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Unsymmetrical basic uncertain information with some decision-making methods

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Abstract. Motivated by a specific decision-making situation, this work proposes the concept and definition of unsymmetrical basic uncertain information which is a further generalization of basic uncertain information and can model uncertainties in some new decision-making situations. We show that unsymmetrical basic uncertain information in some sense can model linguistic hedges such as "at least" and "at most". Formative weighted arithmetic means and induced aggregations are defined for the proposed concept. Rules-based decision making and semi-copula based integral for this concept with some numerical examples are also presented.

Keywords: Aggregation operators, basic uncertain information, evaluation, information fusion, integral, uncertainty, unsymmetrical basic uncertain information

1. Introduction

Uncertainty is pervasive in a wide range of decision making and evaluation problems, from involved information type [1–6] to decision methodology and result [7, 8]. Some typical quantitative uncertain information includes interval information, probability information, fuzzy information [9], and their related extension or generalization forms [10–13].

The uncertainties involved, contained in or related to the above mentioned information types can have different embodiments. For example, in an interval information granule $[a, b] \subseteq [0, 1]$, the uncertainty involved can be expressed by the amount $b - a$; in probability information, variance is usually a suitable gauge to measure the involved uncertainty; and in fuzzy information, the membership/non-membership degrees often directly serve as a yardstick to show the certainties/uncertainties concerned. In addition, more types of known, unknown or unexplainable types of uncertainties can be found in decision making practices.

Diverse forms of uncertainties may result in incompatibility and complexity in decision making. Hence, using an integrated, comprehensive and generalized concept and normative form to express and model more uncertain information is very practical and

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appealing. Recently, [14] and [15] proposed the concept of basic uncertain information (BUI) which can well model and handle a large class of quantitative uncertain information including the ones aforementioned. BUI has soon been further developed and applied in several areas [16–22].

A standard BUI granule has a pair form (x, c) in which $x \in [0, 1]$ is a concerned *evaluation value* and $c \in [0, 1]$ is the *certainty degree* of x; and $1 - c \in [0, 1]$ is the uncertainty degree of x. In practice, certainty degrees may represent the degrees to which decision makers are confident, sure, certain or definite of the concerned evaluation values, while uncertainty degrees may indicate the extents to which they are unconfident, unsure, uncertain or indefinite of the concerned evaluation values. Note that when $c = 1$, $(x, 1)$ indicates the full certainty over evaluation value x and thus it can be understood to be equivalent to the real value x; when $c = 0$, $(x, 0)$ shows the full uncertainty over evaluation value x, and the situation that every value between [0,1] can be considered as a true value, and thus no substantial and effective information can be available.

For any BUI granule (x, c), an intuitively acceptable transformation T can be applied to transform it into an interval value $T(x, c) = [a, b] = [cx, x + (1 - c)(1 - x)]$. It can be observed from the convex combination form $T(x, c) = c[x, x] + (1 - c)[0, 1]$ that the "dilation process" actually is toward the two ends of unit interval [0,1] with a same extent $1 - c$. The dilation can be also seen as "symmetrical" not in sense of "$x - a = b - x$" but in the sense that certainty degree c determine the same proportion with which x is far from the two ends of [0,1].

In practice, such symmetry might not always be obtained or appropriately derived. The following decision making situation shows this fact.

Assume an examination score for a student is known to be $x \in [0, 1]$ but with uncertainty because in general a score may not accurately or correctly represent the true performance of that student. Suppose the uncertainty is further interpreted by a simple statistics from a team of teachers: suppose $100y_U\%$ of all the teachers think the true performance of that student should be higher than x; $100y_L\%$ of them think it should be lower than x; in addition, $100y\%$ of them cannot judge whether or not the score x is accurate; and finally $100z\% = 100(1 - y_U - y_L - y)\%$ of them believe x in general exactly embodies the true performance of that student. BUI is clearly also not suitable to be applied in this satiation without any adaptation or extension.

The original BUI granule with its existing evaluation models apparently cannot fit well to model or cope with the above mentioned evaluation situation. Against this background, we will adapt the original BUI and propose an unsymmetrical form of it which will then be able to well model those situations.

The remainder of this work is arranged as follows. Section 2 gives the strict definition for unsymmetrical basic uncertain information and shows some theoretical and practical features. Section 3 discusses formative weighted arithmetic means and induced aggregations for UBUI. After that, in Section 4 some related rules-based decision making and semi-copula based integral are analyzed with examples. Section 5 concludes and remarks this work.

