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On the Structure of Acyclic Binary Relations

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Abstract. We investigate the structure of acyclic binary relations from different points of view. On the one hand, given a nonempty set we study real-valued bivariate maps that satisfy suitable functional equations, in a way that their associated binary relation is acyclic. On the other hand, we consider acyclic directed graphs as well as their representation by means of incidence matrices. Acyclic binary relations can be extended to the asymmetric part of a linear order, so that, in particular, any directed acyclic graph has a topological sorting.

Keywords: Acyclic binary relations · Functional equations
Acyclic directed graphs · Arborescences · Incidence matrices
Total preorders · Numerical representability
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1 Introduction

Acyclic binary relations are crucial in the mathematical analysis of Decision Making and Social Choice, as well as in Theoretical Computer Science. To put

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only an example, binary relations that model preferences of agents are often asked to be compulsorily acyclic, in order to avoid incoherences. By this reason, theoretical studies on the structure, main properties and scope in possible applications of acyclic binary relations should be welcome as the grounds that support many aspects of Decision Making.

The origin of the study addressed in the present paper comes from an analysis of those binary relations $\mathcal{R}$ on a nonempty set X that appear through a bivariate real-valued function $F : X \times X \rightarrow \mathbb{R}$ such that $x\mathcal{R}y \Leftrightarrow F(x,y) > 0$. In some appealing particular cases, the special kind of binary relation considered is characterized by the fact of the function F being the solution of some functional equation (e.g. the Sincov's one $F(x,y) + F(y,z) + F(z,x) = 0 \quad (x,y,z \in X)$, see [10], closely related to representable total preorders). Surprisingly as it may appear at first glance, the types of binary relations that have already been characterized this way correspond either to very simple situations (namely, reflexivity, irreflexivity and asymmetry) or to sophisticated ones as representable total preorders, interval orders and semiorders. Intermediate situations as transitivity or acyclicity among others remain as open problems. At that stage, we did not have at hand yet any characterization of acyclicity by means of suitable functional equations. Nor we had characterized binary relations that give rise to an acyclic graph, or to a tree –that is also a directed graph– or to a finite union of trees among others. Nevertheless, in some particular situations (e.g., on countable sets) a few characterizations of acyclicity can actually be encountered in the literature (see [3,9]). Also, there are techniques that detect if a binary relation on a finite set is actually an arborescence, as the well-known Kruskal's algorithm (see [8]). However, they have not been built in terms of functional equations but using other techniques (see e.g. [1,3]).

The structure of the manuscript goes as follows: We analyze the relationship between functional equations and acyclicity in Sect. 3. Next we study particular situations where the set on which the binary relations are defined is finite. In that case, alternative mathematical tools to deal with binary relations are graph theory and incidence matrices (see Sect. 4).

2 Preliminaries

Definition 1. A binary relation $\mathcal{R}$ on a nonempty set X is a subset of the Cartesian product $X^2 = X \times X$. Given two elements $x, y \in X$, we will use the standard notation $x\mathcal{R}y$ to express that the pair (x, y) belongs to $\mathcal{R}$.

Naturally associated to a binary relation $\mathcal{R}$ on a set X, we will also deal with the binary relations $\mathcal{R}^c$ and $\mathcal{R}^{-1}$ on X, respectively given by $\mathcal{R}^c = X^2 \setminus \mathcal{R}$, and by $x\mathcal{R}^{-1}y \iff y\mathcal{R}x, \quad (x, y \in X)$.

A binary relation $\mathcal{R}$ defined on a set X is said to be

(i) *reflexive* if $\Delta \subseteq \mathcal{R}$, with $\Delta = \{(x,x) : x \in X\}$ (here Δ stands for the *diagonal* of X^2),
(ii) *irreflexive* if $\mathcal{R} \cap \Delta = \emptyset$,
(iii) *symmetric* if $\mathcal{R}$ and $\mathcal{R}^{-1}$ coincide,

(iv) *antisymmetric* if $\mathcal{R} \cap \mathcal{R}^{-1} \subseteq \Delta$,
(v) *asymmetric* if $\mathcal{R} \cap \mathcal{R}^{-1} = \emptyset$,
(vi) *total* (or *complete*) if $\mathcal{R} \cup \mathcal{R}^{-1} = X^2$,
(vii) *transitive* if $x\mathcal{R}y \wedge y\mathcal{R}z \Rightarrow x\mathcal{R}z$ for every $x, y, z \in X$,
(viii) *negatively transitive* if $\mathcal{R}^c$ is transitive.

Given two binary relations $\mathcal{R}$, $\mathcal{S}$ on X, its *composition* $\mathcal{R} \circ \mathcal{S}$ is a new binary relation on X, defined as follows: For any pair $(x, y) \in X^2$, we declare that $x\ (\mathcal{R} \circ \mathcal{S})\ y$ holds true –equivalently, we say that the pair (x, y) belongs to $\mathcal{R} \circ \mathcal{S} \subseteq X \times X$– whenever there exists $z \in X$ such that (x, z) belongs to $\mathcal{R} \subseteq X \times X$, whereas (z, y) belongs to $\mathcal{S} \subseteq X \times X$. The composition of binary relations is associative. Given a natural number n, we will use the standard notation $\mathcal{R}^n$ to denote the composition $\mathcal{R} \circ \ldots$ (n-times) $\ldots \circ \mathcal{R}$.

