Information Fusion 97 (2023) 101815

Contents lists available at ScienceDirect

Information Fusion

journal homepage: www.elsevier.com/locate/inffus

Full length article

A framework for generalized monotonicity of fusion functions

Mikel Sesma-Sara [a,b,*], Adam Šeliga [c], Michał Boczek [d], LeSheng Jin [e], Marek Kaluszka [d], Martin Kalina [c], Humberto Bustince [a,b], Radko Mesiar [c,f]

[a] *Departamento de Estadística, Informática y Matemáticas, Universidad Pública de Navarra (UPNA), Campus Arrosadia, 31006, Pamplona, Spain*
[b] *Institute of Smart Cities (ISC), Universidad Pública de Navarra (UPNA), Campus Arrosadia, 31006, Pamplona, Spain*
[c] *Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, Bratislava, Slovakia*
[d] *Institute of Mathematics, Lodz University of Technology, 90-924 Lodz, Poland*
[e] *School of Business, Nanjing Normal University, Nanjing, China*
[f] *Institute for Research and Applications of Fuzzy Modelling, University of Ostrava, 30. dubna 22, Ostrava 1, Czech Republic*

A R T I C L E I N F O

Keywords:
Fusion function
Aggregation function
Monotonicity
Generalized monotonicity
Sentiment analysis
Text classification

A B S T R A C T

The relaxation of the property of monotonicity is a trend in the theory of aggregation and fusion functions and several generalized forms of monotonicity have been introduced, most of which are based on the notion of directional monotonicity. In this paper, we propose a general framework for generalized monotonicity that encompasses the different forms of monotonicity that we can find in the literature. Additionally, we introduce various new forms of monotonicity that are not based on directional monotonicity. Specifically, we introduce dilative monotonicity, which requires that the function increases when the inputs have increased by a common factor, and a general form of monotonicity that is dependent on a function T and a subset of the domain Z. This two new generalized monotonicities are the basis to propose a set of different forms of monotonicity. We study the particularities of each of the new proposals and their links to the previous relaxed forms of monotonicity. We conclude that the introduction of dilative monotonicity complements the conditions of weak monotonicity for fusion functions and that (T, Z)-monotonicity yields a condition that is slightly stronger than weak monotonicity. Finally, we present an application of the introduced notions of monotonicity in sentiment analysis.

1. Introduction

The task of representing a collection of numerical data by a single number is common in many data science processes. Historically, several classes of functions have been proposed to address the problem, among which means are a prominent example [1]. Nowadays, this task is approached by aggregation functions, which, since the last decades of the past century, have become a theory of study of their own [2–4]. To specify, an aggregation function A is a function that takes n values from the real unit interval, outputs a number in the same interval and satisfies some fundamentals: it must satisfy the boundary conditions $A(0, \dots, 0) = 0$ and $A(1, \dots, 1) = 1$ and it must be increasing with respect to all its arguments. Aggregation functions are an active research topic, both from the theoretical and applied perspectives [5–7].

One of the tendencies in the theory of aggregation is the relaxation of the monotonicity condition [8]. The existence of functions that do not satisfy the monotonicity property but are undoubtedly valid to fuse information, such as the Lehmer mean [9] or the mode function, motivated the introduction of weaker forms of monotonicity. The first attempt was the introduction of weak monotonicity [10], which focuses on the case in which all the inputs increase by the same amount. Then, this concept was generalized by directional monotonicity [11] by considering increasingness along a fixed real direction $\mathbf{r} \in \mathbb{R}^n$. Later on, directional monotonicity has become the basis of several new relaxed forms of monotonicity that are, to some extent, relative to directions. For example, cone monotonicity deals with monotonicity along all the directions within a cone [12], ordered directional [13] and strengthened ordered directional monotonicity [14] deal with monotonicity according to a direction that is dependent of the relative order of the inputs, pointwise directional monotonicity [15] studies directional monotonicity from a local perspective and $(\mathcal{F}, \mathcal{V})$-monotonicity studies monotonicity with respect to a family of vectors [16]. Some of these notions of monotonicity have been applied to computer vision [17] and

* Corresponding author at: Departamento de Estadística, Informática y Matemáticas, Universidad Pública de Navarra (UPNA), Campus Arrosadia, 31006, Pamplona, Spain.
E-mail addresses: mikel.sesma@unavarra.es (M. Sesma-Sara), seliga@math.sk (A. Šeliga), michal.boczek.1@p.lodz.pl (M. Boczek), jls1980@163.com (L. Jin), marek.kaluszka@p.lodz.pl (M. Kaluszka), kalina@math.sk (M. Kalina), bustince@unavarra.es (H. Bustince), mesiar@math.sk (R. Mesiar).

https://doi.org/10.1016/j.inffus.2023.101815
Received 11 July 2022; Received in revised form 17 March 2023; Accepted 17 April 2023
Available online 23 April 2023
1566-2535/

fuzzy rule based classification systems [18,19] and, currently, the relaxation of monotonicity is still a trend in the theory of aggregation [20–22].

In this work, our objective is twofold. On the one hand, we aim at obtaining a formal generalization of monotonicity for n-ary fusion functions $F : [0,1]^n \to [0,1]$, that extends the relaxed forms of monotonicity that can be found in the literature. On the other hand, we propose some new forms of relaxed monotonicity that are not based on directional monotonicity and do not necessarily rely on a direction. Specifically, we study the situation in which the inputs of the function have increased by a constant factor. This originates the concept of dilative monotonicity, which, in turn, is the basis for reversed dilative and directional dilative monotonicity. We also propose a more general form of monotonicity, its relation to the rest of forms of monotonicity and its potential role in applications. Moreover, we illustrate the applicability of the proposed types of monotonicity in an example of a sentiment analysis (text classification) problem.

This work is organized in the following manner. In Section 2 we recall the definitions of some of the forms of monotonicity that we can find in the literature. In Section 3 we provide the general framework for generalized monotonicity and we relate it to the existing forms of relaxed monotonicity. In Section 4 we introduce the concepts of dilative, reversed dilative and directional dilative monotonicity and study their properties. In Section 5 we present a general form of monotonicity that relies on a function T and a subset $Z \subset [0,1]^n$, we study its properties paying special attention to the case in which T is defined in terms of a binary disjunctive aggregation function and a binary conjunctive aggregation function. We finish this work with some conclusions of our findings and a remark about future works in Section 7.

