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A convergent version of Watson's lemma for double integrals

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ABSTRACT

A modification of Watson's lemma for Laplace transforms $\int_0^\infty f(t)\,\mathrm{e}^{-zt}\,\mathrm{d}t$ was introduced in [Nielsen, 1906], deriving a new asymptotic expansion for large $|z|$ with the extra property of being convergent as well. Inspired in that idea, in this paper we derive asymptotic expansions of two-dimensional Laplace transforms $F(x,y) := \int_0^\infty \int_0^\infty f(t,s)\,\mathrm{e}^{-xt-ys}\,\mathrm{d}t\,\mathrm{d}s$ for large $|x|$ and $|y|$ that are also convergent. The expansions of $F(x,y)$ are accompanied by error bounds. Asymptotic and convergent expansions of some special functions are given as illustration.

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1. Introduction

The double Laplace transform of a function $f(t,s)$ of two real positive variables t and s is defined by means of the double integral

$$\int_0^\infty \int_0^\infty f(t,s)\,\mathrm{e}^{-xt-ys}\,\mathrm{d}t\,\mathrm{d}s,$$

for the values of x and y for which the integral exits.

The double Laplace transform has important applications in the resolution of functional, integral and partial differential equations. We can find several examples that prove its utility to solve a wide class of equations of the Mathematical Physics. Eltayeb and Kiliçman applied double Laplace transform to solve wave, Laplace's and heat equations with convolutions terms, and general linear and partial integro-differential telegraph equations [1]. Debnath discussed the properties and convolution theorem of the double Laplace transform, and applied this theorem to functional, integral and partial differential equations [2]. Futher, in [3] the authors applied the double Laplace transform technique for solving linear partial integro-differential equations with a convolution kernel, and in [4] for solving linear partial differential equations subject to initial and boundary conditions, like the advection-diffusion equation, the reaction-diffusion equation, the Klein-Gordon and the Euler-Bernoulli equations. More recently, these authors have proposed an iterative method

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based on the double Laplace transform for solving nonlinear partial differential equations [5].

On the other hand, the double Laplace transform is not just a problem-solving technique, but also a representation form of special functions; as several special functions or combinations of special functions can be written in the form of a double Laplace transform [6, Chapter 3].

For the sake of generality, we include the possibility of a branch point of the integrand at the origin, and consider the more general double Laplace transform

$$F(x,y) := \int_0^\infty \int_0^\infty \mathrm{e}^{-xt-ys} t^{\alpha-1} s^{\beta-1} f(t,s)\,\mathrm{d}t\,\mathrm{d}s, \quad \Re\alpha > 0,\ \Re\beta > 0, \tag{1.1}$$

with $\Re x > x_0$, $\Re y > y_0$ for certain $x_0, y_0 \in \mathbb{R}$. This integral is well-defined for locally integrable functions $f(t,s)$ on $[0,\infty) \times [0,\infty)$ that grow, at the infinity, not faster than an exponential (and then the integral exists for appropriate values of x_0 and y_0). When $f(t,s)$ is analytic at $(0,0)$, it has an asymptotic expansion at $(0,0)$ of the form

$$f(t,s) = \sum_{k=0}^{n-1} \sum_{l=0}^{k} a_{k-l,l} t^{k-l} s^l + f_n(t,s), \quad a_{k,l} := \frac{1}{k!l!} \frac{\partial^{k+l} f(0,0)}{\partial t^k \partial s^l}, \tag{1.2}$$

with $f_n(t,s) = \mathcal{O}((t+s)^n)$ when $(t,s) \to (0,0)$. This expansion converges in the product of disks $D_r(0) \times D_r(0) := \{(t,s); |t| < r, |s| < r\}$ for a certain $r > 0$. When we replace this expansion in the above integral (1.1) and interchange sum and integral we obtain

$$F(x,y) = \sum_{k=0}^{n-1} \sum_{l=0}^{k} a_{k-l,l} \frac{\Gamma(k-l+\alpha)\Gamma(l+\beta)}{x^{k-l+\alpha} y^{l+\beta}} + R_n(x,y), \tag{1.3}$$

with

$$R_n(x,y) := \int_0^\infty \int_0^\infty \mathrm{e}^{-xt-ys} t^{\alpha-1} s^{\beta-1} f_n(t,s)\,\mathrm{d}t\,\mathrm{d}s = \mathcal{O}((x^{-1}+y^{-1})^n x^{-\alpha} y^{-\beta})$$

as $|x|$ and $|y| \to \infty$. The terms of the expansion (1.3) are of the order $\mathcal{O}((x^{-1} + y^{-1})^k x^{-\alpha} y^{-\beta})$. Therefore, the right hand side of (1.3) is an asymptotic expansion of $F(x,y)$ for large $|x|$ and $|y|$. This is a straightforward generalization of the well-known one-dimensional Watson lemma [7, Chapter 2], [8, Chapter 1] to two variables [8, Chapter 8]. The key point is that, for large positive $\Re x$ and $\Re y$, the dominant contribution to the integral (1.1) comes from the corner point $(t,s) = (0,0)$ of the integration domain $[0,\infty) \times [0,\infty)$. Then, only the value of $f(t,s)$ around the asymptotically relevant point $(t,s) = (0,0)$, that is, the approximation (1.2), is relevant for the asymptotic behaviour of $F(x,y)$ when $\Re x$ and $\Re y$ are large.

In general, expansion (1.3) in not convergent, as we have derived the expansion by interchanging a series with an integral. Take for example the following integral representation of a certain combination of the sine and cosine integrals given in [6, Chapter 3, Section 1.5, Equation (45)],

$$\sin(xy)\mathrm{ci}(xy) - \cos(xy)\mathrm{si}(xy) = \int_0^\infty \int_0^\infty \mathrm{e}^{-xt-ys} \cos(ts)\, \mathrm{d}s\, \mathrm{d}t,$$

where $\mathrm{ci}(z)$ and $\mathrm{si}(z)$ are the cosine and sine integrals functions respectively [7, Section 6.2, Equations (6.2.9), (6.2.11)]. We have that $f(t,s) = \cos(ts) = \sum_{n=0}^{\infty}(-1)^n (ts)^{2n}/(2n)!$, and formula (1.3) becomes

$$\sin(xy)\mathrm{ci}(xy) - \cos(xy)\mathrm{si}(xy) \sim \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(xy)^{2n+1}}. \tag{1.4}$$

The series on the right hand side of (1.4) is not convergent for any value of $(x,y) \in \mathbb{C}^2$, although it is an asymptotic expansion of $\sin(xy)\mathrm{ci}(xy) - \cos(xy)\mathrm{si}(xy)$ for large $|x|$ and $|y|$. Despite the fact that the function $f(t,s)$ is an entire function, the interchange of series and integral gives an expansion (1.4) of the function $\sin(xy)\mathrm{ci}(xy) - \cos(xy)\mathrm{si}(xy)$ that is not convergent.