2. Unsymmetrical basic uncertain information

Some related notations and expressions are fixed or reviewed as follows. A real vector of dimension n is denoted by $\mathbf{x} = (x_i)_{i=1}^n \in [0, 1]^n$. All of the closed intervals (or called interval values, interval numbers) $[a, b] \subseteq [0, 1]$ are denoted by $\mathcal{I}$. In addition, $[a, a]$ is sometimes identified with real number a. For intervals, review the lattice $(\mathcal{I}, \leq_{Int})$ with the standard partial order $\leq_{Int}$ such that $[a_1, b_1] \leq_{Int} [a_2, b_2]$ if and only if $a_1 \leq a_2$ and $b_1 \leq b_2$; denote $[a_1, b_1] <_{Int} [a_2, b_2]$ if and only if $[a_1, b_1] \leq_{Int} [a_2, b_2]$ and $[a_1, b_1] \neq [a_2, b_2]$.

The set of all BUI granules (x, c) is denoted by $\mathcal{B}$. A BUI vector is denoted by $(\mathbf{x}, \mathbf{c}) = (x_i, c_i)_{i=1}^n \in \mathcal{B}^n$ where $\mathbf{x} = (x_i)_{i=1}^n \in [0, 1]^n$ represents an evaluation vector while $\mathbf{c} = (c_i)_{i=1}^n \in [0, 1]^n$ is the certainty vector corresponding to x. Unless otherwise noted, the weight vector or probability vector (of dimension n) used in this work is with the form $\mathbf{w} = (w_i)_{i=1}^n \in [0, 1]^n$ such that $\sum_{i=1}^n w_i = 1$; and the space of such vectors is denoted by $\mathcal{W}^{(n)}$.

Next, we give the definition of unsymmetrical basic uncertain information as follows.

Definition 1. (i) (unsymmetrical basic uncertain information) A granule of unsymmetrical basic uncertain information (UBUI) is expressed by the form $(x, (c_L, c_U))$, in which $x \in [0, 1]$ is the concerned evaluation value and $(c_L, c_U) \in [0, 1]^2$ the *certainty pair* of x, measuring the degree of being trusted, convincing or believable etc., of input value x. In (c_L, c_U), c_U is called the *upper certainty* (of

x) and c_L *lower certainty* (of x). Correspondingly, $(1 - c_L, 1 - c_U)$ is called the *uncertainty pair* of x, in which $1 - c_U$ is called the *upper uncertainty* (of x) and $1 - c_L$ *lower uncertainty* (of x).

(ii) (UBUI vector) The set of all UBUI granules is denoted by $\mathcal{UB}$. An UBUI vector is denoted by $(\mathbf{x}, (\mathbf{c}_L, \mathbf{c}_U)) = (x_i, (c_{Li}, c_{Ui}))_{i=1}^n \in (\mathcal{UB})^n$ where $\mathbf{x} = (x_i)_{i=1}^n \in [0, 1]^n$ represents an evaluation vector while $\mathbf{c}_U = (c_{Ui})_{i=1}^n \in [0, 1]^n$ and $\mathbf{c}_L = (c_{Li})_{i=1}^n \in [0, 1]^n$ are the *upper certainty* vector and *lower certainty vector* corresponding to $\mathbf{x}$, and $(\mathbf{c}_L, \mathbf{c}_U) = (c_{Li}, c_{Ui})_{i=1}^n$ is called the *certainty pair vector*.

Remark 1. For an UBUI granule $(x, (c_L, c_U))$, when $c_L = c_U$, it clearly degenerates into a BUI granule (x, c_L).

Return to the decision making situation discussed in Introduction which cannot be directly or suitably handled by BUI. We next to show that UBUI can well tackle it.

We can reasonably obtain an UBUI by considering the known value x and the statistics (a probability vector (y_L, y_U, y, z)) with a transformation function $S : [0, 1] \times \mathcal{W}^{(4)} \to \mathcal{UB}$:

$$S(x, (y_L, y_U, y, z)) = (x, (y_L + y, y_U + y)) \quad (1)$$

With any obtained UBUI granule $(x, (c_L, c_U))$, we use the following formula $T^* : \mathcal{UB} \to \mathcal{I}$ to transform it into an interval which sometime can be much easier to handle in decision making.

$$T^*(x, (c_L, c_U)) = [c_L x, x + (1 - c_U)(1 - x)] \quad (2)$$

Using standard partial order $\leq_{Int}$ or some further well defined linear orders [24] can make some comparisons that in general cannot be obtained even in UBUI environment.