2. Preliminaries

In this section we recall the definition of an aggregation function as well as the formal definitions of the different forms of relaxed monotonicity in the literature.

Let us start with the concept of standard monotonicity. From here on, let $n \in \mathbb{N} = \{1,2,\dots\}$.

Definition 2.1. A function $F : [0,1]^n \to [0,1]$ is increasing (resp. decreasing), whenever for all $\mathbf{x},\mathbf{y} \in [0,1]^n$ such that $x_i \le y_i$ for all $i \in \{1,\dots,n\}$, it holds that $F(\mathbf{x}) \le F(\mathbf{y})$ (resp. $F(\mathbf{x}) \ge F(\mathbf{y})$).

In this work we deal with fusion functions. In general, a fusion function can be seen as a n-ary grupoid, assigning to n real values from $[0,1]$ a unique output from the same interval. Whenever certain additional properties are fulfilled, we can recall the concept of an aggregation function.

Definition 2.2. A fusion function $A : [0,1]^n \to [0,1]$ is an aggregation function if the following conditions hold:

- $A(\mathbf{0}) = 0$, where $\mathbf{0} = (0,\dots,0)$;
- $A(\mathbf{1}) = 1$, where $\mathbf{1} = (1,\dots,1)$;
- A is increasing.

Additionally, we say that an aggregation function A is conjunctive if $A(\mathbf{x}) \le \min(\mathbf{x})$ for all $\mathbf{x} \in [0,1]^n$, it is disjunctive if $A(\mathbf{x}) \ge \max(\mathbf{x})$ and it is averaging if $\min(\mathbf{x}) \le A(\mathbf{x}) \le \max(\mathbf{x})$ for all $\mathbf{x} \in [0,1]^n$.

A function that satisfies the boundary condition is known as a semi-aggregation function [23].

Definition 2.3 (*[23]*). A function $F : [0,1]^n \to [0,1]$ is a semi-aggregation function if

- $F(\mathbf{0}) = 0$;
- $F(\mathbf{1}) = 1$.

In this work, we deal with functions that meet the boundary conditions, i.e., they are semi-aggregation functions, and also satisfy some kind of relaxed form of monotonicity.

Weak monotonicity was the first attempt at relaxing the monotonicity condition for aggregation functions. Thus, it is less restrictive than standard monotonicity but ensures that the output increases whenever all the inputs have increased by the same amount.

Definition 2.4 (*[10]*). A function $F : [0,1]^n \to [0,1]$ is weakly increasing (resp. weakly decreasing), if for all $c > 0$ and $(x_1,\dots,x_n) \in [0,1]^n$ such that $0 \le x_i + c \le 1$ for all $i \in \{1,\dots,n\}$, it holds that $F(x_1,\dots,x_n) \le F(x_1+c,\dots,x_n+c)$ (resp. $F(x_1,\dots,x_n) \ge F(x_1+c,\dots,x_n+c)$).

This property studies monotonicity along the direction defined by the fixed vector $\mathbf{1}$. Directional monotonicity arose from considering an arbitrary vector $\mathbf{0} \ne \mathbf{r} \in \mathbb{R}^n$.

Definition 2.5 (*[11]*). Let $\mathbf{0} \ne \mathbf{r} \in \mathbb{R}^n$. A function $F : [0,1]^n \to [0,1]$ is $\mathbf{r}$-increasing (resp. $\mathbf{r}$-decreasing), if for all $c > 0$ and $\mathbf{x} \in [0,1]^n$ such that $\mathbf{x} + c\mathbf{r} \in [0,1]^n$, it holds that $F(\mathbf{x}) \le F(\mathbf{x}+c\mathbf{r})$ (resp. $F(\mathbf{x}) \ge F(\mathbf{x}+c\mathbf{r})$).

Weak and directional monotonicity refer to a function property, but they can be also studied in a pointwise manner, i.e., focusing on a specific point of the domain.

Definition 2.6 (*[15]*). Let $\mathbf{0} \ne \mathbf{r} \in \mathbb{R}^n$. A function $F : [0,1]^n \to [0,1]$ is $\mathbf{r}$-increasing (resp. $\mathbf{r}$-decreasing) at $\mathbf{x}$, if for all $c > 0$ such that $\mathbf{x} + c\mathbf{r} \in [0,1]^n$, it holds that $F(\mathbf{x}) \le F(\mathbf{x}+c\mathbf{r})$ (resp. $F(\mathbf{x}) \ge F(\mathbf{x}+c\mathbf{r})$).

This pointwise condition can be used to characterize the general condition of directional monotonicity (and other types of monotonicity, see [15]).

There are some other relaxed forms of monotonicity that rely on increasingness with respect to more than a single vector. For example, the concept of cone monotonicity [12] was originally presented for positive cones $C \subset (\mathbb{R}_+)^n$. In general, it can be defined for any cone $C \subset \mathbb{R}^n$.

Definition 2.7 (*[12]*). Let $C \subset \mathbb{R}^n$ be a nonempty cone. A function $F : [0,1]^n \to [0,1]$ is cone increasing (resp. cone decreasing) with respect to C if F is directionally increasing (resp. directionally decreasing) with respect to any direction $\mathbf{r} \in C$.

Moreover, directional monotonicity has been further generalized by $(\mathcal{F},\mathcal{V})$-monotonicity [16], which is defined with respect to a family of functions $\mathcal{F}$ and a family of vectors $\mathcal{V}$.

Definition 2.8 (*[16]*). Let $\mathcal{F} = \{g_j : D \to [0,1] \mid D \subseteq [0,1]^2 \text{ and } j \in \{1,\dots,n\}\}$ be a family of functions and $\mathcal{V} \subseteq [0,1]^n \setminus \{\mathbf{0}\}$ a family of vectors. A function $F : [0,1]^n \to [0,1]$ is $(\mathcal{F},\mathcal{V})$-increasing (resp. $(\mathcal{F},\mathcal{V})$-decreasing) if for every $\mathbf{r} = (r_1,\dots,r_n) \in \mathcal{V}$ it holds that

$$F(g_1(r_1,x_1),\dots,g_n(r_n,x_n)) \ge F(x_1,\dots,x_n)$$

(resp. $F(g_1(r_1,x_1),\dots,g_n(r_n,x_n)) \le F(x_1,\dots,x_n)$), where $g_j \in \mathcal{F}$ and $(r_i,x_i) \in D$ for any i.

