y,z)) = H(I(x,y), I(x,z)).$$

We can say that H is I-**homogeneous**. This equation has been deeply investigated – c.f. Baczyński and Jayaram [52, §7.2.3 and §7.2.4]. Another occurrences of distributivity involves T-norms over T-conorms and vice-versa; for more details see [9]. Therefore, it is straightforward that the terminology "g-homogeneity" generalizes distributivity in Fuzzy Logic. For example, observe the following proposition:

Proposition 43 *Let N_Z be the standard fuzzy negation and the function $\Pi(x,y) = x \cdot y$. If $F : [0,1]^2 \to [0,1]$ is Π-homogeneous, then F_{N_Z} is Π_{N_Z}-homogeneous.*

Proof: $\Pi_{N_Z}(x,y) = x+y-xy$. If F is Π-homogeneous, then

$$\begin{aligned}
&F_{N_Z}(\Pi_{N_Z}(\lambda,x),_{N_Z}(\lambda,y))\\
&= F_{N_Z}(\lambda+x-\lambda\cdot x, \lambda+y-\lambda\cdot y)\\
&= 1-F(1-(\lambda+x-\lambda\cdot x), 1-(\lambda+y-\lambda\cdot y))\\
&= 1-F((1-\lambda)(1-x),(1-\lambda)(1-y))\\
&= 1-(1-\lambda)\cdot F(1-x,1-y) \text{ by } \Pi\text{-homogeneity}\\
&= 1-(F(1-x,1-y)-\lambda\cdot F(1-x,1-y))\\
&= 1-F(1-x,1-y)+\lambda\cdot F(1-x,1-y))
\end{aligned}$$

On the other hand:

$$\begin{aligned}
&\Pi_{N_Z}(\lambda, F_{N_Z}(x,y)) = \lambda + F_{N_Z}(x,y) - \lambda\cdot F_{N_Z}(x,y)\\
&= \lambda+1-F(1-x,1-y)-\lambda\cdot(1-F(1-x,1-y))\\
&= \lambda+1-F(1-x,1-y)-\lambda+\lambda\cdot F(1-x,1-y)\\
&= 1-F(1-x,1-y)+\lambda\cdot F(1-x,1-y).
\end{aligned}$$

□

In the case of T**-norms** the equation:

$$T(g(\lambda,x),g(\lambda,y)) = g(\lambda,T(x,y))$$

requires $g(\lambda,1)=1$.

Theorem 44 *Given a T-norm T and a function $g:[0,1]^2\to[0,1]$ which has 1 as identity and is increasing in the second argument, then T is g-homogeneous if and only if $T=\min$.*

Proof: Let T be a T-norm and $g:[0,1]^2\to[0,1]$ which has 1 as identity and is increasing in the second argument. Suppose T is g-homogeneous, then by corollary 51, T is idempotent and hence $T=\min$ (the unique idempotent T-norm). Suppose $T=\min$, let $x,y,\lambda\in[0,1]$, case $x\le y$, then $T(g(\lambda,x),g(\lambda,y)) = \min(g(\lambda,x),g(\lambda,y)) = g(\lambda,x) = g(\lambda,T(x,y))$. The other case is analogous. □

Corollary 45

1) *Given two T-norms T_1 and T_2, T_1 is T_2-homogeneous if and only if $T_1=\min$.*
2) *The only T-norm T which is T-homogeneous is the minimum.*
3) *Let T be a T-norm and S be a T-conorm. Then T is S-homogeneous if and only if T is the minimum and S is T-homogeneous if and only if S is the maximum.*

Proof: Minimum is the unique idempotent T-norm whereas maximum is the unique idempotent T-conorm – c.f. [9]. □

Proposition 46 *Let $g:[0,1]^2\to[0,1]$ be a function such that for all $\lambda\in[0,1]$, $g(\lambda,x)=1$ implies $x=1$. The **Drastic Product** T_D is g-homogeneous if and only if $g(\lambda,1)\in\{0,1\}$ and $g(\lambda,0)=0$.*

Proof: $g(\lambda,1) = g(\lambda,T_D(1,1)) = T_D(g(\lambda,1),g(\lambda,1)) \overset{\text{def}}{=} \begin{cases}1 \text{ if } g(\lambda,1)=1\\ 0 \text{ otherwise.}\end{cases}$

Since $g(\lambda,x)=1$ implies that $x=1$ then $g(\lambda,0)<1$. So $g(\lambda,0)\overset{\text{def}}{=} g(\lambda,T_D(0,0)) = T_D(g(\lambda,0),g(\lambda,0)) = 0$.

□

C. Analytical and algebraic properties

Let be $\mathcal{GH}_g^n = \{F : [0,1]^n\to[0,1] : F \text{ is } g\text{-homogeneous}\}$ and, given $F:[0,1]^n\to[0,1]$, $\mathcal{H}(F) = \{g:[0,1]^2\to[0,1] : F \text{ is } g\text{-homogeneous}\}$. Then we can start assuring that, for any F, $\mathcal{H}(F)$ is not empty.

