dC_F-integrals: generalizing C_F-integrals by means of restricted dissimilarity functions

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***Abstract*—The Choquet integral (CI) is an averaging aggregation function that has been used, e.g., in the Fuzzy Reasoning Method (FRM) of Fuzzy Rule-Based Classification Systems (FRBCS's) and in multi-criteria decision making in order to take into account the interactions among data/criteria. Several generalizations of the CI have been proposed in the literature in order to improve the performance of FRBCS's, and also to provide more flexibility in the different models by relaxing both the monotonicity requirement and averaging conditions of aggregation functions. An important generalization are the C_F-integrals, which are pre-aggregation functions that may present interesting non-averaging behavior depending on the function F adopted in the construction and, in this case, offering competitive results in classification. Recently, the concept of d-Choquet integrals was introduced as a generalization of the CI by Restricted Dissimilarity Functions (RDFs), improving the usability of CIs, as when comparing inputs by the usual difference may not be viable. The objective of this paper is to introduce the concept of dC_F-integrals, which is a generalization of C_F-integrals by RDFs. The aim is to analyze whether the usage of dC_F-integrals in the FRM of FRBCS's represents a good alternative towards the standard C_F-integrals that just consider the difference as a dissimilarity measure. For that, we consider six RDFs combined with five fuzzy measures, applied with more than twenty functions F. The analysis of the results are based on statistical tests, demonstrating their efficiency. Additionally, comparing the applicability of dC_F-integrals versus C_F-integrals, the range of the good generalizations of the former is much larger than that of the latter.**

***Index Terms*—CF-integrals, d-Choquet integrals, restricted dissimilarity functions, fuzzy rule based classification systems, pre-aggregation functions**

I. INTRODUCTION

An aggregation function (AF) [1] is a special type of function that fuses different values into a single one, which represents all the considered values. The arithmetic mean, the Product t-norm [2], the Ordered Weight Average [3] and the Choquet integral (CI) [4] are examples of AFs.

Aggregation functions have an important role in Fuzzy Rule-Based Classification Systems (FRBCS's) [5], since they are responsible for aggregating information in several stages of the Fuzzy Reasoning Method (FRM) [6]. While the FRM of Winning Rule (WR) [7] takes into account only the fuzzy rule having the largest compatibility with the example, the usage of the CI in the FRM allows to model the relation among the fired rules by considering a fuzzy measure [8]. In fact, Barrenechea et al. [9] introduced a FRM considering the CI, and obtained an improvement in the performance of the classifier when associated to the power measure.

The CI was generalized in many ways see, e.g., [10]) and some of those generalizations were used in the FRM of FRBCS's, such as the C_T-integrals [11] (also applied in MCDM [12]), CC-integrals [13] (also used in motor-imagery-based brain computer interface systems [14] and group MCDM [15]), C_F-integrals [16] (also used in image processing [17]) and C_{F1F2}-integrals [18], all of them introduced by Lucca et al. Also, a well known generalization of the CI is the fuzzy t-conorm integral $\mathfrak{S}$ (called fuzzy t-integral by Murofushi & Sugeno [19], or generalized t-conorm integral by Narukawa & Torra [20]) for a t-system $(\perp_1, \perp_2, \perp_3, \boxdot)$, where $\perp_1, \perp_2$, $\perp_3$ are continuous t-conorms which are the maximum or Archimedean, and $\boxdot$ is an increasing function satisfying special constraints [19, Def. 2.1]. See also the gC_{F1F2}-integrals by Dimuro et al. [21] and the C_F^m-integrals by Horanska & Šipošová [22].

The main features of those generalizations are that some of them may be neither aggregation functions (since they may not be increasing in the standard sense) nor averaging (i.e., the output of the "aggregation" operator is not bounded by the minimum or the maximum of the inputs). Table I shows an overview of such characteristics, which depend on specific properties of the functions used in the generalization, where T is a t-norm [2], C is a copula [26] and F, F_1 and F_2 are more general functions.

Recently, Bustince et al. [27] introduced the concept of d-Choquet integrals by replacing the difference operator in the definition of the CI by restricted dissimilarity functions (RDFs) [28], [29]. This interesting generalization can improve the usability of the standard CI in some contexts, since it can be applied when the comparison of inputs using the usual difference is not possible/viable, as in the case of intervals [30]. Moreover, since there are several ways of defining

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TABLE I: Main features of the generalizations of the CI

Integral	Incr. (AF)	D. Incr. (PAF)	OD incr. -	Aver.	Non-aver.
CI	✓	✓	✓	✓	
C_T*		✓		✓	
CC	✓	✓	✓	✓	
C_F		✓		✓	✓
$C_{F_1F_2}$**			✓	✓	✓
$gC_{F_1F_2}$	✓	✓	✓	✓	✓
C_F^m***	✓	✓	✓	✓	✓
$\mathfrak{S}$****	✓	✓	✓		

* When T is different from the product t-norm;
** when F_1 and F_2 are not copulae;
*** under certain constraints [22, Props. 6 and 10];
**** whenever $(1 -_{\perp_1} 0) \boxdot 1 = 1$;
Incr.: increasing; D. Incr.: directional increasing [24];
OD incr.: ordered directional increasing [25];
AF: aggregation function [1]; PAF: pre-aggregation function [11];
Aver.: averaging [1]; Non-aver.: Non averaging [1].

dissimilarity functions, one can adopt the one that best fits the faced problem, providing more flexibility to the model.

Then, in an attempt to improve both the performance and flexibility of C_F-integrals in FRBCS's, the general objective of this paper is to introduce the concept of dC_F-integrals, which is a generalization of the Choquet-based C_F-integrals by replacing the difference operator by RDFs. For that, we have two specific goals: (i) a theoretical study, showing the main features of this new aggregation-like function according to both, the function F and the restricted dissimilarity functions used in its construction and (ii) the application of dC_F-integrals in the FRM of a FRBCS, performing an extensive analysis of its behaviour and performance. In this sense, we aim at answering the following research questions:

1. Is it useful to substitute the classical difference by restricted dissimilarity functions in C_F integrals when applied to tackle classification problems?
2. Which combinations of functions F, restricted dissimilarity functions and fuzzy measures provide better performance?
3. Do dC_F-integrals enlarge the flexibility of C_F-integrals?

In order to present a complete and robust study, we consider 33 different datasets selected from KEEL dataset repository [31]. We combine 21 different functions F with six different restricted dissimilarity functions. All these combinations are also tested with five different fuzzy measures. The performance of the dC_F-integrals are measured using the accuracy rate and the results are supported and analyzed considering statistical tests.

The organization of this paper follows this structure. Section II presents the preliminary concepts. In Section III, we introduce the concept of dC_F-integrals as well as a theoretical study. The new FRM is presented in Section IV. The experimental framework is described in Section V. After that, the obtained results are analysed in Section VI. Finally, the conclusions are drawn in Section VII.

II. Preliminaries

A function $F : [0,1]^2 \to [0,1]$ with 0 as *left annihilator element* (0-LAE), that is, $F(0,y) = 0,\ \forall y \in [0,1]$, is said to be left 0-absorbent. If $F(x,1) = x$, for any $x \in [0,1]$, then we say that it has 1 as *right neutral element* (1-RNE). Also, when $F(x,y) \leq x$, $\forall x, y \in [0,1]$, we say that F follows the *Left Conjunctive Property* (LC) [16].

Since we are working with generalizations of the CI, two definitions are essential. The first one is the definition of aggregation functions [1]: let $A : [0,1]^n \to [0,1]$ be an n-ary function, if A satisfies:

(A1) Increasingness in each argument: $\forall i \in \{1,\ldots,n\}$: if $x_i \leq y$ then $A(x_1,\ldots,x_n) \leq A(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_n)$;
(A2) Boundary conditions: $A(0,\ldots,0) = 0$, $A(1,\ldots,1) = 1$;

then A is an *aggregation function* (AF).

The second is a more generic definition, where we ask the function to be increasing only in a pre-defined direction, that is, to be *directional monotonic* [24]. Let H be an n-ary function and $\boldsymbol{r} = (r_1,\ldots,r_n)$ an n-dimensional vector, with $\boldsymbol{r} \neq \mathbf{0} = (0,\ldots,0)$. We say that H is $\boldsymbol{r}$*-increasing* if, for all $\boldsymbol{x} \in [0,1]^n$ and $c>0$ such that $(\boldsymbol{x} + c\boldsymbol{r}) \in [0,1]^n$, it holds that

$$H(x_1 + cr_1,\ldots,x_n + cr_n) \geq H(x_1,\ldots,x_n).$$

If $\boldsymbol{r} = \mathbf{1} = (1,\ldots,1)$, H is said to be weak increasing [32]. If H is $\boldsymbol{r}$-increasing, for some $\boldsymbol{r} \neq \mathbf{0}$, and satisfies the boundary conditions (A2), then H is an $\boldsymbol{r}$*-pre-aggregation function* ($\boldsymbol{r}$-PAF) [11], [33].

By working with fuzzy integrals we also work with *fuzzy measures* [4], that is, $m : 2^N \to [0,1]$ such that, for all $X, Y \subseteq N = \{1,\ldots,n\}$, the following properties holds:

(m1) Increasingness: if $X \subseteq Y$, then $m(X) \leq m(Y)$;
(m2) Boundary conditions: $m(\emptyset) = 0$ and $m(N) = 1$.

The fuzzy measures considered in this study, are the same as those used in [9], whose performances were analyzed in [34]. Their definitions are the following, where $X \subseteq N$:

- *Cardinality or uniform measure*: $m_C(X) = |X|/n$.
- *Dirac's measure*: For a fixed $i \in N$,

$$m_D(X) = \begin{cases} 1 & \text{if } i \in X \\ 0 & \text{if } i \notin X. \end{cases}$$

- *Weighted mean (Wmean)*: Let $(w_1,\ldots,w_n) \in [0,1]^n$ be a weight vector, such that $\sum_{i=1}^n w_i = 1$. Define: $m(\{1\}) = w_1,\ldots,m(\{n\}) = w_n$ and then the Wmean is given by: $m_{WM}(X) = \sum_{i\in X} m(\{i\})$, which is a probability measure on N, being the uniform measure a particular case.
- *Ordered Weighted Averaging (OWA)*: Let m be a symmetric fuzzy measure and derive a weight vector $(w_1,\ldots,w_n)\in[0,1]^n$ as $w_i = m(A_{n-i+1}) - m(A_{n-i})$, for $i\in\{1,\ldots,n\}$, A_i any subset with $|A_i| = i$. Define $m_{OWA}(\{i\}) = w_j$, with j being the i-th biggest component of X, and: $m_{OWA}(X) = \sum_{i\in X} m_{OWA}(\{i\})$.
- *Power Measure (PM)*: $m_P(X) = (|X|/n)^q$, with $q > 0$.

In this study, for the PM, we stress out that the value of the exponent q is learned by means of a genetic algorithm. In fact, as we have as many fuzzy measures as classes, we learn as many values for the parameter q as classes. This approach follows the idea introduced in [9] and widely used by the different generalizations of the CI (see [11], [13], [16], [18]).