Other forms of monotonicity, such as OD, SOD monotonicity or directional monotonicity in a more general framework, can be found in [13,14,22].

Additionally, let us recall the concept of positive homogeneity.

Definition 2.9. A function $F : \mathbb{R}^n \to \mathbb{R}$ is positively homogeneous if for any $\alpha \ge 0$ and any $\mathbf{x} \in [0,1]^n$, it holds that $F(\alpha\mathbf{x}) = \alpha F(\mathbf{x})$.

3. Generalized monotonicity

In this section, we provide a formal generalization of all the defined types of monotonicity for n-ary fusion functions of the form $F : [0,1]^n \to [0,1]$. To that end, we define monotonicity with respect to a binary relation $\mathcal{R}$ on $[0,1]^n$.

Definition 3.1. Let $\mathcal{R} \subset [0,1]^n \times [0,1]^n$ be a binary relation. A function $F : [0,1]^n \to [0,1]$ is $\mathcal{R}$-increasing (resp. $\mathcal{R}$-decreasing) if for any $(\mathbf{x}, \mathbf{y}) \in \mathcal{R}$ it holds that $F(\mathbf{x}) \le F(\mathbf{y})$ (resp. $F(\mathbf{x}) \ge F(\mathbf{y})$).

Definition 3.1 provides a framework to generalize all the aforementioned forms of monotonicity. For instance, setting

$$\mathcal{R} = \{(\mathbf{x}, \mathbf{y}) \in [0,1]^n \times [0,1]^n \mid x_i \le y_i \ \forall i \in \{1, \dots, n\}\},$$

we recover standard monotonicity (see Definition 2.1).

Remark 3.2. The following are two special cases of binary relation:

- the universal relation: $\mathcal{R}^* = [0,1]^n \times [0,1]^n$;
- the empty relation: $\mathcal{R}_* = \emptyset$.

Note that, on the one hand, a function $F : [0,1]^n \to [0,1]$ is $\mathcal{R}^*$-increasing if and only if F is a constant function, i.e., $F(\mathbf{x}) = c \in [0,1]$ for all $\mathbf{x} \in [0,1]^n$. The same result holds for $\mathcal{R}^*$-decreasingness.

On the other hand, all functions $F : [0,1]^n \to [0,1]$ are trivially $\mathcal{R}_*$-increasing and $\mathcal{R}_*$-decreasing.

Similarly, all functions $F : [0,1]^n \to [0,1]$ are trivially $\mathcal{R}$-increasing and $\mathcal{R}$-decreasing for $\mathcal{R} = \{(\mathbf{x}, \mathbf{x}) \mid \mathbf{x} \in [0,1]^n\}$.

Moreover, if we consider two relations $\mathcal{R}_1$ and $\mathcal{R}_2$ such that $\mathcal{R}_1 \subset \mathcal{R}_2$, then every fusion function that is $\mathcal{R}_2$-increasing is also $\mathcal{R}_1$-increasing.

The next results show how Definition 3.1 generalizes the rest of the discussed forms of monotonicity.

Proposition 3.3. *Let $F : [0,1]^n \to [0,1]$. Then, the following items hold:*

(a) F is weakly increasing if and only if F is $\mathcal{R}$-increasing with

$$\mathcal{R} = \{(\mathbf{x}, \mathbf{y}) \in [0,1]^n \times [0,1]^n \mid \mathbf{y} = \mathbf{x} + (c, \dots, c) \ for\ some\ c \ge 0\}.$$

(b) Let $\mathbf{0} \neq \mathbf{r} \in \mathbb{R}^n$. F is $\mathbf{r}$-increasing if and only if F is $\mathcal{R}$-increasing with

$$\mathcal{R} = \{(\mathbf{x}, \mathbf{y}) \in [0,1]^n \times [0,1]^n \mid \mathbf{y} = \mathbf{x} + c\mathbf{r} \ for\ some\ c \ge 0\}.$$

(c) Let $C \subseteq \mathbb{R}^n$ be a nonempty cone. F is cone increasing with respect to C if and only if F is $\mathcal{R}$-increasing with

$$\mathcal{R} = \{(\mathbf{x}, \mathbf{y}) \in [0,1]^n \times [0,1]^n \mid \mathbf{y} = \mathbf{x} + c\mathbf{r} \ for\ some\ c \ge 0\ and\ some\ \mathbf{r} \in C\}.$$

(d) Let $\mathcal{F} = \{g_j : D \to [0,1] \mid D \subseteq [0,1]^2$ and $j \in \{1, \dots, n\}\}$ be a family of functions and $\mathcal{V} \subseteq [0,1]^n \setminus \{\mathbf{0}\}$ a family of vectors. F is $(\mathcal{F}, \mathcal{V})$-increasing if and only if F is $\mathcal{R}$-increasing with

$$\mathcal{R} = \{(\mathbf{x}, \mathbf{y}) \in [0,1]^n \times [0,1]^n \mid \mathbf{y} = \left(g_1(r_1, x_1), \dots, g_n(r_n, x_n)\right) \ for\ some\ \mathbf{r} \in \mathcal{V}\ where\ (r_i, x_i) \in D\ for\ any\ i\}.$$

Proof. Items (a)–(d) are straightforward from Definition 3.1 and from Definitions 2.4, 2.5, 2.7 and 2.8, respectively. □

Note that Proposition 3.3 holds similarly for the cases of decreasingness instead of increasingness, taking into account each type of monotonicity.

4. Some new classes of generalized monotonicity based on dilation

Besides defining a framework for all the relaxed forms of monotonicity that can be found in the literature, Definition 3.1 enables to define some new classes of relaxed monotonicity.