In Section 4 we introduce an asymptotic method for double Laplace transforms (1.1) that, in contrast to the standard method, gives an asymptotic expansion of the double Laplace transform that is also convergent. The method is inspired in the idea introduced in [7, Section 17.3], that justifies the use of inverse factorial series in [9] to derive an asymptotic expansion of the one-dimensional Laplace transform that is also convergent. In order to pave the way for the analysis of Section 4, in the next section we design a convergent and asymptotic method for double Mellin transforms of analytic functions on the square $[0,1] \times [0,1]$. In Section 3 we show that we can relax the conditions required in Section 2 for the integrand of the double Mellin transform and allow an integrable singularity at the boundary of the integration square $[0,1] \times [0,1]$. Then, in Section 4, we introduce a change of the integration variables that transforms the double Laplace integral (1.1) into the double Mellin transform considered in Section 3, deriving in this way an asymptotic and convergent method for double Laplace transforms.

In the remaining of the paper, when we state $|x|$ and $|y| \to \infty$, we assume that they go at the same speed, that is, $|x| = \gamma|y|$, with fixed positive γ; also, we assume that they go along fixed rays in the half complex plane $\Re x > 0$, $\Re y > 0$. Unless stated otherwise, all the disks $D_r(u_0)$ centred at the point u_0 and of radius r mentioned throughout the paper are considered closed disks: $D_r(u_0) := \{u \in \mathbb{C}, |u - u_0| \le r\}$. We also consider the principal value $\arg(z) \in (-\pi, \pi]$ for the argument of any complex variable z.

2. A convergent and asymptotic method for compact double Mellin transforms of analytic functions

In this section we consider Mellin transforms on the compact region $[0,1] \times [0,1]$ of functions of the form $(1-u)^{\alpha-1}(1-v)^{\beta-1}g(u,v)$, where $g(u,v)$ is analytic in the cartesian product of two disks $D_r(1) \times D_r(1)$ of radius $r > 1$:

$$F(x,y) = \int_0^1 \int_0^1 u^{x-1} v^{y-1} (1-u)^{\alpha-1}(1-v)^{\beta-1} g(u,v)\, \mathrm{d}u\, \mathrm{d}v, \quad \Re x, \Re y, \Re\alpha, \Re\beta > 0. \tag{2.1}$$

The integral (2.1) may be interpreted as the double Mellin transform of a function of compact support on the positive first quadrant of the plane: $(1-u)^{\alpha-1}(1-v)^{\beta-1}g(u,v)\chi_{[0,1]\times[0,1]}(u,v)$. For the sake of generality, and for later convenience, we allow branch point singularities in the integrand of the form $(1-u)^{\alpha-1}(1-v)^{\beta-1}$.

The asymptotic features of the integral (2.1) for large $|x|$ and $|y|$ are similar to those of the integral (1.1): the dominant contribution of the integrand to the integral comes from the top right corner of the integration region $(u,v)=(1,1)$; and the branch point at $(1,1)$ introduced by the factor $(1-u)^{\alpha-1}(1-v)^{\beta-1}$ has not any influence in the asymptotic analysis. But the integration region here is compact, and this fact has an important consequence on the convergence of the asymptotic expansion of the integral (2.1) that we detail in the following theorem.

Theorem 2.1: *Assume that $g(u,v)$ is analytic in the cartesian product of two disks $D_r(1)\times D_r(1)$ with $r>1$, and consider the region $\mathcal{D}:=\{(x,y)\in\mathbb{C}\times\mathbb{C}:\min\{\Re x,\Re y\}\geq\Lambda\}$ for arbitrary fixed $\Lambda>0$. Then, for $(x,y)\in\mathcal{D}$, $\Re\alpha,\Re\beta>0$, and $n=1,2,3,\ldots$,*

$$F(x,y)=\int_0^1\int_0^1 u^{x-1}(1-u)^{\alpha-1}v^{y-1}(1-v)^{\beta-1}g(u,v)\,\mathrm{d}u\,\mathrm{d}v$$
$$=\sum_{k=0}^{n-1}\sum_{l=0}^{k}\frac{(-1)^k}{(k-l)!l!}\frac{\partial^k g(1,1)}{\partial u^{k-l}\partial v^l}B(\alpha+k-l,x)\,B\left(\beta+l,y\right)+R_n(x,y,\alpha,\beta),\quad(2.2)$$

where $B(u,v)$ is the beta function [10, Section 5.12]. *The remainder term is bounded in the form*

$$|R_n(x,y,\alpha,\beta)|\leq\frac{M}{r^n}\sum_{k=0}^{n}B(n-k+\Re\alpha,\Lambda)\,B(k+\Re\beta,\Lambda),\qquad(2.3)$$

where M is a positive constant independent on n, x and y. The right-hand side of (2.2) is an asymptotic expansion of $F(x,y)$ for large $|x|$ and $|y|$, as $B(\alpha+k-l,x)B(\beta+l,y)=\mathcal{O}((x^{-1}+y^{-1})^k x^{-\alpha}y^{-\beta})$ and $R_n(x,y,\alpha,\beta)=\mathcal{O}((x^{-1}+y^{-1})^n x^{-\alpha}y^{-\beta})$ for fixed n. It is also uniformly convergent for $(x,y)\in\mathcal{D}$, with an exponential order of convergence: $R_n(x,y,\alpha,\beta)=\mathcal{O}(n^{-x-y}r^{-n})$ when $n\to\infty$ with fixed x and y.

Proof: For large $|x|$ and $|y|$, the asymptotically relevant point in the integral (2.1) is the point $(u,v)=(1,1)$, as it is the point of the integration domain $[0,1]\times[0,1]$ where $|u^xv^y|$ attains its maximum value. Then, only the behaviour of the function $g(u,v)$ at the point $(1,1)$ is relevant for the asymptotic analysis of this integral. Therefore, we consider the Taylor expansion of the function $g(u,v)$ at the point $(1,1)$,

$$g(u,v)=\sum_{k=0}^{n-1}\sum_{l=0}^{k}\frac{1}{(k-l)!l!}\frac{\partial^k g(1,1)}{\partial u^{k-l}\partial v^l}(u-1)^{k-l}(v-1)^l$$
$$+g_n(u,v),\quad (u,v)\in D_r(1)\times D_r(1),\qquad(2.4)$$

with

$$g_n(u,v) = \sum_{k=0}^{n} \frac{(u-1)^{n-k}(v-1)^k}{(2\pi i)^2} \oint\oint \frac{g(z,w)\,\mathrm{d}z\,\mathrm{d}w}{(z-1)^{n-k}(w-1)^k(z-u)(w-v)}. \tag{2.5}$$