The binary relation $\mathcal{R}$ is said to be *acyclic* if $\mathcal{R}^n \cap \Delta = \emptyset$ holds true for every natural number n. The *transitive closure* $\bar{\mathcal{R}}$ of a binary relation $\mathcal{R}$ is defined as $\bar{\mathcal{R}} = \bigcup_{n=1}^{\infty} \mathcal{R}^n$. It is plain that $\bar{\mathcal{R}}$ is transitive.

In the particular case of dealing with *orderings* on X, the standard notation is different. We include it here for sake of completeness.

Definition 2. A *preorder* $\precsim$ on a nonempty set X is a binary relation on X which is reflexive and transitive. An antisymmetric preorder is said to be a *partial order*. A *total preorder* $\precsim$ on a set X is a preorder such that if $x, y \in X$ then $x \precsim y$ or $y \precsim x$ holds. An antisymmetric total preorder is said to be a *total order*. A total order is also called a *linear order*.

If $\precsim$ is a preorder on X, then as usual we denote the associated *asymmetric* relation by $\prec$ and the associated *equivalence* relation by $\sim$ and these are defined, respectively, by $x \prec y \iff (x \precsim y) \wedge \neg(y \precsim x)$ and by $x \sim y \iff (x \precsim y) \wedge (y \precsim x)$. The asymmetric part of a linear order (respectively, of a partial order, of a total preorder) is said to be a *strict linear order* (respectively, a *strict partial order*, a *strict total preorder*).

A total preorder $\precsim$ on a set X is said to be *representable* if there exists a real-valued map $u : X \to \mathbb{R}$ such that, for any $x, y \in X$, we have $x \precsim y \Leftrightarrow u(x) \leq u(y)$. The map u is said to be a *utility function* or an *order-isomorphism*.

Definition 3. Let X be a nonempty set. Let $F : X \times X \to \mathbb{R}$ be a real-valued bivariate function defined on X. The function F satisfies the *Sincov functional equation* if $F(x, y) + F(y, z) = F(x, z)$ holds for every $x, y, z \in X$ (see [4,10]).

The following easy result arises (see e.g. [10]).

Proposition 1. *A bivariate function $F : X \times X \to \mathbb{R}$ satisfies the Sincov functional equation if and only if there exists a real-valued function $G : X \to \mathbb{R}$ such that $F(x, y) = G(y) - G(x)$ holds for all $x, y \in X$.*

Given a binary relation $\mathcal{R}$ on a nonempty set X, we may immediately interpret $\mathcal{R}$ through a bivariate real-valued function $F : X \times X \to \mathbb{R}$. To do so, it is enough to consider the characteristic function of the binary relation $\mathcal{R} \subseteq X \times X$, namely $F(x,y) = 1 \Leftrightarrow (x,y) \in \mathcal{R}$ and $F(x,y) = 0$ otherwise. However, this F may fail to satisfy suitable additional properties, as, for instance, to be the solution of some classical functional equation. Paying attention to the converse situation, we begin with a bivariate map $F : X \times X \to \mathbb{R}$, and we define its associated binary relation $\mathcal{R}_F$ by declaring that $(x,y) \in \mathcal{R}_F$ holds true if and only if $F(x,y) > 0$. It is clear that if F satisfies certain additional properties, its associated binary relation $\mathcal{R}_F$ will a fortiori feature some related special characteristics. To put an obvious example, we may notice that if F vanishes on the diagonal Δ, then $\mathcal{R}_F$ is irreflexive. In this direction, the following result arises. Its proof is straightforward and follows from the corresponding definitions.

Proposition 2. *Let X denote a nonempty set and $F : X \times X \to \mathbb{R}$ a bivariate map. Let $\mathcal{R}_F$ the binary relation defined on X by means of F, as follows: $x\mathcal{R}_F y \Leftrightarrow F(x,y) > 0 \quad (x,y \in X)$. The following statements hold true:*

(i) If $F(x,x) > 0$ holds for every $x \in X$ then $\mathcal{R}_F$ is reflexive.
(ii) If $F(x,x) \leq 0$ holds for every $x \in X$ then $\mathcal{R}_F$ is irreflexive.
(iii) If $F(x,y) + F(y,x) = 0$ holds for every $x, y \in X$ then $\mathcal{R}_F$ is asymmetric.
(iv) If F satisfies the Sincov functional equation, then $\mathcal{R}_F$ is asymmetric and negatively transitive. It is actually a strict total preorder.

For the particular case of representable total preorders, the following well-know result stated in Proposition 3 above plays a crucial role (see e.g. [4]).

Proposition 3. *Let X be a nonempty set. Let $\precsim$ be a total preorder on X. Then the following statements are equivalent:*

(i) The total preorder $\precsim$ is representable by means of a utility function $u : X \to \mathbb{R}$ such that $x \precsim y \Leftrightarrow u(x) \leq u(y) \quad (x, y \in X)$.
(ii) There exists a real-valued bivariate map $F : X \times X \to \mathbb{R}$ that satisfies the Sincov functional equation and, in addition, $x \prec y \Leftrightarrow F(x,y) > 0$ holds true for every $x, y \in X$.

3 Acyclic Binary Relations vs. Functional Equations

Definition 4. Given a nonempty set X endowed with a binary relation $\mathcal{R}$, we say that another binary relation $\mathcal{Q}$ is an *extension* of $\mathcal{R}$ if $x\mathcal{R}y \Rightarrow x\mathcal{Q}y$ holds true for every $x, y \in X$. In other words, as subsets of the Cartesian product $X \times X$, this means that $\mathcal{R} \subseteq \mathcal{Q} \subseteq X \times X$.