4.1. Dilation based monotonicity

We propose a monotonicity condition that is defined whenever the inputs are dilated by a factor.

Definition 4.1. A function $F : [0,1]^n \to [0,1]$ is dilative increasing (resp. decreasing) if it is $\mathcal{R}_d$-increasing (resp. $\mathcal{R}_d$-decreasing) for

$$\mathcal{R}_d = \{(\mathbf{x}, \mathbf{y}) \in [0,1]^n \times [0,1]^n \mid \mathbf{y} = c\mathbf{x} \text{ for some } c \ge 1 \text{ or } \mathbf{x} = \mathbf{0}\}.$$

Clearly, if a function $F : [0,1]^n \to [0,1]$ is increasing, then it is also $\mathcal{R}_d$-increasing, which puts this notion of monotonicity as a relaxed form of monotonicity.

Remark 4.2. The concept of dilative monotonicity is similar to weak monotonicity, considering the product by a constant rather than the addition. Indeed, the definition could be given in the following terms: A function $F : [0,1]^n \to [0,1]$ is dilative increasing (resp. decreasing) if for all $c \ge 1$ and $\mathbf{x} \in [0,1]^n$ such that $c\mathbf{x} \in [0,1]^n$, it holds that $F(\mathbf{x}) \le F(c\mathbf{x})$ (resp. $F(\mathbf{x}) \ge F(c\mathbf{x})$).

However, dilative monotonicity is different from weak monotonicity. The following is an example of a function that is weakly increasing but not dilative increasing.

Example 4.3. Let $f : [0,1]^n \to [0,1]$ be an averaging aggregation function and let $F : [0,1]^n \to [0,1]$ be a function given by

$$F(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} = \mathbf{1}; \\ f(\mathbf{x}), & \text{if } \mathbf{x} \neq \mathbf{1} \text{ and } \max(\mathbf{x}) - \min(\mathbf{x}) < 0.5; \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

Note that F is not increasing in general. Indeed, let $n = 2$ and f be the arithmetic mean. Then, consider $\mathbf{x} = (0.4, 0)$ and $\mathbf{y} = (0.5, 0)$. Clearly, $\mathbf{x} \le \mathbf{y}$, but $F(\mathbf{x}) = 0.2 > 0 = F(\mathbf{y})$.

Moreover, F is weakly increasing but not $\mathcal{R}_d$-increasing. The fact that F is weakly increasing is easy to verify since f is increasing and whenever $\max(\mathbf{x}) - \min(\mathbf{x}) < 0.5$, then $\max(\mathbf{x} + c\mathbf{1}) - \min(\mathbf{x} + c\mathbf{1}) < 0.5$. However, considering f to be the arithmetic mean, if we take $\mathbf{x} = (0.4, 0.1)$ and $c = 2$, then

$$F(\mathbf{x}) = F(0.4, 0.1) = 0.25 > 0 = F(0.8, 0.2) = F(2\mathbf{x}).$$

We can find also functions that are dilative increasing but not weakly increasing.

Example 4.4. Let $g : [0,1]^n \to [0,1]$ be an averaging aggregation function and let $G : [0,1]^n \to [0,1]$ be a function given by

$$G(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} = \mathbf{1}; \\ g(\mathbf{x}) & \text{if } \mathbf{x} \notin \{\mathbf{0}, \mathbf{1}\} \text{ and } \min(\mathbf{x})/\max(\mathbf{x}) < 0.5; \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

Function G is not increasing. Indeed, if we set $\mathbf{x} = (0.5, 0.1)$ and $\mathbf{y} = (0.5, 0.5)$. Clearly, $\mathbf{x} \le \mathbf{y}$, but $G(\mathbf{x}) > 0 = G(\mathbf{y})$.

Moreover, we can check that G is not weakly increasing, but it is $\mathcal{R}_d$-increasing. Indeed, the fact that G is $\mathcal{R}_d$-increasing is easy to verify since g is increasing and whenever $\min(\mathbf{x})/\max(\mathbf{x}) < 0.5$, then $\min(c\mathbf{x})/\max(c\mathbf{x}) < 0.5$. To check that G is not weakly increasing, if we consider g to be the arithmetic mean, we can take $\mathbf{x} = (0.2, 0)$. Thus, $G(\mathbf{x}) = 0.1$. Now, if we set $\mathbf{r} = (0.5, 1)$ and $c = 0.4$, we can see that

$$G(\mathbf{x} + c\mathbf{r}) = G(0.2 + 0.4 \cdot 0.5, 0 + 0.4 \cdot 1) = G(0.4, 0.4) = 0.$$

Therefore, G is not weakly increasing.

We can also find functions that are both weakly increasing and dilative increasing, but not increasing in the standard sense.

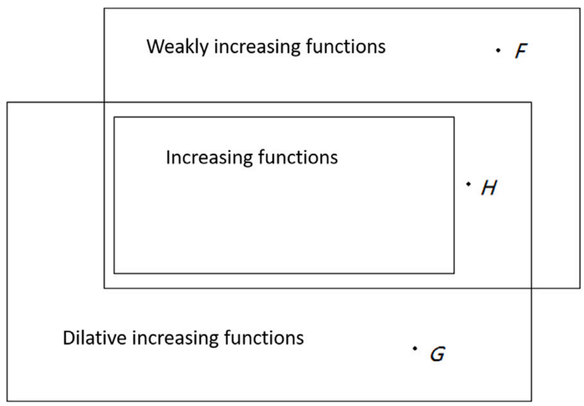

Fig. 1. Illustration of the relation of the set of increasing functions, weakly increasing functions and dilative increasing functions, with points representing functions (1), (2), and (3).

Example 4.5. Let $h : [0,1]^n \to [0,1]$ be an averaging aggregation function and let $H : [0,1]^n \to [0,1]$ be a function given by

$$H(\mathbf{x}) = \begin{cases} h(\mathbf{x}), & \text{if } \mathbf{x} \neq \mathbf{0} \text{ and } \min(\mathbf{x})/\max(\mathbf{x}) \geq 0.5; \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

The function H is not increasing. Indeed, if we set $\mathbf{x} = (0.2, 0.2)$ and $\mathbf{y} = (0.6, 0.2)$. Clearly, $\mathbf{x} \leq \mathbf{y}$, but $H(\mathbf{x}) = 0.2 > 0 = H(\mathbf{y})$.