Proposition 47 *Let $P_2(x,y)=y$ be the projection on the second component. Then, any function $F:[0,1]^n\to[0,1]$ is homogeneous with respect to P_2.*

Proof: Straightforward. □

Proposition 48 *Let $g:[0,1]^2\to[0,1]$ be an aggregation function. Then the following statements are equivalent: (1) Every $F:[0,1]^n\to[0,1]$ is g-homogeneous and (2) $g(x,y)=P_2(x,y)$.*

Proof: The fact that (2) implies (1) follows from the previous proposition. So assume that (1) holds but $g(x,y)\ne P_2(x,y)$. This means that there exist $x,y_0,y_1\in[0,1]$ such that $g(x,y_0)=y_1\ne y_0$. Consider the constant function $F(x_1,\dots,x_n)=y_0$, then $g(x,F(y_0,\dots,y_0)) = g(x,y_0) = y_1$, whereas $F(g(x,y_0),\dots,g(x,y_0)) = y_0$. Since $y_0\ne y_1$, the result follows. □

Note that, for each $\lambda\in[0,1]$ and a function $g:[0,1]^2\to[0,1]$ we can define the mapping: $g_\lambda:[0,1]\to[0,1]$ given by:

$$g_\lambda(t) = g(\lambda,t) \tag{2}$$

Then a function F is g-homogeneous whenever

$$F(g_\lambda(x_1),\dots,g_\lambda(x_n)) = g_\lambda(F(x_1,\dots,x_n))$$

for every $x_1,\dots,x_n,\lambda\in[0,1]$. If g_λ is bijective, we can state the following.

Proposition 49 *Let $F:[0,1]^n\to[0,1]$ be a function and $g:[0,1]^2\to[0,1]$ be an aggregation function such that g_λ at equation (2) is a bijection for every $\lambda\in[0,1]$. Then the following statements are equivalent:*

1) *F is g-homogeneous with respect to g;*
2) $g_\lambda^{-1}(F(g_\lambda(x_1),\dots,g_\lambda(x_n))) = F(x_1,\dots,x_n)$.

Proof: Straightforward. □

Lemma 50 *Let $F:[0,1]^n\to[0,1]$ be a g-homogeneous function and e the identity of F; i.e. $F(e,\dots,e,x,e,\dots,e)=x$. If $\varphi:[0,1]\to[0,1]$ is a bijective function and $\varphi(x)=g(x,e)$, then F is idempotent.*

Proof: Let $x \in [0,1]$ and $y = \varphi^{-1}(x)$, then $F(x,\ldots,x) = F(g(y,e),\ldots,g(y,e)) = g(y,F(e,\ldots,e)) = g(y,e) = x$. □

Corollary 51 *Let $F : [0,1]^n \to [0,1]$ be a g-homogeneous function. If F has identity e and $g(x,e) = x$, for all x, then F is idempotent.*

Proof: It is straightforward from the previous proposition, since the mentioned function φ is precisely the identity function. □

V. Self homogeneous functions

Let us recall example 32. If we consider $g(x,y) = M(x,y) = \frac{x+y}{2}$ (the arithmetic mean). Then, it follows that M is M-homogeneous. In this case, we say that M is self-homogeneous. On the other hand, the product $\Pi_2(x,y) = xy$ is s.t. $\Pi_2(\lambda,\Pi_2(x,y)) = \lambda xy$, whereas $\Pi_2(\Pi_2(\lambda,x),\Pi_2(\lambda,y)) = \lambda^2 xy$. So Π_2 is not self-homogeneous. In what follows we investigate the situation in which a function is self-homogeneous.

Definition 52 *Let $g : [0,1]^2 \to [0,1]$ be a function, g is said to be self-homogeneous if g is g-homogeneous.*

The following proposition shows sufficient conditions to ensure that a function g is self-homogeneous.

Proposition 53 *Every associative, commutative and idempotent function $g : [0,1]^2 \to [0,1]$ is self-homogeneous.*

Proof: Let $g : [0,1]^2 \to [0,1]$ be an associative, commutative and idempotent function. By associativity

$$g(g(\lambda,u),g(\lambda,v)) = g(\lambda,g(u,g(\lambda,v))).$$

Commutativity and associativity lead to

$$g(\lambda,g(u,g(\lambda,v))) = g(\lambda,g(g(u,v),\lambda)) = g(\lambda,g(\lambda,g(u,v))).$$

Again, by associativity and idempotency: $g(\lambda,g(\lambda,g(u,v))) = g(g(\lambda,\lambda),g(u,v)) = g(\lambda,g(u,v))$. Therefore, $g(g(\lambda,u),g(\lambda,v)) = g(\lambda,g(u,v))$, so we have the result. □

Corollary 54

1) *The only t-norm which is self-homogeneous is the minimum.*
2) *The only t-conorm which is self-homogeneous is the maximum.*

Example 55 *The converse of proposition 53 does not hold in general. For instance:*

1) *According to proposition 47, the second projection is self-homogeneous, it is also associative and idempotent but it is not commutative.*
2) *The geometric mean, $g(x,y) = \sqrt{xy}$ is self-homogeneous, idempotent and commutative, but it is not associative.*
3) *The smallest aggregation function A_* is self-homogeneous, commutative and associative, but it is not idempotent.*

Proposition 56 *Let $F : [0,1]^2 \to [0,1]$ be a self-homogeneous function. Then, for every $x,y \in [0,1]$*

$$F(x,F(x,y)) = F(F(x,x),F(x,y)) \tag{3}$$

If F is also injective, then it is idempotent.

Proof: If F is self-homogeneous, then for every $x,y,\lambda \in [0,1]$, $F(\lambda,F(x,y)) = F(F(\lambda,x),F(\lambda,y))$. So, taking $\lambda = x$, we have $F(x,F(x,y)) = F(F(x,x),F(x,y))$. If F is also injective, then by equation (3) it is straightforward to say that it is also idempotent. □

Regarding the converse of Prop. 53 we can state the following.