In this direction, a classical extension theorem was obtained by E. Szpilrajn in 1930. That theorem will be an important key in this Sect. 3.

Lemma 1 *(Szpilrajn extension theorem, 1930). Let X be a nonempty set. Let $\prec$ stand for an irreflexive and transitive binary relation defined on X. Then $\prec$ can be extended to a strict linear order.*

Proof. See [12]. For some related results, see also [11]. □

Using Szpilrajn extension theorem as a tool, we may prove now, as a direct consequence of it, the following result on extension of acyclic binary relations.

Theorem 1. *Let X be a nonempty set. Let $\mathcal{R}$ be an acyclic binary relation defined on X. Then $\mathcal{R}$ can be extended to a strict linear order.*

Proof. Let $\bar{\mathcal{R}}$ be the transitive closure of the given relation $\mathcal{R}$. It is plain that $\bar{\mathcal{R}}$ is transitive, by its own definition, and it is also irreflexive because $\mathcal{R}$ is acyclic. Moreover, $\bar{\mathcal{R}}$ is an extension of $\mathcal{R}$. Since $\bar{\mathcal{R}}$ is irreflexive and transitive, by Lemma 1 (Szpilrajn extension theorem), it can actually be extended to a linear order defined on X. Obviously, such linear order is also an extension of the former acyclic binary relation $\mathcal{R}$. □

Parallel to Szpilrajn extension theorem, the following result is also classical.

Theorem 2 *(Hansson extension theorem, 1968)*. *Let X be a nonempty set. Let $\precsim$ be a preorder defined on X. Then $\precsim$ can be extended to a total preorder defined on X, so that the asymmetric part of that total preorder is also an extension of $\prec$, the asymmetric part of $\precsim$.*

Proof. See [5]. For generalizations, see [11]. □

Remark 1. Matching Hansson extension theorem and Lemma 1 (Szpilrajn extension theorem) we can prove again Theorem 1. To do so, we may observe that given an acyclic binary relation $\mathcal{R}$, and $\bar{\mathcal{R}}$ its transitive closure, the binary relation $\mathcal{Q} = \Delta \cup \bar{\mathcal{R}}$ is a preorder whose asymmetric part is $\bar{\mathcal{R}}$. By Theorem 2, $\mathcal{Q}$ can be extended to a total preorder $\precsim$ whose asymmetric part $\prec$ extends $\bar{\mathcal{R}}$ and consequently $\mathcal{R}$. Finally, by Lemma 1, $\prec$ can be extended to a linear order.

Definition 5. Let X be a nonempty set. Let $\mathcal{S}$ be a binary relation defined on X. Associated to $\mathcal{S}$, let $\mathcal{T}$ be the binary relation defined as $x\mathcal{T}y \Leftrightarrow x\mathcal{S}y \wedge y\mathcal{S}^c x \quad (x, y \in X)$. Given a natural number $n \geq 2$, a n-tuple $(x_1, x_2, \ldots, x_n) \in X^n$ is called a *$\mathcal{TS}$-cycle of order n* if we have $x_1\mathcal{T}x_2\mathcal{S}\ldots\mathcal{S}x_n\mathcal{S}x_1$. Then we say that $\mathcal{S}$ is *consistent* if no $\mathcal{TS}$-cycle of order n appears, for any natural number $n \geq 2$.

Theorem 3 *(Suzumura extension theorem, 1976)*. *Let X be a nonempty set. Let $\mathcal{S}$ be a binary relation defined on X. Associated to $\mathcal{S}$, let $\mathcal{T}$ be the binary relation defined as $x\mathcal{T}y \Leftrightarrow x\mathcal{S}y \wedge y\mathcal{S}^c x \quad (x, y \in X)$. Then, there exists a total preorder $\precsim$ on X that extends $\mathcal{S}$, and with its asymmetric part $\prec$ extending $\mathcal{T}$ too, if and only if the binary relation $\mathcal{S}$ is consistent.*

Proof. See Theorem 3 in [11]. □

Remark 2. A weaker version of Theorem 1 appears now as a corollary of Suzumura extension theorem. As a matter of fact, if $\mathcal{P}$ is an acyclic binary relation on X, the associated binary relation $\mathcal{T}$ defined as $x\mathcal{T}y \Leftrightarrow x\mathcal{P}y \wedge y\mathcal{P}^c x \quad (x, y \in X)$ coincides with $\mathcal{P}$ since $\mathcal{P}$ is acyclic, hence asymmetric. Therefore, for the relation $\mathcal{P}$, the condition of being consistent directly follows from acyclicity. So $\mathcal{P}$ can be extended to the asymmetric part of a total preorder.

Moreover, if we use now Szpilrajn extension theorem (Lemma 1) again, it follows that from this weaker version of Theorem 1 we may retrieve the whole version, since every asymmetric part of a total preorder is indeed irreflexive and transitive.

Definition 6. An acyclic binary relation $\mathcal{R}$ defined on a nonempty set X is said to be *representable* if it is extendable to the asymmetric part of a representable total preorder.

Definition 7. Given a binary relation $\mathcal{R}$ defined on a nonempty set X, a real-valued function $u : X \to \mathbb{R}$ is said to be a *pseudoutility* for $\mathcal{R}$ if $x\mathcal{R}y \Rightarrow u(x) < u(y)$ holds true for every $x, y \in X$.