However, H is both weakly increasing and $\mathcal{R}_d$-increasing. Indeed, note that if $\frac{\min(\mathbf{x})}{\max(\mathbf{x})} \geq 0.5$, then for any $c > 0$ such that $\mathbf{x} + c\mathbf{1} \in [0,1]^n$, it holds that

$$\frac{\min(\mathbf{x} + c\mathbf{1})}{\max(\mathbf{x} + c\mathbf{1})} = \frac{\min(\mathbf{x}) + c}{\max(\mathbf{x}) + c} \geq 0.5.$$

Then, the fact that H is weakly increasing is easy to check.

Similarly, if $\frac{\min(\mathbf{x})}{\max(\mathbf{x})} \geq 0.5$, then for any $c \geq 1$ such that $c\mathbf{x} \in [0,1]^n$, it holds that

$$\frac{\min(c\mathbf{x})}{\max(c\mathbf{x})} = \frac{\min(\mathbf{x})}{\max(\mathbf{x})} \geq 0.5.$$

Thus, it is easy to check that H is also $\mathcal{R}_d$-increasing.

By Examples 4.3–4.5 we have shown that the set of functions that are weakly increasing and the set of functions that are dilative increasing are different sets but they have a nonempty intersection (see Fig. 1).

Another interesting property of dilative monotonicity is its relation with positive homogeneity. Dilative monotonicity is related to positive homogeneity in the same way that weak monotonicity is related to shift-invariance [10]. Namely, if a function $F : [0,1]^n \to [0,1]$ is positive homogeneous, then it is dilative increasing. Indeed, if F is positive homogeneous and we consider $c \geq 1$ and $\mathbf{x} \in [0,1]^n$ such that $c\mathbf{x} \in [0,1]^n$. Then, it holds that $F(c\mathbf{x}) = cF(\mathbf{x}) \geq F(\mathbf{x})$, since $c \geq 1$.

However, the converse does not hold.

Example 4.6. Let $F : [0,1]^n \to [0,1]$ be given by

$$F(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i^2.$$

Clearly, F is an aggregation function and F is dilative increasing. However, F is not positive homogeneous. Indeed, let us consider $n = 4$, then

$$0.5 \cdot F(1,1,1,1) = 0.5 \neq 0.25 = F(0.5, 0.5, 0.5, 0.5).$$

The fact that positive homogeneity implies dilative increasingness enables to construct dilative increasing functions, as the following construction method shows.

Theorem 4.7. *Let $f : [0,1] \to [0,1]$ be a function such that $f(0) = 0$ and $f(1) = 1$. Then, $F : [0,1]^n \to [0,1]$, given by*

$$F(\mathbf{x}) = \begin{cases} 0, & if\ \mathbf{x} = \mathbf{0}; \\ f\left(\frac{\min(\mathbf{x})}{\max(\mathbf{x})}\right)\max(\mathbf{x}), & otherwise, \end{cases} \quad (4)$$

is dilative increasing.

Proof. Given $\alpha > 0$, it holds that $F(\alpha\mathbf{x}) = \alpha F(\mathbf{x})$, therefore F is positive homogeneous and, hence, F is dilative increasing. □

Remark 4.8. The construction (4) can be generalized by considering three positively homogeneous functions $A, B, C : [0,1]^n \to [0,1]$ such that, for all $\mathbf{x} \in [0,1]^n$, it holds that $A(\mathbf{x}) \leq B(\mathbf{x})$. Then, we can construct a dilative increasing function $F : [0,1]^n \to [0,1]$ given by

$$F(\mathbf{x}) = \begin{cases} 0, & \text{if } \mathbf{x} = \mathbf{0}; \\ f\left(\frac{A(\mathbf{x})}{B(\mathbf{x})}\right)C(\mathbf{x}), & \text{otherwise,} \end{cases}$$

with the convention $\frac{0}{0} = 0$.

Moreover, we can derive some properties of a function constructed as in (4).

Proposition 4.9. *Let $F : [0,1]^n \to [0,1]$ be constructed as in (4). Then, F is idempotent.*

Proof. Let $t \in [0,1]$. Let us show that $F(t, \dots, t) = t$. If $t = 0$, then clearly $F(0, \dots, 0) = 0$. If $t > 0$, then

$$F(t, \dots, t) = tf\left(\frac{t}{t}\right) = tf(1) = t. \quad \square$$

There are other properties that are dependent on the unary function f that we use for construction.

Proposition 4.10. *Let $F : [0,1]^n \to [0,1]$ be constructed as in (4). Then, F is continuous if and only if f is continuous.*

Proof. First of all, assume F is continuous and f is not. Thus, on the one hand, there exists $x_0 \in [0,1]$ such that

$$\lim_{x \to x_0} f(x) \neq f(x_0).$$

On the other hand, let $\{\mathbf{x}_j\}$ be a sequence given by $\mathbf{x}_j = (x_{1j}, 1, \dots, 1)$ such that $\lim_{j\to\infty} x_{1j}$. Since F is continuous,

$$\lim_{j\to\infty} F(\mathbf{x}_j) = F(x_0, 1, \dots, 1).$$

But,

$$\lim_{j\to\infty} F(\mathbf{x}_j) = \lim_{j\to\infty} F(x_{1j}, 1, \dots, 1) = \lim_{j\to\infty} f(x_{1j}),$$

and $F(x_0, 1, \dots, 1) = f(x_0)$, which is a contradiction.

For the converse, if f is continuous, then in the case that $\mathbf{x} \neq \mathbf{0}$, F is a composition of continuous functions. To show that F is continuous everywhere, let us consider a sequence $\{\mathbf{x}_j\}$ such that $\lim_{j\to\infty} x_{ji} = 0$ for every $i \in \{1, \dots, n\}$. Thus, since $\max(\mathbf{x}_j)$ tends to 0 when all the components of $\mathbf{x}_j$ tend to 0 and f is bounded, it holds that

$$\lim_{j\to\infty} F(x_{j1}, \dots, x_{jn}) = 0 = F(0),$$

which implies that F is continuous. □

Regarding the monotonicity of F, it is not possible to characterize the monotonicity of F in terms of the monotonicity of f, we only achieve a partial result.