In this formula, the integration contours are two circles of radii $r > 1$ centred at $z = 1$ and $w = 1$ respectively, that is, $|z-1| = |w-1| = r$, and traversed once in the positive sense. The point $z = u$ is inside the first circle and the point $w = v$ is inside the second one. When we replace the expansion (2.4) into the integral on the right-hand side of (2.1) and interchange sum and integral, we obtain (2.2) with

$$R_n(x,y,\alpha,\beta) := \int_0^1\int_0^1 u^{x-1}(1-u)^{\alpha-1}v^{y-1}(1-v)^{\beta-1}g_n(u,v)\,\mathrm{d}u\,\mathrm{d}v. \tag{2.6}$$

The function $g(u,v)$ is analytic on the above mentioned circles. Therefore, using that $|z-1| = |w-1| = r$ in the double integral on the right-hand side of (2.5), it is straightforward to show that the remainder $g_n(u,v)$ may be bounded in the form $|g_n(u,v)| \le Mr^{-n}\sum_{k=0}^{n}(1-u)^{n-k}(1-v)^k$, with M a positive constant independent on n, u and v. Using this bound in (2.6) we get

$$|R_n(x,y,\alpha,\beta)| \le \frac{M}{r^n}\sum_{k=0}^{n} B(n-k+\Re\alpha,\Re x)B(k+\Re\beta,\Re y).$$

Using that, for any $a > 0$, the beta function $B(a,x)$ is a decreasing function of the positive variable x, we obtain (2.3). Finally, from [10, Equation 5.11.12], we have that, when $|x|$ and $|y| \to \infty$, $R_n(x,y,\alpha,\beta) = \mathcal{O}((x^{-1}+y^{-1})^n x^{-\alpha}y^{-\beta})$ and

$$\sum_{l=0}^{k} \frac{(-1)^k}{(k-l)!l!}\frac{\partial^k g(1,1)}{\partial u^{k-l}\partial v^l} B(\alpha+k-l,x)B(\beta+l,y) \sim x^{-\alpha}y^{-\beta}(x^{-1}+y^{-1})^k.$$

■

Example 2.1: Consider the second Appell function [11, Equation 16.15.2],

$$F_2(a;x,y;x+\alpha,y+\beta;b,c) = \frac{\Gamma(x+\alpha)\Gamma(y+\beta)}{\Gamma(x)\Gamma(y)\Gamma(\alpha)\Gamma(\beta)} \times \int_0^1\int_0^1 \frac{u^{x-1}(1-u)^{\alpha-1}v^{y-1}(1-v)^{\beta-1}}{(1-b\,u-c\,v)^a}\,\mathrm{d}u\,\mathrm{d}v, \tag{2.7}$$

with $\Re\alpha, \Re\beta, \Re x, \Re y > 0$, large $|x|$ and $|y|$, and fixed α, β, a, b and c. The above integral has the form considered in Theorem 2.1 with $g(u,v) = (1-b\,u-c\,v)^{-a}$. When $\Re b + \Re c \in \mathbb{C}\setminus[1,\infty)$, the function $g(u,v)$ is analytic in $[0,1]\times[0,1]$, and from (2.2) we derive the following asymptotic expansion for large $|x|$ and $|y|$ that is also convergent

$$F_2(a;x,y;x+\alpha,y+\beta;b,c) = \sum_{k=0}^{\infty}\sum_{l=0}^{k} \frac{(-1)^k}{(k-l)!l!}\frac{(a)_k b^{k-l}c^l}{(1-b-c)^{a+k}}\frac{(\alpha)_{k-l}(\beta)_l}{(x+\alpha)_{k-l}(y+\beta)_l}, \tag{2.8}$$

where $(a)_k$ is the Pochhammer symbol [10, Section 5.2(iii)]. Table 1 contains some numerical experiments that illustrate the accuracy of approximation (2.8).

Table 1. Relative errors (r.e.) in the approximation of the integral given in (2.7) for $\alpha = 0.3$, $\beta = 2.1$, $a = 0.95$, $b = 0.2$, $c = -1.3$, by using (2.8) with the infinite series truncated at $k = n$ (after $(n+1)(n+2)/2$ terms).

$n = 1$							
(x, y)	$(8e^{i\frac{\pi}{3}}, 12)$	$(70e^{i\frac{\pi}{3}}, 50)$	$(90e^{i\frac{\pi}{3}}, 80)$	$(100e^{i\frac{\pi}{3}}, 100)$	$(200e^{i\frac{\pi}{3}}, 300)$	$(300e^{i\frac{\pi}{3}}, 400)$	$(500e^{i\frac{\pi}{3}}, 600)$
r.e.	0.089	0.024	0.015	0.012	4.e−3	3.e−3	1.9e−3

$(x, y) = (1.2, 3.1)$										
n	5	10	15	20	25	30	35	40	45	50
r.e.	4.18e−3	8.3e−5	2.7e−6	1.1e−7	5.28e−9	2.79e−10	1.59e−11	9.6e−13	6.e−14	3.9e−15

Note: The asymptotic and convergent behaviour is shown in the above and below subtables respectively. In this table and the remaining tables of the paper, the computations have been carried out with the symbolic manipulator *Wolfram Mathematica 12.2*; in particular, the command `NIntegrate` has been used to compute the 'exact' value of the double integral involved in the definition of the Appell function F_2 and to compute double integrals in later examples.

3. A convergent and asymptotic method for compact double Mellin transforms of analytic functions. A more general case

We consider in this section functions $g(u, v)$ that may have integrable singularities in two sides of the integration square in (2.1), say in $\mathcal{T} := \{(0, v) \cup (u, 0)\}$, $u, v \in [0, 1]$. That is, we consider functions $g(u, v)$ analytic in $D_1(1) \times D_1(1) \setminus \mathcal{T}$. This extension is interesting on its own and is the subject of this section. But moreover, it is necessary to derive the main result of the paper in the next section.

In this case, we could repeat step by step the proof of Theorem 2.1, considering a region $D_r(1) \times D_r(1)$ with $r < 1$, where $g(u, v)$ is analytic. We would derive the same expansion that we have obtained in Theorem 2.1, with the same asymptotic property. But we would not be able to show the convergence of that expansion, as the parameter r in formula (2.3) would not be large enough: $r < 1$.

Still, it is possible to extend the technique described in the previous section to this kind of integrand $g(u, v)$, obtaining an asymptotic expansion of the compact Mellin transform $F(x, y)$ (2.1) that is also convergent. To this end, we have to relax the hypotheses of the above theorem, at the expense of obtaining a slower speed of convergence of the expansion (2.2). We have the following theorem.