Proposition 57 *Let $F : [0,1]^2 \to [0,1]$ be a self-homogeneous continuous function such that $F(0,0) = 0$ and $F(1,1) = 1$. Then, if $F(0,1) = 0$ or $F(1,0) = 1$, it follows that F is idempotent.*

Proof: Case $F(0,1) = 0$, let's consider the function $f(\lambda) = F(\lambda,1)$. Clearly, in our hypothesis, $f(0) = 0$ and $f(1) = 1$. Moreover, f is surjective (due to the continuity). From the self-homogeneity, we have $F(F(\lambda,1),F(\lambda,1)) = F(\lambda,F(1,1)) = F(\lambda,1)$. Now, for every $t \in [0,1]$, there exists $\lambda(t) \in [0,1]$ such that $F(\lambda(t),1) = t$. So, $F(t,t) = F(F(\lambda(t),1),F(\lambda(t),1)) = F(\lambda(t),1) = t$.

The proof is analogous for $F(1,0) = 1$. □

Although there is a unique idempotent t-norm (the minimum), there are uncountable idempotent overlap functions [14]. The next corollary shows that there is a whole family of self-homogeneous idempotent overlaps.

Proposition 58 *Let f be an overlap function. If f is self-homogeneous, then it is also idempotent. The same applies if f is a grouping function.*

Example 59 *Take the overlap $O(x,y) = \sqrt{x \cdot y}$ and $G(x,y) = 1 - \sqrt{(1-x)(1-y)}$.*

VI. Abstract Homogeneity and Aggregations

Aggregation operators are applied in many fields, like: statistics, image processing, etc. The notion of invariant aggregation operators, i.e. aggregations which do not depend on the given scale of measurement is a powerful concept and also has applications in many fields. One type of such functions are those which are invariant with respect to the multiplication by a constant. They are known as *homogeneous aggregation functions*.

Tatiana and Roman Rückschlossová [58] proposed a way to build homogeneous operators from families of aggregation functions. In this section we generalize their work to the setting of abstract homogeneous functions. To achieve that we introduce the notion of g**-pairs** which are structures that together with associative aggregations provide us a family of abstract homogeneous functions with respect to g by using a function of the form $A : \bigcup_{n\in\mathbb{N}^+} [0,1]^n \to [0,1]$ – see Theorem 62.

Definition 60 *Given a function $g : [0,1]^2 \to [0,1]$, a g**-pair** is a structure $\langle h_g, f_g\rangle$, such that:*

1) *$h_g : [0,1]^n \to [0,1]$ is an abstract g-homogeneous function.*
2) *$f_g : [0,1]^2 \to [0,1]$ is a function such that:*
$$f_g(g(u,v), g(u,w)) = g(f_g(u,u), f_g(v,w)). \tag{4}$$
3) *for all $y, z \in [0,1]$,*
$$g(f_g(y,y), f_g(z, h_g(x_1,\ldots,x_n))) = f_g(z, h_g(x_1,\ldots,x_n)). \tag{5}$$

Example 61

1) *Given the aggregation $g(x,y) = x^q \cdot y$, for $q \in \mathbb{N}^+$, the following functions are g-pairs:*
   a) $\varphi_1(x,y) = \langle \max(x_1,\ldots,x_n), \min(1, \frac{x}{y})\rangle$
   b) $\varphi_2(x,y) = \langle G_1(x_1,\ldots,x_n), \min(1, \frac{x}{y})\rangle$.
   c) $\varphi_3(x,y) = \langle M_p(x_1,\ldots,x_n), \min(1, \frac{x}{y})\rangle$*; for* $p \in]0,+\infty[$ *and* $M_p(x_1,\ldots,x_n) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}}$.
2) *Let $\mathbf{S}(x,y) = y \cdot \sin(x \cdot \frac{\pi}{2})$. Then $\langle M(x_1,\ldots,x_n), \min(1,\frac{x}{y})\rangle$, φ_1, φ_2, and φ_3 are $\mathbf{S}$-pairs.*
3) *Any pair of functions $\langle f_g, h_g\rangle$ is a g-pair for $g(x,y) = y$.*

Theorem 62 *Given a g-pair $\langle h_g, f_g\rangle$, if g is **associative**, then for every function $A : [0,1]^n \to [0,1]$, the function:*

$$H^A(\vec{x}) = g(h_g(\vec{x}), A(f_g(x_1, h_g(\vec{x})), \ldots, f_g(x_n, h_g(\vec{x}))))$$

is g-homogeneous.

Proof: Let g be an associative function together with a g-pair $\langle h_g, f_g\rangle$ and a function $A : [0,1]^n \to [0,1]$. Without loss of generality we demonstrate just for two arguments. For readability we use the notation: $\vec{\kappa} = (g(\lambda, x_1), g(\lambda, x_2))$ and $\omega = g(\lambda, h_g(x_1, x_2))$.

$H^A(\vec{\kappa}) \stackrel{\text{def}}{=} g(h_g(\vec{\kappa}), A(f_g(g(\lambda,x_1), h_g(\vec{\kappa})), f_g(g(\lambda,x_2), h_g(\vec{\kappa}))))$. Since h_g is g-homogeneous, then $H^A(\vec{\kappa}) = g(\omega, A(f_g(g(\lambda,x_1),\omega), f_g(g(\lambda,x_2),\omega)))$. By equation (4) $H^A(\vec{\kappa}) =$
$g(\omega, A(g(f(\lambda,\lambda), f_g(x_1, h_g(x_1,x_2))), g(f_g(\lambda,\lambda), f_g(x_2, h_g(x_1,x_2)))))$.

