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ONTO-SEMIOTIC ANALYSIS OF DIAGRAMMATIC REASONING

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Abstract

Diagrams and in general the use of visualization and manipulative material, play an important role in mathematics teaching and learning processes. Although several authors warn that mathematics objects should be distinguished from their possible material representations, the relations between these objects are still conflictive both from an epistemological point of view as well as an educational one. In this paper we apply theoretical tools of the onto-semiotic approach of mathematics knowledge to analyze the diversity of objects and processes implied in mathematics activity, which is carried out with diagrammatic representations. This enables us to appreciate the synergic relations between ostensive (visual and sequential languages) and non-ostensive objects (abstract and mental entities) overlapping in mathematics practices. The analysis of the characteristics of diagrammatic reasoning and its interpretation in onto-semiotic terms is contextualised by means of the analysis of solving a problem about fractions by applying three procedures that involve diagrams.

Keywords: diagrammatic reasoning, epistemic analysis, fraction concept, onto-semiotic configuration, visualization

Introduction

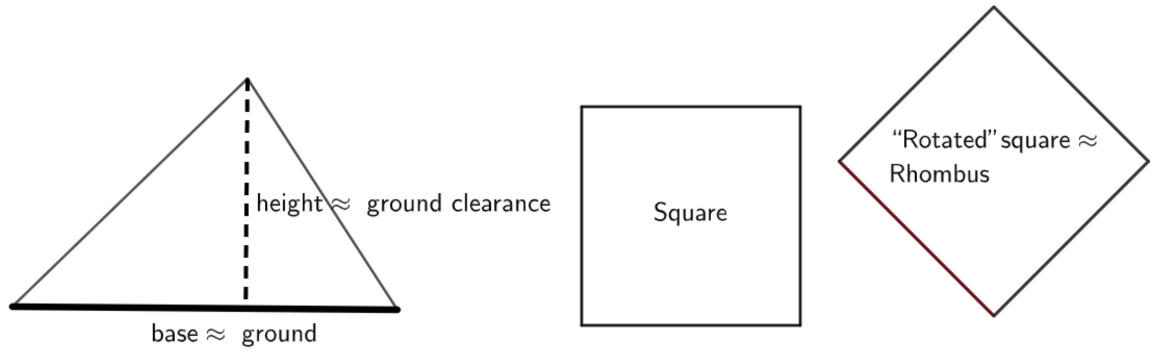
The use of different representations, visualizations, diagrams, manipulative materials are proposed to favour mathematics learning by assuming the supposition that such materials make up *models* of mathematics concepts and of the structures in which they are organised. It is supposed that the use of material representations is necessary, not only to communicate mathematical ideas but also for their own construction. However, the relations between mathematics and the real world are still conflictive[1].

Thus, Geometry is not the 'measurement of the Earth' as is etymologically established but a model of space, that is, a simplified representation which permits prediction and action. So, in mathematics the identification and description of mathematics objects are necessary, as is the recognition of their specific nature. "Any didactic theory, at one moment or another (unless it voluntarily wants to confine itself to a kind of naïve position), must clarify its ontological and epistemological position" (Radford, 2008, p. 221).

Moreover, because "mathematical objects cannot be directly apprehended by the senses, the role of mediating signs (representations of some kind) is crucial in all mathematical activity, including its teaching and learning" (Presmeg, 2006, p. 19). Mathematics objects are abstract whereas diagrams are specific and perceptible. It is necessary not to confuse them but the synergy relations between both types of objects are not dealt with explicitly. Therefore, it is essential to have tools that allow the comprehensible description of mathematical activity (Iori, 2016). In fact, in learning, a key aspect is the difference between the ostensive representation of the geometric object (*figure*) and the object itself (*as a set of properties*). Thus, novice students who do not differentiate representation and object, for example, point out that a triangle has only one base ("horizontal segment") and only one height ("from the horizontal segment to the opposite vertex"), or that a "rotated" square is just a rhombus (Figure 1).

Figure 1

Ostensive representation of geometric objects



In this paper we intend to progress in the identification of the objects involved in diagrammatic reasoning and in the description of its nature. To do this, we will use the semiotic and anthropological perspective proposed by the "onto-semiotic approach" (OSA) of mathematics knowledge and instruction (Font, Godino, & Gallardo, 2013; Godino, Burgos, & Gea, 2021).

> The issue for mathematics education then is what does it mean to know something about mathematical objects and how does the learner develop or construct that knowledge? The answer to this question will to some extent depend on the ontological and epistemological status that is ascribed to those mathematical objects. (Dörfler, 2002, p. 337)

The epistemological and semiotic problem that interests us is to clarify the relationship between the visual, diagrammatic, or whatever other representation (ostensive objects) and immaterial, ideal or abstract mathematical objects (non-ostensive objects) that necessarily accompany them. We are also interested in clarifying the dialect between the different types of languages by being aware of the limitations of the diagrammatic representation that should be compensated by the sequential languages although recognising the epistemic and cognitive possibilities of the visual means of expression. The educational objective is to show that the application of the onto-semiotic configuration tool can help to understand the students' difficulties in mathematics learning by revealing the network of ostensive objects (material objects) and non-ostensive objects (immaterial objects) that intervene in mathematics activity and the synergic relations between the same. In the OSA ontology, the term ‘object’ is used in a broad meaning to refer to any entity involved some way in mathematical practice and that can be identified as a unit. The use of object is metaphoric, since a mathematical concept, is usually conceived as an ideal or abstract entity, and not something tangible, such as a rock, a drawing, or a manipulative (Godino et al., 2021).

In the following section we mention some characteristic features of the visualization and the diagrammatic reasoning that point out the problem described, that is the gap between the representation and the mathematics object represented. Then, we summarise the notion of onto-semiotic configuration of practices, objects, and processes which will be the theoretical tool that we will use to analyse the diagrammatic reasoning displayed when solving a problem about fractions using three different methods. Next, we highlight the synergic relations there are between the different types of languages and the non-ostensive objects necessarily overlapping in mathematics practice. In the final section, we include some reflections about the type of understanding, which the onto-semiotic approach of mathematics knowledge can provide to visualization, and diagrammatic reasoning in addition to some implications about the learning and training of mathematics teachers.

Visualisation and Diagrammatic Reasoning

Visualization and Types of Languages

Arcavi (2003, p. 217) describes the visualization in very general terms:

> Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing

> previously unknown ideas and advancing understandings.

Likewise, this author considers that mathematics, as a human and cultural creation that deals with objects and entities which are very different from any physical phenomena, strongly supports visualization in its different forms and levels, not only in the field of geometry. Through visualization any organisation can be synoptically understood as a configuration making all that is not accessible to the human eye by providing a global apprehension of any organisation of relations (Duval, 2002).