Representable acyclic binary relations can be characterized in terms of a suitable modification of Sincov functional equation, as follows.

Theorem 4. *Let X be a nonempty set. Let $\mathcal{R}$ be an acyclic binary relation defined on X. The following statements are equivalent:*

(i) *$\mathcal{R}$ is representable,*
(ii) *there exist bivariate functions $F : X \times X \to \mathbb{R}$ and $G : X \times X \to \{0, 1\}$ such that F satisfies the Sincov functional equation and $x\mathcal{R}y \Leftrightarrow F(x, y) \cdot G(x, y) > 0$ holds true for every $x, y \in X$,*
(iii) *there exists a pseudoutility function u for the given binary relation $\mathcal{R}$.*

Proof. To prove that (i) $\Rightarrow$ *(ii)* we take a representable total preorder $\precsim$ on X, whose asymmetric part $\prec$ extends $\mathcal{R}$. By Proposition 3, there is a function $F : X \times X \to \mathbb{R}$ that satisfies the Sincov functional equation and $x \prec y \Leftrightarrow F(x, y) > 0 \ \ (x, y \in X)$. Define now $G : X \times X \to \{0, 1\}$ as $G(x, y) = 1 \Leftrightarrow x\mathcal{R}y$ and $G(x, y) = 0$ otherwise $\ \ (x, y \in X)$. We have that $x\mathcal{R}y \implies x \prec y \implies F(x, y) > 0$. Also $x\mathcal{R}y \implies G(x, y) = 1$. Therefore $x\mathcal{R}y \Rightarrow F(x, y) \cdot G(x, y) > 0$ holds true for every $x, y \in X$. Conversely, given $x, y \in X$, if $F(x, y) \cdot G(x, y) > 0$ it follows that $G(x, y) = 1$ by definition of G, so that $x\mathcal{R}y$ holds true. Hence $x\mathcal{R}y \Leftrightarrow F(x, y) \cdot G(x, y) > 0 \ \ (x, y \in X)$.

To prove that (ii) $\Rightarrow$ *(iii)*, let $F : X \times X \to \mathbb{R}$ and $G : X \times X \to \{0, 1\}$ be such that F satisfies the Sincov functional equation and $x\mathcal{R}y \Leftrightarrow F(x, y) \cdot G(x, y) > 0 \ \ (x, y \in X)$. Consider the binary relation $\precsim$ defined on X by declaring that $x \precsim y \Leftrightarrow F(x, y) \geq 0$. Since F satisfies the Sincov functional equation, we have that $F(x, x) = 0 = F(x, y) + F(y, x)$ holds true for every $x, y \in X$. Thus $\precsim$ is reflexive and total. Moreover, the fact $F(x, y) + F(y, z) = F(x, z) \ \ (x, y, z \in X)$ immediately implies that $\precsim$ is transitive, hence it is indeed a total preorder. By definition, its asymmetric part $\prec$ satisfies that for any $x, y \in X$, $x \prec y$ holds if and only if $y \precsim x$ does not hold. Equivalently, $x \prec y$ if and only if $F(y, x) < 0$. This last fact, jointly with $F(x, y) + F(y, x) = 0$, is equivalent to say that $F(x, y) > 0$. By Proposition 3, the total preorder $\precsim$ is representable by a utility function $u : X \to \mathbb{R}$. Therefore, for any $x, y \in X$ we have that $x\mathcal{R}y \implies x \prec y \implies u(x) < u(y)$.

Finally, to prove that (iii) $\Rightarrow$ (i), we consider a pseudoutility u for $\mathcal{R}$. Observe now that the binary relation $\precsim$ on X given by $x \precsim y \Leftrightarrow u(x) \leq u(y) \quad (x, y \in X)$ is a total preorder, whose asymmetric part $\prec$ satisfies that $x \prec y \Leftrightarrow u(x) < u(y) \quad (x, y \in X)$. Hence $\prec$ is actually an extension of the given binary relation $\mathcal{R}$. Thus $\mathcal{R}$ is representable. □

Remark 3. An acyclic binary relation $\mathcal{R}$ defined on a nonempty set X may fail to be representable. A clear example is the asymmetric part of a non-representable linear order. By the way, the structure of non-representable linear orders has been analyzed in depth in [2]. Whenever X is finite or countable, any acyclic binary relation defined on X is representable because any total preorder on a finite or countable set is actually representable (see e.g. Theorem 1.4.8 in [3], or else [2] for further details).

Consider now a nonempty *finite* set X.

Definition 8. Let $\mathcal{R}$ be a binary relation on X. We say that $\mathcal{R}$ is an *arborescence* if the following conditions hold:

(i) $\mathcal{R}$ is irreflexive,
(ii) there exists a unique element $x_0 \in X$, called *root*, such that $x\mathcal{R}x_0$ does not hold for any $x \in X$,
(iii) for any $x \in X$ with $x \neq x_0$, there exists a *unique* $(k+1)$-tuple $(x_0, x_1, \ldots, x_k = x) \in X^{k+1}$, for some suitable $k \in \mathbb{N}$, such that $x_0\mathcal{R}x_1\mathcal{R}\ldots\mathcal{R}x_k$ holds true.

