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# From Restricted Equivalence Functions on $L^n$ to Similarity measures between fuzzy multisets

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***Abstract*—Restricted equivalence functions are well-known functions to compare two numbers in the interval between 0 and 1. Despite the numerous works studying the properties of restricted equivalence functions and their multiple applications as support for different similarity measures, an extension of these functions to an n-dimensional space is absent from the literature. In this paper, we present a novel contribution to the restricted equivalence function theory, allowing to compare multivalued elements. Specifically, we extend the notion of restricted equivalence functions from $L$ to $L^n$ and present a new similarity construction on $L^n$. Our proposal is tested in the context of color image anisotropic diffusion as an example of one of its many applications.**

***Index Terms*—Restricted equivalence function, fuzzy multiset, similarity measure, Color image, Anisotropic diffusion.**

## I. Introduction

Comparison operators have been a matter of study in last years [1], [2]. In fact comparison is one of the most basic operations on data, toghether with equality and sorting, which has boosted long effort on modeling those operators. In a general manner, the research effort has often been tailored to accomodate behaviours as close as possible to human behaviour.

A large part of literature on comparison operators is devoted to metrics, as well as to tighly related classes of operators (as pseudometrics [3], and quasimetrics [4]), in which the metric axioms are either imposed or even tightened, as it is the case of ultrametrics [5]. However, it is unclear whether humans actually behave according to the triangle inequality [2]. Many different counterexamples can be found in specific contexts [6]. For this reason, researchers have attempted to build paradigms of comparison which are neither based nor inspired by metrics. A sensible taxonomy is that by Tversky, discriminating spatial and geommetrical strategies for data comparison [1], in which metrics being listed within the spatial ones. Still, some approaches in literature fit none of both strategies, a relevant example being some of the comparison operators in the Fuzzy Set Theory.

Within the Fuzzy Set Theory, comparison has been tackled in different manners. While a significant part of this effort has been devoted to the idea of fuzzy metrics [7], [8], some researchers have opted out by designing operators based on different inspirations and axioms. One of the most relevant ones is the Restricted Equivalence Functions (REFs), presented in [9] for the comparison of membership degrees in $[0,1]$ by adapting the original axioms proposed by Fodor and Roubens [10]. Since its introduction, the concept of REF has been further adapted to compare non-scalar data. Relevant examples are the Interval-valued REFs (IV-REFs), designed to compare interval-valued membership degrees [11], or the Radial REFs (RREFs), tailored to scalar data in radial setups [12]. A critical need in the adaptation of REFs to scenarios other than its original one relate to the modelling of the monotonicity in data, which is critically used in the axiomatic definition of REFs. Unlike metric-inspired comparison operators, REFs do not rely on triangle inequality. Instead, they model similarity on the basis on data ordering. While monotonicity and ordering is trivially modelled in $[0,1]$, it is not straightforward in other scenarios, as interval-valued or radial data. Hence, dedicated studies are devoted to the understanding and modelling of monotonicity in such scenarios in order to design context-specific REFs.

This work presents an adaptation of REFs to multivariate data, which we denote as $L^n$. Specifically, we present the idea of $L^n$-REFs, and develop a set of axioms these operators must fulfill, with special focus on data ordering. Also, we present construction methods for $L^n$-REFs able to accommodate different interpretations in multivariate ordering. Our proposals are aligned with the classical taxonomy of orderings, as described in the seminal work by Barnett [13]. Note that imposing some order for multivariate data is necessarily an arbitrary, context-dependent task, since *there is no natural order for multivariate data* [13]. As a proof-of-concept, $L^n$-REFs are put to the test in color comparison for anisotropic diffusion. As long as color information is normally represented as multivariate data (irrespective of the specific color space), $L^n$-REFs can be used for color comparison in image processing tasks.

The remainder of this paper is organized as follows. Section II recaps some basic mathematical notions that would be used throughout the rest of the paper. Section III introduces the extension of REFs on $L^n$. Section IV tackles the construction of a similarity measures between fuzzy multisets,

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while Section VI presents an application of these operators to content-aware smoothing in color images. Finally, Section VII presents the conclusions of the paper.

## II. Preliminaries

This section collects the mathematical definitions applied in upcoming sections.

In this paper we consider the lattice $(L, \leq)$ where $L = [0, 1]$ and $\leq$ is the natural order on real numbers. We refer to the elements in $L^n$ with capital letters, that is $X = (x_1, \ldots, x_n) \in L^n$. There is a partial order $\leq_P$ induced by $\leq$, given as follows: $X \leq_P Y$ iff $x_i \leq y_i$ for all $i \in \{1, \ldots, n\}$. The definition of $i$ is the same throughout the text for short; otherwise, it will be explicitly redefined for some exceptions if required.

We denote $\mathbf{0} = (0, \ldots, 0) \in L^n$ and $\mathbf{1} = (1, \ldots, 1) \in L^n$.

**Definition II.1.** *[14]–[16] A function $G : L^n \to L$ is called a $n$-ary aggregation function if it satisfies the following properties:*

(G1) $G(\mathbf{0}) = 0$
(G2) $G(\mathbf{1}) = 1$
(G3) *$G$ is increasing in each variable.*

Some properties that are later used and that can be fulfilled by the aggregation functions are the following:

(Giff1) $G(x_1, \ldots, x_n) = 0$ iff $x_1 = \ldots = x_n = 0$.
(Giff2) $G(x_1, \ldots, x_n) = 1$ iff $x_1 = \ldots = x_n = 1$.

An example of aggregation function is a $n$-ary Weighted Arithmetic Mean (WAM) on $L$ with normalized weights $w_1, \ldots, w_n \in L$. Given $X = (x_1, \ldots, x_n)$, a WAM is a mapping $\omega : L^n \to L$ defined by $\omega(X) = w_1x_1 + \ldots + w_nx_n$ where $w_1 + \ldots + w_n = 1$.

**Definition II.2.** *An automorphism is a continuous, strictly increasing function $\varphi : L \to L$, such that $\varphi(0) = 0$ and $\varphi(1) = 1$. Moreover, the identity on $L$ is denoted by $Id$.*

**Definition II.3.** *[9] A function $R : L \times L \to L$ is called a Restricted Equivalence Function (REF), if it satisfies:*

(R1) *$R(x, y) = 1$ iff $x = y$.*
(R2) *$R(x, y) = 0$ iff $\{x, y\} = \{0, 1\}$.*
(R3) *$R(x, y) = R(y, x)$ for all $x, y \in L$.*
(R4) *If $x \leq y \leq z$, then $R(x, y) \leq R(x, z)$ and $R(x, z) \leq R(y, z)$ for all $x, y, z \in L$.*

In [17], a construction method for REFs in terms of automorphisms is introduced.

**Proposition II.4.** *[17] If $\varphi_1, \varphi_2$ are two automorphisms of $L$, then the function $R : L \times L \to L$ defined by*

$$R(x, y) = \varphi_1^{-1} \left(1 - |\varphi_2(x) - \varphi_2(y)|\right) ,$$

*is a REF.*

**Definition II.5.** *A function $f : (L^n)^m \to L^n$ is called representable if there exist functions $f_1, \ldots, f_n : L^m \to L$ such that*

$$f(X_1, \ldots, X_m) = (f_1(x_{11}, \ldots, x_{m1}), \ldots, f_n(x_{1n}, \ldots, x_{mn})) , \quad (1)$$

*for all $X_1, \ldots, X_m \in L^n$ with $X_i = (x_{i1}, \ldots, x_{in})$ for all $i \in \{1, \ldots, m\}$.*

**Definition II.6.** *Let $U$ be a non-empty universe, then a fuzzy set on $U$ is a function $\mathfrak{A} : U \to L$. The set of all fuzzy sets on $U$ is denoted by $F(U)$.*

**Definition II.7.** *Let $U$ be a non-empty universe, then a fuzzy multiset on $U$ is a function $\mathcal{A} : U \to L^n$. The set of all fuzzy multisets on $U$ is denoted by $LF(U)$. For each $u \in U$, we denote $\mathcal{A}(u) = (\mathcal{A}(u)_1, \ldots, \mathcal{A}(u)_n)$.*

We consider the partial order (inclusion) on $LF(U)$ for all $\mathcal{A}, \mathcal{B} \in LF(U)$ given by $\mathcal{A} \subseteq \mathcal{B}$ iff $\mathcal{A}(u) \leq_P \mathcal{B}(u)$ for all $u \in U$.