Using a fuzzy measure $m : 2^N \to [0,1]$, the discrete *Choquet integral* (CI) [4] with respect to m, is the function $\mathfrak{C}_m : [0,1]^n \to [0,1]$, defined, for all $\boldsymbol{x} \in [0,1]^n$, by

$$\mathfrak{C}_m(\boldsymbol{x}) = \sum_{i=1}^{n} \left(x_{(i)} - x_{(i-1)}\right) \cdot m\left(A_{(i)}\right),$$

where $(x_{(1)}, \ldots, x_{(n)})$ is an increasing permutation of $\boldsymbol{x}$, $x_{(0)} = 0$ and $A_{(i)} = \{(i), \ldots, (n)\}$ is the subset of indices of $n-i+1$ largest components of $\boldsymbol{x}$.

As discussed in the Introduction, several generalizations of the CI may be found in the literature [10]. Recently, Lucca et al. [16] introduced the concept of C_F-integral (which is similar to the F-based discrete Choquet-like integral [23]). Let $F : [0,1]^2 \to [0,1]$ be a bivariate function. The C_F-*integral* with respect to a fuzzy measure $m : 2^N \to [0,1]$ is the function $\mathfrak{C}_m^F : [0,1]^n \to [0,1]$ defined, for all $\boldsymbol{x} \in [0,1]^n$, by

$$\mathfrak{C}_m^F(\boldsymbol{x}) = \min\left\{1,\ \sum_{i=1}^{n} F\left(x_{(i)} - x_{(i-1)},\ m\left(A_{(i)}\right)\right)\right\},$$

where $x_{(i)}$ and $A_{(i)}$ were defined in the previous paragraph for the CI. For examples of functions F, see Table II.

As a key concept in this work, a *restricted dissimilarity function* [28], [29] is a function $\delta : [0,1]^2 \to [0,1]$ that satisfies, for all $x, y, z \in [0,1]$, the following conditions:

(d1) $\delta(x,y) = \delta(y,x)$;
(d2) $\delta(x,y) = 1$ if and only if $\{x,y\} = \{0,1\}$;
(d3) $\delta(x,y) = 0$ if and only if $x = y$;
(d4) if $x \leq y \leq z$, then $\delta(x,y) \leq \delta(x,z)$ and $\delta(y,z) \leq \delta(x,z)$.

By replacing the difference operator in the definition of the CI by a restricted dissimilarity function, Bustince et al. [27] introduced the d-Choquet integral (d-integral, for short). A discrete *d-Choquet integral* with respect to a fuzzy measure $m : 2^N \to [0,1]$ and a restricted dissimilarity function $\delta : [0,1]^2 \to [0,1]$ is a mapping $C_{m,\delta} : [0,1]^n \to [0,n]$, defined, for all $\boldsymbol{x} \in [0,1]^n$, by:

$$C_{m,\delta}(\boldsymbol{x}) = \sum_{i=1}^{n} \delta\left(x_{(i)}, x_{(i-1)}\right) \cdot m\left(A_{(i)}\right)$$

where $x_{(i)}$ and $A_{(i)}$ were defined previously. For examples of restricted dissimilarity functions, see Table III (functions δ).

III. dC_F-INTEGRALS

This section introduces the definition of dC_F-integral, analysing the most important properties.

Definition 1 (dC_F-integral)**.** *Let $F : [0,1]^2 \to [0,1]$ be a function satisfying (0-LAE), $\delta : [0,1]^2 \to [0,1]$ be a restricted dissimilarity function and $m : 2^N \to [0,1]$ be a fuzzy measure. Then, the generalization of the CI by the function F, with respect to δ and m, called dC_F-integral, is the function $\mathfrak{C}_{F,m,\delta} : [0,1]^n \to [0,n]$, defined, for all $\boldsymbol{x} \in [0,1]^n$, by:*

$$\mathfrak{C}_{F,m,\delta}(\boldsymbol{x}) = x_{(1)} + \sum_{i=2}^{n} F\left(\delta(x_{(i)}, x_{(i-1)}),\ m(A_{(i)})\right) \quad (1)$$

where $(x_{(1)}, \ldots, x_{(n)})$ is an increasing permutation on the input $\boldsymbol{x}$ and $A_{(i)} = \{(i), \ldots, (n)\}$.

TABLE II: $(1,0)$-increasing functions F satisfying (0-LAE).

Definition	Description
$T_M(x,y) = \min\{x,y\}$	Minimum t-norm
$T_P(x,y) = xy$	Algebraic product
$T_Ł(x,y) = \max\{0, x+y-1\}$	Łukasiewicz
$T_{DP}(x,y) = \begin{cases} x & \text{if } y = 1 \\ y & \text{if } x = 1 \\ 0 & \text{otherwise.} \end{cases}$	Drastic Product
$T_{NM}(x,y) = \begin{cases} \min\{x,y\} & \text{if } x+y > 1 \\ 0 & \text{otherwise.} \end{cases}$	Nilpotent Minimum
$T_{HP}(x,y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$	Hamacher Product
$O_B(x,y) = \min\{x\sqrt{y}, y\sqrt{x}\}$	[35], Cuadras-Augé copula [36]
$O_{mM}(x,y) = \min\{x,y\}\max\{x^2,y^2\}$	[37], [38], [39]
$O_\alpha(x,y) = xy(1+\alpha(1-x)(1-y))$, with $\alpha \in [-1,1] \setminus \{0\}$	[26], Farlie-Gumbel-Morgenstern copula family
$O_{Div}(x,y) = \frac{xy+\min\{x,y\}}{2}$	[26], [13]
$GM(x,y) = \sqrt{xy}$	Geometric Mean, [40]
$HM(x,y) = \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0 \\ \frac{2}{\frac{1}{x}+\frac{1}{y}} & \text{otherwise.} \end{cases}$	Harmonic Mean, [40]
$Sin(x,y) = \sin\left(\frac{\pi}{2}(xy)^{\frac{1}{4}}\right)$	Sine, [40]
$O_{RS}(x,y) = \min\left\{\frac{(x+1)\sqrt{y}}{2}, y\sqrt{x}\right\}$	
$C_F(x,y) = xy + x^2y(1-x)(1-y)$	[2], [13]
$C_L(x,y) = \max\{\min\{x, \frac{y}{2}\}, x+y-1\}$	[26], [13]
$F_{GL}(x,y) = \sqrt{\frac{x(y+1)}{2}}$	
$F_{BPC}(x,y) = xy^2$	[1]
$F_{BD1}(x,y) = \min\{x, 1-x+\min\{x,y^q\}\}$, with $0 < q \leq 1$	[16], [18]
$F_{NA}(x,y) = \begin{cases} x & \text{if } x \leq y \\ \min\{\frac{x}{2}, y\} & \text{otherwise.} \end{cases}$	[16], [18]
$F_{NA2}(x,y) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{x+y}{2} & \text{if } 0 < x \leq y \\ \min\{\frac{x}{2}, y\} & \text{otherwise.} \end{cases}$	[16], [18]

Proposition 1. *$\mathfrak{C}_{F,m,\delta}$ is well defined.*

Proof. It is immediate that, for any $\boldsymbol{x} \in [0,1]^n$, $0 \leq \mathfrak{C}_{F,m,\delta}(\boldsymbol{x}) \leq n$. Take an input $\boldsymbol{x} \in [0,1]^n$, for which there may be different increasing permutations (i.e., $\boldsymbol{x}$ has repeated elements). For the sake of simplicity, but without loss of generality, consider that there exist $r, s \in \{1, \ldots, n\}$ such that $x_r = x_s = z \in [0,1]$ and, for all $i \in \{1, \ldots, n\}$, with $i \neq r, s$, it holds that $x_i \neq x_r, x_s$. Two possible increasing permutations are:

$$(x_{(1)}, \ldots, x_{(k-1)} = x_r, x_{(k)} = x_s, \ldots, x_{(n)}) \quad (2)$$
$$(x_{(1)}, \ldots, x_{(k-1)} = x_s, x_{(k)} = x_r, \ldots, x_{(n)}) \quad (3)$$

Denote by $m_{(i)}^{(1)} = m^{(1)}(A_{(i)})$ and $m_{(i)}^{(2)} = m^{(2)}(A_{(i)})$, with $i \in \{1, \ldots, n\}$, the fuzzy measures of the subsets of $A_{(i)}$ of indices corresponding to the $n-i+1$ largest components of $\boldsymbol{x}$ with respect to the permutations (2) and (3), respectively. Then, for all $i \neq k$, it holds that

$$m_{(i)}^{(1)} = m_{(i)}^{(2)}, \text{ and} \quad (4)$$
$$m_{(k)}^{(1)} = m(\{s, (k+1), \ldots, (n)\}) \quad (5)$$

$$m^{(2)}_{(k)} \;=\; m(\{r,(k+1),\ldots,(n)\}), \tag{6}$$

which means that it may be the case that $m^{(1)}_{(k)} \neq m^{(2)}_{(k)}$. Denote by $\mathfrak{C}^{(1)}_{F,m,\delta}$ and $\mathfrak{C}^{(2)}_{F,m,\delta}$ the dC_F-integrals with respect to the permutations (2) and (3), respectively, and suppose that

$$\mathfrak{C}^{(1)}_{F,m,\delta}(\boldsymbol{x}) \neq \mathfrak{C}^{(2)}_{F,m,\delta}(\boldsymbol{x}). \tag{7}$$

From Eqs. (4), (5), (6), whenever $k \neq 1$, one has that:

$$\begin{aligned}
&\mathfrak{C}^{(1)}_{F,m,\delta}(\boldsymbol{x}) - \mathfrak{C}^{(2)}_{F,m,\delta}(\boldsymbol{x})\\
&\quad = F\left(\delta(x_{(k)},x_{(k-1)}),\ m^{(1)}_{(k)}\right) - F\left(\delta(x_{(k)},x_{(k-1)}),\ m^{(2)}_{(k)}\right)\\
&\quad = F\left(\delta(x_s,x_r),\ m(\{s,(k+1),\ldots,(n)\})\right) -\\
&\qquad F\left(\delta(x_r,x_s),\ m(\{r,(k+1),\ldots,(n)\})\right)\\
&\quad = F\left(\delta(z,z),\ m(\{s,(k+1),\ldots,(n)\})\right) -\\
&\qquad F\left(\delta(z,z),\ m(\{r,(k+1),\ldots,(n)\})\right)\\
&\quad = F\left(0,\ m(\{s,(k+1),\ldots,(n)\})\right) -\\
&\qquad F\left(0,\ m(\{r,(k+1),\ldots,(n)\})\right) \text{ by (d3)}\\
&\quad = 0 \text{ by (0-LAE)}
\end{aligned}$$

which is a contradiction with (7). Analogous result can be shown for $k=1$. The result can be generalized for any subsets of repeated elements in the input $\boldsymbol{x}$. Then, for any different increasing permutations of the same input $\boldsymbol{x}$ one always get the same result $\mathfrak{C}_{F,m,\delta}(\boldsymbol{x})$. □

Remark 1. *Observe that the first element of the summation in the definition of $\mathfrak{C}_{F,m,\delta}$ is just $x_{(1)}$ instead of*

$$F\left(\delta(x_{(1)},x_{(0)}),\ m(A_{(1)})\right).$$

This is considered to avoid the initial discrepant behavior of non-averaging functions in the initial phase of the aggregation process, as pointed out in [18]. For example, consider a vector with only one component $\boldsymbol{x}=(0.1)$, $\delta_1(x,y)=|x-y|$ and

$$F_{NA2}(x,y) = \begin{cases} 0 & \text{if } x=0\\ \frac{x+y}{2} & \text{if } 0<x\leq y\\ \min\{\frac{x}{2},y\} & \text{otherwise.}\end{cases}$$

If we included the first element in the summation of the integral the result would be:

$$\begin{aligned}
\mathfrak{C}_{F,m,\delta_1}(0.1) \;&=\; F_{NA2}\left(\delta_1(x_{(1)},x_{(0)}),\ m(A_{(1)})\right)\\
&=\; F_{NA2}(0.1-0,1) = \frac{0.1+1}{2} = 0.55.
\end{aligned}$$

Observe here the large discrepancy of the result (a relative error of 450%), since one expects that the aggregated value would be 0.1. Using our definition of dC_F-integral (Equation (1)), this unexpected behavior is avoided and the result is 0.1.