Theorem 3.1: *Consider again the region $\mathcal{D} := \{(x, y) \in \mathbb{C} \times \mathbb{C} : \min\{\Re x, \Re y\} \geq \Lambda\}$ for arbitrary fixed $\Lambda > 0$ and assume that $g(u, v)$ is analytic in the cartesian product of two disks $D_1(1) \times D_1(1) \setminus \mathcal{T}$, with $\mathcal{T} := \{(0, v) \cup (u, 0),\ u, v \in [0, 1]\}$. Moreover, we assume that the singularities of $g(u, v)$ at $\mathcal{T}$ are integrable, that is, the function $u^{\sigma_1} v^{\sigma_2} g(u, v)$ is bounded on $\mathcal{T}$, for certain $\sigma_1, \sigma_2 > 0$ with $\max\{\sigma_1, \sigma_2\} < \min\{1, \Lambda\}$. Then, the thesis of Theorem 2.1 holds true for $(x, y) \in \mathcal{D}$ and $\Re\alpha, \Re\beta > 0$. That is, for $n = 1, 2, 3, \ldots$,*

$$
\begin{aligned}
F(x, y) &= \int_0^1 \int_0^1 u^{x-1}(1-u)^{\alpha-1} v^{y-1}(1-v)^{\beta-1} g(u, v)\, \mathrm{d}u\, \mathrm{d}v \\
&= \sum_{k=0}^{n-1} \sum_{l=0}^{k} \frac{(-1)^k}{(k-l)!\,l!} \frac{\partial^k g(1,1)}{\partial u^{k-l} \partial v^l} B(\alpha + k - l, x)\, B(\beta + l, y) + R_n(x, y, \alpha, \beta), \quad (3.1)
\end{aligned}
$$

but with a different bound for the remainder

$$|R_n(x,y,\alpha,\beta)| \le M \sum_{k=0}^{n} B(n-k+\Re\alpha, \Lambda-\sigma_1)B(k+\Re\beta, \Lambda-\sigma_2). \tag{3.2}$$

In this formula, M is a positive constant independent on n, m, x and y. The right-hand side of (3.1) is an asymptotic expansion of $F(x,y)$ for large $|x|$ and $|y|$: $B(\alpha+k-l,x)B(\beta+l,y) = \mathcal{O}((x^{-1}+y^{-1})^k x^{-\alpha}y^{-\beta})$ and $R_n(x,y,\alpha,\beta) = \mathcal{O}((x^{-1}+y^{-1})^n x^{-\alpha}y^{-\beta})$ when $|x|,|y| \to \infty$ with fixed n. But moreover, it is also uniformly convergent for $(x,y) \in \mathcal{D}$, with a power type order of convergence: $R_n(x,y,\alpha,\beta) = \mathcal{O}(n^{\sigma_1+\sigma_2-(x+y)})$ when $n \to \infty$ with fixed x and y.

Proof: As in Theorem 2.1, the function $g(u,v)$ in the integral (3.1) has the expansion (2.4), but now with a smaller convergence radius $r<1$. As in Theorem 2.1, when we introduce this expansion into the integral (3.1) and interchange sum and integral, we obtain the expansion on the right-hand side of (3.1) with $R_n(x,y,\alpha,\beta)$ given in (2.6). Therefore, bound (2.3) also holds true now, but it is useless, as $r<1$ and then it does not prove the convergence of the expansion (3.1). To prove convergence, we need a sharper bound for the remainder $R_n(x,y,\alpha,\beta)$. To this end we are going to make a more careful analysis of the Cauchy integral representation of the remainder $g_n(u,v)$,

$$g_n(u,v) = \sum_{k=0}^{n} \frac{(u-1)^{n-k}(v-1)^k}{(2\pi i)^2} \oint\oint \frac{g(z,w)\,dz\,dw}{(z-1)^{n-k}(w-1)^k(z-u)(w-v)}, \quad u,v \in (0,1]. \tag{3.3}$$

In this formula, the integration paths are the circles $C_1 := \{w \in \mathbb{C}; |w-1| = r\}$ and $C_2 := \{z \in \mathbb{C}; |z-1| = r\}$ with $r = 1-\varepsilon < 1$, $\varepsilon > 0$ as small as we wish. These paths encircle the points $z=u$ and $z=1$, and $w=v$ and $w=1$ respectively in the positive direction, see Figure 1(a) for the circle C_1; it is similar for the circle C_2. The function $g(z,w)$ is analytic inside $C_1 \times C_2$, that is, the closed contours C_1 and C_2 do not contain the points $(0,w)$ and $(z,0)$ inside, nor any other singularity of the function $g(z,w)$. On the other hand, by Cauchy's theorem, the above integral is a constant function of ε. Moreover, it is defined for $\varepsilon = 0$ $(r=1)$ and it is continuous as a function of ε, since it is the integral of an integrable function. Hence, we can take the limit $\varepsilon \to 0$ and consider that $r=1$. Moreover, using the fact that $z^{\sigma_1}w^{\sigma_2}g(z,w)$ is bounded on $C_1 \times C_2$ we have

$$|g_n(u,v)| \le M_0 \sum_{k=0}^{n} (1-u)^{n-k}(1-v)^k \oint_{C_1}\oint_{C_2} \frac{|z^{-\sigma_1}w^{-\sigma_2}|}{|z-u||w-v|}\,dz\,dw, \quad u,v \in (0,1],$$

with M_0 independent of u, v, n and m. Consider u, $v>0$. After the change of variables $z \mapsto z\,u$ and $w \mapsto w\,v$ we find

$$|g_n(u,v)| \le \frac{M_0}{u^{\sigma_1}v^{\sigma_2}} \sum_{k=0}^{n} (1-u)^{n-k}(1-v)^k \times \oint_{C_1/u}\oint_{C_2/v} \frac{|z^{-\sigma_1}w^{-\sigma_2}|}{|z-1||w-1|}\,dz\,dw, \quad u,v \in (0,1], \tag{3.4}$$

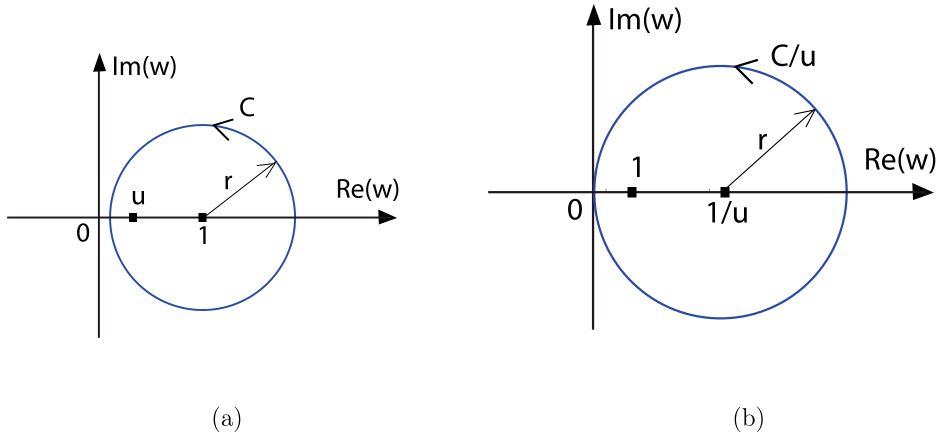

Figure 1. (a) The integration contour C_1 in (3.3) is a circle $|w-1|=r$ centred at $w=1$ and radius $r<1$; it encloses the points $w=1$ and $w=u$. The integration contour C_2 is similar. Then, the integration region in (3.3) is contained inside the domain of analyticity of $g(z,w)$. (b) The first integration contour in (3.4) is a circle $|z-1/u|=1/u$ centred at $z=1/u$ and radius $1/u$; it encloses the points $z=1/u$ and $z=1$, and becomes the imaginary axis traversed downwards when $u\to 0$. The second integration contour is similar. Therefore, the integration region in (3.4) is contained inside the domain of analyticity of $g(u\,z,v\,w)$.