Remark 2. For an UBUI granule $(x, (c_L, c_U))$, it may have or convey more practical meanings. For example, it is interesting to find that $(x, (1, 0))$ may indicate a usually used linguistic term: (a value is) "at least x"; similarly, $(x, (0, 1))$ can represent linguistic term: "at most x". Besides, $T^*(x, (1, 0)) = [x, 1]$ and $T^*(x, (0, 1)) = [0, x]$.

3. Formative weighted arithmetic means and induced aggregations for UBUI

The weighted arithmetic means for aggregating UBUI granules can be formatively made in some piecewise ways. More importantly, with such definitions we can conveniently carry out induced aggregations.

Definition 2. The interval weighted arithmetic mean (with weight vector $\mathbf{w}$) (ItWAM) $\mathsf{ItWAM}_\mathbf{w} : \mathcal{I}^n \to \mathcal{I}$ is defined such that

$$\mathsf{ItWAM}_\mathbf{w}([\mathbf{a}, \mathbf{b}]) = \left[\sum\nolimits_{i=1}^n w_i a_i, \sum\nolimits_{i=1}^n w_i b_i\right] \quad (3)$$

Definition 3. [14] The BUI weighted arithmetic mean (with weight vector $\mathbf{w}$) (BWAM) $\mathsf{BWAM}_\mathbf{w} : \mathcal{B}^n \to \mathcal{B}$ is defined such that

$$\mathsf{BWAM}_\mathbf{w}(\mathbf{x}, \mathbf{c}) = \left(\sum\nolimits_{i=1}^n w_i x_i, \sum\nolimits_{i=1}^n w_i c_i\right) \quad (4)$$

In a similar way, we can directly have the weighted arithmetic means for UBUI vectors because $\mathcal{UB}$ is clearly convext.

Definition 4. The UBUI weighted arithmetic mean (with weight vector $\mathbf{w} \in \mathcal{W}^{(n)}$) (UBWAM) $\mathsf{UBWAM}_\mathbf{w} : (\mathcal{UB})^n \to \mathcal{UB}$ is defined such that

$$\mathsf{UBWAM}_\mathbf{w}(\mathbf{x}, (\mathbf{c}_L, \mathbf{c}_U)) = \left(\sum\nolimits_{i=1}^n w_i x_i, (\sum\nolimits_{i=1}^n w_i c_{Li}, \sum\nolimits_{i=1}^n w_i c_{Ui})\right) \quad (5)$$

Definition 5. For two UBUI granules $(x_1, (c_{L1}, c_{U1}))$ and $(x_2, (c_{L2}, c_{U2}))$,
$(x_1, (c_{L1}, c_{U1}))$ is said to be with certainty not higher than $(x_2, (c_{L2}, c_{U2}))$ if and only if $c_{L1} \leq c_{L2}$ and $c_{U1} \leq c_{U2}$.

Some related simple properties can be obtained as follows.

Proposition 1.

(i) For two UBUI granules $(x, (c_{L1}, c_{U1}))$ and $(x, (c_{L2}, c_{U2}))$, if $c_{L1} \leq c_{L2}$ and $c_{U1} \leq c_{U2}$, then $T^*(x, (c_{L2}, c_{U2})) \subseteq T^*(x, (c_{L1}, c_{U1}))$; if $c_{L1} \leq c_{L2}$ and $c_{U1} \geq c_{U2}$, then $T^*(x, (c_{L1}, c_{U1})) \leq_{Int} T^*(x, (c_{L2}, c_{U2}))$.

(ii) For two UBUI granules $(x_1, (c_L, c_U))$ and $(x_2, (c_L, c_U))$, if $x_1 \leq x_2$, then $T^*(x_1, (c_L, c_U)) \leq_{Int} T^*(x_2, (c_L, c_U))$.