By equation (5) $H^A(\vec{\kappa}) =$
$g(g(\lambda, h_g(x_1,x_2)), A(f_g(x_1, h_g(x_1,x_2)), f_g(x_2, h_g(x_1,x_2))))$.
Since g is associative, then $H^A(\vec{\kappa}) =$
$g(\lambda, g(h_g(x_1,x_2), A(f_g(x_1, h_g(x_1,x_2)), f_g(x_2, h_g(x_1,x_2)))))$
i.e. $H^A(\vec{\kappa}) = g(\lambda, H^A(x_1,x_2))$. □

Remark 63 *Given an associative function g, each g-pair $\varphi = \langle h_g, f_g\rangle$ provides a family of g-homogeneous function $H(\varphi) \stackrel{\text{def}}{=} \{H^A \mid A : [0,1]^n \to [0,1]\}$.*

Example 64

1) *Let $\Pi(x,y) = x \cdot y$ and the g-pairs stated in example 61.1. Then $H(\varphi_1)$, $H(\varphi_2)$ and $H(\varphi_3)$ are families of Π-homogeneous functions.*
2) *Let be $g(x,y) = y$, then any g-pair φ provides a family $H(\varphi)$ of g-homogeneous functions. See example 61.3.*

Lemma 65 *Let g be an associative function and $\varphi = \langle h_g, f_g\rangle$ a g-pair. If $A : [0,1]^n \to [0,1]$ is a function such that $A(\vec{0}) = 0$ and $A(\vec{1}) = 1$, $g(x,0) = g(0,x) = 0$ for every $x \in [0,1]$, $h_g(\vec{1}) = 1$ and $f_g(1,1) = 1$, then $H^A(\vec{0}) = 0$ and $H^A(\vec{1}) = 1$.*

Proof:

1) $H^A(\vec{0}) \stackrel{\text{def}}{=}$
$g(h_g(\vec{0}), A(g(0, f_g(0, h_g(\vec{0}))), \ldots, g(0, f_g(0, h_g(\vec{0}))))) \stackrel{\text{hip}}{=}$
$g(h_g(\vec{0}), A(\vec{0})) \stackrel{\text{hip}}{=} g(h_g(\vec{0}), 0) \stackrel{\text{hip}}{=} 0$
2) $H^A(\vec{1}) \stackrel{\text{def}}{=}$
$g(h_g(\vec{1}), A(g(1, f_g(1, h_g(\vec{1}))), \ldots, g(1, f_g(1, h_g(\vec{1})))))$
$\stackrel{\text{hip}}{=} g(1, A(g(1, f_g(1,1)), \ldots, g(1, f_g(1,1))))$
$\stackrel{\text{hip}}{=} g(1, A(g(1,1), \ldots, g(1,1)))$
$= g(1, A(\vec{1})) = g(1,1) = 1.$

□

Example 66 *Let A be any extended aggregation and the g-pairs stated in example 61.1, then $H^A(\vec{0}) = 0$ and $H^A(\vec{1}) = 1$; e.g. $H^{\mathbf{M}}$, $H^{\mathbf{G}}$, $H^{\mathbf{S}}$, etc. Note that although 0 is not an annihilator for $g(x,y) = y$, we also have $H^A(\vec{0}) = 0$ and $H^A(\vec{1}) = 1$.*

The next theorem establishes sufficient conditions for H^A be an aggregation.

Theorem 67 *Let g be an associative aggregation, $\varphi = \langle h_g, f_g\rangle$ a g-pair such that h_g is first-place non decreasing*[1] *and $\vec{x} \le \vec{y}$ implies $h_g(\vec{x}) \le y_k$, for all k.*

Consider $x_g = h_g(\vec{x})$ and $y_g = h_g(\vec{y})$. If A is an aggregation that satisfies the following condition:

$$\begin{aligned} &g(y_g, A(f_g(y_1,y_g),\ldots,f_g(y_n,y_g))) \ge \\ &g(x_g, A(f_g(y_1,x_g),\ldots,f_g(y_n,x_g))) \end{aligned} \tag{6}$$

then, H^A is a non decreasing g-homogeneous function. If 0 is an annihilator for g, $h_g(\vec{1}) = 1$ and $f_g(1,1) = 1$, then H^A is an aggregation.

Proof: By Theorem 62, H^A is g-homogeneous function. Suppose $\vec{x} \le \vec{y}$, since f_g is first place non decreasing, then $f_g(x_k, x_g) \le f_g(y_k, x_g)$. Since A and g are both aggregations, then $H^A(\vec{x}) \stackrel{\text{def}}{=} g(x_g, A(f_g(x_1,x_g),\ldots,f_g(x_n,x_g))) \le g(x_g, A(f_g(y_1,x_g),\ldots,f_g(y_n,x_g)))$. By transitivity and condition (6) $H^A(\vec{x}) \stackrel{\text{def}}{=} g(x_g, A(f_g(x_1,x_g),\ldots,f_g(x_n,x_g))) \le$

[1] i.e. $x \le y$ implies $h_g(x,z) \le h_g(y,z)$.

$g(y_g, A(f_g(y_1,y_g),\ldots,f_g(y_1,y_g))) \overset{\text{def}}{=} H^A(\vec{y})$. Therefore, H^A is non decreasing.

Moreover, if 0 is an annihilator for g, $h_g(\vec{1}) = 1$ and $f_g(1,1) = 1$, then by lemma 65, H^A is an aggregation. □

Example 68 *The g-pair* $\langle \max(\vec{x}), \min(1, \frac{x}{y})\rangle$ *together with the aggregations* $g_1(x,y) = x \cdot y$ *and* $g(x,y) = y$ *satisfy Theorem 67.*

The next section shows that abstract homogeneity can be used to provide a new paradigm of multi-expert decision making systems called *consistent influenced/disturbed multi-expert decision making systems*. We provide a toy example and a toy algorithm to illustrate our paradigm.

VII. Abstract homogeneity and Consistently influenced Multi-expert decision making

This section introduces a new type of decision-making approach. It shows that *abstract homogeneous functions* can be used to model the situation in which a *consensus relation* of a multi-expert decision making is *consistently influenced*. Before we proceed, we provide an overview of what we mean by a decision making system with an adaptation of one of its phase in order to encompass pre-aggregations.