Remark 4. Notice that the uniqueness restriction arising in condition (iii), with respect to the $(k+1)$-tuple $(x_0, x_1, \ldots, x_k = x) \in X^{k+1}$, avoids that a given point x could be reached from x_0 by two different "sequences of branches".

Proposition 4. *Any arborescence is acyclic.*

Proof. Let $\mathcal{R}$ be an arborescence on X. Suppose that there is a n-cycle $y_1\mathcal{R}y_2\mathcal{R}\ldots\ \mathcal{R}y_n\mathcal{R}y_1$ in X as regards $\mathcal{R}$. Then, the condition iii) for x_0 and $x = y_1$ is no longer true, since for any $(k+1)$-tuple $(x_0, x_1, \ldots, x_k = y_1)$ with $x_0\mathcal{R}x_1\mathcal{R}\ldots\mathcal{R}x_k = y_1$ we have, repeating now the cycle, that the $(k+n+1)$-tuple $(x_0, x_1, \ldots, x_k = y_1, y_2, \ldots, y_n, y_1)$ also satisfies $x_0\mathcal{R}x_1\mathcal{R}\ldots\mathcal{R}x_k = y_1\mathcal{R}y_2\mathcal{R}\ldots\mathcal{R}y_n\mathcal{R}y_1$, in contradiction with the hypothesis of uniqueness. □

We introduce another equivalent way to define the notion of arborescence.

Proposition 5. *Let $\mathcal{R}$ be a binary relation on a nonempty finite set X with at least two elements. Then $\mathcal{R}$ is an arborescence if and only if the following conditions hold:*

(i) $\mathcal{R}$ is irreflexive,
(ii) there exists a unique element $x_0 \in X$ such that $x\mathcal{R}x_0$ does not hold for any $x \in X$ (in particular, the relation $\mathcal{R}$ is nonvoid),
(iii) for any $x, y, z \in X$, it holds true that $(y\mathcal{R}x \wedge z\mathcal{R}x) \Rightarrow y = z$.

Proof. Assume that $\mathcal{R}$ is an arborescence. If there exist $x, y, z \in X$ such that $y\mathcal{R}x \wedge z\mathcal{R}x$ holds true with $y \neq z$, then taking a $(k+1)$-tuple $(x_0, x_1, \ldots, x_k = y)$ with $x_0\mathcal{R}x_1\mathcal{R}\ldots\mathcal{R}x_k = y$ and another $l+1$-tuple $(x_0, y_1, \ldots, y_l = z)$ with $x_0\mathcal{R}y_1\mathcal{R}\ldots\mathcal{R}y_l = z$, we may construct two *different* tuples from x_0 to x, namely the $(k+2)$-tuple $(x_0, x_1, \ldots, x_k = y, x)$ for which we have $x_0\mathcal{R}x_1\mathcal{R}\ldots\mathcal{R}x_k = y\mathcal{R}x$ and the $(l+2)$-tuple $(x_0, y_1, \ldots, y_l = z, x)$ satisfying that $x_0\mathcal{R}y_1\mathcal{R}\ldots\mathcal{R}y_l = z\mathcal{R}x$. But this contradicts condition (iii) in Definition 8.

Conversely, suppose now that $\mathcal{R}$ satisfies the conditions in the statement of Proposition 5. Given $x \in X$ with $x \neq x_0$, by conditions (ii) and (iii) there exists a unique element $y \in X$ such that $y\mathcal{R}x$ holds true. If $y = x_0$ we are done. And if $y \neq x_0$, then with the same argument, there exists a unique element $z \in X$ for which $z\mathcal{R}y$ holds true. Again if $z = x_0$ we are done. Also, if $z \neq x_0$, there exists a unique element $t \in X$ for which $t\mathcal{R}z$ holds true. This process goes on until we arrive at x_0. This must compulsorily happen by condition (ii) and the fact of X being finite. So it is clear that condition (iii) in Definition 8 must hold true, too. This concludes the proof. □

Definition 9. Let X be a finite set and $\mathcal{R}$ a binary relation on X. Then $\mathcal{R}$ is said to be a *forest* if X can be split as a union of pairwise disjoint subsets, say $X = \bigcup_{i=1}^{n} X_n$, accomplishing the following conditions:

(i) The restriction of $\mathcal{R}$ to X_i is an arborescence, por any $i \in \{1, \ldots, n\}$,
(ii) If $i \neq j$ then $x_i\mathcal{R}x_j$ does not hold, for any $x_i \in X_i$, $x_j \in X_j$.

Proposition 6. *Any forest –and, in particular, any arborescence– is a representable acyclic binary relation.*

Proof. The fact of being acyclic is a direct consequence of Proposition 4 and Definition 9 (of the concept of a forest). Since the support set X is finite, the results follows now from Remark 3. In addition, a different alternative argument to prove the representability follows from Theorem 1, since $\mathcal{R}$ can be extended to the asymmetric part of a linear order on X. That linear order is a fortiori representable because X is finite (see e.g. Theorem 1.2.1 in [3]). Therefore $\mathcal{R}$ is also representable, by Definition 6. □

Now we analyze which conditions should be added to those in the statement of Theorem 4 in order to characterize arborescences and forests among acyclic binary relations, by using some functional equation.