In the following, consider all fuzzy measures $m: 2^N \to [0,1]$, functions $F:[0,1]^2 \to [0,1]$ satisfying (0-LAE) and restricted dissimilarity functions $\delta:[0,1]^2 \to [0,1]$.

Since the ranges of dC_F-integrals are in $[0,n]$, there is no sense to talk about their boundary conditions in general, unless one just deals with increasing dC_F-integrals. Then, in the context of this paper, the boundary conditions of AF and PAF (conditions (A2)), are referred just by $0,1$-conditions.

Proposition 2 ($0,1$-conditions)**.** *$\mathfrak{C}_{F,m,\delta}$ satisfies the $0,1$-conditions.*

Proof. (i) Take $\boldsymbol{x}=\boldsymbol{0}=(0,\ldots,0)$. Then:

$$\begin{aligned}
\mathfrak{C}_{F,m,\delta}(\boldsymbol{0}) &= 0+\sum_{i=2}^{n} F\left(\delta(0,0),\ m(A_{(i)})\right)\\
&= \sum_{i=2}^{n} F\left(0,\ m(A_{(i)})\right) \qquad \text{by (d3)}\\
&= 0 \qquad \text{(by 0-LAE)}
\end{aligned}$$

(ii) For $\boldsymbol{x}=\boldsymbol{1}=(1,\ldots,1)$, we have:

$$\begin{aligned}
\mathfrak{C}_{F,m,\delta}(\boldsymbol{x}) &= 1+\sum_{i=2}^{n} F\left(\delta(1,1),\ m(A_{(i)})\right)\\
&= 1+\sum_{i=2}^{n} F\left(0,\ m(A_{(i)})\right) \qquad \text{by (d3)}\\
&= 1 \quad \text{by (0-LAE)}
\end{aligned}$$

□

In what follows, denote the range of a dC_F-integral $\mathfrak{C}_{F,m,\delta}$ by $\mathrm{Ran}(\mathfrak{C}_{F,m,\delta})$.

Remark 2. *If the range of a dC_F-integral is $[0,1]$, then the $0,1$-conditions are equivalent to the boundary conditions (A2). Additionally, whenever a dC_F-integral is increasing and satisfies the $0,1$-conditions then its range is $[0,1]$. Now, whenever a dC_F-integral is not increasing, then, even if it satisfies the $0,1$-conditions, there may exist $\boldsymbol{y}\in[0,1]^n$, $\boldsymbol{0}<\boldsymbol{y}<\boldsymbol{1}$ such that $\mathfrak{C}_{F,m,\delta}(\boldsymbol{y})>1$, as it was shown in [27, Example 3.6 (iii)], which is the particular case of a dC_F-integral for $F=T_P$ (the product t-norm) (in fact, the standard d-Choquet integral).*

Proposition 3. *$Ran(\mathfrak{C}_{F,m,\delta})\subseteq[0,1]$ if F satisfies (LC) and the following condition holds, for all $\boldsymbol{x}\in[0,1]^n$:*

$$\sum_{i=2}^{n}\delta(x_{(i)},\ x_{(i-1)}) \leq 1-x_{(1)}. \tag{8}$$

Proof. For any $\boldsymbol{x}\in[0,1]^n$, $\mathfrak{C}_{F,m,\delta}(\boldsymbol{x})\geq 0$ and

$$\begin{aligned}
\mathfrak{C}_{F,m,\delta}(\boldsymbol{x}) &= x_{(1)}+\sum_{i=2}^{n} F\left(\delta(x_{(i)},\ x_{(i-1)}),\ m(A_{(i)})\right)\\
&\leq x_{(1)}+\sum_{i=2}^{n}\delta(x_{(i)},\ x_{(i-1)}) \quad \text{by (LC)}\\
&\leq 1 \quad \text{by (8).}
\end{aligned}$$

□

Theorem 1 (Directional monotonicity)**.** *If F is $(1,0)$-increasing and, for all $a,b\in[0,1]$, with $a\geq b$, and $h>0$ such that $a+h, b+h\in[0,1]$, it holds that:*

$$\delta(a+h,b+h)\geq\delta(a,b), \tag{9}$$

then $\mathfrak{C}_{F,m,\delta}$ is $\boldsymbol{1}$-increasing

Proof. For any $\boldsymbol{x}\in[0,1]^n$, $\boldsymbol{c}=(c,\ldots,c)$, with $c>0$ and $\boldsymbol{x}+\boldsymbol{c}\in[0,1]^n$, consider that Eq. (9) holds whenever $h=$

c, $a = x_{(i)}$ and $b = x_{(i-1)}$, for any $i = 2, \ldots, n$, that is, $\delta(x_{(i)} + c,\ x_{(i-1)} + c) \geq \delta(x_{(i)},\ x_{(i-1)})$. Since F is $(1,0)$-increasing, then we have that $F\left(\delta(x_{(i)},\ x_{(i-1)}),\ m(A_{(i)})\right) - F\left(\delta(x_{(i)} + c,\ x_{(i-1)} + c),\ m(A_{(i)})\right) < 0$. Thus:

$$\sum_{i=2}^{n} F\left(\delta(x_{(i)},\ x_{(i-1)}),\ m(A_{(i)})\right) - \sum_{i=2}^{n} F\left(\delta(x_{(i)} + c,\ x_{(i-1)} + c),\ m(A_{(i)})\right) < 0 < c.$$

Therefore:

$$(x_{(1)} + c) + \sum_{i=2}^{n} F\left(\delta(x_{(i)} + c,\ x_{(i-1)} + c),\ m(A_{(i)})\right) > x_{(1)} + \sum_{i=2}^{n} F\left(\delta(x_{(i)},\ x_{(i-1)}),\ m(A_{(i)})\right),$$

thus $\mathfrak{C}_{F,m,\delta}(\boldsymbol{x} + \boldsymbol{c}) > \mathfrak{C}_{F,m,\delta}(\boldsymbol{x})$, and $\mathfrak{C}_{F,m,\delta}$ is $\mathbf{1}$-increasing. □

It is immediate that:

Theorem 2 (PAF). *If F is $(1,0)$-increasing and (LC), and also both Condition (8) of Proposition 3 and Condition (9) of Theorem 1 hold, then $\mathfrak{C}_{F,m,\delta}$ is an $\mathbf{1}$-PAF.*

Theorem 3 (Monotonicity). *$\mathfrak{C}_{F,m,\delta}$ is increasing if and only if the following conditions hold:*
(i) For all $a, b \in [0,1]$, with $a \leq b$, $c \in Ran(m)$ and $h \in [0, b-a]$ it holds that:

$$F(\delta(a,b),c) - F(\delta(a+h,b),c) \leq h; \tag{10}$$

(ii) For all $a_1, a_2, b_1, b_2 \in [0,1]$, there exist $h_1, h_2 \geq 0$, with $a_1 + h_1, a_2 + h_2 \in [0,1]$ such that: If $b_2 \leq b_1$ and $h_2 \leq h_1$ then:

$$F(a_1 + h_1, b_1) - F(a_2 + h_2, b_2) \geq F(a_1, b_1) - F(a_2, b_2). \tag{11}$$

Proof. ($\Leftarrow$) Take $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$ such that, for some $k \in \{1, \ldots, n\}$ and $\lambda \geq 0$, it holds that $x_{(k)} = y_{(k)} + \lambda$, and, for all $i \neq k$, $x_{(i)} = y_{(i)}$, such that:

$$x_{(k-1)} = y_{(k-1)} \leq x_{(k)} = y_{(k)} + \lambda \leq x_{(k+1)} = y_{(k+1)}. \tag{12}$$

Then, one has the following possibilities:
(a) $k = 1$: In this case, $x_{(1)} = y_{(1)} + \lambda$. Denote $a = y_{(1)}$, $b = y_{(2)}$, $c = m(A_{(2)}) \in (0,1]$ and $h = \lambda \in [0, b-a]$. Since (d1) holds, it follows that:

$$\begin{aligned}
\mathfrak{C}_{F,m,\delta}(\boldsymbol{x}) &= (y_{(1)} + \lambda) + F(\delta(y_{(2)}, y_{(1)} + \lambda), m(A_{(2)})) \\
&\quad + \sum_{i=3}^{n} F\left(\delta(y_{(i)}, y_{(i-1)}),\ m(A_{(i)})\right) \\
&= a + h + F(\delta(b, a+h), c) \\
&\quad + \sum_{i=3}^{n} F\left(\delta(y_{(i)}, y_{(i-1)}),\ m(A_{(i)})\right) \\
&\geq a + h + F(\delta(b,a),c) - h \\
&\quad + \sum_{i=3}^{n} F\left(\delta(y_{(i)}, y_{(i-1)}),\ m(A_{(i)})\right) \text{ by (10)} \\
&= y_{(1)} + F(\delta(y_{(2)}, y_{(1)}), m(A_{(2)})) \\
&\quad + \sum_{i=3}^{n} F\left(\delta(y_{(i)}, y_{(i-1)}),\ m(A_{(i)})\right) \\
&= \mathfrak{C}_{F,m,\delta}(\boldsymbol{y}).
\end{aligned}$$

(b) $1 < k < n$: Observe that, by (d4), it holds that:

$$\delta(y_{(k)} + \lambda, y_{(k-1)}) \geq \delta(y_{(k)}, y_{(k-1)}) \tag{13}$$

$$\delta(y_{(k+1)}, y_{(k)}) \geq \delta(y_{(k+1)}, y_{(k)} + \lambda). \tag{14}$$