A *multi-expert decision making problem based on preference relations* can be summarized in the following way: We have a set of p alternatives $X = \{x_1,\ldots,x_p\}$, with $p > 2$, and a set of n experts $E = \{e_1,\ldots,e_n\}$, $(n > 2)$. Each of the experts provides his/her preferences on the alternatives. We assume that the expert e_t (with $t \in \{1,\ldots,n\}$) expresses his/her preferences by means of a relation (matrix)

$$R^t = \begin{pmatrix} \cdot & R^t_{12} & \ldots & R^t_{1p} \\ R^t_{21} & \cdot & \ldots & R^t_{2p} \\ \ldots & \ldots & \ldots & \ldots \\ R^t_{p1} & R^t_{p2} & \ldots & \cdot \end{pmatrix}$$

where $R^t_{ij} \in [0,1]$ expresses the preference of expert e_t on alternative x_i over alternative x_j. Note that we do not impose any additional condition for R^t.

We must find a solution, either an alternative or a set of alternatives, which is (are) the most accepted one(s) by the experts.

The literature proposes two steps to solve a problem of multi-expert decision making – c.f. [59].

1) Uniform representation of information. In this phase, the heterogeneous information for the problem (the information can be represented by means of preference orderings or utility functions or fuzzy preference relations) is translated into a homogeneous information by means of different transformation functions. We assume that this step has already been fulfilled when the preference relations R^t are built.
2) Application of a *selection procedure*. This procedure consists of two phases:
   - Aggregation phase. A collective preference relation is built from the set of individual preference relations.
   - Exploitation phase. A given method is applied to the collective preference structure to obtain a selection of alternatives.

We focus on **Aggregation phase**. However, since the name: "**Aggregation phase**" induces the reader to think about the use of *aggregation functions* and we want include the application of other functions, we suggest new names for this phase, namely: **Amalgamation phase** and **amalgamator**. In what follows, we provide a mathematical description for this phase (amalgamation):

A. Abstract Homogeneity and Amalgamation phase

The reduction of all the given preference relations R^t into one single collective preference relation R^C is done in this phase using *pre-aggregation functions* which we call **amalgamators**. In other words, given a pre-agregation function (amalgamator) $A : [0,1]^n \rightarrow [0,1]$ and the preference relations: $R^1,\ldots,R^n$, the *Amalgamation Phase* (by using A) can be seen as a function $\widehat{A} : \mathcal{R}^n_{p\times p} \rightarrow \mathcal{R}_{p\times p}$ s.t:

$$\widehat{A}(R^1,\ldots,R^n)_{ij} = \begin{cases} 0.5 & \text{, if } i = j \\ A(R^1_{ij},\ldots,R^n_{ij}) & \text{, otherwise} \end{cases}$$

where $\mathcal{R}_{p\times p}$ is the set of all preference relations on p alternatives.

In what follows *we propose a toy algorithm for the amalgamation phase* of a multi-expert decision making system. In this case we use the $mode$ (which is not an aggregation function) as the basic function to **amalgamate** the data. We follow with an illustrative application.

Algorithm 1:

Input: n preference relations:

$$R^t = \begin{pmatrix} \cdot & R^t_{12} & \ldots & R^t_{1p} \\ R^t_{21} & \cdot & \ldots & R^t_{2p} \\ \ldots & \ldots & \ldots & \ldots \\ R^t_{p1} & R^t_{p2} & \ldots & \cdot \end{pmatrix}, \text{ for } t \in \{1,\ldots,n.\}$$

Output: A collective preference relation:

$$R^C = \begin{pmatrix} \cdot & R^C_{12} & \ldots & R^C_{1p} \\ R^C_{21} & \cdot & \ldots & R^C_{2p} \\ \ldots & \ldots & \ldots & \ldots \\ R^C_{p1} & R^C_{p2} & \ldots & \cdot \end{pmatrix}$$

1 **for** $i = 1$ to p **do**
2   **for** $j = 1$ to p **do**
3   **if** $i = j$ **then**
4             $R^C_{ii} = 0.5$
5   **else**
6             $R^C_{ij} \longleftarrow \max(mmode(R^1_{ij},\ldots,R^n_{ij}))$
7   **end**
8   **end**
9 **end**

Obs: Step **6** returns the composition of $\max$ with the choice function $mmode$. According to example 21, this composition is an internal pre-aggregation. Observe that the user can replace $\max$ by any function f which chooses an element of $mmode$ s.t. $f \circ mmode$ is a pre-aggregation.

B. Illustrative example

Consider a multi-expert decision making problem with three alternatives (a_1, a_2, a_3) and six experts $(e_1, \ldots, e_6)$. Each expert provides his/her preference relations (Table I). Each entry R^t_{ij} of the relations R^t of Table I, where $t = 1, \ldots, 6$, indicates the preference of expert e_t on alternative a_i over alternative a_j, where $i, j = 1, 2, 3$.

Table II(a) shows the multimodes of components R^t_{ij}, with $t = 1, \ldots, 6$. The multimodes are calculated using Def. 19, considering all preference relations of Table I. For example, for R^t_{23} (i.e., the various preferences of alternative a_2 over alternative a_3 according to the six experts), using Def. 19, one has that $\vec{x} = (R^1_{23}, \ldots, R^6_{23}) = (0.2, 0.4, 0.6, 0.2, 0.3, 0.4)$ and $k(1, \vec{x}) = \#\{1, 5\} = 2$, $k(2, \vec{x}) = \#\{2, 4\} = 2$, $k(3, \vec{x}) = \#\{3\} = 1$ and $k(5, \vec{x}) = \#\{5\} = 1$. Then, it holds that $m = \max\{1, 2\} = 2$ and $mmode(R^1_{23}, \ldots, R^6_{23}) = \{0.2, 0.4\}$.