Theorem 5. *Let X be a nonempty finite set. Let $\mathcal{R}$ be an acyclic binary relation defined on X. The following statements are equivalent:*

(i) $\mathcal{R}$ is an arborescence,
(ii) there exist bivariate functions $F : X \times X \to \mathbb{R}$ and $G : X \times X \to \{0, 1\}$ such that the following conditions are met:
- *(a) F satisfies the Sincov functional equation and $x\mathcal{R}y \Leftrightarrow F(x, y) \cdot G(x, y) > 0$ holds true for every $x, y \in X$,*
- *(b) there exists a unique $x_0 \in X$ such that $F(x, x_0) \cdot G(x, x_0) \leq 0$ for every $x \in X$,*

– *(c) for every* $x, y, z \in X$ *we have that* $F(y,x) \cdot F(z,x) \cdot G(y,x) \cdot G(z,x) = [F(y,x) \cdot G(y,x)]^2 \cdot \delta(y,z)$*, where* δ *stands here for the Kronecker delta function, that is, given* $(a,b) \in X^2$*, we have that* $\delta(a,b) = 1$ *if* $a = b$*, whereas* $\delta(a,b) = 0$ *otherwise.*

Proof. To prove that (i) $\implies$ (ii) we argue as in Theorem 4, so that again we consider a representable total preorder $\precsim$ on X, whose asymmetric part $\prec$ extends $\mathcal{R}$. Once more, by Proposition 3, there is a function $F : X \times X \to \mathbb{R}$ that satisfies the Sincov functional equation and $x \prec y \Leftrightarrow F(x,y) > 0 \quad (x, y \in X)$. Let now $G : X \times X \to \{0,1\}$ be given as $G(x,y) = 1 \Leftrightarrow x\mathcal{R}y$ and $G(x,y) = 0$ otherwise $(x, y \in X)$. Since $\mathcal{R}$ is acyclic, condition (ii)-(a) directly follows from the proof of Theorem 4. In addition, since $\mathcal{R}$ is an arborescence, there exists $x_0 \in X$ such that $x\mathcal{R}x_0$ never holds, for any $x \in X$. In other words, by definition of G, we have $G(x, x_0) = 0$ for every $x \in X$, so that condition (ii)-(b) is also satisfied. Finally, given $x, y, z \in X$, if $y = z$ the condition (ii)-(c) trivially follows. If $y \neq z$, we have that $\delta(y,z) = 0$. By condition (iii) in the statement of Proposition 5 we have that $y\mathcal{R}x$ or $z\mathcal{R}x$ fails to be true, so that $G(y,x).G(z,x) = 0$. Therefore the condition (ii)-(c) is always met.

Let us prove now that (ii) $\implies$ (i): The binary relation $\mathcal{R}$ is obviously irreflexive, since it is acyclic. By condition (ii)-(b) we have that there exists a unique x_0 such that $F(x,x_0) \cdot G(x,x_0) \leq 0$ holds for every $x \in X$. Equivalently, there exists a unique element $x_0 \in X$, for which $x\mathcal{R}x_0$ does not hold for any $x \in X$.

Finally, given any $x, y, z \in X$ such that both $y\mathcal{R}x$ and $z\mathcal{R}x$ hold true, we have that $F(y,x) \cdot G(y,x) > 0$ and also $F(z,x) \cdot G(z,x) > 0$. Hence, by condition (ii)-(c) it follows that $F(y,x) \cdot F(z,x) \cdot G(y,x) \cdot G(z,x) = [F(y,x) \cdot G(y,x)]^2 \cdot \delta(y,z)$, so that by simplifying we arrive at $F(z,x) \cdot G(z,x) = F(y,x) \cdot G(y,x)\delta(y,z)$. Thus $\delta(y,z) = 1$ a fortiori, since $F(z,x) \cdot G(z,x) > 0$. So we conclude that $y = z$. Therefore $\mathcal{R}$ is an arborescence by Proposition 5. □

4 Directed Acyclic Graphs and Incidence Matrices

Each result on binary relations of a finite set can immediately be interpreted in terms of Graph Theory, a branch of Discrete Mathematics. Basically, a *graph* consists of a finite set of vertices or points –also known as *nodes*– that are connected by arcs or lines –also known as *edges*–. In fact, some of the nodes can be pairwise related (or not), and we say that each pair of related nodes constitutes an edge of the graph. In addition, the edges may be directed or undirected, giving rise to the so-called *directed graphs*, where the edges have an orientation and are also said to be *directed edges* or *arrows*, as well as to *undirected graphs*, in which edges have no orientation at all.

Now we may observe that if X is a nonempty finite set and $\mathcal{R}$ is a binary relation on X, we can schematically represent $\mathcal{R}$ as a graph in which each node corresponds to each element in X, and an arrow is drawn from the node that represents the element $x \in X$ to the node that corresponds to $y \in X$ if and only if $x\mathcal{R}y$ holds true. Conversely, if we are given a directed graph, we immediately can interpret it as a binary relation on a finite set.

Definition 10. A *cycle* in a directed graph is an ordered tuple of nodes $(x_1, \ldots, x_k)$ such that there is an arrow from x_i to x_{i+1} for every $i < k$ and there is also one arrow from x_k to x_1. In the particular case in which $k = 1$ a 1-cycle is said to be a *loop*. A *directed acyclic graph* is a directed graph with no cycles.

(Notice that every directed acyclic graph can be interpreted as an acyclic binary relation on a nonempty finite set, and viceversa).