Godino et al. (2012) analyse the notion of visualization by applying the tools of the onto-semiotic approach of mathematics knowledge (Godino et al., 2007; Font et al., 2013) and propose to distinguish between the “visual practices” and the “non-visual practices” or symbolical/analytical. They focus their attention on whether the types of linguistic objects and artefacts considered to be visual that intervene in a practice, stake the visual perception. On the understanding that the symbolic representations (natural language or formal language), although they consist of visible inscriptions, are not considered as exactly visual inscriptions, but as analytical or sentential.

The sequential *languages* (for example, symbolic logic, natural languages) only use the relation of linking to represent the relation between objects. To the contrary, spatial relations are used to represent other relations in the diagrams.

> The idea is that sentential languages are based on acoustic signals which are sequential in nature, and so must have a complex syntax in order to express certain relationships - whereas diagrams, being two-dimensional, are able to display some relationships without the intervention of a complex syntax. (Shin & Lemon, 2008, section 3)

The Role of Diagrams in Mathematical Work

Different conceptions are proposed about the use of diagrams in the works of research analysed in the field of mathematics education. Arcavi includes them as just another visual resource which coordinates with visualization; however according to the literature about diagrammatic reasoning, the diagrams, understood in the framework of *Peircean semiotics* (Dörfler, 2005; Bakker & Hoffmann, 2005; Rivera, 2011), form an essential resource of mathematics reasoning in addition to other fields and scientific disciplines (Bechtel, 2017; Giardino, 2017; Shin & Lemon, 2008).

We found that research on diagrammatic reasoning presents a double conception of the notion of diagram. One wider conception in which any type of inscription that makes use of the spatial positioning in two or three dimensions (right, left; forward, backward; up, down; inclusion, intersection, separation; accumulation, …) is a diagram (geometric figures; cartesian graphs; matrixes; graphs; conceptual maps; organization charts; sketches and maps, …). Another more restricted conception requires us to be able to carry out specific transformations, combinations, and constructions with these representations according to certain specific syntactic and semantic rules. The parts which make up a diagram can be any type of inscription such as letters, numbers, special signs, or geometrical figures.

Peirce includes the algebraic formulas in the notion of diagram since they are understood as icons of relations between their constituent elements. One feature which distinguishes the icons is when directly observing them, other facts relative to the objects which are different from those which are sufficient to determine their construction, can be discovered. This ability to reveal unexpected facts is precisely where the use of algebraic formulas lies and so its iconic-diagrammatic nature is what prevails. So, when we affirm: "the expression $y = x^2 - 2x + 1$ is a parabola" we are informing of the essential properties of the mathematics object. However, the letters of the algebraic expression, taken randomly, are not icons but indices: each letter is an index of quantity. To the contrary, the signs +, =, /, etc., are symbols in the sense of Peirce.

As Filloy and colleagues point out "algebraic expressions are thus an example of the imbrication of three kinds of signs in mathematical writing: the letters are indices; the signs

+, =, etc. are symbols; and the expression taken as a whole is an icon” (Filloy et al., 2008, p. 47).

Diagrammatic reasoning

Dörfler (2003, p. 41) identifies some of the characteristics of the diagrams and the diagrammatic reasoning:

- Diagrammatic inscriptions have a structure consisting of a specific spatial arrangement of and spatial relationships among their parts and elements.
- Based on this diagrammatic structure there are rule-governed operations on and with the inscriptions by transforming, composing, decomposing, combining them (calculations in arithmetic and algebra, constructions in geometry, derivations in formal logic).
- Another type of conventionalized rule governs the application and interpretation of the diagram within and outside of mathematics, i.e., what the diagram can be taken to denote or model.
- The diagrammatic inscriptions have a generic aspect, which permits to construct arbitrary instances of the same type of diagram.
- Diagrammatic reasoning is a rule-based inventive and constructive manipulation of diagrams to investigate their properties and relationships.
- Diagrammatic reasoning is not mechanistic or purely algorithmic, it is imaginative and creative.
- In diagrammatic reasoning the focus is on the diagrammatic inscriptions irrespective of what their referential meaning might be. The objects of diagrammatic reasoning are the diagrams themselves and their already established properties.
- Efficient and successful diagrammatic reasoning presupposes intensive and extensive experience with manipulating diagrams. A widespread ‘inventory’ of diagrams, their

properties and relationships support and occasions the creative and inventive usage of diagrams.

Diagrammatic reasoning involves three steps (Bakker & Hoffmann, 2005, p. 340): the first step is to construct a diagram (or diagrams) by means of a representational system; the second step of diagrammatic reasoning is to experiment with the diagram (or diagrams); the third step is to observe the results of experimenting and reflect on them. Any experimentation with a diagram is being carried out within a system of representation and is a rule or activity situated within a practice. From this experimentation and observation, Peirce highlights that we can “discover unnoticed and hidden relations among the parts of a diagram” (CP 3.363). In this sense Rivera (2011), points out that “with the aid of diagrammatic reasoning, the focus switches to detecting, constructing, and establishing regularities and invariant relationships that eventually take the shape of concepts and theorems that are themselves diagrams in some other format” (Rivera, 2011, p. 229).

Registers of Representation and Diagrams

Duval (2006) attributes an essential role not only to the use of different systems of semiotic representation (SSR) for mathematics work but also to the treatment of the signs within each system and the conversion between different SSR:

> The role that signs play in mathematics is not to be substituted by objects but by other signs! What matters are not representations but their transformation. Unlike the other areas of scientific knowledge, signs and semiotic representation transformation are at the heart of mathematics activity. (Duval, 2006, p. 107)

Dörfler (2005) recognises that the diagrams can make up a register of autonomous representation to represent and produce mathematics knowledge in certain specific fields, however it is not complete. It requires to be complemented by conceptual-verbal language to express notions like continuity and differentiability; impossibility that specific objects exist; or situations of the use of quantifiers *for all, each one and there are.*

The relations between the physical objects, the diagrams and other visualizations used in mathematics practice and mathematics objects (concepts, propositions, procedures, …) are conflictive. Radford (2003, p. 43) highlights the problem of the impossibility of any direct access to mathematical objects and the ensuing need for means to render them sensible. Duval (2006, p. 129) conceives the mathematics object as "the invariant of a set of phenomena or the invariant of some multiplicity of possible representations" and insists on not confusing the mathematics object with its different representations. This leads to explaining the cognitive paradox of mathematics learning:

> The crucial problem of mathematics comprehension for learners, at each stage of the curriculum, arises from the cognitive conflict between these two opposite requirements: how can they distinguish the represented object from the semiotic representation used if they cannot get access to the mathematical object apart from the semiotic representations? (Duval, 2006, p. 107)

Another problem related to the use of diagrams is that pointed out by Shin, & Lemon (2008, section 4.1), which consists of moving from what is particular to what is general:

> A central issue, if not the central issue, was the generality problem. The diagram that appears with a Euclidean proof provides a single instantiation of the type of geometric configurations the proof is about. Yet properties seen to hold in the diagram are taken to hold of all the configurations of the given type. What justifies this jump from the particular to the general?