Finally, Table II(b) contains the resulting collective preference relation R^C based on the choice function $\max$, calculated using Algorithm 1. That is, for each entry of Table II(a), one takes the maximum. In the example of the previous paragraph, for R^t_{23}, we have that the collective preference of alternative a_2 over alternative a_3 is $R^C_{23} = \max(mmode(R^1_{ij}, \ldots, R^6_{ij})) = \max\{0.2, 0.4\} = 0.4$.

(a) R^1

	a_1	a_2	a_3
a_1	0.5	0.4	0.8
a_2	0.6	0.5	0.2
a_3	0.2	0.8	0.5

(b) R^2

	a_1	a_2	a_3
a_1	0.5	0.3	0.8
a_2	0.7	0.5	0.4
a_3	0.2	0.6	0.5

(c) R^3

	a_1	a_2	a_3
a_1	0.5	0.1	0.8
a_2	0.9	0.5	0.6
a_3	0.2	0.4	0.5

(d) R^4

	a_1	a_2	a_3
a_1	0.5	0.2	0.7
a_2	0.8	0.5	0.2
a_3	0.3	0.8	0.5

(e) R^5

	a_1	a_2	a_3
a_1	0.5	0.8	0.9
a_2	0.2	0.5	0.3
a_3	0.1	0.7	0.5

(f) R^6

	a_1	a_2	a_3
a_1	0.5	0.6	0.4
a_2	0.4	0.5	0.4
a_3	0.3	0.6	0.5

Table I
PREFERENCES OF EXPERTS $e_1, \ldots, e_6$

(a) $mmode(R^1_{ij}, \ldots, R^6_{ij})$

$mmode$	a_1	a_2	a_3
a_1	{0.5}	{0.1,0.2,0.3,0.4,0.6,0.8}	{0.8}
a_2	{0.2,0.4,0.6,0.7,0.8,0.9}	{0.5}	{0.2,0.4}
a_3	{0.2}	{0.6,0.8}	{0.5}

(b) Collective Preference Relation R^C

$\max \circ mmode$	a_1	a_2	a_3
a_1	0.5	0.8	0.8
a_2	0.9	0.5	0.4
a_3	0.2	0.8	0.5

Table II
MULTI-MODES AND THE COLLECTIVE PREFERENCE RELATION R^C.

Now we show what we mean by *consistent influence/disturbance* and the role of abstract homogeneity in this new concept.

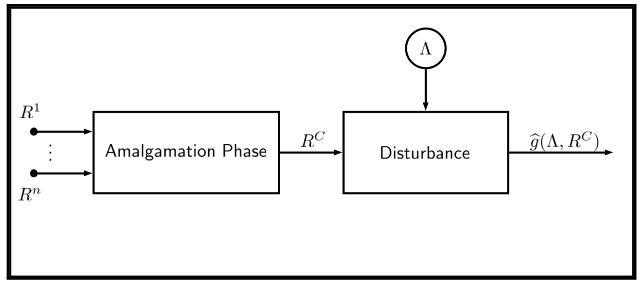

Figure 1. Disturbance of the consensus preference relation

C. Abstract homogeneity and consistent influence/disturbance on decision making processes

Suppose we have applied a decision making process and we want to influence the resulting collective preference relation R^C, by using an extra-opinion given by a new preference relation Λ, called the matrix of influence/disturbance. For example, suppose R^C and Λ are preference relations of the form:

$$R^C = \begin{pmatrix} 0.5 & R^C_{12} & \ldots & R^C_{1p} \\ R^C_{21} & 0.5 & \ldots & R^C_{2p} \\ \ldots & \ldots & \ldots & \ldots \\ R^C_{p1} & R^C_{p2} & \ldots & 0.5 \end{pmatrix} \text{ and } \Lambda = \begin{pmatrix} 0.5 & \lambda_{12} & \ldots & \lambda_{1p} \\ \lambda_{21} & 0.5 & \ldots & \lambda_{2p} \\ \ldots & \ldots & \ldots & \ldots \\ \lambda_{p1} & \lambda_{p2} & \ldots & 0.5 \end{pmatrix}$$

Consider a function $g : [0,1]^2 \to [0,1]$ and a collective preference relation R^C. The influenced (disturbed) collective preference relation based on (Λ, g) is given by:

$$\widehat{g}(\Lambda, R^C) = \begin{pmatrix} 0.5 & g(\lambda_{12}, R_{12}) & \ldots & g(\lambda_{1p}, R_{1p}) \\ g(\lambda_{21}, R_{21}) & 0.5 & \ldots & g(\lambda_{2p}, R_{2p}) \\ \ldots & \ldots & \ldots & \ldots \\ g(\lambda_{p1}, R_{p1}) & g(\lambda_{p2}, R_{p2}) & \ldots & 0.5 \end{pmatrix} \quad (7)$$

The function g is called the influence (disturbance) method.

The matrix $\widehat{g}(\Lambda, R^C)$ is obtained by applying the mapping $\widehat{g} : \mathcal{R}^2_{p\times p} \to \mathcal{R}_{p\times p}$, defined by:

$$\widehat{g}(\Lambda, R^C)_{ij} = \begin{cases} 0.5 & \text{, if } i = j \\ g(\lambda_{ij}, R^C_{ij}) & \text{, otherwise.} \end{cases} \quad (8)$$

This process is summarized in Figure 1.

Another possibility is to disturb the preference relations of experts individually with this new preference relation Λ (using the influence (disturbance) method g) and then apply the amalgamation phase to obtain the collective (disturbed) preference relation, as illustrated in Figure 2.