Throughout this work, different $f : (L^n)^m \to L^n$ (as presented in Def. II.5) can be applied to a point $X_i = (x_{i1}, \ldots, x_{in}) \in L^n$, that is, $f(X_1, \ldots, X_m)$. Furthermore, in this work some $f$ functions are constructed with different WAMs for $m = n$; in these cases the function $f$ takes the form

$$f(X_1, \ldots, X_n) = (w_1(X), \ldots, w_n(X)) ,$$

which can be rewritten as a lineal transformation of a point $X \in L^n$ to another point $X' \in L^n$ by action of a matrix multiplication as

$$(f(X_1, \ldots, X_n))^T = \begin{pmatrix} w_1(X) \\ \vdots \\ w_n(X) \end{pmatrix} = \begin{pmatrix} w_{11}x_1 & + \cdots + & w_{1n}x_n \\ \vdots & \vdots & \vdots \\ w_{n1}x_1 & + \cdots + & w_{nn}x_n \end{pmatrix} = \begin{pmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & & \vdots \\ w_{n1} & \cdots & w_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = WX^T = X'^T , \quad (2)$$

where the $i$-th row of the matrix $W$ is composed of the weights of the $i$-th WAM $\omega_i$. This form of representation allows us to treat many future operations as simple algebraic linear operations.

## III. Restricted Equivalence Functions on $L^n$

This section introduces the extension of REFs on $L^n$ allowing to compare multivalued objects. Depending on how opposite multivalued objects are defined, two different classes of REFs are presented: type 1 (Section III-A) and type 2 (Section III-B). Then, we study the combination of both types of REFs (Section III-C).

### A. Restricted equivalence function on $L^n$ of type 1

If we only consider the elements $\mathbf{0}$ and $\mathbf{1}$ as opposite, leading to the minimum similarity value, a REF on $L^n$ of type 1, $R_{L_1}$, is defined as follows.

**Definition III.1.** *Let $n$ be a positive integer. A function $R_{L_1} : L^n \times L^n \to L^n$ is called a REF on $L^n$ of type 1, if it satisfies the following properties:*

($RL1_1$) *$R_{L_1}(X, Y) = \mathbf{1}$ iff $X = Y$.*
($RL2_1$) *$R_{L_1}(X, Y) = \mathbf{0}$ iff $\{X, Y\} = \{\mathbf{0}, \mathbf{1}\}$.*

(RL3$_1$) $R_{L_1}(X,Y) = R_{L_1}(Y,X)$ *for all* $X, Y \in L^n$.
(RL4$_1$) *If* $X \leq_P Y \leq_P Z$, *then* $R_{L_1}(X,Z) \leq_P R_{L_1}(X,Y)$ *and* $R_{L_1}(X,Z) \leq_P R_{L_1}(Y,Z)$ *for all* $X, Y, Z \in L^n$.

Next we present a construction method for REF on $L^n$ of type 1 according to Def. III.1.

**Theorem III.2.** *Let $n$ be a positive integer and $\omega_i : L^n \to L$ be n-ary WAMs with normalized weighting vectors $(w_{i1}, \ldots, w_{in})$ such that the vectors are linearly independent and there exists $k \in \{1, \ldots, n\}$ with $w_{kj} \neq 0$ for all $j \in \{1, \ldots, n\}$. Let* $\mathbf{R} = (R_1, \ldots, R_n)$ *be a sequence of REFs on L. Then a function $R_{L_1} : L^n \times L^n \to L^n$ given by*

$$R_{L_1}(X,Y) = (R_1(\omega_1(X), \omega_1(Y)), \ldots, R_n(\omega_n(X), \omega_n(Y))) \quad (3)$$

*is a REF on $L^n$ of type 1 for all $X, Y \in L^n$.*

*Proof.* (RL1$_1$) Sufficiency follows from Eq. (3). With respect to necessity, let $R_{L_1}(X,Y) = \mathbf{1}$; then,

$$R_i(\omega_i(X), \omega_i(Y)) = 1, \text{ for all } i \in \{1, \ldots, n\} \quad (4)$$

By (R1), Eq. (4) is satisfied iff $\omega_i(X) = \omega_i(Y)$, for all $i \in \{1, \ldots, n\}$ which implies $X = Y$, by the linearity of the weighting vectors.

(RL2$_1$) Again, we only prove necessity. Let $R_{L_1}(X,Y) = \mathbf{0}$; then we have

$$R_i(\omega_i(X), \omega_i(Y)) = 0, \text{ for all } i \in \{1, \ldots, n\} \quad (5)$$

By (R2), Eq. (5) is satisfied iff $\{\omega_i(X), \omega_i(Y)\} = \{0, 1\}$. Since there exists $k$ such that $w_{kj} \neq 0$ for all $j \in \{1, \ldots, n\}$, and so it follows that $\omega_k$ satisfies (Giff1) and (Giff2), it follows that $\{X, Y\} = \{\mathbf{0}, \mathbf{1}\}$.

(RL3$_1$) The proof is straightforward.

(RL4$_1$) From $x_i \leq y_i \leq z_i$ we have $\omega_j(X) \leq \omega_j(Y) \leq \omega_j(Z)$ for all $j$ and, consequently, $R_j(\omega_j(X), \omega_j(Z)) \leq R_j(\omega_j(X), \omega_j(Y))$, from which it follows that $R_{L_1}(X,Z) \leq R_{L_1}(X,Y)$. The proof for $R_{L_1}(X,Z) \leq_P R_{L_1}(Y,Z)$ is similar. □

We introduce a construction method for REF on $L^n$ of type 1 rewriting the REFs $(R_1, \ldots, R_n)$ in Theorem III.2 in terms of automorphisms.

**Corollary III.3.** *Under the assumptions of Thm. III.2, let $\varphi_{ij}$ for $i \in \{1, \ldots, n\}$ and $j \in \{1, 2\}$, be automorphisms of $L$. Then a function $R_{L_1} : L^n \times L^n \to L^n$ given by*

$$R_{L_1}(X,Y) = \big(\varphi_{11}^{-1}(1 - |\varphi_{12}(\omega_1(X)) - \varphi_{12}(\omega_1(Y))|), \ldots, \varphi_{n1}^{-1}(1 - |\varphi_{n2}(\omega_n(X)) - \varphi_{n2}(\omega_n(Y))|)\big), \quad (6)$$

*is a REF on $L^n$ of type 1, for all $X, Y \in L^n$.*

Note that it is possible to rewrite this expression in terms of Eq. (2) if all $\omega_i$ are WAMs such that

$$R_{L_1}(X,Y) = \mathbf{R}\left((WX^T)^T, (WY^T)^T\right), \quad (7)$$

where the matrix $W$ is composed of the weights of the $\omega_i$ WAMs and $\mathbf{R}$ is a sequence of REFs on $L$.

### *B. Restricted equivalence function on $L^n$ of type 2*

Instead considering the elements **0** and **1** as opposite, but any complementary pair of crisp elements, leading to the minimum similarity value, a REF on $L^n$ of type 2 is defined as follows.

**Definition III.4.** *Let $n$ be a positive integer. A function $R_{L_2} : L^n \times L^n \to L^n$ is called a REF on $L^n$ of type 2, if it satisfies the following properties:*

(RL1$_2$) $R_{L_2}(X,Y) = \mathbf{1}$ *iff* $X = Y$.
(RL2$_2$) $R_{L_2}(X,Y) = \mathbf{0}$ *iff* $\{x_i, y_i\} = \{0, 1\}$.
(RL3$_2$) $R_{L_2}(X,Y) = R_{L_2}(Y,X)$ *for all* $X = (x_1, \ldots, x_n), Y = (y_1, \ldots, y_n) \in L^n$.
(RL4$_2$) *If* $\min(x_i, z_i) \leq y_i \leq \max(x_i, z_i)$, *then* $R_{L_2}(X,Z) \leq_P R_{L_2}(X,Y)$ *and* $R_{L_2}(X,Z) \leq_P R_{L_2}(Y,Z)$ *for all* $X = (x_i, \ldots, x_n), Y = (y_i, \ldots, y_n), Z = (z_i, \ldots, z_n) \in L^n$.

Next we present a construction method for REF on $L^n$ of type 2 according to Def. III.4.

**Theorem III.5.** *Let $n$ be a positive integer and $\omega_i : L^n \to L$ be n-ary WAMs with normalized weighting vectors $(w_{i1}, \ldots, w_{in})$ such that the vectors are linearly independent and there exists $k \in \{1, \ldots, n\}$ with $w_{kj} \neq 0$ for all $j \in \{1, \ldots, n\}$. Let* $\mathbf{R} = (R_1, \ldots, R_n)$ *be a sequence of REFs on L. Then a function $R_{L_1} : L^n \times L^n \to L^n$ given by*

$$R_{L_2}(X,Y) = (\omega_1(R_1(x_1,y_1), \ldots, R_1(x_n,y_n)), \ldots, \omega_n(R_n(x_1,y_1), \ldots, R_n(x_n,y_n))), \quad (8)$$

*is a REF on $L^n$ of type 2, for all $X, Y \in L^n$.*

*Proof.* (RL1$_2$) Sufficiency follows from Eq. (8). With respect to necessity, let $R_{L_2}(X,Y) = \mathbf{1}$; then,

$$\omega_i(R_i(x_1,y_1), \ldots, R_i(x_n,y_n)) = 1, \text{ for all } i \in \{1, \ldots, n\}$$

Since there exists $k$ such that $w_{kj} \neq 0$ for all $j \in \{1, \ldots, n\}$, it follows that $\omega_k$ satisfies (Giff2), thus $R_k(x_1,y_1) = \ldots = R_k(x_n,y_n) = 1$ from which it follows that $X = Y$.