Proposition 4.11. *Let $F : [0,1]^n \to [0,1]$ be constructed as in (4). Then, if F is increasing, then f is increasing.*

Proof. Let $x, y \in [0,1]$ such that $x \leq y$ and set $\mathbf{x} = (x, 1, \dots, 1)$ and $\mathbf{y} = (y, 1, \dots, 1)$. Clearly, $\mathbf{x} \leq \mathbf{y}$ and, since F is increasing, it holds that $F(\mathbf{x}) \leq F(\mathbf{y})$. Now, we have that $F(\mathbf{x}) = f(x)$ and $F(\mathbf{y}) = f(y)$ and, hence, f is increasing. □

However, the converse of Proposition 4.11 does not hold. In the next example we show a function F as in (4) which is not increasing, although f is.

Example 4.12. Let $f : [0,1] \to [0,1]$ be given by $f(x) = x^2$, which is increasing. Let $n = 2$ and F be constructed as in (4) using the defined function f. Thus,

$$F(x_1, x_2) = \begin{cases} 0, & \text{if } (x_1, x_2) = (0,0) \\ \dfrac{\min(x_1, x_2)^2}{\max(x_1, x_2)}, & \text{otherwise.} \end{cases}$$

Note that $(0.2, 0.5) \leq (0.2, 0.8)$, but

$$F(0.2, 0.5) = 0.08 > 0.05 = F(0.2, 0.8),$$

and therefore F is not increasing.

Although the notion of dilative monotonicity is essentially different from directional monotonicity, it can also be characterized in terms of pointwise directional monotonicity [15].

Theorem 4.13. *Let $F : [0,1]^n \to [0,1]$ be a function. Then, F is dilative increasing if and only if F is pointwise directionally increasing in the sense that it is $\mathbf{r_x}$-increasing at $\mathbf{x}$ for all $\mathbf{x} \in [0,1]^n$ with $\mathbf{r_x} = (x_1, \dots, x_n)$.*

Proof. Let $F : [0,1]^n \to [0,1]$ be a dilative increasing function and let $\mathbf{x} \in [0,1]^n$. If we consider $\mathbf{r_x} = (x_1, \dots, x_n)$ and $c > 0$ such that $\mathbf{x} + c\mathbf{r_x} \in [0,1]^n$, it holds that

$$F(\mathbf{x} + c\mathbf{r_x}) = F(\mathbf{x} + c\mathbf{x}) = F((1+c)\mathbf{x}),$$

and by dilative increasingness, it holds that $F((1+c)\mathbf{x}) \geq F(\mathbf{x})$. Therefore, $F(\mathbf{x} + c\mathbf{r_x}) \geq F(\mathbf{x})$.

For the converse, let F be $\mathbf{r_x}$-increasing at $\mathbf{x}$ for all $\mathbf{x} \in [0,1]^n$ with $\mathbf{r_x} = (x_1, \dots, x_n)$. Since $F(\mathbf{x} + c\mathbf{r_x}) = F((1+c)\mathbf{x})$, to show that $F(d\mathbf{x}) \geq F(\mathbf{x})$, it suffices to take $c = d - 1$. □

4.2. Reversed dilative monotonicity

An alternate type of monotonicity comes from considering the dual form to dilation, i.e., instead of considering monotonicity in the direction of $\mathbf{x}$ (as stated in Theorem 4.13), we could consider directions $\mathbf{1} - \mathbf{x}$, which leads to reversed dilative monotonicity (see Theorem 4.17).

Definition 4.14. A function $F : [0,1]^n \to [0,1]$ is reversed dilative increasing (resp. decreasing) if it is $\mathcal{R}_{rd}$-increasing (resp. $\mathcal{R}_{rd}$-decreasing) for

$$\mathcal{R}_{rd} = \{(\mathbf{x}, \mathbf{y}) \in [0,1]^n \times [0,1]^n \mid \mathbf{y} = \mathbf{x} + s(\mathbf{1} - \mathbf{x}) \text{ for some } s \in [0,1]\}.$$

As with dilative increasingness, if a function $F : [0,1]^n \to [0,1]$ is increasing, then it is also $\mathcal{R}_{rd}$-increasing. Therefore, reversed dilative monotonicity is another relaxed form of monotonicity.

Reversed dilative monotonicity is related to dilative monotonicity and we can construct reversed dilative increasing functions from dilative increasing functions. This property can be achieved by means of dual functions. Given a function $F : [0,1]^n \to [0,1]$, we define its dual $F^d : [0,1]^n \to [0,1]$ as

$$F^d(\mathbf{x}) = 1 - F(\mathbf{1} - \mathbf{x}).$$

Theorem 4.15. *Let $F : [0,1]^n \to [0,1]$ be a function. Then, F is dilative increasing if and only if F^d is reversed dilative increasing.*

Proof. ($\Rightarrow$) Let $s \in [0,1[$ (note that the case $s = 1$ is trivial) and, thus,

$$\begin{aligned} F^d(\mathbf{x} + s(\mathbf{1} - \mathbf{x})) &= 1 - F(\mathbf{1} - (\mathbf{x} + s(\mathbf{1} - \mathbf{x}))) \\ &= 1 - F((1-s)(\mathbf{1} - \mathbf{x})). \end{aligned}$$

Now, since F is dilative increasing, we can take $c = \frac{1}{1-s} > 1$ and it holds that

$$F((1-s)(\mathbf{1} - \mathbf{x})) \leq F(c(1-s)(\mathbf{1} - \mathbf{x})) = F(\mathbf{1} - \mathbf{x}).$$

Therefore,

$$1 - F((1-s)(\mathbf{1} - \mathbf{x})) \geq 1 - F(\mathbf{1} - \mathbf{x}),$$

which proves that F^d is reversed dilative increasing.