where the integration contours C_1/u and C_2/v are, respectively, the scaled circles $C_1/u=\{z\in\mathbb{C};|z-1/u|=1/u\}$ and $C_2/v=\{w\in\mathbb{C};|w-1/v|=1/v\}$ traversed in the positive direction, see Figure 1(b) for the circle C_1/u; it is similar for the circle C_2/v. In the limit $u,v\to 0$ both scaled circles become the imaginary axis traversed downwards, and the above double integral on these paths is finite. Then, the double integral on the right-hand side of (3.4) can be bounded uniformly for $u,v\in(0,1]$. Therefore,

$$|g_n(u,v)|\le Mu^{-\sigma_1}v^{-\sigma_2}\sum_{k=0}^{n}(1-u)^{n-k}(1-v)^k,$$

with M a positive constant independent on n, m, u and v. Introducing this bound in (2.6) we obtain, after straightforward computations,

$$|R_n(x,y,\alpha,\beta)|\le M\sum_{k=0}^{n}B(n-k+\Re\alpha,\Re x-\sigma_1)B(k+\Re\beta,\Re y-\sigma_2).$$

From the asymptotic behaviour of the gamma function [10, Equation 5.11.13] and the definition of the beta function [10, Equation 5.12.1] we immediately derive $B(\alpha+k-l,x)B(\beta+l,y)=\mathcal{O}((x^{-1}+y^{-1})^k x^{-\alpha}y^{-\beta})$ and $R_n(x,y,\alpha,\beta)=\mathcal{O}((x^{-1}+y^{-1})^n x^{-\alpha}y^{-\beta})$ when $|x|$ and $|y|\to\infty$ with fixed n. An also that $R_n(x,y,\alpha,\beta)=\mathcal{O}(n^{(\sigma_1+\sigma_2)-(x+y)})$ when $n\to\infty$ with fixed x,y. Furthermore, since the right-hand side above is a decreasing function of $\Re x$ and $\Re y$, (3.2) holds true and the expansion (3.1) is also uniformly convergent for $(x,y)\in\mathcal{D}$.

■

Observation 3.1: When the singularities of $g(u,v)$ at the border $\mathcal{T}$ are stronger than what we have assumed in Theorem 3.1, we can still apply Theorem 3.1 at the expense of reducing the convergence region $\mathcal{D}$. Suppose that $u^{\sigma_1}v^{\sigma_2}g(u,v)$ is not bounded at $\mathcal{T}$ for any $0 < \sigma_1, \sigma_2 < 1$, but $u^{\sigma_1+\delta}v^{\sigma_2+\delta}g(u,v)$ is bounded for a certain $\delta > 0$. Theorem 3.1 above can still be applied at the smaller region $\mathcal{D}_\delta := \{(x,y) \in \mathbb{C} \times \mathbb{C} : \Re x, \Re y \geq \Lambda + \delta\} \subset \mathcal{D}$. To show this, just write $u^{x-1}v^{y-1}g(u,v) = u^{\bar{x}-1}v^{\bar{y}-1}\bar{g}(u,v)$, with $\bar{x} := x - \delta, \bar{y} := y - \delta$ and $\bar{g}(u,v) := u^\delta v^\delta g(u,v)$. We can apply Theorem 3.1 with (x,y) replaced by $(\bar{x},\bar{y})$ and $g(u,v)$ replaced by $\bar{g}(u,v)$.

Example 3.1: Consider the following derivative of the second Appell function,

$$\frac{\partial^2}{\partial x \partial y}\left[\frac{\Gamma(x)\Gamma(y)\Gamma(\alpha)\Gamma(\beta)}{\Gamma(x+\alpha)\Gamma(y+\beta)}F_2(a;x,y;x+\alpha,y+\beta;b,c)\right]$$
$$= \int_0^1\int_0^1 u^{x-1}(1-u)^{\alpha-1}v^{y-1}(1-v)^{\beta-1}\log u \log v(1-b\,u-c\,v)^a\,\mathrm{d}u\,\mathrm{d}v, \tag{3.5}$$

where $\Re\alpha, \Re x, \Re\beta, \Re y > 0$. It has the form considered in Theorem 3.1 with $g(u,v) = (1 - b\,u - c\,v)^{-a}\log u \log v$, and σ_1 and σ_2 any positive numbers as close to 0 as we wish. When $\Re b + \Re c \in \mathbb{C} \setminus [1,\infty)$, the function $g(u,v)$ is analytic in $[0,1] \times [0,1] \setminus \mathcal{T}$ and, from (3.1), we derive the following convergent and asymptotic expansion for large $|x|$ and $|y|$,

$$\frac{\partial^2}{\partial x \partial y}\left[\frac{\Gamma(x)\Gamma(y)\Gamma(\alpha)\Gamma(\beta)}{\Gamma(x+\alpha)\Gamma(y+\beta)}F_2(a;x,y;x+\alpha,y+\beta;b,c)\right]$$
$$= \frac{1}{\Gamma(a)}\sum_{k=0}^{\infty}\sum_{l=0}^{k+2}(-1)^l(l-1)!c^{l-1}\,A(k+2,l)(1-b-c)^{a+l-1}$$
$$\times B\left(\alpha+k-l+2,x\right)B\left(\beta+l,y\right), \tag{3.6}$$

with

$$A(k,l) := \sum_{j=0}^{k-l-1}\frac{(-b)^j}{j!(l+j-k)}\frac{\Gamma(a+l+j-1)}{(1-b-c)^j}\,{}_3F_2\left(\left.\begin{array}{c}1,1,1-l\\ 2,2-a-l-j\end{array}\right|1+\frac{b-1}{c}\right),$$

and ${}_3F_2$ a generalized hypergeometric function [11, Section 16.2]. Table 2 contains some numerical experiments that illustrate the accuracy of approximation (3.6).