Then, it is possible to denote $\delta(y_{(k)}, y_{(k-1)}) = a_1$, $\delta(y_{(k)} + \lambda, y_{(k-1)}) = a_1 + h_1$, $\delta(y_{(k+1)}, y_{(k)} + \lambda) = a_2$ and $\delta(y_{(k+1)}, y_{(k)}) = a_2 + h_2$, where $h_1 = \delta(y_{(k)} + \lambda, y_{(k-1)}) - \delta(y_{(k)}, y_{(k-1)}) \geq 0$ and $h_2 = \delta(y_{(k+1)}, y_{(k)}) - \delta(y_{(k+1)}, y_{(k)} + \lambda) \geq 0$. Also denote $b_1 = m(A_{(k)})$ and $b_2 = m(A_{(k+1)})$ and notice that $b_2 \leq b_1$. Then it follows that:

$$\begin{aligned}
&\mathfrak{C}_{F,m,\delta}(\boldsymbol{x}) \\
&= y_{(1)} + \sum_{i=2}^{k-1} F(\delta(y_{(i)},\ y_{(i-1)}),\ m(A_{(i)})) \\
&\quad + F(\delta(y_{(k)} + \lambda,\ y_{(k-1)}),\ m(A_{(k)})) \\
&\quad + F(\delta(y_{(k+1)},\ y_{(k)} + \lambda),\ m(A_{(k+1)})) \\
&\quad + \sum_{i=k+2}^{n} F(\delta(y_{(i)},\ y_{(i-1)}),\ m(A_{(i)})) \\
&= y_{(1)} + \sum_{i=2}^{k-1} F(\delta(y_{(i)},\ y_{(i-1)}),\ m(A_{(i)})) \\
&\quad + F(a_1 + h_1,\ b_1) + F(a_2,\ b_2) \\
&\quad + \sum_{i=k+2}^{n} F(\delta(y_{(i)},\ y_{(i-1)}),\ m(A_{(i)})) \\
&\geq y_{(1)} + \sum_{i=2}^{k-1} F(\delta(y_{(i)},\ y_{(i-1)}),\ m(A_{(i)})) \\
&\quad + F(a_1,\ b_1) + F(a_2 + h_2,\ b_2) \\
&\quad + \sum_{i=k+2}^{n} F(\delta(y_{(i)},\ y_{(i-1)}),\ m(A_{(i)})) \text{ by (13), (14), (11)} \\
&= y_{(1)} + \sum_{i=1}^{k-1} F(\delta(y_{(i)},\ y_{(i-1)}), m(A_{(i)})) \\
&\quad + F(\delta(y_{(k)},\ y_{(k-1)}),\ m(A_{(k)})) \\
&\quad + F(\delta(y_{(k+1)},\ y_{(k)}),\ m(A_{(k+1)})) \\
&\quad + \sum_{i=k+2}^{n} F(\delta(y_{(i)},\ y_{(i-1)}),\ m(A_{(i)})) = \mathfrak{C}_{F,m,\delta}(\boldsymbol{y}).
\end{aligned}$$

(c) $k = n$: In this case, $x_{(n)} = y_{(n)} + \lambda$. By (d4) and condition (ii) of the theorem when $h_2 = 0$, it follows that:

$$\begin{aligned}
\mathfrak{C}_{F,m,\delta}(\boldsymbol{x}) &= y_{(1)} + \sum_{i=2}^{n-1} F\left(\delta(y_{(i)}, y_{(i-1)}),\ m(A_{(i)})\right) \\
&\quad + F(\delta(y_{(n)} + \lambda,\ y_{(n-1)}),\ m(A_{(n)})) \\
&\geq y_{(1)} + \sum_{i=2}^{n-1} F\left(\delta(y_{(i)}, y_{(i-1)}),\ m(A_{(i)})\right) \\
&\quad + F(\delta(y_{(n)},\ y_{(n-1)}),\ m(A_{(n)}))
\end{aligned}$$

$$= \mathfrak{C}_{F,m,\delta}(\boldsymbol{y}).$$

($\Rightarrow$) Since $\mathfrak{C}_{F,m,\delta}$ is increasing, then for all $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$ there is $k \in \{1,\dots,n\}$ and $\lambda \geq 0$ for which $x_{(k)} = y_{(k)} + \lambda \in [0,1]$, and for any $i \in \{1,\dots,n\}$ with $i \neq k$, $x_{(k)} = y_{(k)}$, satisfying Condition (12), it holds that:

$$\begin{aligned} &\mathfrak{C}_{F,m,\delta}(\boldsymbol{x}) - \mathfrak{C}_{F,m,\delta}(\boldsymbol{y}) \geq 0 \\ &\Leftrightarrow x_{(1)} + \sum_{i=2}^{n} F(\delta(x_{(i)},\ x_{(i-1)}),\ m(A_{(i)})) \\ &\quad - y_{(1)} + \sum_{i=2}^{n} F(\delta(y_{(i)},\ y_{(i-1)}),\ m(A_{(i)})) \geq 0. \quad (15) \end{aligned}$$

Here, the only non-zero elements are the ones that contain the k-th element: this induces to the following possibilities:
(a) $k=1$: In this case we have $x_{(1)} = y_{(1)} + \lambda$ and, by (15):

$$\begin{aligned} &(y_{(1)} + \lambda) + F(\delta(y_{(2)}, y_{(1)} + \lambda), m(A_{(2)}) \\ &\quad - y_{(1)} - F(\delta(y_{(2)}, y_{(1)}), m(A_{(2)}) \geq 0 \\ &\Leftrightarrow F(\delta(y_{(2)}, y_{(1)}), m(A_{(2)}) - F(\delta(y_{(2)}, y_{(1)} + \lambda), m(A_{(2)}) \\ &\quad \leq \lambda. \quad (16) \end{aligned}$$

By using the same notation of the item (b) of the ($\Leftarrow$)-part of the proof, Eq. (16) becomes:

$$F(\delta(b,a),c) - F(\delta(b,a+h),c) \leq h,$$

since $a = y_{(1)} \leq b = y_{(2)}$, $c = m(A_{(2)}) \in (0,1]$ and $h = \lambda \in [0, b-a]$. By (d1), the Condition (ii) holds.
(b) $1 < k < n$: By (15), one has that:

$$\begin{aligned} &F(\delta(y_{(k)} + \lambda, y_{(k-1)}), m(A_{(k)})) \\ &\quad + F(\delta(y_{(k+1)}, y_{(k)} + \lambda), m(A_{(k+1)})) \\ &\quad \geq F(\delta(y_{(k)}, y_{(k-1)}), m(A_{(k)})) \\ &\quad + F(\delta(y_{(k+1)}, y_{(k)}), m(A_{(k+1)})) \\ \Leftrightarrow\quad & F(\delta(y_{(k)} + \lambda, y_{(k-1)}), m(A_{(k)})) \\ &\quad - F(\delta(y_{(k+1)}, y_{(k)}), m(A_{(k+1)})) \\ &\quad \geq F(\delta(y_{(k)}, y_{(k-1)}), m(A_{(k)})) \\ &\quad - F(\delta(y_{(k+1)}, y_{(k)} + \lambda), m(A_{(k+1)})) \quad (17) \end{aligned}$$

Since inequalities (13) and (14) hold, and $b_2 = m(A_{(k+1)}) \leq m(A_{(k)}) = b_1$, (17) can be written, using the notation adopted in the item (c) of the ($\Leftarrow$)-part of the proof, as:

$$F(a_1 + h_1, b_1) - F(a_2 + h_2, b_2) \geq F(a_1, b_1) - F(a_2, b_2),$$

where $h_1 = \delta(y_{(k)} + \lambda, y_{(k-1)}) - \delta(y_{(k)}, y_{(k-1)}) \geq 0$ and $h_2 = \delta(y_{(k+1)}, y_{(k)}) - \delta(y_{(k+1)}, y_{(k)} + \lambda) \geq 0$. Then, the Condition (ii) holds.
(c) $k = n$: In this case $x_{(n)} = y_{(n)} + \lambda$ and, by (15):

$$\begin{aligned} &F(\delta(y_{(n)} + \lambda,\ y_{(n-1)}),\ m(A_{(k)})) \\ &\quad - F(\delta(y_{(n)},\ y_{(n-1)}),\ m(A_{(k)})) \geq 0. \end{aligned}$$

By (d4) we have that $\delta(y_{(n)} + \lambda,\ y_{(n-1)}) \geq \delta(y_{(n)},\ y_{(n-1)})$. Now considering $\delta(y_{(n)} + \lambda,\ y_{(n-1)}) = a_1 + \lambda_1$, $\delta(y_{(n)},\ y_{(n-1)}) = a_1$ and $b_1 = m(A_{(k)})$, we then have that

$$F(a_1 + \lambda_1, b_1) - F(a_1, b_1) \geq 0 \Leftrightarrow F(a_1 + \lambda_1, b_1) \geq F(a_1, b_1),$$

which is the case of having $h_2 = 0$ in Condition (ii). □

From Proposition 2 and Theorem 3, it follows that:

Theorem 4 (AF). *$\mathfrak{C}_{F,m,\delta}$ is an aggregation function if and only if the conditions of Theorem 3 hold.*

We point out that any aggregation-like operator is required to present some kind of "increasingness property" in order to guarantee the preservation of the information quality of the output related to the information quality of the inputs, in the light of Domain Theory [41]. In this sense, the higher are the values of the inputs, in some considered direction, the higher should be the aggregated value to the same direction [10], [21]. Observe, in Table III, that there may exist dC_F-integrals that are neither increasing nor directional increasing, which is the case, e.g, of $\mathfrak{C}_{F,\delta_3,m}$ and $\mathfrak{C}_{F,\delta_5,m}$. Nevertheless, they are Ordered Directional (OD) monotone functions [25]. Such functions are monotonic along different directions according to the ordinal size of the coordinates of each input.