(RL2$_2$) Again, we only prove necessity. Let $R_{L_1}(X,Y) = \mathbf{0}$; then,

$$\omega_i(R_i(x_1,y_1), \ldots, R_i(x_n,y_n)) = 0, \text{ for all } i \in \{1, \ldots, n\}.$$

Since there exits $k \in \{1, \ldots, n\}$ such that $w_{kj} \neq 0$ for all $j \in \{1, \ldots, n\}$, it follows that $\omega_k$ satisfies (Giff1), hence $R_k(x_1,y_1) = \ldots = R_k(x_n,y_n) = 0$ and consequently $\{x_1,y_1\} = \ldots = \{x_n,y_n\} = \{0,1\}$.

(RL3$_2$) The proof is straightforward.

(RL4$_2$) Let $\min(x_i, z_i) \leq y_i \leq \max(x_i, z_i)$ for all $i$. Then $R_i(x_j, z_j) \leq R_i(x_j, y_j)$ for all $i, j$, hence $\omega(R_i(x_1,z_1), \ldots, R_i(x_n,z_n)) \leq \omega(R_i(x_1,y_1), \ldots, R_i(x_n,y_n))$ and consequently $R_{L_2}(X,Z) \leq R_{L_2}(X,Y)$. The proof for $R_{L_2}(X,Z) \leq_P R_{L_2}(Y,Z)$ is similar. □

We introduce a construction method for REFs on $L^n$ of type 2 rewriting REFs $(R_1, \ldots, R_n)$ in Thm. III.5 in terms of automorphisms.

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Note that it is possible to rewrite this expression in terms of Eq. (2) if all $\omega_i$ are WAMs such that

$$R_{L_2}(X,Y) = \left(W(\mathbf{R}(X,Y))^T\right)^T \ , \tag{9}$$

where the matrix $W$ is composed of the weights of the $\omega_i$ WAMs and $\mathbf{R}$ is a sequence of REFs on $L$.

**Corollary III.6.** *Under the assumptions of Thm. III.5, let $\varphi_{ij}$ for $i \in \{1,\ldots,n\}$ and $j \in \{1,2\}$, be automorphisms of $L$. Then a function $R_{L_2} : L^n \times L^n \to L^n$ given by*

$$\begin{aligned} R_{L_2}(X,Y) = \big(&\omega_1(\varphi_{11}^{-1}(1-|\varphi_{12}(x_1)-\varphi_{12}(y_1)|), \\ &\ldots,\varphi_{11}^{-1}(1-|\varphi_{12}(x_n)-\varphi_{12}(y_n)|)\big)\,, \\ &\ldots,\omega_n\left(\varphi_{n1}^{-1}(1-|\varphi_{n2}(x_1)-\varphi_{n2}(y1)|),\right. \\ &\left.\ldots,\varphi_{n1}^{-1}(1-|\varphi_{n2}(x_n)-\varphi_{n2}(y_n)|)\right)\big)\ , \end{aligned} \tag{10}$$

*is a REF on $L^n$ of type 2, for all $X,Y \in L^n$.*

### C. Convex combination between REFs

From the definitions of REF on $L^n$ of type 1 and 2 (Defs. III.1 and III.4), the question naturally arises of what type of function would be obtained from the convex combination of both similarity measures.

As proved by the following theorem, any convex combination of the two different types of REFs is again a REF on $L^n$ of type 1. This statement provides a new construction method for REFs on $L^n$ of type 1.

**Theorem III.7.** *Let $R_{L_1} : L^n \times L^n \to$ be a REF on $L^n$ of type 1 and $R_{L_2} : L^n \times L^n \to$ be a REF on $L^n$ of type 2. Then the convex combination $\alpha R_{L_1} + (1-\alpha)R_{L_2}$, for any $\alpha \in ]0,1]$, is a REF on $L^n$ of type 1.*

*Proof.* ($RL1_1$) and ($RL3_1$) are immediate since ($RL1_1$)=($RL1_2$) and ($RL3_1$)=($RL3_2$).

($RL2_1$) Sufficiency directly follows from the fact that $\{X,Y\} = \{\mathbf{0},\mathbf{1}\}$ implies $\{x_i,y_i\} = \{0,1\}$ for each $i$. With respect to necessity, let $\alpha R_{L_1} + (1-\alpha)R_{L_2} = \mathbf{0}$ for some $\alpha \in ]0,1[$; then $R_{L_1} = \mathbf{0}$ and $R_{L_2} = \mathbf{0}$. Implications of $R_{L_1} = \mathbf{0}$ are more restrictive than the implications of $R_{L_2} = \mathbf{0}$. Therefore, from $\alpha R_{L_1} + (1-\alpha)R_{L_2} = \mathbf{0}$ it follows $\{X,Y\} = \{\mathbf{0},\mathbf{1}\}$. The case of $\alpha = 1$ is immediate.

($RL4_1$) Let $X \leq_P Y \leq_P Z$, then $\min(x_i,z_i) \leq y_i \leq \max(x_i,z_i)$ for all $i \in \{1,\ldots,n\}$. □

## IV. Similarity measure between fuzzy multisets

In this section we tackle the construction of a similarity measure between fuzzy multisets. We first introduce aggregation functions on $L^n$ (Section IV-A) to support the similarity measure construction presented at the end (Section IV-B).

### A. Aggregation functions on $L^n$

Aggregating multiples values is *an indispensable tool in each discipline based on data processing* [18]. Aggregation functions [19]–[21] are usually applied to obtain the most representative and significant information from large amounts of data, for example, in the field of statistics or data science.

Generally, aggregation functions are defined on $L$ [22], obtaining as a result one single value. In this work, we define an aggregation function and a construction method for it on $L^n$ as follows.

**Definition IV.1.** *Let $m$ be a positive integer. A function $G_L : (L^n)^m \to L^n$ is called an $m$-ary aggregation function on $L^n$ if it satisfies the following properties:*

(GL1) $G_L(\mathbf{0},\ldots,\mathbf{0}) = \mathbf{0}$.
(GL2) $G_L(\mathbf{1},\ldots,\mathbf{1}) = \mathbf{1}$
(GL3) *$G_L$ is increasing in each variable.*

A construction method for aggregation functions on $L^n$ is given in the following theorem.

**Theorem IV.2.** *Let $m$ be a positive integer and let $\omega_1,\ldots,\omega_n : L^n \to L$ be $n$-ary WAMs. Let $\mathbf{G} = (G_1,\ldots,G_n)$ be a sequence of $m$-ary aggregation functions on $L$. Then a function $G_L : (L^n)^m \to L^n$ given by*

$$\begin{aligned} G_L(X_1,\ldots,X_m) = \\ &(G_1(\omega_1(X_1),\ldots,\omega_1(X_m)), \\ &\ldots,G_n(\omega_n(X_1),\ldots,\omega_n(X_m))) \end{aligned} \tag{11}$$

*is an $m$-ary aggregation function on $L^n$ for all $X_1,\ldots,X_m \in L^n$.*

*Proof.* Properties (A1)-(A3) directly follow from Eq. (11) and the fact that $G_1,\ldots,G_n$ and $\omega_1,\ldots,\omega_n$ are aggregation functions. □

**Proposition IV.3.** *Under the assumption of Theorem. IV.2, let $\omega_i(X) = 0$ iff $X = \mathbf{0}$ and let there exists $j \in \{1,\ldots,n\}$ such that $\omega_j(X) = 1$ iff $X = \mathbf{1}$. Then it holds:*

(i) *If $G_i$ satisfies* (Giff1) *for all $i$, then it holds that $G_L(X_1,\ldots,X_m) = \mathbf{0}$ iff $X_1 = \ldots = X_m = \mathbf{0}$.*
(ii) *If $G_i$ satisfies* (Giff2) *for all $i$, then it holds that $G_L(X_1,\ldots,X_m) = \mathbf{1}$ iff $X_1 = \ldots = X_m = \mathbf{1}$.*

*Proof.* (i) Let $G_L(X_1,\ldots,X_m) = \mathbf{0}$. Then, we have $G_i(\omega_i(X_1),\ldots,\omega_i(X_m)) = 0$ thus, for all $j$, $\omega_i(X_j) = 0$ and consequently $X_1 = \ldots = X_m = \mathbf{0}$.

(ii) The proof is similar to that of item (i). □

### B. Similarity measure construction

In this paper we introduce a similarity measure between fuzzy multisets. Similarity measures have been deeply studied in the literature in a wide variety of applications [23]–[25]. Due to the diversity of elements to compare (e.g. images [26], sets [27] or concepts [28]), different axioms or properties may be required in the similarity measure construction.

In the case of punctual elements, Bustince *et al.* [17] considered the following axioms (Definition IV.4) taking into account the concept of similarity measure for fuzzy sets given by Xuecheng [29] also usually required in multivalued elements comparison (for example in image processing applications) [30]–[33].

**Definition IV.4.** *A similarity measure between fuzzy sets is a mapping $S : F(U) \times F(U) \to L$ such that:*

(S1) $S(\mathfrak{A},\mathfrak{B}) = S(\mathfrak{B},\mathfrak{A})$ *for all* $\mathfrak{A},\mathfrak{B} \in F(U)$.