With a vector of UBUI granules $(\mathbf{x}, (\mathbf{c}_L, \mathbf{c}_U)) = (x_i, (c_{Li}, c_{Ui}))_{i=1}^n$, a reasonable induced weighting method is to allocate larger weight to those x_i's that have larger certainties. For two UBUI granules $(x_1, (c_{L1}, c_{U1}))$ and $(x_2, (c_{L2}, c_{U2}))$, Definition

5 provides a means to compare their certainties with $(c_{L1}, c_{U1}) \leq (c_{L2}, c_{U2})$ if $c_{L1} \leq c_{L2}$ and $c_{U1} \leq c_{U2}$. Sometimes (c_{L1}, c_{U1}) and (c_{L2}, c_{U2}) cannot be compared. Therefore, some lattice based inducing and weighting methods (such as the three-set method [25, 26] can be suitably applied in this setting. Another effective and practical method is to use binary aggregation functions [15] to firstly merge certainty pairs into some corresponding real certainties. For example, recall a semi-copula $(x, y) \mapsto x \circ y$ is a binary aggregation operator which is monotonic non-decreasing w. r. t. each parameter and satisfies $1 \circ x = x \circ 1 = x$.

Correspondingly, some types of preference for high certainties (derived from certainty pairs) are discussed below.

- Preference for high lower certainties c_{Li}, indicating a decision making emphasis on relatively guaranteed lower bounds.
- Preference for high upper certainties c_{Ui}, indicating a decision making emphasis on relatively guaranteed upper bounds.
- Preference for high overall certainties $c_{Li} + c_{Ui}$, indicating a decision making emphasis on relatively accuracy of evaluation values.
- Preference for high aggregated certainties $c_{Li} \circ c_{Ui}$ (with $\circ$ being any semi-copula), indicating another type of decision making emphasis on relatively accuracy of evaluation values.

Remark 3. Ordered weighted averaging (OWA) [27] is a widely applied powerful tool to model bi-polar preferences [28, 29]. We adapt OWA operator to UBUI environment not in a strict mathematical sense, but in a more formal, defining and decision making methodological way. For having a thorough way of defining such OWA aggregation, one may refer to some recent literatures [25, 26]. Hence, the certainty induced UBUI ordered weighted averaging (CIUBOWA) (with weight vector $\mathbf{w}$), $\mathsf{CIUBOWA}_{\mathbf{w}} : (\mathcal{UB})^n \to \mathcal{UB}$, is defined as an UBWAM operator with the same weight vector $\mathbf{w}$, $\mathsf{UBWAM}_{\mathbf{w}} : (\mathcal{UB})^n \to \mathcal{UB}$, where the weight vector $\mathbf{w}$ is must directly related to the certainty pair vector $(\mathbf{c}_L, \mathbf{c}_U) = (c_{Li}, c_{Ui})_{i=1}^n$, while in UBWAM we allow the involved weight vector $\mathbf{w}$ to be either related or not related to the certainty pair vector.

Example 1. Assume we have a vector of UBUI with four pieces of granules

$$(\mathbf{x}, (\mathbf{c}_L, \mathbf{c}_U)) = ((0.5, (1, 0.3)), (0.8, (0.8, 0.5)), (0.3, (0.4, 0.8)), (0.6, (0.6, 0.6)).$$

And suppose a decision maker has emphasis on high aggregated certainties $c_{Li} \cdot c_{Ui}$ to a moderate extent which can derive the weight vector $\mathbf{w}$) from a weight vector $\mathbf{u} = (0.4, 0.3, 0.2, 0.1)$ with a moderately decreasing trend. Since $(c_{Li} \cdot c_{Ui})_{i=1}^4 = (0.3, 0.4, 0.32, 0.36)$, we may accordingly assign a corresponding higher weight in $\mathbf{u} = (0.4, 0.3, 0.2, 0.1)$ to a corresponding higher value in $(c_{Li} \cdot c_{Ui})_{i=1}^4 = (0.3, 0.4, 0.32, 0.36)$. That is, we may assign weight 0.4 to $(c_{L2} \cdot c_{U2})$ (namely to $i = 2$), assign weight 0.3 to $(c_{L4} \cdot c_{U4})$, assign weight 0.2 to $(c_{L3} \cdot c_{U3})$, and assign weight 0.1 to $(c_{L1} \cdot c_{U1})$, which constitutes a desired weight vector $\mathbf{w} = (0.1, 0.4, 0.2, 0.3)$. Then,