Definition 11. We say that a directed graph *admits a topological ordering* (also known as a *topological sorting* in this literature) if there exists a suitable linear order $\prec$ on the nodes of the graph such that it preserves the existing arrows. That is, if there is in the graph an arrow from the node x_i to the node x_j, then $x_i \prec x_j$ must hold true.

The following classical theorem is just a rephrasal of Theorem 1. It is a classical in Graph Theory, where several *sorting algorithms* have been introduced to get a topological sorting on a directed acyclic graph (see e.g. [6,7]). We should notice that the topological sorting on a directed acyclic graph *is not unique*, in general.

Theorem 6. *Any directed acyclic graph admits a topological sorting.*

It is a classical in Graph Theory, where *sorting algorithms* have been introduced to get a topological sorting on a directed acyclic graph (see e.g. [6,7]).

Another alternative way to deal with binary relations on nonempty finite sets comes from Matrix Theory. Thus, given a binary relation $\mathcal{R}$ on a set $X = \{x_1, \ldots, x_n\}$, we can visualize $\mathcal{R}$ by means of a suitable square matrix $n \times n$, called its incidence matrix. Needless to say, from such a matrix we can retrieve the binary relation $\mathcal{R}$ as well as its corresponding directed graph, already considered above. Conversely, from the graph we can easily get the corresponding matrix.

Definition 12. A $n \times n$ square matrix each of whose entries is either 0 or 1 is said to be an *incidence matrix*. Given a nonempty finite set X and a binary relation $\mathcal{R}$ on X, the *incidence matrix relative to the binary relation* $\mathcal{R}$ is the $n \times n$ matrix $M_{\mathcal{R}} = (m_{ij})$ with $m_{ij} = \chi_{\mathcal{R}}(x_i, x_j) \quad (i, j \in \{1, \ldots, n\})$. (Here $\chi_{\mathcal{R}}(x_i, x_j) = 1 \Leftrightarrow x_i \mathcal{R} x_j$. Otherwise $\chi_{\mathcal{R}}(x_i, x_j) = 0$.)

Let us analyze now how some properties of a binary relation $\mathcal{R}$ defined on a nonempty set can directly be observed by looking at its corresponding incidence matrix $M_{\mathcal{R}}$.

Proposition 7. *Let $\mathcal{R}$ be a binary relation defined on a nonempty finite set X. Let $M_{\mathcal{R}}$ be incidence matrix relative to $\mathcal{R}$. The following properties hold true.*

(i) $\mathcal{R}$ *is reflexive if and only if* $m_{ii} = 1$ *for every* $1 \leq i \leq n$.
(ii) $\mathcal{R}$ *is irreflexive if and only if* $m_{ii} = 0$ *for every* $1 \leq i \leq n$.
(iii) $\mathcal{R}$ *is asymmetric if and only if all the entries in the main diagonal of* $M_{\mathcal{R}}^2$ *are zeroes.*
(iv) If the cardinality of X *(henceforward denoted* $\#X$*) is* n, $\mathcal{R}$ *is acyclic if and only if for any natural number* k *with* $1 \leq k \leq n$, *all the entries in the main diagonal of* $M_{\mathcal{R}}^k$ *are zeroes.*
(v) If $\#X = n$, *and* $\mathcal{R}$ *is acyclic, then* $M_{\mathcal{R}}^n$ *is the null matrix.*
(vi) If $\#X = n$ *and* $\mathcal{R}$ *is acyclic, then* $I - M_{\mathcal{R}}$ *is a regular matrix such that* $(I - M_{\mathcal{R}})^{-1} = I + M_{\mathcal{R}} + \ldots + M_{\mathcal{R}}^{n-1}$.

Proof. Parts (i) and (ii) directly follow from the corresponding definitions.

Let us prove part (iii). Assume first that $\mathcal{R}$ is asymmetric. Then for every $i, j \in \{1, \ldots, n\}$ we have that $\chi_{\mathcal{R}}(x_i, x_j) \cdot \chi_{\mathcal{R}}(x_j, x_i) = m_{ij}m_{ji} = 0$. Hence the sum $\Sigma_{j=1}^n m_{ij}m_{ji} = 0$. But this sum is the i-th element in the main diagonal of $M_{\mathcal{R}}^2$. Conversely, if $\Sigma_{j=1}^n m_{ij}m_{ji} = 0$ then it is plain that $\chi_{\mathcal{R}}(x_i, x_j) \cdot \chi_{\mathcal{R}}(x_j, x_i) = 0$ for every $i, j \in \{1, \ldots, n\}$ so that $x_i \mathcal{R} x_j$ forces the negation of $x_j \mathcal{R} x_i$. Hence $\mathcal{R}$ is asymmetric.

To prove part (iv), first we assume that $\mathcal{R}$ is acyclic. Let $k \in$ be such that $1 \leq k \leq n$. Observe now that the $i-th$ term in the main diagonal of $M_{\mathcal{R}}^k$ consists of sums of products of the kind $m_{i_1 i_2} \cdot m_{i_2 i_3} \cdot \ldots \cdot m_{i_k i_{k+1}}$ with $i = i_1$ and also $i_{k+1} = i$. But, being $\mathcal{R}$ is acyclic, it is clear that all these products are null. Conversely, we may notice that the existence of a cycle on k elements, where $1 \leq k \leq n$, $\{x_{i1}, \ldots, x_{ik}\} \subseteq X^k$ such that $x_{i1} \mathcal{R} x_{i2} \mathcal{R} \ldots \mathcal{R} x_{ik} \mathcal{R} x_{i1}$ forces the i_1-th element in the main diagonal of $M_{\mathcal{R}}^k$ to be different from zero, in contradiction with the hypothesis of the statement.