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(S2) $S(\mathfrak{A},\mathfrak{B}) = 0$ *if and only if* $\{\mathfrak{A}(u),\mathfrak{B}(u)\} = \{0,1\}$ *for all* $u \in U$.
(S3) $S(\mathfrak{A},\mathfrak{B}) = 1$ *if and only if* $\mathfrak{A}(u) = \mathfrak{B}(u)$ *for all* $u \in U$.
(S4) *For all* $\mathfrak{A},\mathfrak{B},\mathfrak{C} \in F(U)$ *such that* $\mathfrak{A} \le \mathfrak{B} \le \mathfrak{C}$ *it holds* $S(\mathfrak{A},\mathfrak{C}) \le S(\mathfrak{A},\mathfrak{B})$ *and* $S(\mathfrak{C},\mathfrak{A}) \le S(\mathfrak{C},\mathfrak{B})$.

There are several ways to construct Similarity measures as introduced in Definition IV.4, such as the following, based on REFs.

**Proposition IV.5.** *Let* $R : L\times L \to L$ *be a REF and* $G : L^n \to L$ *an aggregation function satisfying* (Giff1) *and* (Giff2). *The function* $S : F(U) \times F(U) \to L$ *given by*

$$S(\mathfrak{A},\mathfrak{B}) = G(R(\mathfrak{A}(u_1),\mathfrak{B}(u_1)),\ldots,R(\mathfrak{A}(u_n),\mathfrak{B}(u_n))) \tag{12}$$

*is a similarity measure between fuzzy sets.*

Extending this work, we define a similarity measure as follows.

**Remark 1.** *We formulate two types of axioms according to the different ways of understanding complementarity in the settings of* $n$*-tuples. This is in line with the different axioms* ($RL2_1$) *and* ($RL2_2$) *in the definitions of REFs on* $L^n$ *of type 1 and 2, as well as with respect to the antecedents in axioms* ($RL4_1$) *and* ($RL4_2$)*, being* ($RL4_1$) *more restrictive in comparison with* ($RL4_2$) *as well as occurring in axiom* ($RL2_1$) *comparing with* ($RL2_2$)*. These considerations allow to introduce two different types of similarity measures between fuzzy multisets.*

**Definition IV.6.** *Let* $U$ *be a non-empty set. A function* $S_{L_1} : LF(U)\times LF(U) \to L^n$ *is called a similarity measure between two fuzzy multisets of type 1, if it satisfies the following axioms according to the definition of REF on* $L^n$ *of type 1:*

($SL1_1$) $S_{L_1}(\mathcal{A},\mathcal{B}) = \mathbf{1}$ *iff* $\mathcal{A} = \mathcal{B}$.
($SL2_1$) $S_{L_1}(\mathcal{A},\mathcal{B}) = \mathbf{0}$ *iff* $\mathcal{A}$ *and* $\mathcal{B}$ *are* $\{\mathcal{A}(u),\mathcal{B}(u)\} = \{\mathbf{0},\mathbf{1}\}$*, for all* $u \in U$.
($SL3_1$) $S_{L_1}(\mathcal{A},\mathcal{B}) = S_{L_1}(\mathcal{B},\mathcal{A})$ *for all* $\mathcal{A}$ *and* $\mathcal{B} \in LF(U)$.
($SL4_1$) *If* $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{D}$ *for all* $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D} \in LF(U)$*, then* $S_{L_1}(\mathcal{A},\mathcal{D}) \le_P S_{L_1}(\mathcal{B},\mathcal{C})$.

**Definition IV.7.** *Let* $U$ *be a non-empty set. A function* $S_{L_2} : LF(U)\times LF(U) \to L^n$ *is called a similarity measure between two fuzzy multisets of type 2, if it satisfies the following axioms according to the definition of REF on* $L^n$ *of type 2:*

($SL1_2$) $S_{L_2}(\mathcal{A},\mathcal{B}) = \mathbf{1}$ *iff* $\mathcal{A} = \mathcal{B}$.
($SL2_2$) $S_{L_2}(\mathcal{A},\mathcal{B}) = \mathbf{0}$ *iff* $\mathcal{A}(u) \in \{0,1\}^n$ *and* $\mathcal{B}(u) = \mathbf{1} - \mathcal{A}(u)$ *for all* $u \in U$ *(*$\mathcal{A}$ *and* $\mathcal{B}$ *are complementary crisp sets).*
($SL3_2$) $S_{L_2}(\mathcal{A},\mathcal{B}) = S_{L_2}(\mathcal{B},\mathcal{A})$ *for all* $\mathcal{A}$ *and* $\mathcal{B} \in LF(U)$.
($SL4_2$) *If* $\min(\mathcal{A}(u)_i,\mathcal{D}(u)_i) \le \mathcal{B}(u)_i$ *and* $\max(\mathcal{A}(u)_i,\mathcal{D}(u)_i) \ge \mathcal{C}(u)_i$*, for all* $u \in U$ *and for all* $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D} \in LF(U)$*, then* $S_{L_2}(\mathcal{A},\mathcal{D}) \le_P S_{L_2}(\mathcal{B},\mathcal{C})$ *(where* $\mathcal{A}(u)_i$ *denotes the* $i$*th coordinates of a* $n$*-tuple* $\mathcal{A}(u)$*).*

Next, we introduce a construction method for similarity measures between fuzzy multisets based on the aggregation of REFs on $L^n$. We also prove that we obtain a similarity measure of type 1 when aggregating REFs on $L^n$ of type 1 and a similarity measure of type 2 when aggregating REFs on $L^n$ of type 2.

**Theorem IV.8.** *Let* $m$ *be a positive integer and* $U = \{u_1,\ldots,u_m\}$*. Let* $R_{L_1} : L^n \times L^n \to L^n$ *be a REF on* $L^n$ *of type 1 and* $R_{L_2} : L^n \times L^n \to L^n$ *be a REF on* $L^n$ *of type 2. Let* $G_L : (L^n)^m \to L^n$ *be an* $m$*-ary aggregation on* $L^n$ *such that, for all* $X_1,\ldots,X_m \in L^n$*,* $G_L(X_1,\ldots,X_m) = \mathbf{0}$ *implies* $X_1 = \ldots = X_m = \mathbf{0}$ *and* $G_L(X_1,\ldots,X_m) = \mathbf{1}$ *implies* $X_1 = \ldots = X_m = \mathbf{1}$*. Then it holds:*

(i) $S_{L_1} : LF(U) \times LF(U) \to L^n$ *given by*

$$S_{L_1}(\mathcal{A},\mathcal{B}) = G_L\left(R_{L_1}(\mathcal{A}(u_1),\mathcal{B}(u_1)),\ldots,R_{L_1}(\mathcal{A}(u_m),\mathcal{B}(u_m))\right), \tag{13}$$

*is a similarity measure between fuzzy multisets of type 1.*

(ii) $S_{L_2} : LF(U) \times LF(U) \to L^n$ *given by*

$$S_{L_2}(\mathcal{A},\mathcal{B}) = G_L\left(R_{L_2}(\mathcal{A}(u_1),\mathcal{B}(u_1)),\ldots,R_{L_2}(\mathcal{A}(u_m),\mathcal{B}(u_m))\right), \tag{14}$$

*is a similarity measure between fuzzy multisets of type 2.*

*Proof.* (i) ($SM1_1$), ($SM2_1$), ($SM3_1$) and ($SM4_1$) follow from Eq. (13), the assumptions on $G_L$ and ($RL1_1$), ($RL2_1$), ($RL3_1$) and ($RL4_1$) respectively.

(ii) The proof is similar to that of item (i). □

Considering the construction of REFs on $L^n$ of type 1 and 2 (Thm. III.2 and Thm. III.5 respectively) and the aggregation functions on $L^n$ (Thm. IV.2), we obtain a construction method for similarity measures of type 1 and type 2 on $LF(U)$ respectively.

**Remark 2.** *By Eqs. 13 and 14 we obtain a wide class of similarity measures of type 1 and 2 on* $LF(U)$*. For both we follow the next procedure:*

- *Choose* $n$*-ary WAMs* $\omega_i : L^n \to L$*, such that the vectors are linearly independent and there exists* $k \in \{1,\ldots,n\}$ *with* $w_{kj} \ne 0$ *for all* $j \in \{1,\ldots,n\}$.
- *Choose a sequence* $\mathbf{R} = (R_1,\ldots,R_n)$ *of REFs on* $L$ *and construct:*
  (i) *REF on* $L^n$ *of type 1 given by Eq.* (3)*, if the goal is to construct a similarity measure of type 1.*
  (ii) *REF on* $L^n$ *of type 2 given by Eq.* (8)*, if the goal is to construct a similarity measure of type 2.*
- *Choose a sequence* $\mathbf{G} = (G_1,\ldots,G_n)$ *of* $m$*-ary aggregation functions on* $L$ *such that* $G_i$ *satisfies* (Giff1) *and* (Giff2) *for all* $\{1,\ldots,n\}$*. Construct the* $m$*-ary aggregation function* $G_L$ *on* $L^n$ *given by Eq.* (11)*. Some examples of aggregation functions* $G_i$ *satisfying the above assumptions are the arithmetic mean, a WAM with nonzero weights or the Choquet integral [34] with respect to a fuzzy measure* $\mu$ *on* $\{1,\ldots,m\}$*, satisfying* $\mu(A) = 0$ *iff* $A = \emptyset$ *and* $\mu(A) = 1$ *iff* $A = \{1,\ldots,m\}$.