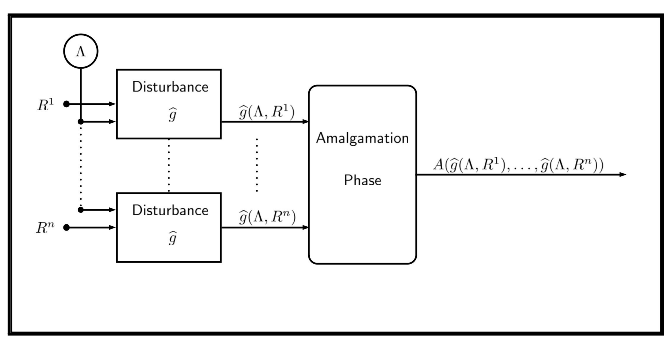

Figure 2. Consensus of the individually disturbed expert preference relations.

A good property for such disturbance in a decision making process is that both methods produce the same output matrix. This is what we call *consistent influence/disturbance*. In what follows, we define precisely what we mean.

Definition 69 *Given: (1) a vector* $(R^1_{ij},\ldots,R^n_{ij})$ *which represents the* ij*-preference of* n*-experts; (2) a bivariate function* g*; a pre-aggregation* A *and (3) a factor* λ_{ij}*, which will influence the* ij*-preferences. The function* g *consistently influences/disturbs the consensus matrix* R^C*, if it does not matter if it is applied on each individual preference* R^k_{ij}*, or on the final consensus preference* R^C_{ij}*. In other words, if the following equation is satisfied:*

$$R^C_{ij} = A(g(\lambda_{ij}, R^1_{ij}),\ldots,g(\lambda_{ij}, R^n_{ij})) = g(\lambda_{ij}, A(R^1_{ij},\ldots,R^n_{ij})).$$

This means that the amalgamator A must be g-homogeneous. Figure 3 illustrates what we mean.

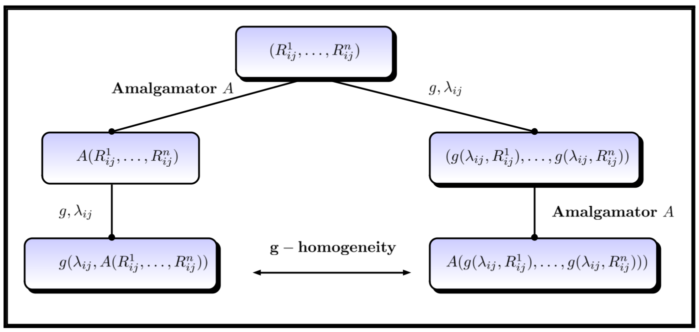

Figure 3. g-homogeneity and influence/disturbance scheme

In other words, whenever the resulting collective preference relation $R^C = \widehat{A}(R^1,\ldots R^n)$ is influenced by the *extra* opinion Λ by using g, the resulting (influenced) collective preference relation $\widehat{g}(\Lambda, \widehat{A}(R^1,\ldots R^n))$ coincides with the collective preference relation which rises from $\widehat{A}$ applied on all disturbed experts preference relation (by using g and Λ) $\widehat{A}(\widehat{g}(\Lambda, R^1),\ldots,\widehat{g}(\Lambda, R^n))$.

In what follows we show that our toy algorithm illustrate this situation. To achieve that, we use as the influence (disturbance) method the weighted average function $g_a(x,y) = a\cdot x + (1-a)\cdot y$ from Example 33, for $a = 0.5$, and a matrix Λ. Since $\max\circ mmode$ is g_a-homogeneous (c.f. Example 33) we obtain a consistently influenced/disturbed system.

1) Revisiting the illustrative example: Consider the multi-expert decision making situation exposed in subsection VII-B and a new preference relation Λ of a *new expert* e_0. He/She has a separate judgment (his/her own preference relation) and a influence (disturbance) method g to influence the resulting collective preference relation R^C given by algorithm 1. Imagine that the preferences of experts $e_1,\ldots,e_n$ are based on *technical criteria* whereas the preferences of e_0 are based on *political/strategical criteria* and he/she want to influence R^C with Λ and g as if the experts took into account his/her criteria in their opinions. In other words, the new R^C (denoted here by R^C_d) should be equal to the output collective preference relation R_C provided by algorithm 1 whenever the experts took into account the same criterion as e_0 (together with the influence (disturbance) method g) to provide their preference relation R^t. In other words, R^C must be *consistently influenced/disturbed* by (Λ, g). To achieve that the function $\max\circ mmode$ provided at step 6 must be g-homogeneous (c.f. Figure 3).

For example, our expert e_0 provides the influence (disturbance) method $g_{0.5}(x,y) = 0.5\cdot x + (1-0.5)\cdot y$ (i.e., the arithmetic mean) to influence R^C. In fact, according to example 33, for any $a\in[0,1]$ and any weighted average function $g_a(x,y) = a\cdot x+(1-a)\cdot y$, the function $\max\circ mmode$ is g_a-homogeneous.