Definition 2. *[25] Consider a function $Od : [0,1]^n \to [0,1]$ and let $\boldsymbol{r} = (r_1,\dots,r_n)$ be a real n-dimensional vector, $\boldsymbol{r} \neq \boldsymbol{0}$. Od is said to be ordered directionally (OD) $\boldsymbol{r}$-increasing if, for each $\boldsymbol{x} \in [0,1]^n$, any permutation $\sigma : \{1,\dots,n\} \to \{1,\dots,n\}$ with $x_{\sigma(1)} \geq \dots \geq x_{\sigma(n)}$, and $c > 0$, with $x_{\sigma(i)} + cr_i \in [0,1]$, for $i \in \{1,\dots,n\}$, such that $1 \geq x_{\sigma(1)} + cr_1 \geq \dots \geq x_{\sigma(n)} + cr_n$, it holds that $Od(\boldsymbol{x} + c\boldsymbol{r}_{\sigma^{-1}}) \geq Od(\boldsymbol{x})$, where $\boldsymbol{r}_{\sigma^{-1}} = (r_{\sigma^{-1}(1)},\dots,r_{\sigma^{-1}(n)})$. Similarly, one defines an ordered directionally (OD) $\boldsymbol{r}$-decreasing function.*

Theorem 5. *For any $k > 0$, the dC_F-integral is an (OD) $(k,0,\dots,0)$-increasing function.*

Proof. For all $\boldsymbol{x} \in [0,1]^n$ and permutation $\sigma : \{1,\dots,n\} \to \{1,\dots,n\}$, with $x_{\sigma(1)} \geq \dots \geq x_{\sigma(n)}$, and $c > 0$, with $x_{\sigma(i)} + cr_i \in [0,1]$, for $i \in \{1,\dots,n\}$, and $1 \geq x_{\sigma(1)} + cr_1 \geq \dots \geq x_{\sigma(n)} + cr_n$, for $\boldsymbol{r}_{\sigma^{-1}} = (r_{\sigma^{-1}(1)},\dots,r_{\sigma^{-1}(n)})$, one has that:

$$\begin{aligned} &\mathfrak{C}_{F,m,\delta}(\boldsymbol{x} + c\boldsymbol{r}_{\sigma^{-1}}) \\ &= x_{(1)} + c \cdot r_{\sigma^{-1}(1)} \\ &\quad + \sum_{i=2}^{n-1} F(\delta(x_{(i)} + cr_{\sigma^{-1}(i)}, x_{(i-1)} + cr_{\sigma^{-1}(i-1)}), m(A_{(i)})) \\ &\quad + F(\delta(x_{(n)} + cr_{\sigma^{-1}(n)}, x_{(n-1)} + cr_{\sigma^{-1}(n-1)}), m(A_{(n)})) \\ &= x_{(1)} + \sum_{i=2}^{n-1} F(\delta(x_{(i)}, x_{(i-1)}), m(A_{(i)})) \\ &\quad + F(\delta(x_{(n)} + ck, x_{(n-1)}), m(A_{(n)})) \\ &\geq x_{(1)} + \sum_{i=2}^{n-1} F(\delta(x_{(i)}, x_{(i-1)}), m(A_{(i)}) \\ &\quad + F(\delta(x_{(n)}, x_{(n-1)}), m(A_{n)}) \quad \text{by (d4)} \\ &= \mathfrak{C}_{F,m,\delta}(\boldsymbol{x}). \end{aligned}$$

□

Lastly, some other important properties are studied:

Proposition 4. *$\mathfrak{C}_{F,m,\delta}$ is idempotent.*

Proof. Consider $\boldsymbol{x} = (x,\dots,x) \in [0,1]^n$. Then:

$$\mathfrak{C}_{F,m,\delta}(\boldsymbol{x}) = x + \sum_{i=2}^{n} F\left(\delta(x,x),\ m(A_{(i)})\right)$$

TABLE III: Properties of the dC_F-integral for various F satisfying (0-LAE) and restricted dissimilarity functions, based on the results presented in this paper. Here, m means that $\mathfrak{C}_{F,m,\delta}(\boldsymbol{x}) \geq \min(\boldsymbol{x})$.

	$\delta_0(x,y)=\lvert x-y\rvert$					$\delta_1(x,y)=(x-y)^2$					$\delta_2(x,y)=\sqrt{\lvert x-y\rvert}$				
Function	Agg.	1-inc	1-PA	OD-(k,_)-inc	Ave	Agg.	1-inc	1-PA	OD-(k,_)-inc	Ave	Agg.	1-inc	1-PA	OD-(k,_)-inc	Ave
T_M		✓	✓	✓	✓		✓	✓	✓	✓		✓		✓	m
T_P	✓	✓	✓	✓	✓		✓	✓	✓	✓		✓		✓	m
$T_Ł$		✓	✓	✓	✓		✓	✓	✓	✓		✓		✓	m
T_{DP}		✓	✓	✓	✓		✓	✓	✓	✓		✓		✓	m
T_{NM}		✓	✓	✓	✓		✓	✓	✓	✓		✓		✓	m
T_{HP}		✓	✓	✓	✓		✓	✓	✓	✓		✓		✓	m
O_B		✓	✓	✓	✓		✓	✓	✓	✓		✓		✓	m
O_{mM}		✓	✓	✓	✓		✓	✓	✓	✓		✓		✓	m
O_α		✓	✓	✓	✓		✓	✓	✓	✓		✓		✓	m
O_{Div}		✓	✓	✓	✓		✓	✓	✓	✓		✓		✓	m
GM		✓		✓	m		✓		✓	m		✓		✓	m
HM		✓		✓	m		✓		✓	m		✓		✓	m
Sin		✓		✓	m		✓		✓	m		✓		✓	m
O_{RS}		✓		✓	m		✓		✓	m		✓		✓	m
C_F		✓	✓	✓	✓		✓	✓	✓	✓		✓		✓	m
C_L		✓	✓	✓	✓		✓	✓	✓	✓		✓		✓	m
F_{GL}		✓		✓	m		✓		✓	m		✓		✓	m
F_{BPC}	✓	✓	✓	✓	✓		✓	✓	✓	✓		✓		✓	m
F_{BD1}		✓	✓	✓	✓		✓	✓	✓	✓		✓		✓	m
F_{NA}		✓	✓	✓	✓		✓	✓	✓	✓		✓		✓	m
F_{NA2}		✓		✓	m		✓		✓	m		✓		✓	m

	$\delta_3(x,y)=\lvert\sqrt{x}-\sqrt{y}\rvert$					$\delta_4(x,y)=\lvert x^2-y^2\rvert$					$\delta_5(x,y)=(\sqrt{x}-\sqrt{y})^2$				
Function	Agg.	1-inc	1-PA	OD-(k,_)-inc	Ave	Agg.	1-inc	1-PA	OD-(k,_)-inc	Ave	Agg.	1-inc	1-PA	OD-(k,_)-inc	Ave
T_M				✓	m		✓		✓	m				✓	✓
T_P				✓	m		✓		✓	m				✓	✓
$T_Ł$				✓	m		✓		✓	m				✓	✓
T_{DP}				✓	m		✓		✓	m				✓	✓
T_{NM}				✓	m		✓		✓	m				✓	✓
T_{HP}				✓	m		✓		✓	m				✓	✓
O_B				✓	m		✓		✓	m				✓	✓
O_{mM}				✓	m		✓		✓	m				✓	✓
O_α				✓	m		✓		✓	m				✓	✓
O_{Div}				✓	m		✓		✓	m				✓	✓
GM				✓	m		✓		✓	m				✓	m
HM				✓	m		✓		✓	m				✓	m
Sin				✓	m		✓		✓	m				✓	m
O_{RS}				✓	m		✓		✓	m				✓	m
C_F				✓	m		✓		✓	m				✓	✓
C_L				✓	m		✓		✓	m				✓	✓
F_{GL}				✓	m		✓		✓	m				✓	m
F_{BPC}				✓	m		✓		✓	m				✓	✓
F_{BD1}				✓	m		✓		✓	m				✓	✓
F_{NA}				✓	m		✓		✓	m				✓	✓
F_{NA2}				✓	m		✓		✓	m				✓	m

$$
\begin{aligned}
&= x + \sum_{i=2}^{n} F\left(0,\ m(A_{(i)})\right) \quad \text{by (d3)} \\
&= x \quad \text{by (0-LAE).}
\end{aligned}
$$

Therefore, $\mathfrak{C}_{F,m,\delta}(\boldsymbol{x})$ is idempotent. □

Proposition 5. $\mathfrak{C}_{F,m,\delta}(\boldsymbol{x}) \geq \min(\boldsymbol{x})$*, for all* $\boldsymbol{x} \in [0,1]^n$.

Proof. It follows that

$$
\begin{aligned}
&\mathfrak{C}_{F,m,\delta}(\boldsymbol{x}) = x_{(1)} \\
&\qquad + \sum_{i=2}^{n} F\left(\delta(x_{(i)}, x_{(i-1)}),\ m(A_{(i)})\right) \geq x_{(1)} = \min(\boldsymbol{x}).
\end{aligned}
$$

□

Proposition 6. *If F satisfies (LC) and δ satisfies the condition*

$$
\sum_{i=2}^{n} \delta(a_i,\ a_{i-1}) \leq a_n - a_1 \tag{18}
$$

for any $0 \leq a_1 \leq \ldots \leq a_n \leq 1$, then $\mathfrak{C}_{F,m,\delta}(\boldsymbol{x}) \leq \max(\boldsymbol{x})$, for all $\boldsymbol{x} \in [0,1]^n$.

Proof. Consider $\boldsymbol{x} \in [0,1]^n$. Then:

$$
\begin{aligned}
\mathfrak{C}_{F,m,\delta}(\boldsymbol{x}) &= x_{(1)} + \sum_{i=2}^{n} F\left(\delta(x_{(i)}, x_{(i-1)}),\ m(A_{(i)})\right) \\
&\leq x_{(1)} + \sum_{i=2}^{n} \delta(x_{(i)}, x_{(i-1)}) \text{ by (LC)} \\
&\leq x_{(n)} = \max(\boldsymbol{x}) \quad \text{by (18).}
\end{aligned}
$$

Therefore, $\mathfrak{C}_{F,m,\delta}(\boldsymbol{x}) \leq \max(\boldsymbol{x})$. □

From Propositions 5 and 6, it is immediate that:

Proposition 7. *If F satisfies (LC) and the condition (18) holds then $\mathfrak{C}_{F,m,\delta}$ is averaging.*

Table III shows examples of combinations of functions F and δ that satisfy the following properties: aggregation (Agg.), **1**-increasiness (1-inc), **1**-pre-aggregation (1-PAF), OD-(k,0, ...,0)-increasing (OD-(k,_)-inc) and averaging (Ave.). Notice that the only combinations of functions F and δ satisfying the conditions necessary for the dC_F-integral to be an aggregation function are the pairs T_P and δ_0, and F_{BPC} and δ_0. Just two studied dC_F-integrals are not directional increasing, namely, the ones based on the restricted dissimilarity functions δ_3 and δ_5. Nevertheless, not all the reminder dC_F-integrals are PAFs. Some of them, although **1**-increasing, do not have their ranges equal to the unit interval, which clearly depends on the considered function F, as the dC_F-integrals based on δ_0 or δ_1, and the functions GM, HM, sin, O_{RS}, F_{GL} or F_{NA2}. Finally, all dC_F-integrals are OD-(k,0, ...,0)-increasing.

Remark 3. *Notice that all RDFs presented in Table III are derived from δ_0. In fact, they were constructed according to [29, Prop. 2]. It follows that, for $i \in \{1, \ldots, 5\}$ and $x_1, \ldots, x_n \in [0,1]$: $\mathfrak{C}_{F,m,\delta_i}(x_1, \ldots, x_n) - x_{(1)} = \mathfrak{C}_{F_{\alpha_i},m,\delta_0}(x_1^{\beta_i}, \ldots, x_n^{\beta_i}) - x_{(1)}^{\beta_i}$, where $F_{\alpha_i}(u,v) = F(u^{\alpha_i}, v)$, for $u,v \in [0,1]$ and $\alpha_i, \beta_i \geq 0$. Nevertheless, it is possible to define an RDF that is not derived from δ_0, such as $\delta : [0,1]^2 \to [0,1]$ given, for all $x,y \in [0,1]$ and $c \in (0,1)$, by*

$$\delta(x,y) = \begin{cases} 1, & \text{if } \{x,y\} = \{0,1\}, \\ 0, & \text{if } x = y, \\ c, & \text{otherwise}. \end{cases}$$

The respective $\mathfrak{C}_{F,m,\delta}$ is $\mathbf{1}$-increasing (but not a $\mathbf{1}$-PAF) and OD $(k, 0, \ldots, 0)$-increasing. It also holds that $\mathfrak{C}_{F,m,\delta}(\boldsymbol{x}) \geq \min(\boldsymbol{x})$, for all $\boldsymbol{x} \in \mathbb{R}^n$, although, it is not averaging.