Two examples of the application of similarity measures between fuzzy multisets in images are presented below.

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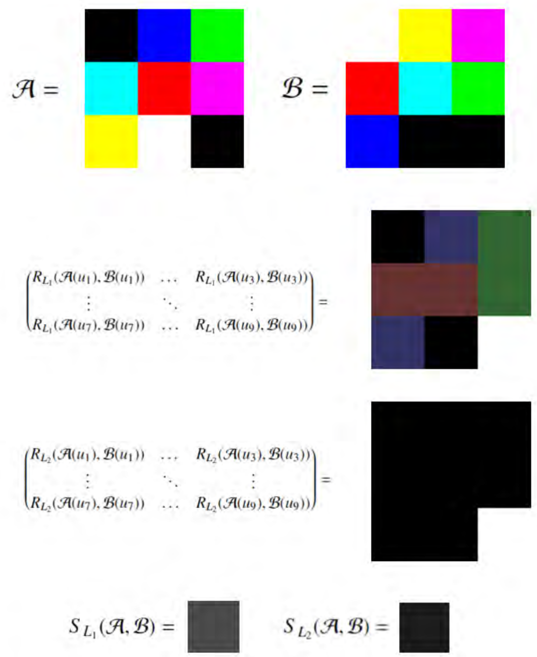


Fig. 1: Comparison between two images using similarity measure between multisets. A and B represent 3x3 pixel images, which have been amplified for visualization ease.

**Example IV.9.** *We have the following matrices $A$ and $B$:*

$$A = \begin{pmatrix} (0,0,0) & (0,0,1) & (0,1,0) \\ (0,1,1) & (1,0,0) & (1,0,1) \\ (1,1,0) & (1,1,1) & (0,0,0) \end{pmatrix} \quad (15)$$

$$B = \begin{pmatrix} (1,1,1) & (1,1,0) & (1,0,1) \\ (1,0,0) & (0,1,1) & (0,1,0) \\ (0,0,1) & (0,0,0) & (0,0,0) \end{pmatrix} \quad (16)$$

*These $A, B$ matrices can be considered as multisets where each of the elements are degrees of membership of fuzzy multisets:*

$$\mathcal{A} = \{(u_1, (0,0,0), \dots, (u_9, (0,0,0)\}$$
$$\mathcal{B} = \{(u_1, (1,1,1), \dots, (u_9, (0,0,0)\}$$

*In this sense, these multisets are shown as images (Fig. 1). Notice that 8 out of 9 values of both A and B represent complementary colours, while the remaining element is equal. That is, 8 out of 9 elements are crisp complementary fuzzy multisets, while the remaining one is equal. In order to compare the images, we apply the Restricted Equivalence Functions on $L^n$ pixel-wise, i.e., REFs on $L^3$. Then, the values obtained by comparing each pixel are fused with a function $G_L : (L^3)^m \to L^3$.*

*Taking into account the Remark 2, we first choose 3 3-ary WAMs. The matrix $W$ representing the linearly independent vectors is the following.*

$$W = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{pmatrix} \quad (17)$$

*We choose a sequence $\mathbf{R} = (R_1, R_2, R_3)$ of REFs on L, where $R_i(x,y) = 1 - |x-y|$, for all $i \in \{1,2,3\}$ and we construct a REF on $L^n$ of type 1 given by Eq. (3) to construct a similarity measure of type 1 given by Eq. (13) and a REF on $L^n$ of type 2 given by Eq. (8) to construct a similarity measure of type 2 given by Eq. (14).*

*Finally, we choose a sequence $\mathbf{G} = (G_1, G_2, G_3)$ of $m$-ary aggregation functions where $G_i(x_1, \dots, x_m) = \frac{1}{m}\sum_{k=1}^{m} x_k$ for all $i \in \{1,2,3\}$ in order to construct an $m$-ary aggregation function $G_L$ on $L^3$. Given $X_1, \dots, X_m \in L^3$ where $X_i = (x_{i1}, x_{i2}, x_{i3})$ for all $i \in \{1, \dots, m\}$, we use the expression $G_L(X_1, \dots, X_m) = (G_1(x_{11}, \dots, x_{m1}), G_2(x_{12}, \dots, x_{m2}), G_3(x_{13}, \dots, x_{m3}))$.*

*The images representing the values obtained in the application of the restricted equivalence functions on $L^3$ of type 1 and type 2, are shown in Fig. 1. In the following, the similarity between these multisets is calculated by the application of $G_L$ to the values obtained by using $R_{L_1}$ and $R_{L_2}$, which is also shown in Fig. 1.*

$$S_{L_1}(\mathcal{A}, \mathcal{B}) = \left(\frac{13}{45}, \frac{13}{45}, \frac{13}{45}\right), \qquad S_{L_2}(\mathcal{A}, \mathcal{B}) = \left(\frac{1}{9}, \frac{1}{9}, \frac{1}{9}\right)$$

*As previously introduced, by definition, $R_{L_1}(X,Y) = \mathbf{0}$ iff $\{X,Y\} = \{\mathbf{0}, \mathbf{1}\}$ and $R_{L_2}(X,Y) = \mathbf{0}$ iff $\{x_i, y_i\} = \{0,1\}$. This difference in the axiomatisation of each of the definitions can be clearly seen by comparing the two results of this example. When we use a REF on $L^n$ of type 1 in the comparison of complementary colours, this function is only $\mathbf{0}$ (black) when we compare white ($\mathbf{1}$) versus black ($\mathbf{0}$). On the other hand, when we use a REF on $L^n$ of type 2, all comparisons between complementary colours are worth $\mathbf{0}$ (black), such as the comparison between red $(1,0,0)$ and cyan $(0,1,1)$. As can be seen, this has an impact on the value of each type of similarity measure between multisets.*

**Example IV.10.** *In this case, we compare a noisy version of that same image (Fig. 2). Applying the two types of REFs on $L^3$ we obtain the results shown in Fig. 2. The different similarity measures between multisets are the following:*

$$S_{L_1}(\mathcal{A}, \mathcal{B}) = (0.5340, 0.7668, 0.5332)$$
$$S_{L_2}(\mathcal{A}, \mathcal{B}) = (0.5008, 0.8334, 0.4997)$$

*The image represented by the multiset $\mathcal{B}$ is noisy with a filter with a magenta-dominated component ($(1,0,1)$). As we can see, in both types of similarity measurement between both images the green channel ($(0,1,0)$), which is the complementary colour to magenta, predominates.*

## V. Anisotropic Diffusion based on $L^n$-REFs

In this work, we intend to test whether the different parameterizations of $L^n$-REFs actually lead to variable results when incorporated into an already existing data-processing procedure. That is, whether the flexibility in $L^n$-REFs can be used to optimize the results in a data comparison-based procedure. In order to do so, we have selected as case of study the Perona-Malik Anisotropic diffusion (PMAD) model, a well-known image processing algorithm for content-aware smoothing. As most content-aware the smoothing strategies,

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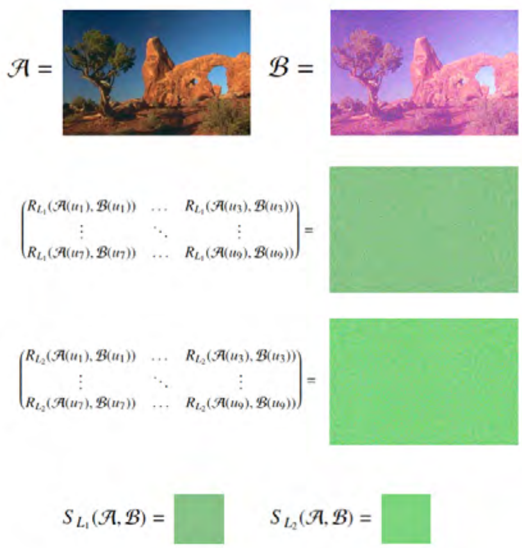


Fig. 2: Comparison between two images using similarity measure between multisets. An image is compared with a noisy version of itself. Both similarity measures return a "green" result as comparison, which indicates that the added noise has a predominant magenta component (green and magenta are complementary colours).

the PMAD model relies on semi-local data comparison to estimate the need for smoothing around each pixel of the image. Hence, its application to color images requires the design or selection of comparison operators for multivalued (color) data.

In section V-A, with first introduce the PMAD model, as it was initially designed for monochromatic images. Then, Section V-B present an evolution of the classical PMAD model to color images using $L^n$-REFs.