Diagrams and Abstract Objects

Other authors (Bakker & Hoffmann, 2005), following Peirce, in addition to assigning a central role to the operations to be carried out on diagrammatic inscriptions assume an understanding of mathematics objects condensed in the *hypostatic abstraction*. According to this conception a certain characteristic of a set of objects is considered as a new object; a name is assigned to a specific predicate and thus creating an abstract object. "In mathematics, a collection is a hypostatic abstraction. And the cardinal numbers are hypostatic abstractions derived from predicate of a collection” (Peirce, CP 5.535).

To think of mathematics objects as qualities of collections of objects or invariants of a set of phenomena or representations, that are converted into new objects (abstracts) for the simple reason of having been given a name with specific terminology, is to adopt a position not exempt from philosophical, cognitive and also educational problems. This supposes not considering the linguistic revolution that Wittgenstein provided about mathematics activity and the resulting product of this activity. Thus, from a formalist perspective, the mathematical object is its definition. The definition is not only a way of understanding or describing the object, “it is itself the object”. From Wittgenstein’s perspective, “language games” imply that the meaning of mathematical objects is conditioned by their use in a context. There is no longer a single answer to the question “what is such a mathematical object?”, and, consequently, the statements “Tom, Dick, or Harry knows what that mathematical object is” are meaningless.

Sherry (2009) adopts an anthropological perspective on the role of diagrams in mathematics argumentation that differs from the Peircean semiotic perspective, according to which the diagrams are an essential means in the process of hypostatic abstraction. Sherry analyses the role of the diagrams in mathematics reasoning (geometric and numerical – algebraic) without resorting to the introduction of abstract objects and relying on a Wittgeinsteinian perspective of mathematics. “Recognizing that a diagram is just one among other physical objects is the crucial step in understanding the role of diagrams in mathematical argument” (Sherry, 2009, p. 65).

The position of this author is based on observing the way in which mathematics is applied to specific objects. The experience with diagrams should provide the students with the opportunity to see the relationship, which is mutually determining between the construction of an inferential rule and the development of mathematics knowledge. This is a question of avoiding recurring to abstract conceptions which are conceived in an empirical-realistic way (hypostatic abstraction) to understand them as socially agreed grammatical rules, about the use of languages through which we describe our worlds (material or immaterial).

Onto-Semiotic Configurations

The OSA is a theoretical framework that has developed principles and methodological tools to address the epistemological, ontological, cognitive, instructional, and ecological problems inherent to the processes of teaching and learning mathematics (Godino et al., 2007; Font et al., 2013). “It is assumed that mathematics is a human activity (anthropological postulate) and that the entities involved in this activity come or emerge from the actions and discourse through which they are expressed and communicated (semiotic postulate)” (Font et al., 2013, p. 107).

In this paper we use a specific OSA tool that allows for detailed analyses of mathematical activity. This is the onto-semiotic configuration (Figure 2), which helps to identify the objects and processes involved in the mathematical practices carried out to solve mathematical problems. The following typology of primary mathematics objects is proposed:

- *Languages* (terms, expressions, notations, graphs) in different registers (written, oral, gesture, etc.).
- *Situations-problems* (extra - mathematics applications, exercises).
- *Concepts - definition* (introduced using definitions or descriptions) (straight-line, dot, number, average, function).
- *Propositions* (statements about concepts).
- *Procedures* (algorithms, operations, calculation techniques).

- *Arguments* (statements used to justify or explain the propositions and procedures, whether they are deductive or otherwise).

Mathematical objects intervening in mathematical practices or emerging from them, depend on the language game in which they take part, and can be considered from the following dual dimensions or facets (Godino et al., 2007, p. 131).

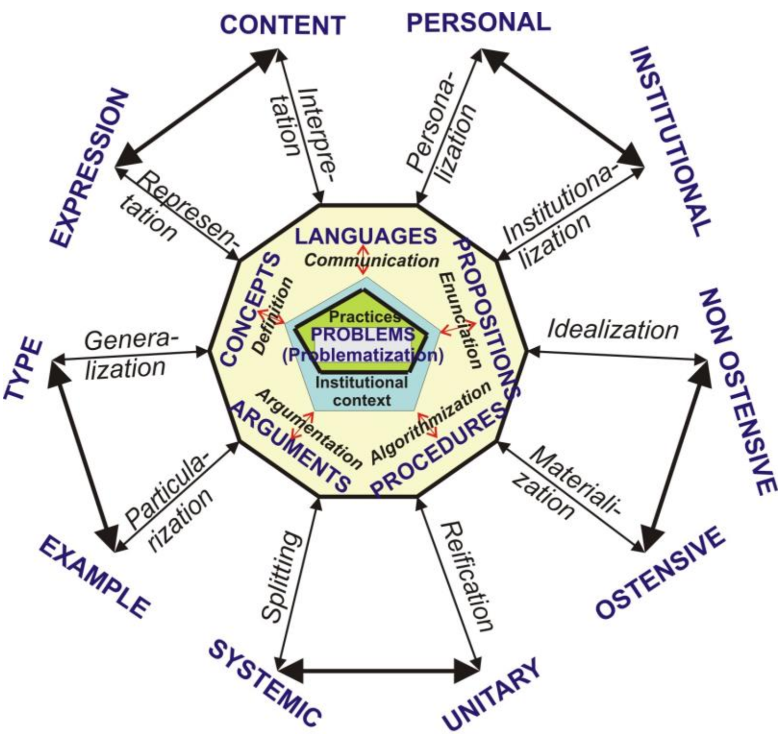
- *Personal–institutional.* Institutional objects emerge from systems of practices shared within an institution, while personal objects emerge from specific practices from a person.
- *Ostensive–non-ostensive.* Mathematical objects (both at personal and institutional facets) are, in general, non-perceptible. However, they are used in public practices through their associated ostensive notations, symbols, graphs, etc. The distinction between ostensive and non-ostensive is relative to the language game in which they take part. Ostensive objects can also be thought, imagined by a subject or be implicit in the mathematical discourse (for example, the multiplication sign in algebraic notation).
- *Extensive–intensive* (example–type). An extensive object is used as a particular case (a specific example, i.e., the function $y = 2x + 1$), of a more general class (i.e., the family of functions $y = mx + n$), which is an intensive object. This duality allows to focus the attention on the dialectic between the particular and the general, which is a key issue in the construction and application of mathematical knowledge.
- *Unitary–systemic.* In some circumstances mathematical objects are used as unitary entities (they are supposed to be previously known), while in other circumstances they are seen as systems that could be decomposed to be studied.
- *Expression–content*. They are the antecedent and consequent of semiotic functions. Mathematical activity is essentially relational, since the different objects described are

not isolated, but they are related in mathematical language and activity by means of semiotic functions. Each type of object can play the role of antecedent or consequent (signifier or signified) in the semiotic functions established by a subject (person or institution).