4. A convergent and asymptotic expansion of double Laplace transforms

The main obstruction to the convergence of the expansion (1.3) is that the integration region $[0,\infty) \times [0,\infty)$ in (1.1) cannot be contained in any convergence region given by the cartesian product of disks $D_r(0) \times D_r(0)$ of the expansion (1.2) of $f(t,s)$ regardless how large r is. Then, when we replace $f(t,s)$ in (1.1) by its Taylor expansion (1.2) and interchange series and integral, the convergence of the resulting series (1.3) is not guaranteed. It is not guaranteed even when $f(t,s)$ is an entire function of its two variables (and $r = \infty$).

Inspired by the idea introduced in [7, Section 17.3], we consider a change of integration variables $t \to u$, $s \to v$ on the right-hand side of (1.1) that compacts the unbounded

Table 2. Relative errors in the approximation of the integral given in (3.5), for $\alpha = 0.3, \beta = 2.1, a = 1.05, b = -1.1, c = -0.5$, by using (3.6) with the infinite series truncated at $k = n$.

	$n = 1$						
(x, y)	(5.2, 4.5)	(10, 16)	(25, 30)	(40, 50)	(60, 75)	(100, 100)	(200, 300)
r.e.	0.46	0.23	0.1	0.07	0.048	0.03	0.013

	$(x, y) = (15.2, 13.5)$									
n	5	10	15	20	25	30	35	40	45	50
r.e.	8.67e−3	2.39e−5	4.55e−7	2.27e−8	2.e−9	2.58e−10	4.38e−11	9.15e−12	2.25e−12	6.25e−13

Note: The asymptotic and convergent behaviour is shown in the above and below subtables respectively.

region $[0, \infty) \times [0, \infty)$ into a bounded region for the new variables u, v. More precisely, we consider the change of variables $t = -\log u, s = -\log v$ that gives rise to the bounded integration region $(0, 1] \times (0, 1]$ in the variables u, v considered in the previous sections. After this change of variables, the integral (1.1) has the form considered in Theorem 3.1:

$$F(x, y) := \int_0^1 \int_0^1 u^{x-1}(1-u)^{\alpha-1} v^{y-1}(1-v)^{\beta-1} g(u, v)\, \mathrm{d}u\, \mathrm{d}v, \tag{4.1}$$

with

$$g(u, v) := \left(\frac{\log u}{u-1}\right)^{\alpha-1} \left(\frac{\log v}{v-1}\right)^{\beta-1} f(-\log u, -\log v).$$

With this change of variables, we are introducing logarithmic singularities in the set $\mathcal{T} = \{(0, v) \cup (u, 0), u, v \in [0, 1]\}$. But, as we have seen in Section 3, the effect of integrable singularities on the set $\mathcal{T}$ is not very painful. The effect is a slower speed of convergence of the expansion with respect to the regular case analysed in Section 2. In order to better understand the effect of these logarithmic singularities on $\mathcal{T}$, it is necessary to take a closer look to the mappings $u \to t = -\log u, v \to s = -\log v$. Under these transformations, the $u, v-$disks $D_r(1)$, $0 < r \le 1$, become the respective $t, s-$ regions

$$S_r := \{t, s = -\log(1 + \rho\, \mathrm{e}^{i\theta}), 0 \le \rho \le r, -\pi < \theta \le \pi\}$$

(see Figure 2 for $r = 1$ and Figure 3 for $r < 1$).

Under the inverse maps $u = \mathrm{e}^{-t}, v = \mathrm{e}^{-s}$, the end $t, s-$points $t, s = \infty$ in (1.1) are transformed into the singular $u, v-$points u, $v = 0$ in (4.1). More generally, the unbounded region S_r around the $t, s-$integration region $[0, \infty) \times [0, \infty)$ is transformed into the cartesian product of two u, v-disks: $\{|u - 1| \le 1\} \times \{|v - 1| \le 1\}$. This means that, if the function $f(t, s)$ is analytic in the region $S_1 \times S_1$, then the function $g(u, v)$ is analytic in the region $D_1(1) \times D_1(1) \setminus \mathcal{T}$ (with logarithmic branch points at $\mathcal{T}$). Then, we can use Theorem 3.1 for the integral (4.1) if $f(t, s)$ is analytic in the region $S_1 \times S_1$ and does not grow too fast at the infinity. This idea is summarized in the following theorem.

Theorem 4.1: *Consider the region $\mathcal{D} := \{(x, y) \in \mathbb{C} \times \mathbb{C} : \min\{\Re x, \Re y\} \ge \Lambda\}$ for arbitrary fixed $\Lambda > 0$. Assume that $f(t, s)$ is analytic in the region $S_1 \times S_1$, with S_r defined above and that $f(t, s) = \mathcal{O}(\mathrm{e}^{\sigma_1 t}\, \mathrm{e}^{\sigma_2 s})$, for certain $\sigma_1, \sigma_2 > 0$ with* $\max\{\sigma_1, \sigma_2\} < \min\{1, \Lambda\}$

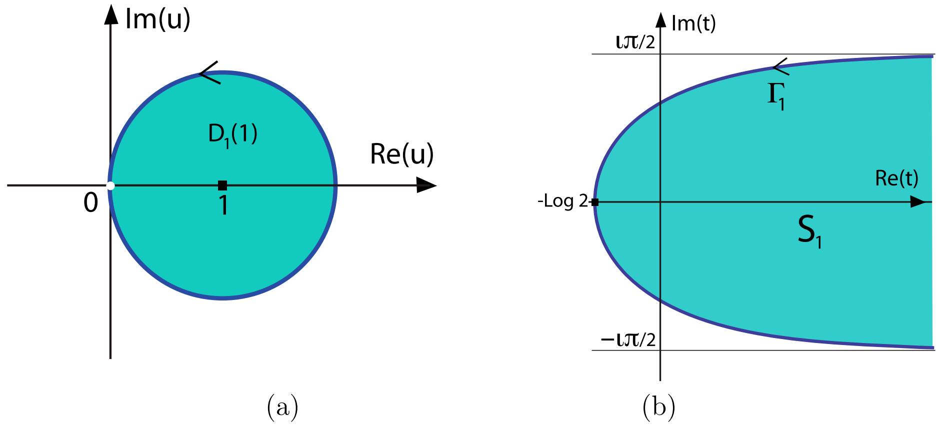

Figure 2. The image under the map $t = -\log u$ of the disk $|u-1| \leq 1$ depicted on the left figure is the unbounded $t-$region S_1 depicted on the right figure, whose contour is the curve $t = -\log(1 + \mathrm{e}^{i\theta})$, $-\pi < \theta < \pi$.