To prove (v), notice that any entry in $M_{\mathcal{R}}^n$ consists of sums of products of the type $m_{i_1 i_2} \cdot m_{i_2 i_3} \cdot \ldots \cdot m_{i_n i_{n+1}}$. Since $\#X = n$ in the tuple $(i_1, i_2, \ldots, i_{n+1})$ a repetition occurs, so giving rise to a part of that tuple of the kind $(j_1, j_2, \ldots, j_k)$ with $k \leq n$ and $j_1 = j_k$. Therefore the product $m_{j_1 j_2} \cdot m_{j_2 j_3} \cdot \ldots \cdot m_{j_{k-1} j_k}$ is zero, and so is $m_{i_1 i_2} \cdot m_{i_2 i_3} \cdot \ldots \cdot m_{i_n i_{n+1}}$.

Let us conclude by proving part (vi). Since $M_{\mathcal{R}}^n$ is the null matrix by part (v), it follows that $(I - M_{\mathcal{R}}) \cdot (I + M_{\mathcal{R}} + \ldots + M_{\mathcal{R}}^{n-1}) = (I + M_{\mathcal{R}} + \ldots + M_{\mathcal{R}}^{n-1}) - (M_{\mathcal{R}} + \ldots + M_{\mathcal{R}}^{n-1} + M_{\mathcal{R}}^n) = I - M_{\mathcal{R}}^n = I$. So $I - M_{\mathcal{R}}$ is a regular matrix whose inverse equals $I + M_{\mathcal{R}} + \ldots + M_{\mathcal{R}}^{n-1}$. □

Theorem 7. *Let* $\mathcal{R}$ *be an acyclic binary relation defined on a nonempty finite set* X. *Then* $\mathcal{R}$ *is an arborescence if an only if there is a unique* $i \in \{1, \ldots, n\}$ *such that all the entries in the* i*-th column of* $M_{\mathcal{R}}$ *are zeroes, while all the entries in the* i*-th row of* $(I - M_{\mathcal{R}})^{-1}$ *equal* 1.

Proof. Assume first that $\mathcal{R}$ is an arborescence. Let $X = \{x_1, \ldots, x_n\}$. By Definition 8, there exists an element $x_i \in X$ such that $x_j \mathcal{R} x_i$ does not hold for any $x_j \in X$. Therefore $m_{ji} = 0$ for every $1 \leq j \leq n$, so that the i-th column of $M_{\mathcal{R}}$ is null. Moreover, given $j \neq i$, there is a unique $k + 1$-tuple $(x_i = x_{i0}, x_{i1}, \ldots, x_{ik} = x_j) \in X^{k+1}$, for some suitable $k \in \mathbb{N}$, such that $x_i = x_{i0} \mathcal{R} x_{i1} \mathcal{R} \ldots \mathcal{R} x_{ik} = x_j$ holds true. Therefore the entry in the row i and column j of $M_{\mathcal{R}}^k$ must be 1 by the uniqueness hypothesis. Since that k is also unique, we have that all the entries in the i-th row of $(I + M_{\mathcal{R}} + \ldots + M_{\mathcal{R}}^{n-1})$ are 1, so that by parts (v) and (vi) of Proposition 7 we conclude that the all the terms in the i-th row of $(I - M_{\mathcal{R}})^{-1}$ equal 1.

To prove the converse, first we notice that condition (i) in Definition 8 is trivially met because $\mathcal{R}$ is acyclic and, in particular, irreflexive. In addition, since all the entries in the i-th column of $M_{\mathcal{R}}$ are zeroes for a unique $i \in \{1, \ldots, n\}$, the condition (ii) in Definition 8 is accomplished by taking $x_0 = x_i$. Moreover, because all the entries in the in the i-th row of $(I - M_{\mathcal{R}})^{-1}$ equal 1, and taking into account that, by part vi) in Proposition 7, the equality $(I - M_{\mathcal{R}})^{-1} = I + M_{\mathcal{R}} + \ldots + M_{\mathcal{R}}^{n-1}$ holds true, we observe that being $x_0 = x_i$ and $x_j = x$, there exists a unique $1 \leq k \leq n$ such that the entry in the i-th row and j-th column of $M_{\mathcal{R}}^k$ equals 1, whereas for any other $l \neq k$ the entry in the i-th row and j-th column of $M_{\mathcal{R}}^k$ equals 0. Hence, there exists a unique $(k+1)$-tuple $(x_i = x_{i0}, x_{i1}, \ldots, x_{ik} = x_j) \in X^{k+1}$, with $x_{i0}\mathcal{R}x_{i1}\mathcal{R}\ldots\mathcal{R}x_{ik}$ holding true. So the condition iii) in Definition 8 is also accomplished. □

5 Concluding Remarks

Acyclic binary relations have been considered under different points of view, paying an special attention to the use of some suitable functional equation. When the set on the relations are considered is finite, the parallelism between binary relations, graphs and incidence matrix has been shown. Here, any result arising in one of those approaches –namely: abstract binary relations, directed acyclic graphs, and incidence matrices– immediately has a translation into any other one of those settings.

Among open problems within this theory, we point out that, as far as we know, the question of characterizing all those acyclic binary relations on a set that fail to admit a pseudoutility representation has not been solved yet.

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