These facets or dualities are grouped in pairs that are dually and dialectically complementary. The dualities as well as the objects can be analysed from the process-product perspective. The emergence of the primary objects (problems, definitions, propositions, procedures, and arguments) take place with the respective mathematics communication processes, problematization, definition, expression, procedure forming (algorithmization, routinization, …), and argumentation. On the other hand, the dualities lead to the following cognitive/epistemic processes: institutionalization-personalization; generalization-particularization; analysis/splitting-synthesis/reification; materialization/precision-idealization/abstraction; expression /representation-meaning.

Figure 2

Onto-semiotic configuration of practices, objects, and processes



An abstract object (ideal or hypostatic) is understood in the OSA as an entity:

- Immaterial (non-ostensive).
- General (intensive).
- That can be considered in the following way:
  - unitary (as a rule) or systemic (onto-semiotic configuration of practices, objects and processes);
  - personal (mental) or institutional (sociocultural);
  - antecedent (signifying) or consequent (meaning) in a semiotic function.

The process of abstraction by which the abstract objects emerge or are built entails the combination of other more basic cognitive-epistemic processes: generalization, idealization (understood as dematerialization), unitarization (reification), giving meaning, representation. This anthropological way of understanding abstraction, that is, the emergence of general and immaterial objects forming mathematical structures, has important consequences for mathematics education since mathematics learning should take place through a progressive participation of the students in the mathematics language games used in the heart of the mathematics practices communities (institutions or sociocultural groups). Thus, anthropology as "epistemology of use in context within a community" establishes that knowledge has an essentially social dimension. In this way, dialogue and social interaction take on an important role, in comparison with the mere manipulation and visualization of ostensive objects.

The anthropological vision of Wittgenstein is assumed, according to which the concepts, propositions and mathematics procedures are no other than empirical propositions which have been socially reified as rules. Sherry clearly and synthetically describes this Wittgeinsteinian conception of mathematics objects:

> For an empirical proposition to harden into a rule, there must be overwhelming agreement among people, not only in their observations, but also in their reactions to them. This

> agreement reflects, presumably, biological, and anthropological facts about human beings. An empirical proposition that has hardened into a rule very likely has practical value, underwriting inferences in commerce, architecture, etc. (Sherry, 2009, p 66)

Behind diagrammatic reasoning, from the use of visualizations and manipulative teaching materials which help in mathematics learning, there is an implicit adoption of an empirical–realistic position about the nature of mathematics and which does not grant the essential role to language and to social interaction in the emergence of mathematics objects. To a certain extent, it is supposed that the mathematical object 'is seen', is hypostatically detached from empirical qualities of the collections of things. Faced with this position, coming from the epistemology and Peircean semiotics, the anthropological conception of mathematics is found and according to which concepts and mathematics propositions should be understood, not as hypostatic abstractions of noticeable quality, but as regulations of the operative and discursive practices carried out by people to describe and act in the social and empirical world in which we live.

In previous research works, Godino and Cols. apply a semiotic analysis technique of mathematics practices through which the mathematics objects used in the practices are to be revealed. In Godino (2002) a first approximation to this technique is carried out by analysing a lesson from a textbook about the median; in Godino, Font, and Wilhelmi (2006) the onto-semiotic analysis is carried out from a lesson about addition and subtraction; and in Godino, Font, Wilhelmi, and Lurduy (2011) the responses of a child to a task related to learning of tens are analysed. In this article, we summarize the semiotic analysis using Tables 1 to 3 to show the configuration of mathematics practices, objects and processes used when solving a problem on fractions.

Onto-Semiotic Configurations Implied in Diagrammatic Reasoning

In this section we analyse the types of practices, objects and processes that are used when solving a problem about fractions by applying three procedures that involve the use of diagrammatic reasoning. We try to show that by accompanying visual-diagrammatic language the support of sequential-analytical language is necessary and together with the ostensive objects, inherent to both types of languages, a configuration of abstract objects which takes part in the mathematics practices, is always present. Likewise, we will show that solving the problem implies carrying out processes of particularization of abstract objects which have been previously shared and processes of materialization (construction and manipulation of diagrams).

> *Martini cocktail problem (fraction of alcohol):*
>
> *A Martini is a cocktail which is made up of 5 parts gin and 1 part vermouth. Suppose that 2/5 of the gin is alcohol and 1/6 of the vermouth. What percentage of alcohol does a Martini have?*

Solution 1: Use of Area Diagrams to Represent the Fractions

The sequence of area diagrams of Figure 3 is explanatory of the problem-solving process for someone who knows the conventions assumed, as well as the concepts and procedures implied. However, the justification and explanation of the solution requires the following sequence of discursive and operative practices:

1) The unitary quantity of Martini is represented by means of a square (Figure 3A)
2) The square is divided vertically into 6 equal parts (Figure 3B).
3) The fraction of gin is 5/6 of the square unit (red part, Figure 3B).
4) The fraction of vermouth is 1/6 of the said square (white part, Figure 3B).
5) The white rectangle that represents the quantity of vermouth is divided into 6 equal parts of which 1 part corresponds to the quantity of (1/6 of 6) (Figure 3C).

6) The quantity of alcohol of the gin is represented by the two blue columns in Figure 3D (2/5 of 5).

7) The quantities of alcohol in the gin and in the vermouth should be expressed in the same unit of measurement, so, the two blue rectangles that represent the quantity of alcohol in the gin should be divided horizontally into six equal parts (Figure 3E).

8) The total quantity of alcohol in the Martini will be 12 + 1 = 13 small squares (Figure 3E).

9) The total quantity of Martini represented by the initial square should also be measured with the same unit of measurement with which the quantities of alcohol are measured, so the six horizontal lines are prolonged (Figure 3F).