Table III(a) contains the preferences Λ of e_0 and Table III(b) contains R^C disturbed by $(\Lambda, g_{0.5})$, namely, the matrix R^C_d given in Eq. 7. In order to understand how each entry of this matrix is calculated, for example, considering the collective preference relation R^C_{23} obtained in subsection VII-B (that is, the resulting collective preference of alternative a_2 over alternative a_3) and using Eq. 8, we obtain that

$$R^C_{d\ 23} = g(\lambda_{23}, R^C_{23}) = 0.5\cdot 0.7 + (1-0.5)\cdot 0.4 = 0.55.$$

Now we show that the function $\max\circ mmode$ is $g_{0.5}$-homogeneous. Tables VII-C1(a)-(f) contain the six experts' opinions taking into account the point of view of e_0, i.e., each table contains their original opinion disturbed by Λ and $g_{0.5}$, using Eq. 8 in each entry of each preference matrix. For example, considering the preference relation R^1 of expert e_1 given in subsection VII-B, one has that $R^1_{d23} = g(\lambda_{23}, R^1_{23}) = 0.5\cdot 0.7 + (1-0.5)\cdot 0.2 = 0.45$, $R^2_{d23} = g(\lambda_{23}, R^2_{23}) = 0.5\cdot 0.7 + (1-0.4)\cdot 0.2 = 0.55$ and $R^3_{d23} = g(\lambda_{23}, R^2_{23}) = 0.5\cdot 0.7 + (1-0.5)\cdot 0.6 = 0.65$, and, similarly, one obtains $R^4_{d23} = 0.45$, $R^5_{d23} = 0.5$ and $R^6_{d23} = 0.55$.

Then, in Table VII-C1(g), we show the multimodes of components R^t_{dij}, with $t = 1,\ldots,6$. The multimodes are calculated using Def. 19, considering all disturbed preference relations of Table VII-C1(a)-(f). For example, for R^t_{d23} (i.e., the various disturbed preferences of alternative a_2 over alternative a_3), using Def. 19, one has that $\vec{x} = (R^1_{d23},\ldots,R^6_{d23}) = (0.45, 0.55, 0.65, 0.45, 0.5, 0.55)$ and $k(1,\vec{x}) = \#\{1,4\} = 2$, $k(2,\vec{x}) = \#\{2,6\} = 2$, $k(3,\vec{x}) = \#\{3\} = 1$ and $k(5,\vec{x}) = \#\{5\} = 1$. Then, it holds that $m = \max\{1,2\} = 2$ and $mmode(R^1_{d23},\ldots,R^6_{d23}) = \{0.45, 0.55\}$.

Finally, Table IV(h) contains the resulting disturbed collective preference relation R^C_d based on function $\max$, calculated using Algorithm 1. That is, for each entry of Table VII-C1(g), one takes the maximum. In the example of the previous paragraph, for R^t_{d23}, we have that the disturbed collective preference of alternative a_2 over alternative a_3 is $R^C_{d\ 23} = \max(mmode(R^1_{dij},\ldots,R^6_{dij})) = \max\{0.45, 0.55\} = 0.55$. As expected, this table is equal to Table III(b), since $\max\circ mmode$ is $g_{0.5}$-homogeneous.

VIII. Final Remarks

In this paper we have introduced the notion of abstract homogeneity. In our opinion this concept is important in

(a) e_0's Preferences Λ

Λ	a_1	a_2	a_3
a_1	0.5	0.7	0.8
a_2	0.6	0.5	0.7
a_3	0.2	0.8	0.5

(b) $R_d^C = \widehat{g}(\Lambda, R^C)$ (disturbed R^C)

R_d^C	a_1	a_2	a_3
a_1	0.5	0.75	0.8
a_2	0.75	0.5	0.55
a_3	0.2	0.8	0.5

Table III
e_0'S PREFERENCES (Λ) AND R_d^C (THE DISTURBED R^C)

(a) R_d^1

	a_1	a_2	a_3
a_1	0.5	0.55	0.8
a_2	0.6	0.5	0.45
a_3	0.2	0.8	0.5

(b) R_d^2

	a_1	a_2	a_3
a_1	0.5	0.5	0.8
a_2	0.65	0.5	0.55
a_3	0.2	0.7	0.5

(c) R_d^3

	a_1	a_2	a_3
a_1	0.5	0.4	0.8
a_2	0.75	0.5	0.65
a_3	0.2	0.6	0.5

(d) R_d^4

	a_1	a_2	a_3
a_1	0.5	0.45	0.75
a_2	0.7	0.5	0.45
a_3	0.25	0.80	0.50

(e) R_d^5

	a_1	a_2	a_3
a_1	0.5	0.75	0.85
a_2	0.4	0.5	0.5
a_3	0.15	0.75	0.5

(f) R_d^6

	a_1	a_2	a_3
a_1	0.5	0.65	0.6
a_2	0.5	0.5	0.55
a_3	0.25	0.7	0.5

(g) Multi-modes of disturbed R^k's.

$mmode$	a_1	a_2	a_3
a_1	{0.5}	{0.4,0.45,0.5,0.55,0.65,0.75}	{0.8}
a_2	{0.4,0.5,0.6,0.65,0.7,0.75}	{0.5}	{0.45,0.55}
a_3	{0.2}	{0.7,0.8}	{0.5}

(h) Output Matrix, R^C, from disturbed R^k's (namely: R_d^k's)

$\max \circ mmode$	a_1	a_2	a_3
a_1	0.5	0.75	0.8
a_2	0.75	0.5	0.55
a_3	0.2	0.8	0.5

Table IV
g_a-HOMOGENEITY OF $\max \circ mmode$

itself, since it generalizes the notion of homogeneity without imposing any restriction on g, which provides more flexibility than the other generalizations found in the literature. This is reinforced with some occurrences in further fields (as we have shown in sections III, IV.A and IV.b). In these sections we demonstrate some properties of abstract homogeneity related to the corresponding field. Beyond generalization and the occurrence in different fields, abstract homogeneity enable us to introduce a new paradigm for the theory of multi-expert decision making called: consistently influenced/disturbed decision making systems (as we have demonstrated with our toy example).

Future work is concerned with the development of interval g-homogeneity in the light of *interval representation* proposed by Santiago *et. al.* – c.f. [37], [60], [61], [62], inspired by the work by Lima *et. al.* [27]-

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