$$\begin{aligned}
&\mathsf{CIUBOWA}_{\mathbf{w}}(\mathbf{a}, (\mathbf{c}_L, \mathbf{c}_U)) = \\
&\mathsf{UBWAM}_{\mathbf{w}}(\mathbf{a}, (\mathbf{c}_L, \mathbf{c}_U)) \\
&\quad = \left(\sum_{i=1}^4 w_i a_i, (\sum_{i=1}^4 w_i c_{Li}, \sum_{i=1}^4 w_i c_{Ui})\right) \\
&\quad = ((0.05 + 0.32 + 0.06 + 0.18), ((0.1 + 0.32 \\
&\qquad +0.08 + 0.18), (0.03 + 0.2 + 0.16 + 0.18))) \\
&\quad = (0.61, (0.68, 0.57)).
\end{aligned}$$

4. Rules-based decision making and semi-copula based integral in UBUI environment

In contrast to optimal value aimed analytic decision making, rules-based decision making [30–33] can provide with some more efficient, flexible, automatic, and still sensible and effective decision procedures especially in the situations where the conditions for using analytic decision making are very limited. Besides, rules-based decision making, in a heuristic way, can be quite suitable for the multi-objective decision situations with well preset multiple rules.

Through a given UBUI granule $(x, (c_L, c_U))$, next we exemplify some sets of decision rules for performing multiple satisfactions aimed decision making.

Rule set (i): "$x \geq 0.7$", "$c_L \geq 0.8$" and "$c_U \leq 0.3$".

Rule set (ii): "$x \geq 0.9$", and "$c_L \geq 0.3$".

Rule set (iii): "$x \geq 0.4$", and "$c_L \wedge (1 - c_U) \geq 0.9$".

Rule set (iv): "$0.3 \leq x \leq 0.7$", and "$c_L + c_U \geq 1.5$".

Rule set (v): "$c_L = 1$", and "$x + 1.5(1 - c_U) \geq 2$".

To show its further potential in decision making and evaluation, we present the following illustrative example where multiple satisfactions aimed decision making can be carried out by a union of several individual rule sets.

Suppose there is a need to evaluate the comprehensive performance of a university teacher in the past three years of employment term. Therefore, some well predetermined sets of rules seem efficient and objective to perform this evaluation task. For example, we may come up with four sets of rules corresponding to four criteria: (i) academic performance; (ii) teaching spirit and attitude; (iii) teaching effect; (iv) social service done. There are two ways of being qualified for a teacher: (A) criteria (i) is satisfied; and (B) any two of criteria (ii), (iii) and (iv) are satisfied. Assume the individual performances for each criterion have already been transformed into UBUI granules, which is represented by $(\mathbf{x}, (\mathbf{c}_L, \mathbf{c}_U)) = ((0.3, (1, 0.3)), (0.8, (0.1, 0.1)), (0.5, (0.4, 0.4)), (0.4, (0.7, 0.7))$. And assume the rules for satisfactions of the four criteria are formulated below, respectively:

Rule set (i): "$x_1 \geq 0.6$", and "$c_{L1} \geq 0.8$".

Rule set (ii): "$x_2 \geq 0.8$", and "$c_{L2} \wedge (1 - c_{U2}) \geq 0.4$".

Rule set (iii): "$x_3 \geq 0.7$", "$c_{L3} \geq 0.8$" and "$c_{U3} \leq 0.5$".

Rule set (iv): "$x_4 \geq 0.3$", and "$c_{L4} \geq 0.5$".

Clearly, only (iv) has been satisfied for that teacher and thus neither the way (A) nor the way (B) has been fulfilled. Hence, the teacher cannot be judged to be qualified in the current situation where data collected is with some relative high uncertainties. Suppose another teacher gains the following different performance $(\mathbf{x}', (\mathbf{c}'_L, \mathbf{c}'_U)) = ((0.7, (0.8, 0)), (0.8, (0.1, 0.1)), (0.1, (1, 1)), (0.9, (0.4, 0.3))$, then clearly he should be qualified since (A) has been satisfied.

We now consider a variation of fuzzy integral in UBUI environment (the fuzzy integral in BUI environment has been recently thoroughly analyzed in an unpublished work [34]).