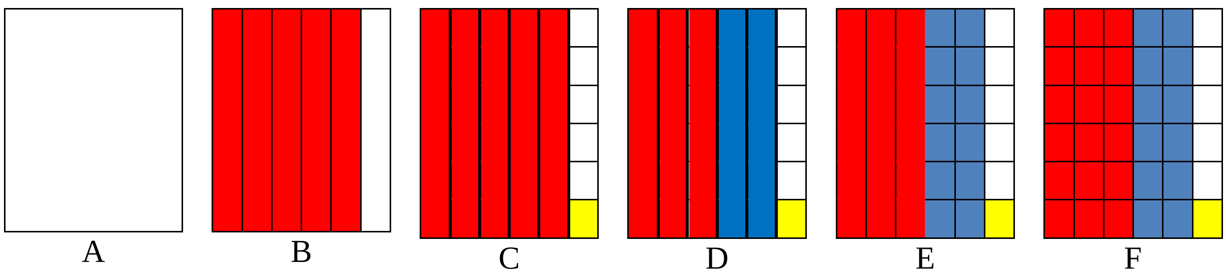
10) The fraction of Martini will be 13/36 (Figure 3F).

11) Since the proportion of alcohol of the Martini is 13/36 ≈ 0,3611, the percentage (approximately) will be 36,11%.

12) Solution represented as follows, in Figure 3:

Figure 3

Area diagrams to solve the Martini problem



Source: Adapted from Giacomone et al. (2019, p. 25)

In terms of Duval´s theory of semiotic representation registers, a conversion is begun, moving from the sequential register of natural language (task statement) to the graphic register (area diagram); specific treatments are carried out within this register to finally move onto the sequential register once again: *The fraction of alcohol of the Martini is 13/36.* However as shown in the sequence of practices 1) to 9), the sequential register must accompany the graphic register. Likewise, the operative and discursive practices used are guided by the process of *non-ostensive* objects and processes that we reveal in the second column of Table 1. In the third column of this table, we indicated the role that each practice plays in the problem-solving process, in addition to its intentionality.

Table 1

Configuration of objects and meanings

Textualized operative and discursive practices	**Non ostensive objects (concepts, propositions, procedures, arguments)**	**Use and purpose of the practices**
Statement		
A Martini is a cocktail, which is made with 5 parts gin and 1 part vermouth.	*Concept:* A whole unit of volume. *Procedure*: Composition of the whole unit from equal parts.	Describe what a Martini is made of
Suppose that 2/5 of the gin is alcohol and 1/6 of the vermouth also.	*Concept:* fraction as part of a whole unit which is divided into equal parts of which one part is individualized. The case of the fractioned composition of the gin (2/5) and of the vermouth, is specified.	Fix the fraction of alcohol in the gin and the vermouth as data.
What percentage of alcohol does a Martini have?	*Concept*: Whole unit; fraction, part of a whole unit divided into equal parts	Express the problematic question of the task
	Solution	
1) The unit of Martini is represented by means of a square, (Figure 3A).	*Concept*: unitary quantity.	Particularize and materialize the concept of unit.
2) The square is vertically divided into 6 equal parts, (Figure 3B).	*Procedure*: division of the unit into equal parts.	Action required to ostensively (by diagram) represent the composition of the Martini in the following practice, considering the statement.
3) The fraction of gin is 5/6 of the square unit. (Figure 3B, red part).	*Concept:* fraction as part of a whole divided into equal parts. *Convention:* the fraction is expressed in two equal ways, with an arithmetic diagram (5/6) and a graph diagram.	Express in fractions the quantity of gin in the Martini.

4) The fraction of vermouth is 1/6 of the square (Figure 3B, white part).	*Concept:* fraction as part of a whole divided into equal parts. *Convention:* the fraction is expressed in two equal forms with an arithmetic diagram (1/6) and a graph.	Express in fractions the quantity of vermouth in the Martini.
5) The white rectangle that represents the quantity of vermouth (...)	*Procedure:* division of a unit in equal parts. *Concept*: fraction as an operator.	Express in fractions the quantity of alcohol in the vermouth.
6) The quantity of alcohol in the gin (...)	*Concept*: fraction as an operator.	Express in fractions the quantity of alcohol in the vermouth.
7) The quantities of alcohol in the gin and the vermouth should be expressed (...)	*Concept*: unit of measurement; measurement. *Procedure:* measure an area with a given unit.	Make possible the measurement of all the quantities with one same unit. This is done using natural arithmetic.
8) The quantity of alcohol in the Martini will be 12 + 1 = 13 small squares (Figure 3E).	*Concept:* volume magnitude (sumable). *Procedures*: counting and adding.	Measure the quantity of alcohol in the Martini with natural numbers (13 units).
9) The total quantity of Martini represented by the initial square (...)	*Procedure:* measure an area with a given unit. *Concept:* Cartesian product of natural numbers.	Make possible the measurement of all the quantities with one same unit. Using natural arithmetic.
10) The fraction of alcohol of the Martini will be 13/36 (Figure 3F).	*Concept:* fraction as part of a whole. *Proposition* The fraction of the alcohol in the Martini is 13/36. *Argumentation*: it is formed by the sequence of steps 1) to 10), supported by the use of arithmetic diagrams, of areas and of natural sequential language.	Response to the fraction question posed.
11) Since the proportion of alcohol in the Martini is 13/36 ≈ 0,3611, the percentage (approximate) will be 36,11%.	*Concepts*: rational number; proportionality; fraction; decimal and percentual approximation. *Procedures:* obtaining the decimal expression using the numerator and the denominator quotient; step to the percentual expression.	Response to the problem and its justification in terms of percentual expression.

In addition to the processes indicated in Table 1, the subject who solves the problem basing his/her reasoning on the use of area diagrams carries out processes of materialization of the concepts and operations with fractions implied in the statement and on the composition of the partial results that he/she obtains. The solution is finally found by using an arithmetic procedure of counting the units of fractions which have been represented in the last diagram by using the process of idealization (the ratio of the number of small blue squares to the total number of small squares is the fraction of alcohol of the Martini).

Solution 2: Use of a Hierarchical Diagram

The tree diagram in Figure 4 explains the problem-solving process for someone who knows the conventions assumed as well as the concepts and procedures implied. However, the justification and explanation of the solution requires to carry out the following sequence of discursive and operative practices:

1) The diagram constructed in Figure 4 expresses the splitting of a unit of volume of Martini in two parts, gin and vermouth at the first level, thus indicating the corresponding fraction in each connector.

2) The splitting of the parts of gin and vermouth, which are now considered as unit quantities, in two parts, alcohol and not alcohol are expressed at the second level thus indicating the corresponding fraction in each connector.