IV. dC_F-INTEGRALS IN THE FRM OF FRBCS'S

In this section, we present the application of the dC_F-integral in the FRM of a FRBCS. Considering a classification problem containing t training examples $\mathrm{x}_p = (x_{p1}, \ldots, x_{pn}, y_p)$, with $p = 1, \ldots, t$, where each x_{pi} is the value of the $i = 1, \ldots, n$ variable, and $y_p \in \mathcal{C} = \{C_1, \ldots, C_M\}$ is the label of the class of the p-th training example, and *M* is the number of classes.

Here, we focus on FRBCS's, specifically, the Fuzzy Association Rule-based Classification model for High Dimensional Problems (FARC-HD) [42] fuzzy classifier. The structure of the fuzzy rules generated by this classifier is:

$$\text{Rule } R_j\text{: If } x_1 \text{ is } A_{j1} \text{ and } \ldots \text{ and } x_n \text{ is } A_{jn} \\ \text{then Class is } C_j \text{ with } RW_j,$$

where R_j is the label of the j-th rule, A_{ji} is a fuzzy set representing a linguistic term modeled by a triangular shaped membership function. C_j is the class label and $RW_j \in [0,1]$ is the rule weight [43], which in this case is computed as the confidence of the fuzzy rule.

Following the same approach used in the previous generalizations of the CI (see [11], [13], [16] and [18]), we modify the classical FRM of FARC-HD to include the dC_F-integrals in its third stage. Thus, the classification soundness degree for all classes of a new example x is computed by:

$$S_k(x) = C_k^{\mathfrak{C}_{F,m,\delta}}\left(b_1^k(x), \ldots, b_L^k(x)\right),$$

where k is related with the class, L is the number of rules, $\left(b_1^k(x), \ldots, b_L^k(x)\right)$ are the association degrees of x with the class of each rule, given by $b_j^k(x) = \mu_{A_j}(x) \cdot RW_j^k$, where $\mu_{A_j}(x) = AG\left(\mu_{A_{j1}}(x_1), \ldots, \mu_{A_{jn}}(x_n)\right)$, $j = 1, \ldots, L$, AG is an aggregation function, and μ is the membership degree of the elements of the fuzzy set A_j. Finally, $C_k^{\mathfrak{C}_{F,m,\delta}}$ is the dC_F-integral that aggregate the fired rules for each class.

V. EXPERIMENTAL FRAMEWORK

In this section, we present the experimental framework used in the study. We start providing the features of the considered datasets. Then, we show the configuration of the proposal and, finally, we discuss the statistical tests that are used to validate the quality of the results.

A. Datasets used in the study

This study is conducted taking into consideration 33 different datasets selected from KEEL dataset repository [31]. We highlight that these datasets are the same ones used in previous studies, such as C_F-integrals [16] and C_{F1F2}-integrals [18]. This allow a comparison with state-of-the-art approaches.

We summarize the datasets in Table IV. For each dataset, we present the corresponding identification (Id), the number of instances (#Inst), attributes (#Atts) and classes (#Class). Additionally, we point out that these datasets do not present monotonic characteristics [44].

We applied a 5-fold cross-validation procedure, which consists in spliting the datasets into five partitions containing 20% of the examples each one. The model is learned using 4 partitions for training and tested in the remaining partition. The general performance of the model is measured according to each testing partition, based on the accuracy rate (the number of correctly classified examples divided by the total number of examples). At the end, after calculating each partition performance, we use the average result of the five testing partitions to generate the output of the algorithm.

B. Configuration of the proposal

The new FRM presented in this paper, considering the concept of dC_F-integrals developed in Section III, is applied in the Fuzzy Association Rule-Based Classification method for High-Dimensional problems (FARC-HD) [42] fuzzy classifier. The configurations used by the algorithms are the same one suggested by the authors and is composed by: linguistic labels per variable (5), conjunction operator (Product t-norm), rule weight (Confidence), minimum support (0.05), minimum confidence (0.8), depth of the search tree (3), number of fuzzy rules that cover each example (2), population size (50), gray codification (30 bits per gene), number of evaluations (20.000).

TABLE IV: Summary of the datasets used in the study.

Id.	Dataset	#Inst.	#Atts.	#Class
App	Appendicitis	106	7	2
Bal	Balance	625	4	3
Ban	Banana	5,300	2	2
Bnd	Bands	365	19	2
Bup	Bupa	345	6	2
Cle	Cleveland	297	13	5
Con	Contraceptive	1,473	9	3
Eco	Ecoli	336	7	8
Gla	Glass	214	9	6
Hab	Haberman	306	3	2
Hay	Hayes-Roth	160	4	3
Ion	Ionosphere	351	33	2
Iri	Iris	150	4	3
Led	led7digit	500	7	10
Mag	Magic	1,902	10	2
New	Newthyroid	215	5	3
Pag	Pageblocks	5,472	10	5
Pen	Penbased	10,992	16	10
Pho	Phoneme	5,404	5	2
Pim	Pima	768	8	2
Rin	Ring	740	20	2
Sah	Saheart	462	9	2
Sat	Satimage	6,435	36	7
Seg	Segment	2,310	19	7
Shu	Shuttle	58,000	9	7
Son	Sonar	208	60	2
Spe	Spectfheart	267	44	2
Tit	Titanic	2,201	3	2
Two	Twonorm	740	20	2
Veh	Vehicle	846	18	4
Win	Wine	178	13	3
Wis	Wisconsin	683	11	2
Yea	Yeast	1,484	8	10

C. Statistical test for performance comparisons

In this paper is considered hypothesis validation techniques to present a statistical analysis of the obtained results [45], [46]. Since the validity conditions of parametric tests are not satisfied, is considered the usage of non-parametric tests [47].

To perform group comparisons, the Aligned Friedman rank test [48] is used. This test uses a reverse raking, that is, the lowest rank is considered as the best one. Additionally, the post-hoc Holm's test [49] is computed to indicate when the approach achieving the less ranking (known as control method) rejects the null hypothesis. To do so, we calculate the Adjusted P-Value (APV) to be able to compare directly the control method, with a level of significance α, versus the other ones.

VI. Performance Analysis

In this section, the results achieved when different dC_F-integrals are applied to aggregate the information in the FRM are presented. The experimental study is developed with a double aim:

1) To analyze if the introduction of the RDFs in the dC_F-integrals allows the system to enhance the results obtained when the classical difference operator is applied, which is considered as baseline of the study. Moreover, we want to check if certain RDFs are more beneficial for the system than others. The results and analyses of this aim are shown in Section V-A.
2) To study if there is a synergy among the best RDFs (found in the first part of the study), the fuzzy measures and the functions F. This study, which is shown in Section V-B, helps to reduce the number of combinations to be tested, since we can suggest a few ones achieving stable and competitive results.

In order to make a complete and robust study, this analysis considers the combinations of 5 different RDFs with 21 generalizations of the CI using 5 different fuzzy measures. All those combinations are applied in 33 datasets. In other words, 525 experiments per dataset have been conducted.

The obtained results are summarized in Tables V, VI and VII, where the rows present the different functions F used for the generalizations. The columns are related with the combination of fuzzy measures and different RDFs. Observe that the dC_F-integrals using δ_0 (the difference operator) are the original C_F-integrals, which are considered as our **baseline**. We should point out that the usage of δ_0 combined with the product t-norm as the function F ($F = T_P$) result in the standard Choquet integral (first column and second row of Tables V, VI and VII). Finally, the value of each cell represents the mean of the accuracy obtained in testing in the 33 considered datasets.

Aiming at extracting the maximum information of the results and to ease their comprehension, for each function F, when comparing the different RDFs for a specific fuzzy measure, we highlight with **boldface** and underline the largest and lowest accuracy mean, respectively. Moreover, the symbol $^+$ indicates for each function F (row), the combination of RDF and fuzzy measure that achieves the largest accuracy among all fuzzy measures. Finally, for each RDF (column) we stress with an * the function F providing the best result. The detailed testing results for the different combinations can be shown in (https://github.com/Giancarlo-Lucca/dCF-integrals).

A. Studying the usefulness of RDFs

In this subsection the usefulness of the substitution of the classical difference by a RDF is studied. To do it, the results obtained by the RDFS are compared against the classical difference. After that, we will analyze whether an specific RDF is able to provide better results than the remainder ones. Performing an initial analysis of the effectiveness of the RDFs in Tables V-VII some important points are found, such as:

- The generalization based on the δ_1 achieves the lowest results for all used functions F in all considered fuzzy measures.
- The usage of the δ_4 in general, present inferior results when compared against the difference operator (δ_0) for all considered functions and fuzzy measures.
- δ_3 presents similar results to the classical difference.
- Generalizations considering δ_2 and δ_5 tend to improve the results obtained by the baseline. In this sense, we highlight that the usage of the δ_5 seems to provide a superior performance.