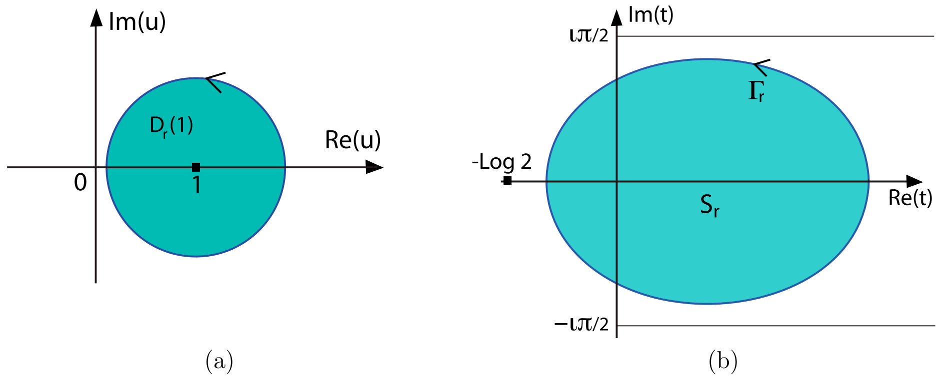

Figure 3. The image under the map $t = -\log u$ of the disk $|u-1| \leq r < 1$ depicted on the left figure is the $t-$region S_r depicted on the right figure, whose contour is the curve $t = -\log(1 + r\,\mathrm{e}^{i\theta})$, $-\pi < \theta \leq \pi$.

as $t, s \to \infty$. Then, for $(x,y) \in \mathcal{D}$, $\Re\alpha > 0, \Re\beta > 0$, and $n = 1, 2, 3, \dots$,

$$\begin{aligned} F(x,y) &= \int_0^\infty \int_0^\infty \mathrm{e}^{-xt-ys} t^{\alpha-1} s^{\beta-1} f(t,s)\,\mathrm{d}t\,\mathrm{d}s \\ &= \sum_{k=0}^{n-1} \sum_{l=0}^{k} A_{k-l,l}(\alpha,\beta) B\left(k-l+\alpha, x\right) B\left(l+\beta, y\right) + R_n(x,y,\alpha,\beta), \end{aligned} \tag{4.2}$$

where

$$A_{0,0}(\alpha,\beta) := a_{0,0} = f(0,0), \quad A_{1,0}(\alpha,\beta) := \frac{\alpha-1}{2} a_{0,0} + a_{1,0},$$

$$A_{0,1}(\alpha,\beta) := \frac{\beta-1}{2} a_{0,0} + a_{0,1}$$

and, for $n, m = 1, 2, 3, \ldots$, the remaining coefficients of the expansion are

$$A_{n,m}(\alpha,\beta) := \sum_{k=0}^{n}\sum_{l=0}^{m} \frac{(k+\alpha-1)(l+\beta-1)}{(n+\alpha-1)(m+\beta-1)} a_{k,l}(-1)^{n+m-k-l} \times \widetilde{B}_{n-k}(n+\alpha-1)\widetilde{B}_{m-l}(m+\beta-1). \tag{4.3}$$

In this formula, $a_{k,l} := (\frac{1}{k!l!})\frac{\partial^{k+l}f(0,0)}{\partial t^k \partial s^l}$ and $\widetilde{B}_m(\alpha)$ are the normalized Nörlund polynomials $\widetilde{B}_m(\alpha) := B_m(\alpha)/m!$, where $B_m(\alpha)$ denote the standard Nörlund polynomials [12, Section 24.16].

The coefficients $A_{n,m}(\alpha,\beta)$ and the remainder $R_n(x,y,\alpha,\beta)$ are bounded in the form $|A_{n,m}(\alpha,\beta)| \le M$ and

$$|R_n(x,y,\alpha,\beta)| \le M \sum_{k=0}^{n} B(n-k+\Re\alpha, \Lambda-\sigma_1)B(k+\Re\beta, \Lambda-\sigma_2), \tag{4.4}$$

where M is a positive constant independent on n, m, x and y. The right-hand side of formula (4.2) is an asymptotic expansion of $F(x,y)$ for large $|x|$ and $|y|$ and fixed n, as we have that $B(k-l+\alpha,x)B(l+\beta,y) = \mathcal{O}((x^{-1}+y^{-1})^k x^{-\alpha}y^{-\beta})$ and $R_n(x,y,\alpha,\beta) = \mathcal{O}((x^{-1}+y^{-1})^n x^{-\alpha}y^{-\beta})$. Moreover it is also uniformly convergent for $(x,y) \in \mathcal{D}$ with a power type order of convergence: $R_n(x,y,\alpha,\beta) = \mathcal{O}(n^{\sigma_1+\sigma_2-x-y})$ when $n \to \infty$ and fixed x, y.

Proof: After the change of variables $t = -\log u$, $s = -\log v$, the integral in formula (4.2) becomes the integral (4.1) considered in Theorem 3.1 with $g(u,v)$ given also in (4.1). Then, formulas (4.2) and (4.4) follow from Theorem 3.1 with

$$A_{n,m}(\alpha,\beta) := \frac{(-1)^{n+m}}{n!m!}\frac{\partial^{n+m}g(1,1)}{\partial u^n \partial v^m},$$
$$g(u,v) := \left(\frac{\log u}{u-1}\right)^{\alpha-1}\left(\frac{\log v}{v-1}\right)^{\beta-1} f(-\log u, -\log v).$$

Therefore we have that

$$A_{n,m}(\alpha,\beta) := \frac{(-1)^{n+m}}{n!m!}\frac{\partial^{n+m}}{\partial u^n \partial v^m}\left[\left(\frac{\log u}{u-1}\right)^{\alpha-1}\left(\frac{\log v}{v-1}\right)^{\beta-1} f(-\log u, -\log v)\right]_{u=1,v=1}$$
$$= \frac{(-1)^{n+m}}{(2\pi i)^2}\oint\oint \frac{g(u,v)}{(u-1)^{n+1}(v-1)^{m+1}}\,\mathrm{d}u\,\mathrm{d}v,$$

where the integration paths are two circles of a certain radius r centred at $u=1$ and $v=1$ respectively: $|u-1| = |v-1| = r$, traversed once in the positive sense. After the change of integration variables $u \to t, v \to s$ given by $u = \mathrm{e}^{-t}, v = \mathrm{e}^{-s}$ we have that

$$A_{n,m}(\alpha,\beta) = \frac{(-1)^{n+m}}{(2\pi i)^2}\int_{\Gamma_r}\int_{\Gamma_r}\frac{(-t)^{\alpha-1}(-s)^{\beta-1}f(t,s)\,\mathrm{e}^{-t-s}}{(\mathrm{e}^{-t}-1)^{n+\alpha}(\mathrm{e}^{-s}-1)^{m+\beta}}\,\mathrm{d}t\,\mathrm{d}s,$$

where the integration contour Γ_r is the path $|\mathrm{e}^{-t}-1| = |\mathrm{e}^{-s}-1| = r$ depicted in Figure 3(b). Then we have that

$$A_{n,m}(\alpha,\beta) = \frac{1}{n!m!}\frac{\partial^{n+m}}{\partial t^n \partial s^m}\left[\left(\frac{t}{1-\mathrm{e}^{-t}}\right)^{n+\alpha}\left(\frac{s}{1-\mathrm{e}^{-s}}\right)^{m+\beta} f(t,s)\,\mathrm{e}^{-t-s}\right]_{t=0,s=0}.$$