### A. Perona-Malik anisotropic diffusion

Image smoothing and regularization is a frequent task in different computer vision processes. It originally intended to produce a regularized version of the image to avoid problems due to noise and signal contamination. However, it soon became clear that such signal regularization had negative effects, mainly the blurring of object boundaries and the disappearance of small objects [35]. Hence, a large effort was devoted to regularization schemes able to regularize the image within objects, while avoiding the regularization at their boundaries. Such efforts rendered in the so-called content-aware smoothing techniques, examples being Bilateral Filtering [36] and Mean Shift regularization [37].

The PMAD model is a content-aware smoothing technique that, inspired by heat diffusion processes, is able to produce non-isotropic smoothing on an image [38]. This technique has been deeply studied in the literature [39], as well as evolved to more intricate models. Evolved models normally tackle either the construction of scale-spaces in the diffusion process [40], [41] or the use of more elaborate information in the diffusion process (see, e.g., the seminal work by Weickert [35]). In this work, we stick to the original definition in [38].

Let $I : \Omega \mapsto \mathbb{R}+$ be an image. In the PMAD model, each pixel is understood to be a body with as an amount of heat equal to its grayscale tone. In order to simulate the heat diffusion across the image, by Fick's equation, certain flux $\phi$ in a given image $I$ is modelled as

$$\phi = -D\nabla I \ , \tag{18}$$

where $D$ is a symmetric, positive definite matrix and $\nabla I$ is the local conductivity of $I$. At time $t$, assuming that image $I$ does not present heat loss, we have

$$\delta_t I = -\mathrm{div}\phi \ , \tag{19}$$

where div is the divergence operator. From the above equations, we can state that

$$\delta_t I = \mathrm{div}(D\nabla I) \ . \tag{20}$$

Perona and Malik defined an smoothing process over discrete time $t$ in an image, based on the idea that

$$\delta_t I = \mathrm{div}(g(|\nabla I|^2)\nabla I) \ , \tag{21}$$

where $g$ is a decreasing function that modulates the amount of heat transfer, which depends of the gradient magnitude. Perona and Malik developed a scalar approximation of Eq. (21), in order to avoid vectorial expressions and to adapt it to discrete scenarios as images. Vectorial expressions can indeed be found in, for example, [35]. According to Perona and Malik, the image $I$ evolves in time as

$$I_{i,j}^{[t+1]} = I_{i,j}^{[t]} + \lambda(z_N \cdot \Psi_N I_{i,j}^{[t]} + z_W \cdot \Psi_W I_{i,j}^{[t]} + z_E \cdot \Psi_E I_{i,j}^{[t]} + z_S \cdot \Psi_S I_{i,j}^{[t]}) \ , \tag{22}$$

where $\lambda \in ]0, 0.25]$ and $\Psi_\gamma$ in each one of the four directions is given by

$$\begin{aligned} \Psi_N I_{i,j} &= I_{i-1,j} - I_{i,j} \\ \Psi_W I_{i,j} &= I_{i,j-1} - I_{i,j} \\ \Psi_E I_{i,j} &= I_{i,j+1} - I_{i,j} \\ \Psi_S I_{i,j} &= I_{i+1,j} - I_{i,j} \ . \end{aligned} \tag{23}$$

The coefficient $z_\gamma$ is a conductivity coefficient that represents the amount of diffusion between each pixel and its environment, given by

$$z_\gamma = g\left(|\Psi_\gamma I_{i,}|\right) \ , \tag{24}$$

where $g$, as stated before, is a decreasing conductivity function and $\gamma$ represents each one of the four different directions. The authors in [38] propose as two different conductivity functions,

$$g_1(x) = e^{-\left(\frac{x}{K}\right)^2} \ , \tag{25}$$

and

$$g_2(x) = \frac{1}{1 + \left(\frac{x}{K}\right)^2} \ , \tag{26}$$

where $K$ is a threshold that limits the diffusion when $|\Psi_\gamma I_{i,j}| > K$.

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### B. Extending the PMAD model to color images

In this work, we present a generalization of the Perona and Malik anisotropic diffusion method [38] for color images. A previous generalization was presented in [42] for grayscale images using Restricted Dissimilarity Functions (RDF) [43], following a similar strategy to the one in this work.

Any content-aware smoothing technique requires some understanding of the local and semilocal information at each pixel or region of an image. It is precisely such understanding what determines how smoothing is locally performed. In the case of the PMAD model, this comes down to pixel-to-pixel comparison. Specifically, in the quantitative comparison of each pixel with its four-point neighbours. In [42], the scalar differences in Eq. (23) were replaced by RDFs. However, color representation requires a multivalued representation of the information at each pixel. Regardless of the color representation chosen for the image (RGB, CieLAB, HSV,...), it will render in a multivalued representation of color. Hence, in order to apply the PMAD model in color images, it is required to find a dissimilarity measure for multivalued elements. In this case, and benefiting from the fact that REFs take values in $[0, 1]$, we will create dissimilarity measures from $L^n$-REFs by inverting its results.

An evolution of the PMAD model able to cope with multichannel (color) images can be achieved by replacing Eq. (23) by:

$$\begin{aligned} \Psi_N I_{i,j} &= \operatorname{sign}(I_{i-1,j} - I_{i,j}) \cdot (\mathbf{1} - R_{L_1}(I_{i-1,j}, I_{i,j})) \\ \Psi_W I_{i,j} &= \operatorname{sign}(I_{i,j-1} - I_{i,j}) \cdot (\mathbf{1} - R_{L_1}(I_{i,j-1}, I_{i,j})) \\ \Psi_E I_{i,j} &= \operatorname{sign}(I_{i,j+1} - I_{i,j}) \cdot (\mathbf{1} - R_{L_1}(I_{i,j+1}, I_{i,j})) \\ \Psi_S I_{i,j} &= \operatorname{sign}(I_{i+1,j} - I_{i,j}) \cdot (\mathbf{1} - R_{L_1}(I_{i+1,j}, I_{i,j})) \end{aligned} \quad (27)$$

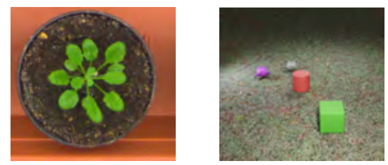
Remarkably, the main difference between Eqs. (23) and (27) is the fact that the former is devoted to monochannels, and hence applies to scalar values, while the latter is prepared for multichannel or color images.

## VI. Experimental setup

While designing comparison operators is similar to designing many other operators within the Fuzzy Set Theory, its quantitative validation poses a series of challenges absent in other types of operators. Specifically, it is hard to model, at a fine quantitative level, the expected behaviour of comparison operators. Ideally, we could compare the values yielded by a comparison operators to the quantitative comparisons performed by one or more humans. However, this is a practical impossibility, since humans are severely inconsistent in providing such quantitative values. While humans are relatively good at ranking or selecting, they can hardly provide consistent, meaningful quantitative evaluations of data similarity. In such a situation, it is necessary to find innovative ways to put comparison operators to the test.

In order to prove the suitability of the $L^n$-REFs as color image comparison measures, and to visually represent the differences between each of their parameterizations, we have applied our color-ready evolution of the PMAD model to two sample images. Moreover, for comparison purposes, we apply to the images two well-known content-aware smoothing techniques, MeanShift [44] and Bilateral [45] filtering.

Images have been extracted from two datasets. Firstly, the Plant Phenotyping Dataset [46], [47] which consists of a collection of real natural annotated images of plants with its segmented mask of leaves, its centers, and bounding boxes. In this work, we only use the leaf mask, which indicates its location (that we consider the object of the image). Secondly, we use the ClevrTxt dataset [48] that consists of a series of synthetic scenes-generated pictures with a large variety of objects, backgrounds, and textures, along with its associated segmented ground-truth image. The images selected for our experiments can be seen in Figure 3



Plant04-frame01 ClevrTex vbg 019810

Fig. 3: Color images from the Plant Phenotyping and ClevrTxt dataset.

The color-ready PMAD model has been configured with the following parameters:

$(i)$ REFs: $R_{L_1}$.
$(ii)$ Automorphisms: $\varphi_{i1}$ and $\varphi_{j2}$.
$(iv)$ Conductivity functions: $g_2$.
$(v)$ Constants: $\lambda$ and $K$.

Note also that the PMAD model is an iterative process with a trivial convergence state. Hence, it needs to be stopped at a certain number of iterations. Results will be displayed, for each combination of parameters, at different numbers of iterations, so as to illustrate the evolution of images w.r.t. such number of iterations.

Moreover, in addition to a visual evolution of the process, we propose to use a homogeneity measure [49] which has been proved helpful in image processing tasks like image segmentation. We use this homogeneity measure to get how similar are the colors in the neighborhood of a pixel, obtaining a homogeneity image that reflects where the regions are homogeneous and where there are relevant changes. Using the ground-truths provided by the datasets, we extract the homogeneity of both the objects and background present in the images (Figures 6, 7). The different homogeneity images of object and background values are obtained at certain iteration times to analyse its evolution.