3) The fraction of alcohol of the gin is 2/5 of the quantity of gin; since this quantity is 5/6 of the quantity of Martini, the fraction of alcohol in the Martini coming from the gin will be the 'the fraction of the fraction', that is,

$$\frac{2}{5} \times \frac{5}{6} = \frac{10}{30} = \frac{1}{3}$$

4) The fraction of alcohol of the vermouth is 1/6 of the quantity of vermouth; since this quantity is 1/6 of the quantity of the Martini, the fraction of alcohol in the Martini coming from the vermouth will be the 'fraction of the fraction', that is,

$$\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

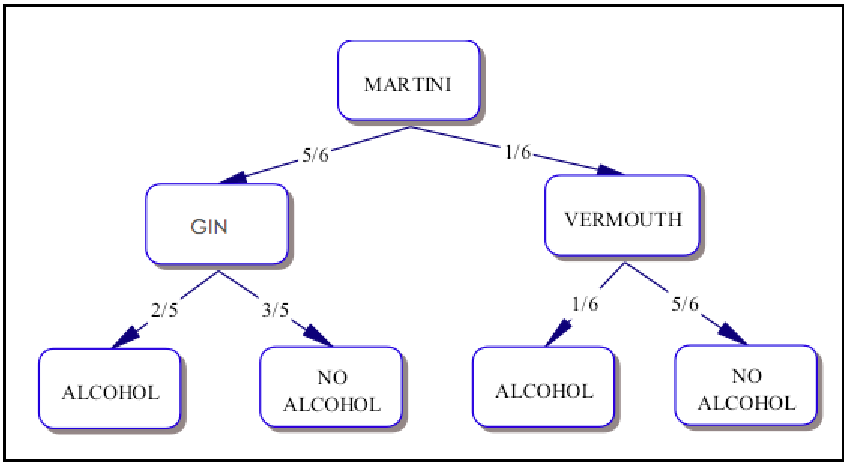
5) The total fraction of alcohol in the Martini will be the sum of the fractions of alcohol coming from the gin and the vermouth, that is,

$$\frac{1}{3} + \frac{1}{36} = \frac{13}{36}$$

6) Given that the fraction of alcohol in the Martini is 13/36 ≈ 0,3611, the percentage (approximate) will be 36,11%.

Figure 4

Solution of the task using a tree diagram



Fraction of alcohol:

$$\frac{2}{5}\times\frac{5}{6}+\frac{1}{6}\times\frac{1}{6}=\frac{13}{36}$$

$13/36 \approx 0{,}3611$

Table 2 includes the configuration of objects and processes at stake in the solution of the problem using the tree diagram in Figure 4.

Table 2

Configuration of objects and meanings

Textualized operative and discursive practices	**Non ostensive objects: concepts, propositions, procedures, arguments**	**Use and purpose of the practices**
	Statement *(The same as the previous case)*	
	Solution	
1) The diagram constructed in Figure 4 expresses (…) at the first level (…)	*Concepts*: first level of a diagram, connector, unit of quantity and fraction. *Procedure:* splitting of a whole into equal parts. *Representation agreement*: the fractions over the connectors refer to the fractionary relation between the quantities connected.	Express in a fractioned diagram the quantity of gin and vermouth in the Martini.
2) The splitting (…) is expressed at the second level (...)	The same as practice 1.	Express in fractions the quantity of alcohol present in the gin and in the vermouth.
3) The fraction of alcohol in the gin is 2/5 of the quantity of gin (...)	*Concepts*: multiplication of fractions (fraction of a fraction); unit quantity. *Procedures:* multiplication of fractions; change of unit when changing from the first to the second level of the diagram (the	Express in fractions the quantity of alcohol in the Martini which comes from the gin.

	volume of gin and vermouth are now considered new units which are fractioned).	
4) The fraction of alcohol in the vermouth is 1/6 (...)	Same as practice 3)	Express in fractions the quantity of alcohol in the Martini which comes from the vermouth.
5) The total fraction of alcohol in the Martini will be (...)	*Concepts:* sum of fraction. *Procedures*: sum of fractions with different denominator. *Proposition:* the fraction of alcohol in the Martini is 13/36. *Argument*: it is formed by the sequence of steps 1) to 5), supported by the use of arithmetic and hierarchical diagrams and by natural sequential language.	Response in fractions to the problem.
6) Given that the fraction of alcohol in the Martini is 13/36 ≈ 0,3611, the percentage (approximate) will be 36,11%.	*Concepts*: rational number; fraction; decimal and percentual approximation. *Procedures:* obtaining the decimal expression by the quotient of the numerator and the denominator, step to the percentual expression.	Response to the problem and its justification in terms of percentual expression.

The analysis of each one of the individualized practices in Table 2 can be done in more detail. Thus, in the first unit of the statement, the systematic application of the notion of the onto-semiotic configuration of practices, objects and processes leads us to recognise that the subject who reads the statement should carry out a process of interpretation (semiosis or attribution of the meaning) of the diagram ⅖, identifying the ‘concept of fraction’ understood here from an institutional point of view as a socially accepted rule: a whole unit is split into equal parts and one or several of these parts are individualized. To follow, a particularization process of the case should be carried out: the whole unit is divided into 5 parts and 2 are considered separately.

In the first two units of the solution (the diagram in Figure 4) the subject should carry out a splitting process of the system of elements that make up the diagram, distinguishing three hierarchical levels, the units that make up the whole unit of each level, connectors, the fractions and the operations with fractions that should be carried out. A splitting process of the partial calculation carried out at each branch of the tree should also be carried out in order to obtain the fraction of alcohol of the Martini and of the materialization of the calculation in the arithmetic-diagrammatic expression,

$$\frac{2}{5}\times\frac{5}{6}+\frac{1}{6}\times\frac{1}{6}=\frac{13}{36}$$

The rest of the discursive and operative practices carried out which are necessarily supported by the sequential-natural language, are essential to establish the connection between both types of diagrams and explain that in the conditions of this problem the fraction of alcohol in the Martini is 13/36.

The structure of the system of practices that must be carried out to solve a problem is shown in the hierarchical diagram in an iconic way. The fraction of fraction (multiplication of fractions) is reflected in the composition of the two inferior levels of the diagram (up, down) whilst the sum of resulting fractions is reflected in the lateral layout of the two branches (left, right).

One feature which distinguishes the use of area diagrams with respect to the hierarchical diagram is that the meaning of the concept of fraction that is used in each case is different: in the areas the fraction intervenes as an operator of a quantity of area whereas in the tree diagram the fraction is the ratio between the parts of a generic whole which is divided into equal parts and the parts which are individualized. The procedure based on diagrams in areas has less generality traits than the hierarchical ones.

Solution 3: Arithmetic Fractioning

The problem can be solved without using graphic type diagrams, although the use of the fractioning expression (which in Peirce´s semiotic is also a diagram) is inevitable. The following sequence of operative and discursive practices establishes the justification and the explanation that the fraction of alcohol in the Martini is 13/36.

1) The fraction of gin that the cocktail contains is 5/6, because the unit of volume of Martini has been divided into 6 equal parts and 5 correspond to the gin.

2) For the same reason the fraction of vermouth will be 1/6.

3) The alcohol contained in the gin is a fraction of the fraction of gin, in this case 2/5 of 5/6.