From this formula, using

$$\left(\frac{u}{1-\mathrm{e}^{-u}}\right)^{n+\alpha}\mathrm{e}^{-u} = \left(\frac{u}{1-\mathrm{e}^{-u}}\right)^{n+\alpha} - \left(\frac{u}{1-\mathrm{e}^{-u}}\right)^{n+\alpha-1} u,$$

and [12, Section 24.16, Equation 24,16.9] we find

$$\begin{aligned} A_{n,m}(\alpha,\beta) = {} & \sum_{k=0}^{n}\sum_{l=0}^{m}(-1)^{n+m-k-l}a_{k,l}[\widetilde{B}_{n-k-1}(n+\alpha-1)+\widetilde{B}_{n-k}(n+\alpha)] \\ & \times [\widetilde{B}_{m-l-1}(m+\beta-1)+\widetilde{B}_{m-l}(m+\beta)], \end{aligned} \tag{4.5}$$

with $\widetilde{B}_{-1}(\alpha) := 0$.

On the other hand, integrating by parts in the integral representation of the generalized Nörlund polynomials (that follows from [5, Section 24.16, Equation 24.16.9]),

$$\widetilde{B}_m(\alpha) = \frac{1}{2\pi i}\oint\left(\frac{t}{\mathrm{e}^t-1}\right)^{\alpha}\frac{\mathrm{d}t}{t^{m+1}},$$

we find that, for $m = 1, 2, 3, \dots,$

$$m\widetilde{B}_m(\alpha) = \alpha\left[\widetilde{B}_m(\alpha) - \widetilde{B}_{m-1}(\alpha) - \widetilde{B}_m(\alpha+1)\right].$$

Formula (4.3) follows from this one and (4.5). ■

Example 4.1: Consider the function given in the introduction section [6, Chapter 3, Section 1.5, Equation (45)]

$$\sin(xy)\mathrm{ci}(xy) - \cos(xy)\mathrm{si}(xy) = \int_0^\infty\int_0^\infty \mathrm{e}^{-xt-ys}\cos(ts)\,\mathrm{d}s\,\mathrm{d}t, \quad \Re x > 0,\ \Re y > 0. \tag{4.6}$$

This integral is the double Laplace transform of $f(t,s) = \cos(ts)$ and has the form considered in Theorem 4.1 with $\alpha = \beta = 1$. From (4.2) we derive the following asymptotic

Table 3. Relative errors in the approximation of the integral (4.6) by using (4.7) with the infinite series truncated after n terms.

	$n=1$						
(x,y)	$(5.3, 4.1)$	$(7.5, 6.2)$	$(15, 22)$	$(30, 30)$	$(50, 60)$	$(110, 120)$	$(200, 300)$
r.e.	4.15e−3	9.2e−4	1.8e−5	2.47e−6	2.22e−7	1.15e−8	5.55e−10

	$(x,y)=(9.2,7.5)$									
n	5	10	15	20	24	30	35	40	45	50
r.e.	2.e−4	3.3e−6	1.14e−7	1.15e−8	2.27e−9	7.28e−10	2.52e−10	9.79e−11	4.2e−11	1.9e−11

Note: The asymptotic and convergent behaviour is shown in the above and below subtables respectively.

expansion for large $|x|$ and $|y|$ that is also convergent,

$$\sin(xy)\mathrm{ci}(xy) - \cos(xy)\mathrm{si}(xy) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{n-k,k}B(n-k+1,x)B(k+1,y), \tag{4.7}$$

where $A_{0,0}=1$, $A_{1,0}=A_{0,1}=0$ and, for $n,m=1,2,3,\ldots$,

$$A_{n,m} = \sum_{j=0}^{\lfloor \frac{\min(n,m)}{2} \rfloor} \frac{(2j)^2}{(2j)!}\frac{(-1)^{n+m+j}}{nm}\widetilde{B}_{n-2j}(n)\widetilde{B}_{m-2j}(m),$$

where $\lfloor a \rfloor$ represents the integer part of the real number a (Table 3).

Example 4.2: Consider the integral

$$H(x,y) := \int_0^{\infty}\int_0^{\infty} \frac{t^{\alpha-1}s^{\beta-1}\,\mathrm{e}^{-xt-ys}}{a+t+s}\,\mathrm{d}s\,\mathrm{d}t, \quad \Re\alpha > 0,\ \Re\beta > 0,\ a > 0, \tag{4.8}$$

for $\Re x > 0$ and $\Re y > 0$. It has the form considered in Theorem 4.1 with $f(t,s) = 1/(a+t+s)$. From (4.2) we derive the following asymptotic expansion for large $|x|$ and $|y|$ that is also convergent,

$$H(x,y) = \sum_{k=0}^{\infty}\sum_{l=0}^{k} A_{k-l,l}B(k-l+1,x)\,B\left(l+1,y\right), \tag{4.9}$$

where $A_{0,0}=1, A_{1,0}=(a(\alpha-1)-2)/2a^2, A_{0,1}=(a(\beta-1)-2)/2a^2$, and for $n,m=1,2,3,\ldots$ (Table 4),

$$A_{n,m} = \frac{(-1)^{n+m}}{a}\sum_{k=0}^{n}\sum_{l=0}^{m}\frac{(k+l)!}{k!l!a^{k+l}}\frac{(k+\alpha-1)(l+\beta-1)}{(n+\alpha-1)(m+\beta-1)} \times \widetilde{B}_{n-k}(n+\alpha-1)\widetilde{B}_{m-l}(m+\beta-1).$$

Table 4. Relative errors in the approximation of the integral (4.8) for $\alpha = 2.3$, $\beta = 1.1$, $a = 1.5$, by using (4.9) with the infinite series truncated after n terms.

	$n = 1$						
(x, y)	$(3.5e^{i\frac{\pi}{4}}, 2.2)$	$(15e^{i\frac{\pi}{4}}, 20)$	$(30e^{i\frac{\pi}{4}}, 40)$	$(50.5e^{i\frac{\pi}{4}}, 50)$	$(90e^{i\frac{\pi}{4}}, 80)$	$(100e^{i\frac{\pi}{4}}, 100)$	$(200e^{i\frac{\pi}{4}}, 300)$
r.e.	0.187	0.032	0.017	0.014	8.6e−3	6.9e−3	2.4e−3

	$(x, y) = (5.2, 4.5)$									
n	5	10	15	20	24	30	35	40	45	50
r.e.	1.9e−4	1.3e−4	1.85e−5	5.3e−6	1.8e−6	7.6e−7	3.5e−7	1.8e−7	1.e−7	6.e−8

The asymptotic and convergent behaviour is shown in the above and below subtables respectively.

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