This initial analysis indicates that the results obtained by the classical difference operator can be improved if the generalizations by the RDFs are used, where δ_2 and δ_5 stand out. To clarify even more these findings, in Table VIII, we show the number of functions F in which the different RDF (columns) achieved the largest result per fuzzy measure (rows). The last row of this table, #Total, is the number of best results

TABLE V: Accuracy mean obtained in tests - Part 1

	Cardinality					
	δ_0	δ_1	δ_2	δ_3	δ_4	δ_5
T_M	79.41	77.69	79.99	79.61	78.57	**80.38**
T_P	79.02	77.84	80.17	78.90	78.02	**80.56**$^{+*}$
$T_Ł$	77.12	77.09	77.17	76.75	77.24	**77.95**
T_{DP}	77.19	77.19	77.19	77.19	77.19	77.19
T_{NM}	77.21	77.08	77.72	77.19	76.97	**78.27**
T_{HP}	79.41	77.70	80.16	79.71	78.36	**80.26**
O_B	79.05	77.59	**80.08**	79.26	78.17	79.97
O_{mM}	78.23	77.15	79.76	78.31	77.47	**79.95**
O_α	78.80	77.55	**80.40**$^{+*}$	78.99	77.91	80.27
O_{div}	78.97	77.44	79.98	79.36	77.87	**80.14**
GM	80.33*	79.13	80.14	**80.40**	79.70	80.00
HM	79.64	78.00	**80.28**	79.75	79.42	80.05
Sin	80.12	80.12*	80.12	80.12	80.12	80.12
O_{RS}	80.17	79.02	80.33	**80.56**$^{+*}$	79.26	80.24
C_F	78.52	77.46	79.88	78.98	77.92	**80.24**$^{+}$
C_L	79.41	77.69	**80.13**	79.18	78.53	79.83
F_{GL}	80.15	79.26	80.26	79.91	80.21	**80.43**
F_{BPC}	77.72	77.24	79.30	78.13	77.44	**79.72**
F_{BD1}	79.60	77.89	**80.29**	79.64	78.67	79.96
F_{NA}	79.10	77.74	**80.38**$^{+}$	79.24	78.39	79.84
F_{NA2}	80.16	80.11	79.98	80.15	**80.34***	79.91
Mean	79.02	78.00	79.70	79.11	78.47	**79.78**
	Dirac					
	δ_0	δ_1	δ_2	δ_3	δ_4	δ_5
T_M	**79.41**	77.44	78.82	78.41	77.78	79.30
T_P	79.02	77.44	78.82	78.41	77.78	**79.30**
$T_Ł$	77.12	77.44	78.82	78.41	77.78	**79.30**
T_{DP}	77.19	77.44	78.82	78.41	77.78	**79.30**$^{+}$
T_{NM}	77.21	77.44	78.82	78.41	77.78	**79.30**
T_{HP}	**79.41**	77.44	78.82	78.41	77.78	79.30
O_B	79.05	77.44	78.82	78.41	77.78	**79.30**
O_{mM}	78.23	77.44	78.82	78.41	77.78	**79.30**
O_α	78.80	77.44	78.82	78.41	77.78	**79.30**
O_{div}	78.97	77.44	78.82	78.41	77.78	**79.30**
GM	**80.33***	78.27	79.67	79.30	78.75	79.69
HM	**79.64**	77.52	79.24	78.59	77.86	79.07
Sin	80.12	80.12*	80.12	80.12	80.12	80.12*
O_{RS}	80.17	78.27	79.67	79.30	78.75	79.69
C_F	78.52	77.44	78.82	78.41	77.78	**79.30**
C_L	**79.41**	77.44	78.82	78.41	77.78	79.30
F_{GL}	80.15	79.12	80.40*	**80.61***	80.19*	80.02
F_{BPC}	77.72	77.44	78.82	78.41	77.78	**79.30**
F_{BD1}	79.60	77.51	**80.13**	79.40	78.60	80.00
F_{NA}	79.10	77.44	78.82	78.41	77.78	**79.30**
F_{NA2}	**80.16**	79.96	**80.16**	80.02	79.89	79.83
Mean	79.02	77.85	79.18	78.81	78.24	**79.46**

TABLE VI: Accuracy mean obtained in tests - Part 2

	OWA					
	δ_0	δ_1	δ_2	δ_3	δ_4	δ_5
T_M	79.17	77.44	79.35	78.97	77.98	**79.46**
T_P	78.64	76.76	**79.49**	78.42	77.19	79.20
$T_Ł$	**77.22**	76.85	77.17	76.79	77.05	77.00
T_{DP}	77.19	77.19	77.19	77.19	77.19	77.19
T_{NM}	**77.23**	76.94	76.89	76.97	77.14	77.02
T_{HP}	79.40	76.97	**79.82**	79.02	77.90	79.53
O_B	78.89	76.61	79.55	79.06	77.37	**79.58**
O_{mM}	77.97	77.12	78.60	77.63	76.76	**78.63**
O_α	78.51	76.68	**79.36**	78.41	77.09	79.15
O_{div}	79.27	77.05	79.26	78.74	77.96	**79.48**
GM	**80.32**	78.46	80.04	79.61	78.96	79.84
HM	79.53	77.12	**80.05**	79.29	78.26	79.72
Sin	80.12	80.12*	80.12	80.12	80.12	80.12*
O_{RS}	**80.12**	77.69	79.72	79.20	78.83	79.66
C_F	78.55	76.79	**79.35**	78.31	77.26	79.29
C_L	79.33	76.83	**79.53**	78.77	77.93	79.42
F_{GL}	**80.50***	79.42	80.40*	80.33*	80.20*	79.99
F_{BPC}	78.09	77.41	**78.74**	77.65	77.13	78.57
F_{BD1}	79.52	78.04	79.88	79.22	78.35	**79.99**
F_{NA}	79.19	77.07	**79.89**	79.00	77.97	79.79
F_{NA2}	**80.21**	79.23	79.64	79.48	79.31	79.61
Mean	79.00	77.51	**79.24**	78.68	78.00	79.15

of each RDF. Also, we provide in the last column, #δ_Total, the number of functions where any RDF (δ_1 to δ_5) enhanced the mean obtained by the classical difference for a specific fuzzy measure (consequently the largest number could be 21).

From the results in Table VIII, it is observable that the usage of the RDFs are suitable since the number of times where they improve the results of the classical difference is high (see the last column of the table). It is noticeable in this analysis that, in 81 out of the 105 combinations (each fuzzy measure is generalized by 21 different functions), the achieved mean by any RDF is superior than that of the classical difference. Among the new RDFs, it is noticeable the superiority of the δ_5 approach, since it provides 43 of these 81 combinations where a RDF is better than the classical difference. A satisfactory number of combinations is also obtained when δ_2 is considered.

Another interesting observation can be noticed when comparing exclusively the standard CI (with δ_0 and T_P) using the different RDFs (see Tables V, VI and VII). Its noticeable that for any fuzzy measure, in all cases we have RDFs that have obtained a superior accuracy mean compared with the CI.

Up to this point, it is clear that the usage of dC_F-integrals is a good alternative when compared with C_F-integrals, which uses the difference operator. However, in order to give a support to the previous findings, a statistical study by applying the Aligned Friedman rank test is performed.

In this test, we compare the performance of the 6 RDFs for each fuzzy measure, analyzing whether a RDF is statistically better than the remainder ones or not. Since this is a large study, in Table IX the results considering exclusively the PM are presented, since this fuzzy measure is the one that achieves the best synergy with the RDFs (see Subsection VI-B). We stress out that the complete statistical analysis, considering all fuzzy measures is also available in the git repository.

In Table IX, for each function F, the RDFs are sorted from the lowest to the highest obtained rank (the lowest one is considered as control method and it is compared with the remaining ones). The APV column indicates if there are statistical differences between the method in the row and the control one. When the obtained APV is inferior than 0.10 it is underlined, indicating that there is a statistical difference in favor to the control method.

To ease the interpretation of the statistical results, a summary is provided in Table X. In this table, the rows are the different RDFs and the columns the fuzzy measures. The value of each cell is the number of times in which the RDF in the row is considered as the control method in the Aligned Friedman rank test (therefore, the best method) for each fuzzy measure. For instance, taking a look at the column of the PM, it is observable that count for δ_0, δ_2 and δ_5 are 3, 8 and 8, respectively, these are the number of times that each RDF, is

TABLE VII: Accuracy mean obtained in tests - Part 3

	Wmean					
	δ_0	δ_1	δ_2	δ_3	δ_4	δ_5
T_M	78.64	77.99	**80.17**	79.54	78.38	79.90
T_P	77.69	77.41	**80.12**	78.94	77.84	79.97
$T_{Ł}$	76.86	76.97	**77.70**	77.51	77.11	77.45
T_{DP}	77.19	77.19	77.19	77.19	77.19	77.19
T_{NM}	77.12	76.88	77.77	77.39	76.92	**78.16**
T_{HP}	78.71	77.66	79.77	79.47	78.60	**80.26**
O_B	78.42	77.62	80.07	79.20	78.17	**80.43***
O_{mM}	77.14	77.39	79.49	78.12	77.53	**79.71**
O_α	77.86	77.64	79.83	79.24	78.32	**80.07**
O_{div}	78.65	77.64	79.70	79.16	78.15	**79.89**
GM	79.90	79.01	80.16	**80.55**$^{+*}$	80.22	80.33
HM	79.37	78.37	**80.16**	79.83	78.92	79.78
Sin	80.12	80.12	80.12	80.12	80.12	80.12
O_{RS}	79.49	78.64	**80.31**	79.97	79.26	80.15
C_F	77.75	77.57	**80.03**	79.06	77.73	79.78
C_L	78.47	77.64	79.84	79.22	78.45	**79.99**
F_{GL}	**80.32***	79.17	80.24	80.13	80.23*	79.89
F_{BPC}	77.10	77.39	79.39	78.12	77.58	**79.45**
F_{BD1}	79.19	77.63	**80.32***	79.80	78.59	79.91
F_{NA}	78.66	77.83	79.93	79.41	78.61	**79.94**
F_{NA2}	80.03	**80.40***	79.90	79.86	80.05	79.84
Mean	78.51	78.01	**79.63**	79.13	78.47	**79.63**
	PM					
	δ_0	δ_1	δ_2	δ_3	δ_4	δ_5
T_M	79.30	77.73	80.40	79.42	78.54	**80.57**$^{+*}$
T_P	79.20	78.06	**80.46**	79.55	78.49	80.10
$T_{Ł}$	78.35	77.10	79.32	78.43	78.07	**79.65**$^{+}$
T_{DP}	77.19	77.19	77.19	77.19	77.19	77.19
T_{NM}	79.02	77.21	**79.83**$^{+}$	78.31	77.81	79.76
T_{HP}	79.74	77.76	80.26	79.47	78.53	**80.33**$^{+}$
O_B	79.44	77.74	**80.49**$^{+}$	79.61	78.71	79.98
O_{mM}	79.19	77.84	**80.06**$^{+}$	78.99	78.64	80.05
O_α	79.25	77.72	80.16	79.87	78.64	**80.19**
O_{div}	79.26	77.77	**80.34**$^{+}$	79.53	78.61	80.24
GM	80.23	79.22	**80.43**	80.17	80.02	80.21
HM	80.28	78.32	80.30	79.82	79.06	**80.36**$^{+}$
Sin	80.12	80.12	80.12	80.12	80.12	80.12
O_{RS}	**80.46**	79.20	80.30	80.10	80.19	80.23
C_F	79.34	77.79	80.05	79.52	78.47	**80.23**
C_L	79.25	77.56	80.11	79.74	78.68	**80.41**$^{+}$
F_{GL}	80.26	79.11	**80.50***	80.15	80.39*	80.34
F_{BPC}	79.19	77.87	**80.25**$^{+}$	79.14	78.21	80.00
F_{BD1}	79.79	77.67	79.98	79.41	78.61	**80.43**$^{+}$
F_{NA}	79.64	77.61	**80.27**	79.43	78.91	79.91
F_{NA2}	**80.55**$^{+*}$	80.36*	79.90	80.36*	80.38	79.96
Mean	79.48	78.14	**80.03**	79.44	78.87	80.01

TABLE VIII: Relation of times that each RDF combined with the fuzzy measures obtained a **bold face** among the analysis

	δ_0	δ_1	δ_2	δ_3	δ_4	δ_5	#δ_Total
Cardinality	0	0	6	2	1	10	19
Dirac	6	0	2	1	0	11	14
OWA	6	0	8	0	0	5	13
Wmean	1	1	7	1	0	9	18
PM	2	0	9	0	0	8	17
#Total	15	1	32	4	1	43	81

considered as control method in Table X[1]. Finally, in the last row, the number of times (#nDiff) in which the δ_0 (baseline) is statistically outperformed by any RDF is provided.