Recall a fuzzy measure (also known as capacity) [35] on set $[n] = \{1, \cdots, n\}$ is a set function $\mu : 2^{[n]} \to [0, 1]$ such that (i) $\mu(\emptyset) = 0$ and $\mu([n]) = 1$; and (ii) $\mu(A) \leq \mu(B)$ whenever $A \subset B$. A semi-copula based integral [24, 35, 36] with fuzzy measure μ defined on $[n]$ and semi-copula $\circ$, $F_{\mu,\circ} : [0, 1]^n \to [0, 1]$, is an aggregation function of dimension n such that

$$F_{\mu,\circ}(\mathbf{x}) = \max_{A \subseteq 2^{[n]} \setminus \{\emptyset\}} \left\{ \min_{i \in A}\{x_i\} \circ \mu(A) \right\} \quad (6)$$

Note that for semi-copula "min" we have a familiar version of Sugeno integral, for semi-copula "product" we obtain a version of Shilkret integral, and for semi-copula that is a strict t-norm, we achieve a version of Weber integral.

However, when the inputs x_i's are with uncertainties and expressed by BUI granules, it may be unreasonable for Equation (6) to be directly carried out. In [34], some special techniques are discussed to cope with such problem and here we extend it to adapt to UBUI environment by proposing the following form $U_\mu : (\mathcal{UB})^n \to [0, 1]$:

$$U_{\mu,\circ}(\mathbf{x}, (\mathbf{c}_L, \mathbf{c}_U)) = \max_{A \subseteq 2^{[n]} \setminus \{\emptyset\}; \min_{i \in A}\{c_{Li}\} \geq \alpha, \min_{i \in A}\{c_{Ui}\} \leq \beta} \left\{ \min_{i \in A}\{x_i\} \circ \mu(A) \right\} \quad (7)$$

where $\alpha, \beta \in [0, 1]$ are fixed thresholds.

Example 2. Still consider the above illustrated example of university teacher performance evaluation. Assume $\mu : 2^{[4]} \to [0, 1]$ represents the interrelated importance between the four criteria: (i) academic performance; (ii) teaching spirit and attitude; (iii) teaching effect; (iv) social service done. μ satisfies $\mu(\emptyset) = \mu(\{2\}) = \mu(\{3\}) = \mu(\{4\}) = 0$, and $\mu(A) = 1$ whenever $A \notin \{\emptyset, \{2\}, \{3\}, \{4\}\}$. Preset the two thresholds $\alpha = 0.6$ and $\beta = 1$, and adopt min $\wedge$ operator for the semi-copula. Predetermine the decision rule: if $U_{\mu,\circ}(\mathbf{x}, (\mathbf{c}_L, \mathbf{c}_U)) \geq 0.5$, then the performance is qualified.

Then, we observe $U_{\mu,\circ}(\mathbf{x}, (\mathbf{c}_L, \mathbf{c}_U)) = \max\{0.3 \wedge 1, 0.4 \wedge 0\} = 0.3$; so the related performance cannot be considered to be qualified with current knowledge. Still, note that $U_{\mu,\circ}(\mathbf{x}', (\mathbf{c}'_L, \mathbf{c}'_U)) = \max\{0.7 \wedge 1, 0.1 \wedge 0\} = 0.7$ and hence the related teacher should be directly given the "qualified" evaluation according to the preset rules.

5. Conclusions

BUI as a power information type can handle a large variety of uncertain decision making and evaluation problems. BUI has different structure and practical meaning to some other types of uncertain information such as intuitionistic fuzzy information and hesitant fuzzy information, to name just a few. For some more situations where BUI cannot be well applied, we proposed an extended form of it called UBUI, which proved to have more uses and can model more situations and even linguistic expressions. UBUI is a generalization of BUI, and when the two certainty parameters in UBUI coincide, it degenerates into BUI. The weighted means for UBUI are discussed but in a more formative form. Nevertheless, induced ordered weighted averaging is still as powerful in UBUI environment as in other data environments.

Rules-based decision making and semi-copula based integral in UBUI environment are also briefly discussed which have good potential to be further investigated and applied. Some exemplified cases in educational evaluation are presented to show it has further application background. Though it is with several advantages and opportunities, UBUI currently still has limitations. For example, how to design effective or reasonable aggregation functions for them, and how to derive and elicit such type of two-parameter uncertainty from individual decision maker rather than from statistics, should be more deeply studied in future works.

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