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Addendum to "uniqueness of unconditional basis of infinite direct sums of quasi-Banach spaces"

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Abstract

After [*Uniqueness of unconditional basis of infinite direct sums of quasi-Banach spaces*, Positivity 26 (2022), Paper no. 35] was published, we realized that Theorem 4.2 therein, when combined with work of Casazza and Kalton (Israel J. Math. 103:141–175, 1998) , solves the long-standing problem whether there exists a quasi-Banach space with a unique unconditional basis whose Banach envelope does not have a unique unconditional basis. Here we give examples to prove that the answer is positive. We also use auxiliary results in the aforementioned paper to give a negative answer to the question of Bourgain et al. (Mem Am Math Soc 54:iv+111, 1985)*Problem 1.11 whether the infinite direct sum $\ell_1(X)$ of a Banach space X has a unique unconditional basis whenever X does.

Keywords Uniqueness of unconditional basis · Quasi-Banach space

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Roughly speaking, the Banach envelope $\widehat{X}$ of a quasi-Banach space X is the Banach space "closest" to X. It is not surprising then that $\widehat{X}$ and X share many important structural features such as having the same dual space. If $\mathcal{X}$ is a normalized unconditional basis of X then $\mathcal{X}$ is a semi-normalized unconditional basis of $\widehat{X}$ (see [2, Section 10]) so it is natural to wonder if the property of having a unique unconditional basis (up to equivalence and permutation) will be transferred to the Banach envelope. This problem is far from trivial since in all known spaces so far, the pattern shows that $\widehat{X}$ has a unique unconditional basis whenever X does. Take, for instance, the classical

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p-Banach spaces ℓ_p, $H_p(\mathbb{T})$, $\ell_p(\ell_1)$ or $\ell_1(\ell_p)$ for $0 < p < 1$, which have a unique unconditional basis (see [4, 10, 14]) and whose Banach envelope ℓ_1 also does [13].

Dealing only with quasi-Banach spaces whose Banach envelope is isomorphic to ℓ_1 is too restrictive. To obtain a better insight into the underlying pattern, we must look at quasi-Banach spaces with a more complicated Banach envelope. If we focus on the mixed-norm matrix spaces $\ell_p(\ell_q)$, $0 < p, q \le \infty$ (where ℓ_∞ means c_0) we realize that the p-Banach spaces $\ell_p(\ell_2)$, $\ell_p(c_0)$, and $c_0(\ell_p)$ for $0 < p < 1$, have a unique unconditional basis (see [5, 12]); since the Banach envelopes of those spaces, namely $\ell_1(\ell_2)$, $\ell_1(c_0)$ and $c_0(\ell_1)$ respectively, also do (see [6]), these examples reinforce the above-mentioned pattern.

Bourgain et al. proved that $c_0(\ell_2)$ has a unique unconditional basis but that, in contrast, the spaces $\ell_2(c_0)$ and $\ell_2(\ell_1)$ *do not*. We observe that while neither $\ell_2(c_0)$ nor $c_0(\ell_2)$ are the Banach envelope of a non-locally convex natural quasi-Banach space with a basis [11], there are non-locally convex spaces such as $\ell_2(\ell_p)$ for $0 < p < 1$ whose Banach envelope is $\ell_2(\ell_1)$. However, no technique specific to non-locally convex spaces has been shown to be effective to determine whether these spaces have a unique unconditional basis.

Classical Banach spaces seem not to provide examples that disprove the conjecture that uniqueness of unconditional basis passes to Banach envelopes, but the non-classical Tsirelson space $\mathcal{T}$ can be used because Casazza and Kalton [8] proved that that $c_0(\mathcal{T})$ does not have a unique unconditional basis even though $\mathcal{T}$ does (see [7, Theorem 5.1]) . The original Tsirelson's space $\mathcal{T}^*$ has also a unique unconditional basis. This can be deduced from the following result in combination with the fact that $\mathcal{T}$ is the dual space of $\mathcal{T}^*$ (see [9]).

Lemma 1 *Let X be a Banach space with an unconditional basis. Suppose that X^* has a unique unconditional basis. Then X has a unique unconditional basis too.*

Applying Lemma 1 with $X = c_0(\mathcal{T})$ gives that its dual space $\ell_1(\mathcal{T}^*)$ does not have a unique unconditional basis in spite of the fact that $\mathcal{T}^*$ does. Notice the tight connection of this example to [6, Problem 11.1], where the question of whether the uniqueness of unconditional basis passes to infinite ℓ_1-sums is raised. In addition, combining our remark with [1, Theorem 4.2] we solve in the negative the above-mentioned conjecture:

Theorem 2 *For each $0 < p < 1$ there exists a p-Banach space X with a unique unconditional basis whose Banach envelope $\widehat{X}$ does not have a unique unconditional basis.*

Indeed, the p-Banach space $\ell_p(\mathcal{T}^*)$, $0 < p < 1$, has a unique unconditional basis (see [1, Example 7.12(ii)]), and its Banach envelope is $\ell_1(\mathcal{T}^*)$.

For the sake of completeness we close this informative note by proving Lemma 1.

Proof of Lemma 1 Since X^* has a basis, it is separable. Therefore, since the property of having a separable dual is inherited by subspaces, X contains no isomorphic copy of ℓ_1. Then, by [3, Corollary 3.3.3], any unconditional basis of X is shrinking. Let $\mathcal{X}$ and $\mathcal{Y}$ be normalized unconditional bases of X. The basic sequences $\mathcal{X}^*$ and $\mathcal{Y}^*$ of their biorthogonal functionals are semi-normalized unconditional bases of X^*. Hence, by assumption, they are permutatively equivalent. By the reflexivity principle for basic sequences in Banach spaces (see [3, Corollary 3.2.4]), $\mathcal{X}$ and $\mathcal{Y}$ are equivalent up to a permutation.

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