In Figures 4-5 we present the anisotropic diffusion results for the example images of the Plant Phenotyping and ClevrTxt datasets in Fig. 3 applying as dissimilarity measure $RD_{L_1}$ along with its homogeneity images, and in Figures 6-7 we show the evolution of background and object homogeneity through the different iteration times. The experiments have been obtained with the conductivity function $g_2(x)$. We set $\varphi_{i1} = \{x^{1/2}, x\}$ and $\varphi_{j2} = \{x^{1/2}, x, x^2\}$, leading to four different combination of automorphisms. The $\lambda$ value is set throughout the experiments to $0.25$ and $K$ is set to $0.005$ so

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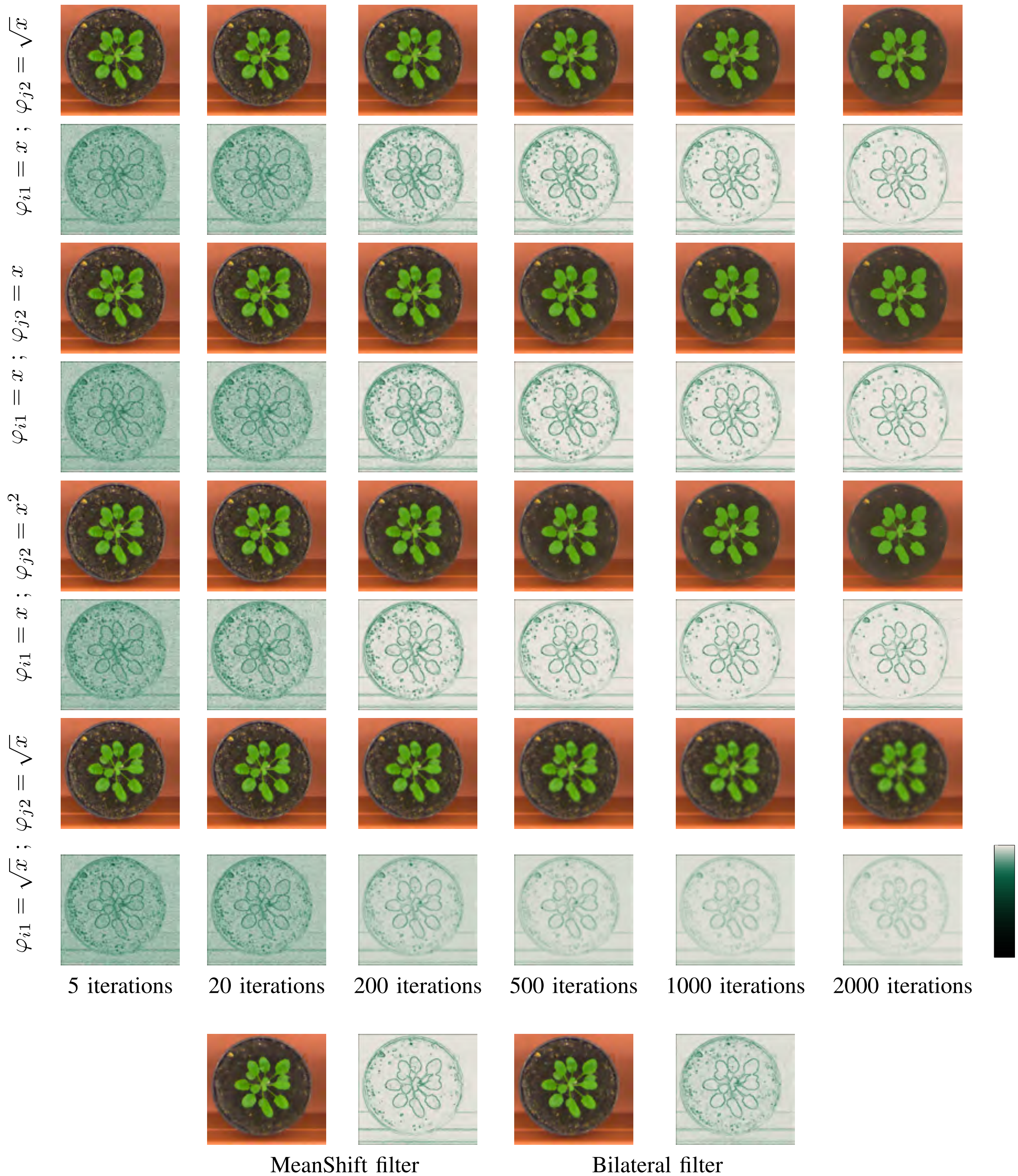


Fig. 4: Smoothing and Homogeneity over image plant04-frame01 with different configurations of our proposal along with the MeanShift and Bilateral filtering

that the diffusion speed remains slow. Results are replicated for 5, 20, 200, 500, 1000 and 2000 iterations, corresponding with each column in the figures.

We can see similar regularization results all along the different images. In Figure 4 we can see that most of the colour variations and textures are removed in the initial iterations, except for the last case (when both $\varphi_{i1}$ and $\varphi_{j2}$ are $\sqrt{x}$) where the regularization is blurry. In general the plant leafs are clearly separated and the edges preserved. Even between the bucket and the brown background. In the case of Figure 5, we see that in the initial steps there is a high variation in the tonal values of the background and our proposal manage to remove it preserving the object edges. Again, in the last combination of automorphisms the blurring effect remains. We clearly see the regularization process of both background and object in the odd rows where a brighter colour indicates the maximum homogeneity.

In terms of homogeneity we see in Figure 6 how both object and background begins at a similar homogeneity value and evolve through iteration on the same path until they stabilizes near the maximum value. In the particular case of the last combination of parameters (d) we can see that the

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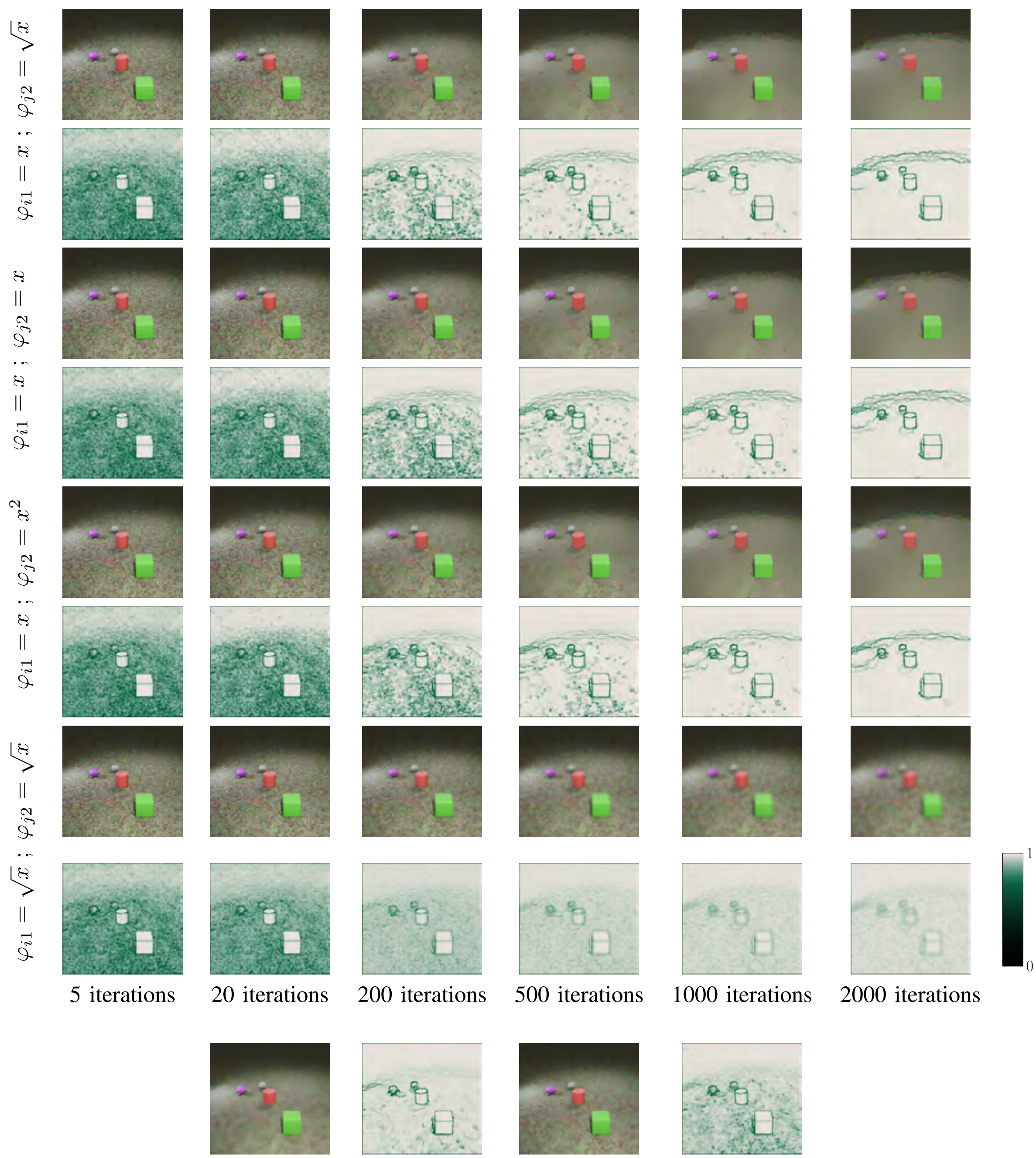


Fig. 5: Smoothing and Homogeneity over image ClevrTex vbg 019810 with different configurations of our proposal along with the MeanShift and Bilateral filtering

homogeneity values are not as high as the other approaches and the distance between object and background is not as easy to perceive. In Figure 7 the object homogeneity remain stable from the beginning of the iterative process and does not move, except for the last case where it becomes worse. In the case of the background we can observe that in the first iteration steps almost all the variations are removed and then the regions remain stable. Comparing the results we can state that in order to obtain a sharp-edge regularization $\varphi_{i1} = x^{1/2}$ must be avoided. Comparing our approach to those obtained with MeanShift and Bilateral filtering we see that the results are similar to those not using the $\varphi_{i1} = x^{1/2}$ automorphism. On the one hand, in terms of object smoothing these methods are equivalent to our proposal, which has a slightly better homogeneity value. On the other hand, background smoothing is worse than our method as it can be seen in the images of Fig. 4 and 5, specially when using the Bilateral filtering.