If the cardinality and PM are used, since they are the fuzzy measures that achieve the best results (see Sect. VI-B), we see that δ_0 is statistically improved in almost half of the cases. Furthermore, in general, δ_5 is the best option, followed by δ_2.

Another observation can be made by taking an exclusive look to the CI, which is the function base of this study, in the statistical analysis. It is observable from the second column of Table IX, that the δ_2 can be considered as statistically superior than the CI since it have a lowest rank and the obtained APV when comparing this two cases is small.

In light of the obtained means and the statistical tests, it is noticeable that the use of dC_F-integrals are an interesting approach in alternative to the C_F-integrals. It is also noticeable that there are many approaches in which there are statistical differences with respect to the δ_0. Therefore, the suitability of the new approach is empirically proved.

B. Analyzing the synergy among the RDFs, functions F and fuzzy measures

In this subsection the synergy among the use of RDFs, functions F and fuzzy measures is analyzed. Taking a look at Table VIII, it can be observed that the number of functions F where RDFs achieve a competitive performance is large. In order to reduce the number of functions and to focus on the best synergies, in this subsection we only provide a study using δ_2 and δ_5 as RDFs and the cardinality and PM as fuzzy measures. This is due to the fact that their application led to a general improvement of the dCF-integrals.

To clarify the synergy of the methods, we show in Table XI for the considered fuzzy measures (rows) and RDFs (columns), the top 3 (where #Top1 is the highest accuracy, #Top2 is the second one and #Top3 the third) functions F that achieved the best averaged behaviours among the 33 considered datasets. Observe that this ranking is obtained by analyzing the respective column (fuzzy measure and RDF) in Tables V-VII.

From the results in Table XI some interesting findings emerge. Considering the functions F, we observe that F_{GL}, T_P and T_M appeared two times, while the remaining functions just once, in specific cases. We highlight that the F_{GL} and T_P appeared for both, δ_2 and δ_5. We also want to stress that T_M appears in both fuzzy measures when combined with δ_5, which clearly shows the good synergy between this function and RDF. In fact, observe that the combination of PM with δ_5 and T_M led to the largest accuracy mean in the study.

VII. Conclusion

In this paper, the concept of dC_F-integrals was introduced. These functions generalize the C_F-integrals [16] by restricted dissimilarity functions δ [29], that is, the difference operator used by the C_F-integrals is replaced by restricted dissimilarity functions. Also, dC_F-integrals can be understood as a generalization of the d-Choquet integral [27] by a function F. Important properties that the dC_F-integrals satisfy, which are based on characteristics of the function F and the restricted dissimilarity functions, were shown.

The dC_F-integrals were applied as the aggregation-like operator in the FRM of a state-of-the-art FRBC, in a large experiment, with different analyses, considering several points of view. Taking into account the obtained results, it is noticeable that dC_F-integrals could be considered as a good alternative

[1] We point out that we do not count the results of both functions T_{DP} and Sin, as all the RDFs are the same, APV = 1.0

TABLE IX: Align Friedman rank tests and APV considering PM as fuzzy measure.

T_M			T_P			$T_Ł$			T_{DP}		
Method	Rank	APV	Method	Rank	APV	Method	Rank	APV	Method	Rank	APV
δ_5	58.82	(–)	δ_2	57.79	(–)	δ_5	60.80	(–)	δ_0 $(\mathfrak{C}^{T_{DP}})$	99.50	(–)
δ_2	69.45	0.45	δ_5	72.91	0.28	δ_2	71.32	0.45	δ_1	99.50	1.00
δ_3	94.20	0.02	δ_3	89.06	0.05	δ_3	104.00	0.00	δ_2	99.50	1.00
δ_0 $(\mathfrak{C}^{T_M})$	100.53	0.00	δ_0 $(\mathfrak{C}^{T_P})$	106.03	0.00	δ_0 $(\mathfrak{C}^{T_Ł})$	105.38	0.00	δ_3	99.50	1.00
δ_4	127.32	0.00	δ_4	132.89	0.00	δ_4	112.06	0.00	δ_4	99.50	1.00
δ_1	146.68	0.00	δ_1	138.32	0.00	δ_1	143.44	0.00	δ_5	99.50	1.00

O_{mM}			O_α			O_{div}			GM		
Method	Rank	APV	Method	Rank	APV	Method	Rank	APV	Method	Rank	APV
δ_5	61.98	(–)	δ_2	71.11	(–)	δ_5	69.14	(–)	δ_2	85.05	(–)
δ_2	65.94	0.77	δ_5	72.21	1.00	δ_2	69.59	0.97	δ_0 $(\mathfrak{C}^{GM})$	93.08	1.00
δ_0 $(\mathfrak{C}^{O_{mM}})$	99.06	0.01	δ_3	79.26	1.00	δ_3	88.53	0.33	δ_5	95.26	1.00
δ_3	107.14	0.00	δ_0 $(\mathfrak{C}^{O_\alpha})$	103.15	0.06	δ_0 $(\mathfrak{C}^{O_{div}})$	100.18	0.08	δ_3	96.32	1.00
δ_4	118.29	0.00	δ_4	122.73	0.00	δ_4	126.74	0.00	δ_4	101.61	0.96
δ_1	144.59	0.00	δ_1	148.55	0.00	δ_1	142.82	0.00	δ_1	125.70	0.01

C_F			C_L			F_{GL}			F_{BPC}		
Method	Rank	APV	Method	Rank	APV	Method	Rank	APV	Method	Rank	APV
δ_5	68.06	(–)	δ_5	68.52	(–)	δ_2	80.82	(–)	δ_2	60.89	(–)
δ_2	77.45	0.50	δ_2	68.89	0.97	δ_5	87.12	1.00	δ_5	68.58	0.58
δ_3	88.85	0.28	δ_3	80.21	0.81	δ_4	90.05	1.00	δ_0 $(\mathfrak{C}^{F_{BPC}})$	95.89	0.03
δ_0 $(\mathfrak{C}^{C_F})$	94.38	0.18	δ_0 $(\mathfrak{C}^{C_L})$	104.80	0.03	δ_0 $(\mathfrak{C}^{F_{GL}})$	95.97	0.84	δ_3	96.06	0.03
δ_4	126.85	0.00	δ_4	122.24	0.00	δ_3	107.86	0.22	δ_4	132.97	0.00
δ_1	141.41	0.00	δ_1	152.33	0.00	δ_1	135.18	0.00	δ_1	142.61	0.00

T_{NM}			T_{HP}			O_B		
Method	Rank	APV	Method	Rank	APV	Method	Rank	APV
δ_2	58.23	(–)	δ_5	68.09	(–)	δ_2	62.14	(–)
δ_5	62.61	0.75	δ_2	69.32	0.93	δ_3	85.06	0.19
δ_0 $(\mathfrak{C}^{T_{NM}})$	84.26	0.13	δ_0 $(\mathfrak{C}^{T_{HP}})$	83.39	0.55	δ_5	85.68	0.19
δ_3	113.92	0.00	δ_3	94.23	0.19	δ_0 $(\mathfrak{C}^{O_B})$	96.02	0.04
δ_4	132.76	0.00	δ_4	131.38	0.00	δ_4	121.38	0.00
δ_1	145.23	0.00	δ_1	150.59	0.00	δ_1	146.73	0.00

HM			Sin			O_{RS}		
Method	Rank	APV	Method	Rank	APV	Method	Rank	APV
δ_0 $(\mathfrak{C}^{HM})$	72.26	(–)	δ_0 $(\mathfrak{C}^{Sin})$	99.50	(–)	δ_0 $(\mathfrak{C}^{O_{RS}})$	81.62	(–)
δ_5	78.06	1.00	δ_1	99.50	1.00	δ_2	93.59	0.83
δ_2	78.68	1.00	δ_2	99.50	1.00	δ_4	94.33	0.83
δ_3	92.89	0.43	δ_3	99.50	1.00	δ_5	96.88	0.83
δ_4	128.30	0.00	δ_4	99.50	1.00	δ_3	99.71	0.79
δ_1	146.80	0.00	δ_5	99.50	1.00	δ_1	130.86	0.00

F_{BD1}			F_{NA}			F_{NA2}		
Method	Rank	APV	Method	Rank	APV	Method	Rank	APV
δ_5	59.65	(–)	δ_2	70.24	(–)	δ_0 $(\mathfrak{C}^{F_{NA2}})$	73.39	(–)
δ_0 $(\mathfrak{C}^{F_{BD1}})$	80.82	0.21	δ_5	79.74	0.51	δ_1	89.58	0.36
δ_2	82.45	0.21	δ_0 $(\mathfrak{C}^{F_{NA}})$	86.21	0.51	δ_4	92.11	0.36
δ_3	98.88	0.01	δ_3	94.23	0.26	δ_3	95.70	0.34
δ_4	125.18	0.00	δ_4	113.08	0.00	δ_2	120.73	0.00
δ_1	150.02	0.00	δ_1	153.50	0.00	δ_5	125.50	0.00

TABLE X: Total of times that each approach is considered as control variable in the Friedman rank test

	Cardinality	Dirac	OWA	Wmean	PM
δ_0	–	5	8	1	3
δ_1	–	–	–	1	–
δ_2	5	2	8	7	8
δ_3	2	1	–	1	–
δ_4	1	–	–	–	–
δ_5	11	11	3	9	8
#nDiff	8	5	0	16	9

to be used instead of C_F-integrals in classification problems, since they improve the performance of the classical difference operator. We highlight the usage of the RDF δ_5 combined with the function T_M and the fuzzy measure PM.

In a broader scenario, our developments showed that the dC_F-integrals can enlarge the flexibility of C_F-integrals, since different combinations of RDFs, functions F and fuzzy measures can be used, so being adapted to each kind of problem.

TABLE XI: Summary of the functions that achieved the top 3 best performance per generalization and fuzzy measure.

	δ_2			δ_5		
	#Top1	#Top2	#Top3	#Top1	#Top2	#Top3
Cardinality	O_α	F_{NA}	O_{RS}	T_P	F_{GL}	T_M
PM	F_{GL}	O_b	T_P	T_M	F_{BD1}	C_L

Future works are in two directions. For the theoretical part, we intend to (i) study the relation between the generalizations of the Choquet integral and the fuzzy t-conorm integral, and (ii) defined the dC_F-integrals in the interval-valued context.

As for the applied part, we want to study: (i) the application in the context of multi-criteria decision making; (ii) to consider methods for learning general fuzzy measures; and (iii) to analyze the behavior of this new approach when considering monotone (or not) datasets.

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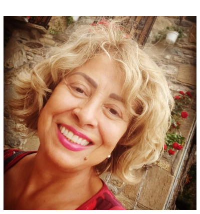

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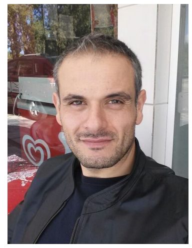

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