4) That is, $\frac{2}{5} \times \frac{5}{6} = \frac{10}{30} = \frac{1}{3}$

5) The alcohol contained in the vermouth is a fraction of the fraction of vermouth, in this case 1/6 of 1/6.

6) That is, $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

7) The fraction of alcohol in the Martini will be the sum of the fractions of alcohol provided by the gin and by the vermouth.

8) That is, $\frac{1}{3} + \frac{1}{36} = \frac{13}{36}$

9) Given that the fraction of alcohol of the Martini is 13/36 ≈ 0,3611, the percentage (approximate) will be 36,11%.

In Table 3 we include the configuration of objects and processes that are involved in the solution of the problem using arithmetic fractioning.

Table 3

Configuration of objects and meanings

Textualized operative and discursive practices	**Non ostensive objects: concepts, propositions, procedures, arguments**	**Use and purpose of the practices**
	Statement *(The same as the previous case)*	
1) The fraction of gin that the cocktail contains is 5/6, because the unit of volume of Martini has been divided into 6 equal parts and 5 correspond to the gin.	*Concept:* fraction, as part of a whole. *Proposition:* the fraction of gin in the cocktail is 5/6. *Argument:* why the Martini has been divided into 6 equal parts and 5 correspond to the gin.	Express in fractions the quantity of gin present in the Martini from the information about the problem.
2) For the same reason the fraction of vermouth will be 1/6.	Same as practice 1)	Express in fractions the quantity of vermouth present in the Martini from the information given about the problem.

3) The alcohol contained in the gin is a fraction of the fraction of gin, in this case, 2/5 of 5/6.	*Concept*: fraction of a fraction (multiplication of fractions).	Establish the relation of alcohol present in the gin to justify practice 4.
4) That is, $\frac{2}{5}\times\frac{5}{6}=\frac{10}{30}=\frac{1}{3}$	*Proposition*: The fraction of alcohol in the gin is 1/3. *Argument:* why the new whole unit (5/6) is divided into 5 equal parts and 2 are taken. *Procedure:* multiplication of fractions; simplification of fractions. Concepts: rational number, prime fraction.	Express in fractions the quantity of alcohol present in the gin.
5) The alcohol contained in the vermouth is a fraction of the fraction of the vermouth, in this case, 1/6 of 1/6.	Same as practice 3)	Establish the relation of alcohol present in the vermouth to justify practice 6).
6) That is, $\frac{1}{6}\times\frac{1}{6}=\frac{1}{36}$	Same as practice 4)	Express in fractions the quantity of alcohol present in the vermouth.
7) The fraction of alcohol in the Martini will be the sum of the fractions of alcohol provided by the gin and the vermouth.	*Concept*: sum of fractions	Interpret the data obtained in the previous practices, in terms of the fractional response to the task to justify practice 9).
8) That is, $\frac{1}{3}+\frac{1}{36}=\frac{13}{36}$	*Proposition:* the fraction of alcohol of the Martini is 13/36. *Argument:* this is the result of the sum of the fractions obtained by applying the corresponding procedure (sum of fractions with a different denominator).	Fractional response to the problem.
9) Given that the fraction of alcohol of the Martini is 13/36 ≈ 0,3611, the percentage (approximate) will be 36,11%.	*Concepts*: rational number; fraction; decimal and percentual approximation. Procedures: obtaining the decimal expression using the quotient of the numerator and the denominator; move to percentual expression.	Response to the problem and its justification in terms of percentual expression.

The arithmetic fractionary solution depends more on sequential language as is shown in practices 1), 2), 3), 5) and 7). By attributing spatial characteristics to the fractionary representation and to the transformations done with them (the number below divides, and the number above multiplies; the denominators that are below and the numerators that are above, are multiplied) the fractionary arithmetic solution also uses diagrammatic reasoning (practices 4), 6) and 8).

Heuristic Power of the Solutions: Particularizations Versus Generalization

In addition to the solutions studied in the previous sections, others which imply the use of different degrees and modalities of visualization, or mixed solutions that combine the diagrammatic solutions with fractionary arithmetics, can be elaborated. For example, a variant of the fractionary arithmetic solution could be the following.

1) Let´s suppose that we prepare 36 liters of Martini.
2) The quantity of gin will be $\left({}^{5}/_{6}\right)(36) = 30$.
3) The quantity of vermouth, $36 - 30 = 6$.
4) The quantity of alcohol in the gin will be, $\left({}^{2}/_{5}\right)(30) = 12$.
5) The quantity of alcohol in the vermouth, $\left({}^{1}/_{6}\right)(6) = 1$.
6) The total quantity of alcohol in the Martini will be, $12 + 1 = 13$.
7) So, the fraction of alcohol in the Martini will be ${}^{13}/_{36}$.

With exception of the fraction expression, which incorporates the disposition of the numerator and the denominator as visual element by indicating the different role that each one plays in the mathematics practices, the rest of the mathematics practices depend on natural sequential language. The reasoning is based, nevertheless on the essential participation of the concept of fraction as operator and as the relation part-all.

One variant of the task statement, which requires a substantial change in the modality of usable diagrams, is the following:

> *Suppose that the Martini cocktail can be prepared with different proportions of gin and vermouth. We want to elaborate a rule (formula) which enables us to determine the fraction of alcohol in the Martini for each possible composition. It is supposed that the fractions of alcohol of the gin and the vermouth do not change (2/5 and 1/6, respectively).*

In this case we hope to carry out the following sequence of operative and discursive practices supported using algebraic diagrams:

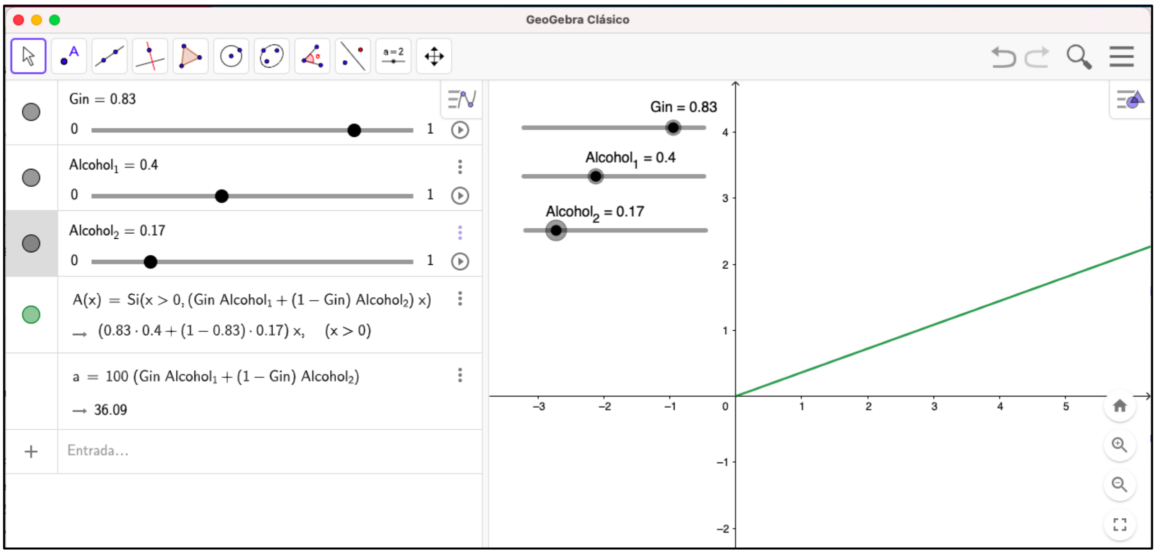
1) We suppose that *g/m* indicates the fraction of gin in the Martini.
2) The fraction of vermouth will be *(m-g)/m.*
3) The fraction of alcohol in the Martini will be,