($\Leftarrow$) Let F be reversed dilative increasing and let us take $\mathbf{x} \in [0,1]^n$ and $c > 1$ such that $d\mathbf{x} \in [0,1]^n$. Thus,

$$F^d(c\mathbf{x}) = 1 - F(\mathbf{1} - c\mathbf{x}),$$

and since F is reversed dilative increasing, we can take $s = 1 - \frac{1}{c} \in]0,1]$, and then

$$\begin{aligned} F(\mathbf{1} - c\mathbf{x}) &\leq F((\mathbf{1} - c\mathbf{x}) + s(\mathbf{1} - (\mathbf{1} - c\mathbf{x}))) \\ &= F(\mathbf{1} - c\mathbf{x} + (c-1)\mathbf{x}) \\ &= F(\mathbf{1} - \mathbf{x}). \end{aligned}$$

Therefore,

$$F^d(c\mathbf{x}) \geq 1 - F(\mathbf{1} - \mathbf{x}),$$

which proves that F^d is dilative increasing. □

Theorem 4.15 serves as construction method of functions satisfying reversed dilative monotonicity. Indeed, it is possible to consider the dual of the function given in Theorem 4.7.

Remark 4.16. We have established the relation between positive homogeneity and dilative increasingness. The case of reversed dilative increasingness is related in a similar way to the dual notion of homogeneity. In [24] homogeneity is generalized by end-point linearity, which studies the property of a function being linear when restricted to line segments. This way, homogeneity corresponds to $\mathbf{0}$-end linearity and, thus, the concept is generalized to $\mathbf{z}$-homogeneity, for $\mathbf{z} \in \mathbb{R}^n$. In our case, reversed dilative increasingness is related to $\mathbf{1}$-homogeneity in the same way dilative increasingness is related to homogeneity ($\mathbf{0}$-homogeneity).

The reader can find construction methods of $\mathbf{0}$- and $\mathbf{1}$-homogeneous functions in [24], which serve as construction methods of dilative and reversed dilative increasing functions, respectively.

Reversed dilative monotonicity can also be characterized in terms of pointwise directional monotonicity.

Theorem 4.17. *Let $F : [0,1]^n \to [0,1]$ be a function. Then, F is reversed dilative increasing if and only if F is pointwise directionally increasing in the sense that it is $\mathbf{r_x}$-increasing at $\mathbf{x}$ for all $\mathbf{x} \in [0,1]^n$ with $\mathbf{r_x} = (1 - x_1, \dots, 1 - x_n)$.*

Proof. Let $F : [0,1]^n \to [0,1]$ be a reversed dilative increasing function and let $\mathbf{x} \in [0,1]^n$. If we consider $\mathbf{r_x} = (1 - x_1, \dots, 1 - x_n)$ and $c > 0$ such that $\mathbf{x} + c\mathbf{r_x} \in [0,1]^n$, it holds that

$$F(\mathbf{x} + c\mathbf{r_x}) = F(\mathbf{x} + c(\mathbf{1} - \mathbf{x})),$$

and by reversed dilative increasingness, it holds that $F(\mathbf{x} + c(\mathbf{1} - \mathbf{x})) \geq F(\mathbf{x})$. Therefore, $F(\mathbf{x} + c\mathbf{r_x}) \geq F(\mathbf{x})$.

For the converse, let F be $\mathbf{r_x}$-increasing at $\mathbf{x}$ for all $\mathbf{x} \in [0,1]^n$ with $\mathbf{r_x} = (1 - x_1, \dots, 1 - x_n)$. Since $F(\mathbf{x} + c\mathbf{r_x}) = F(\mathbf{x} + c(\mathbf{1} - \mathbf{x}))$, to show that $F(\mathbf{x} + s(\mathbf{1} - \mathbf{x})) \geq F(\mathbf{x})$, it suffices to take $c = s$. □

4.3. Directional dilative monotonicity

Similar to the extension of weak monotonicity to directional monotonicity, we can define a form of dilative monotonicity by considering a vector $\mathbf{d} \in \mathbb{R}^n$ instead of a constant c. We call this notion directional dilative monotonicity.

Definition 4.18. Let $\mathbf{d} \in \mathbb{R}^n$ be such that $d_i \geq 1$ for all $i \in \{1, \dots, n\}$. A function $F : [0,1]^n \to [0,1]$ is dilative **d**-increasing (resp. **d**-decreasing) if it is $\mathcal{R}_{\mathbf{d}}$-increasing (resp. $\mathcal{R}_{\mathbf{d}}$-decreasing) for

$$\mathcal{R}_{\mathbf{d}} = \{(\mathbf{x}, \mathbf{y}) \in [0,1]^n \times [0,1]^n \mid \mathbf{y} = c\mathbf{d} \cdot \mathbf{x} \text{ for some } c \geq 0,\ \mathbf{x} \leq \mathbf{y}\},$$

where $\cdot$ denotes the standard scalar product: $\mathbf{d} \cdot \mathbf{x} = (d_1 x_1, \dots, d_n x_n)$.

Remark 4.19. Note that when $\mathbf{d} = (1, \dots, 1)$, dilative **d**-monotonicity coincides with dilative monotonicity.

Directional dilative monotonicity is more restrictive than dilative monotonicity, but it can also be considered a relaxed form of monotonicity. Indeed, if a function is increasing, then it is also directional dilative increasing for any $\mathbf{d} \in \mathbb{R}^n$ be such that $d_i \geq 1$ for all $i \in \{1, \dots, n\}$. However, there exist functions that are directional dilative increasing, but not increasing.

Example 4.20. Let $\mathbf{d} = (1.2, 1.4)$ and $F : [0,1]^2 \to [0,1]$ be a function given by

$$F(x_1, x_2) = \begin{cases} 0, & \text{if } x_2 < 0.3 \text{ or } (x_1, x_2) = (1, 0.3) \\ 1, & \text{otherwise.} \end{cases}$$

Clearly, F is not increasing because $(0.5, 0.3) \leq (1, 0.3)$ and $F(0.5, 0.3) = 1 > 0 = F(1, 0.3)$. However, F is $(0,1)$-increasing because whenever the second argument increases, the value of F increases as well. Moreover, F is dilative **d**-increasing. Indeed, let $\mathbf{x} \in [0,1]^2$ and $c > 0$ such that $\mathbf{x} \leq c\mathbf{d} \cdot \mathbf{x} \in [0,1]^2$, i.e., it holds that

$$x_1 \leq 1.2 c x_1,$$

and

$$x_2 \leq 1.4 c x_2.$$

Suppose, for contradiction, that $F(x_1, x_2) > F(1.2cx_1, 1.4cx_2)$. Since F is $(0,1)$-increasing, the only possibility is that $x_2 = 1.4cx_2$, which implies $c = \frac{1}{1.4}$. But, in that case, $x_1 > 1.2\frac{1}{1.4}x_1$, which is a contradiction. Therefore, F is dilative **d**-increasing.