## VII. Conclusions

In this work we present a novel contribution to the restricted equivalence functions theory, an extension of the Restricted Equivalence Functions (REFs) on $L^n$. Specifically, we present

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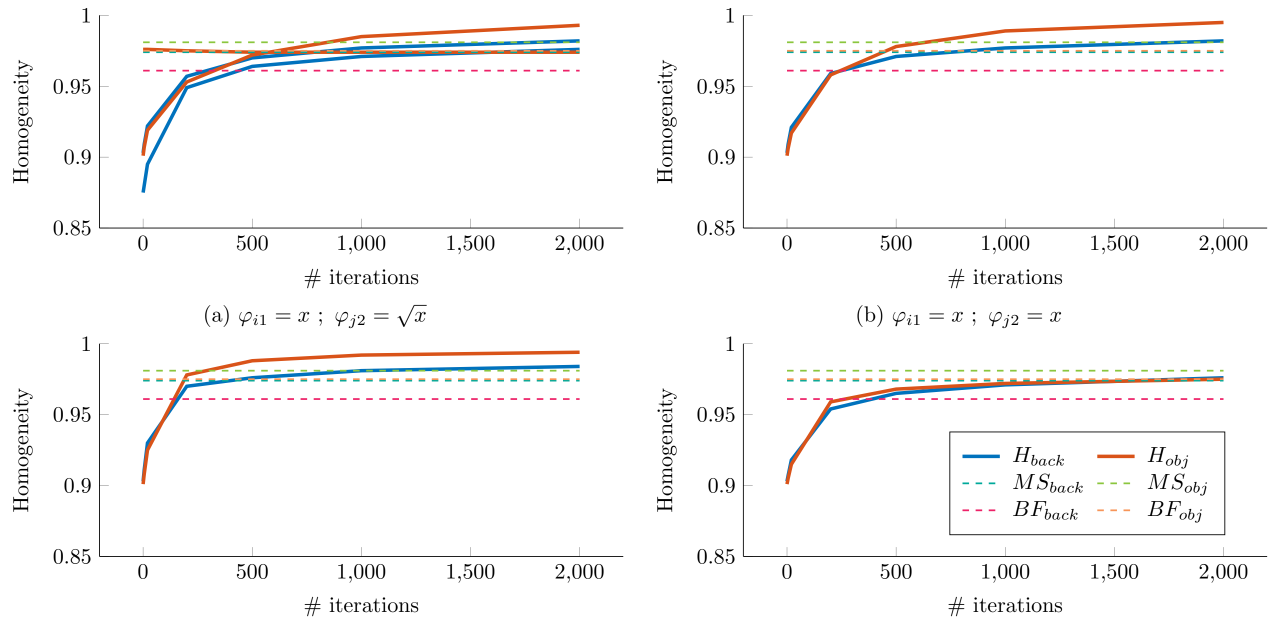


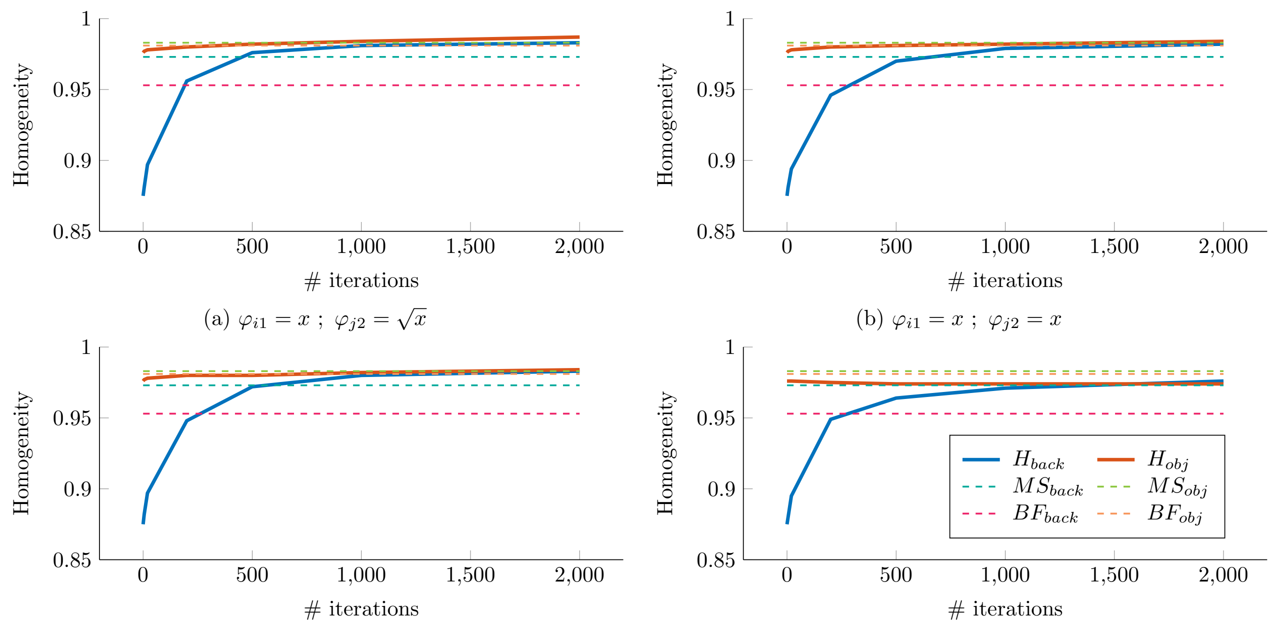

(a) $\varphi_{i1} = x$ ; $\varphi_{j2} = \sqrt{x}$

(b) $\varphi_{i1} = x$ ; $\varphi_{j2} = x$

(c) $\varphi_{i1} = x$ ; $\varphi_{j2} = x^2$

(d) $\varphi_{i1} = \sqrt{x}$ ; $\varphi_{j2} = \sqrt{x}$

Fig. 6: Homogeneity from object and background of image plant04-frame01




(a) $\varphi_{i1} = x$ ; $\varphi_{j2} = \sqrt{x}$

(b) $\varphi_{i1} = x$ ; $\varphi_{j2} = x$

(c) $\varphi_{i1} = x$ ; $\varphi_{j2} = x^2$

(d) $\varphi_{i1} = \sqrt{x}$ ; $\varphi_{j2} = \sqrt{x}$

Fig. 7: Homogeneity from object and background of image ClevrTex vbg 019810

a construction method based on automorphisms and simple algebraic linear operations. We also introduced an extended similarity measure between fuzzy multisets for multivalued elements.

We tested our proposal on an image processing application, anisotropic diffusion [38], a region-based smoothing technique. This technique applies a regularization based on the magnitude of the dissimilarity between neighboring pixels; in order to test the suitability of the REFs on $L^n$ to measure the similarity between two multivalued pixels, we adapted the expression in [38] to color images and replaced the difference between uni-valued pixels by Restricted Dissimilarity Functions (RDFs), a derived expression from the REFs.

As a conclusion, we can state that extended REFs on $L^n$ are a suitable comparison measure between multi-valued elements. We can also affirm that different parameterizations of these functions lead to distinct anisotropic diffusion results in the specific proposed application, verifying the sensitivity of these functions to the parameters settings and their suitability to different scenarios.

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In future work we want to explore more applications of the REFs on $L^n$, specifically on color image processing. Due to the shortage of measures of color image comparison taking into account the three channels at the same time, and not applying single channel measures, our proposed algorithm is a novel proposal with multiple applications. Also, we want to explore the creation of a specific color space for the comparison of two given images, exploiting the maximum information contained in both of them, obtained applying REFs.

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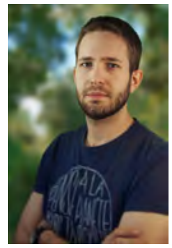

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