$$A = \frac{2}{5} \times \frac{g}{m} + \frac{1}{6} \times \frac{(m-g)}{m} = \frac{7g + 5m}{30m} = \frac{7}{30}\left(\frac{g}{m}\right) + \frac{1}{6}$$

Generalizations can be obtained not only with algebraic diagrams, but the use of algebraic diagrams is one of the main paths in mathematical practices since they allow facing different types of mathematical problems and generalizations. In fact, this algebraic generalization can be implemented with the dynamic geometry software, GeoGebra (Figure 5).

Figure 5

Generalization with GeoGebra: parameters and sliders



Synergy Between the Diagrammatic and Sequential Languages

In previous section, we have shown that there is a narrow overlapping between the objects that intervene in the mathematics activity, specifically between

- the diagrammatic-visual and the sequential languages,
- the ostensive objects (materials) and the non ostensive (immaterial),
- the extensive objects (particulars) and the intensive objects (general).

The use of diagrams in mathematics practice should be accompanied by other means of non visual expressions to achieve the justification and explanation of the mathematics tasks and the operative and discursive practices implied when carrying them out. The genesis of mathematics knowledge is situated halfway between both languages where a mutual interrelation and reinterpretation is necessary. Furthermore, we have shown that the means of expression are empirical ‘artefacts’, which require the implicit use of a system of non-ostensive objects of conceptual, propositional, and argumentative nature. We have also revealed some processes of particularization, generalization; splitting, composition; materialization, idealization which are used in the demonstrative-explanatory process carried out.

Our analysis agrees with and supports Sherry´s position about the use of diagrams in mathematics work: what is most relevant, more than building a specific diagram, is the mathematics knowledge implied, which is not visible anywhere, and is not identified with the diagrams that are used for its representation and manipulation.

> No matter how carefully the diagrams are drawn, the result is not simply read of the diagram. The students may fail to see the implicit result in the diagrams, but the failure will not be because of deficiencies in the constructed diagrams, but rather because of an inability to grasp a conceptual relation. (Sherry, 2009, p. 68)

So, Sherry summarises the role of the diagrams in mathematics reasoning in two aspects:

> In the first place, a diagram serves as the ground for synthesizing a mathematical rule from existing concepts and inference rules. The second role of a geometrical diagram is to warrant further inferences in virtue of its simple empirical characteristics. (Sherry, 2009, p. 69)

The diagram supports or makes possible the necessary process of particularization of the general rule; it makes the conceptual object intervene to participate in a practice from which another new conceptual object will emerge (in our example, the fraction which is the response to the problem posed).

Final Reflections

This paper complements others previously carried in the OSA framework where the role of representations in mathematics education and the potential use of considering the process of non-ostensive objects implicated in the use of these representations, is analysed (Font, Godino, & Contreras, 2008). In this case we also use the notion of onto-semiotic configuration of practices, objects, and processes to dialogue with the research carried out on diagrammatic reasoning and the use of visualizations.

The way diagrams are understood has important consequences for mathematics education whenever the use of these resources penetrates all school mathematics activity. We consider that it is important to surpass ingenuous empiristic positions about the use of manipulatives and visualizations in the processes of mathematics teaching and learning: there is always a cohort of intervening non-material objects which are essential to solve these situations accompanying the necessary materializations that intervene in the situations-problems and the corresponding mathematics practices. This onto-semiotic vision of mathematics practices does not come from an unaccessible world but from this social world in which we live and are involved in our daily practice.

“The sign is a creation between individuals, a creation within a social milieu. Therefore, the item in question [the item to which a sign will refer] must first acquire interindividual significance, and only then can it become an object for sign formation” Voloshinov (1973, p. 22).

Visualization (in general materialization) is useful and necessary in mathematics practice, above all if it is diagrammatic and therefore metaphorically reflects the conceptual mathematics structures. However, this layer of material objects should not prevent seeing the layer of immaterial objects that really make up the conceptual system of institutional mathematics. Both layers are interwoven and to a certain extent are inseparable. There are complex dialectic relations between the ostensive and non ostensive objects since the activity of mathematics production and communication cannot be carried out without the synergic combination between both types of objects.

Finally, it is important to note that this detailed epistemic analysis has robust implications for the mathematics teacher's education. Diagrams are considered essential by many researchers and teachers, as these are fundamental for mathematical reasoning (Giardino, 2017; Kadunz, 2016, Wille, 2020), and for communication and problem solving in science (Hill et al., 2014; Roberts et al., 2008). Authors have evidenced that involving students in diagrammatic reasoning tasks strengthens connections between the different meanings of the mathematical concept represented (for example, fraction concept); nevertheless, the analysis of mathematical tasks and the different ways of approaching them is necessary to understand the potential difficulties and learning obstacles (Wittmann, 2021). Therefore, a challenging task for the teacher is the comprehensible description of the activity shown by the students and the construction of their knowledge when doing mathematics (Kadunz, 2016).

In this sense, the mathematics teacher should have knowledge, understanding and competence to discriminate the different types of objects that intervene in school mathematics practice, based on the use of different systems of representations and being aware of the synergic relations between the same (Burgos, & Godino, 2022; Giacomone, Godino, Wilhelmi, & Blanco, 2018). He/she should be competent to design and manage processes of materialization and idealization of the mathematics objects at the same time as the corresponding processes of particularization and generalization (Calle, Breda, & Font, 2021).

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Declarations

Ethical Approval and Consent to Participate The study has been approved by the research projects of all participating authors. All data generated and analyzed during this study are included in this published article and have been obtained by the original authors. Informed consent was obtained from all participating authors included in this research.

Notes

[1]The following affirmation is attributed to Einstein: "Whenever mathematics propositions which have something to do with reality are not certain and whenever they are certain they have nothing to do with reality".

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