Although directional dilative monotonicity is a relaxed form of monotonicity, it is still quite restrictive. This is, in part, due to the fact that dilative **d**-increasingness implies dilative increasingness with respect to more vectors.

Proposition 4.21. *Let $F : [0,1]^n \to [0,1]$ and $\mathbf{d} \in \mathbb{R}^n$ be such that $d_i \geq 1$ for all $i \in \{1, \dots, n\}$. If F is dilative **d**-increasing, then F is dilative $\mathbf{d}^p$-increasing for every $p \in \mathbb{N}$, where $\mathbf{d}^p = (d_1^p, \dots, d_n^p)$.*

Proof. For induction, the case $p = 1$ coincides with the definition of dilative **d**-increasingness. Assume that F is dilative $\mathbf{d}^p$-increasing for every $p \in \{1, \dots, k\}$. Let us show that F is also dilative $\mathbf{d}^{k+1}$-increasing. Let $\mathbf{x} \in [0,1]^n$ and $c > 0$ such that $\mathbf{x} \leq c\mathbf{d}^{k+1} \cdot \mathbf{x} \in [0,1]^n$. Note that, since $d_i \geq 1$, it holds that

$$\mathbf{x} \leq c\mathbf{d}^k \cdot \mathbf{x} \leq c\mathbf{d}^{k+1} \cdot \mathbf{x} \in [0,1]^n.$$

Since F is dilative $\mathbf{d}^n$-increasing, it holds that $F(\mathbf{x}) \leq F(c\mathbf{d}^k \cdot \mathbf{x})$. Moreover, setting $\mathbf{y} = c\mathbf{d}^k \cdot \mathbf{x}$, since F is dilative **d**-increasing, it holds that

$$F(\mathbf{y}) \leq F(\mathbf{d} \cdot \mathbf{y}) = F(c\mathbf{d}^{k+1} \cdot \mathbf{x}),$$

which completes the proof. □

5. Some new classes of generalized monotonicity based on a general function and a subset of $[0,1]^n$

In this section we introduce another new type of monotonicity that is more general than the ones presented in the previous section. In some sense, this type of monotonicity can generalize directional monotonicity and dilative monotonicity as well.

5.1. General case: Monotonicity based on a function T

We now study monotonicity of functions $F : [0,1]^n \to [0,1]$ according to a certain non-empty set $Z \subset [0,1]^n$ and with respect to a function

$$T : [0,1]^n \times Z \to \mathbb{R}^n,$$

that relates the inputs of F with the elements of Z.

Definition 5.1. Let $\emptyset \neq Z \subset [0,1]^n$ and $T : [0,1]^n \times Z \to \mathbb{R}^n$. A function $F : [0,1]^n \to [0,1]$ is (T, Z)-increasing (resp. (T, Z)-decreasing) if it is $\mathcal{R}_{(T,Z)}$-increasing (resp. $\mathcal{R}_{(T,Z)}$-decreasing) for

$$\mathcal{R}_{(T,Z)} = \{(\mathbf{x}, \mathbf{y}) \in [0,1]^n \times [0,1]^n \mid \mathbf{y} = T(\mathbf{x}, \mathbf{z}) \text{ for some } \mathbf{z} \in Z,\ \mathbf{x} \leq \mathbf{y}\}.$$

If a function $F : [0,1]^n \to [0,1]$ is increasing, then it is also (T, Z)-increasing, making (T, Z)-monotonicity a relaxed form of monotonicity.

Remark 5.2. The concept of (T, Z)-monotonicity introduced in Definition 5.1 may seem similar to the notion of generalized directional monotonicity, or $(\mathcal{F}, \mathcal{V})$-monotonicity introduced in [16] (see Definition 2.8 for its definition and Proposition 3.3(d) for its characterization as a monotonicity with respect to a binary relation $\mathcal{R}$ on $[0,1]^n$).

However, there exist some differences. First of all, $(\mathcal{F}, \mathcal{V})$-monotonicity relies on a family of functions $\mathcal{F} = \{g_j : D \to [0,1] \mid D \subset [0,1]^2 \text{ and } j \in \{1, \dots, n\}\}$, and each of the functions g_j affects one of the inputs. Moreover, each coordinate of the elements in $\mathcal{V}$ affects *via* the corresponding g_j the input j of function F. Contrarily, function T is a function that outputs a n-tuple in $\mathbb{R}^n$ and could mix all the components of the elements in Z with all the inputs of function F to produce an output.

Second of all, in order to consider (T, Z)-increasingness, we require that the transformation $T(\mathbf{x}, \mathbf{z})$ yields a greater tuple than $\mathbf{x}$, i.e., we require $T(\mathbf{x}, \mathbf{z}) \geq \mathbf{x}$ in order to meet the monotonicity condition $F(T(\mathbf{x}, \mathbf{z})) \geq F(\mathbf{x})$. However, this is not the case in the case of Definition 2.8. In fact, $(\mathcal{F}, \mathcal{V})$-increasingness could yield a condition that corresponded to an actual decrease of function F.

As mentioned before, (T, Z)-monotonicity is related to the rest of types of monotonicity.

For example, we can recover a more restrictive version of weak monotonicity, that is not exactly weak monotonicity.

Proposition 5.3. *Let $Z = \{(z, \dots, z) \mid z \in [0,1]\}$ and $T : [0,1]^n \times Z \to \mathbb{R}^n$ be given by*

$$T(\mathbf{x}, \mathbf{z}) = \left(\min(x_1 + z, 1), \dots, \min(x_n + z, 1)\right). \tag{5}$$

If a function $F : [0,1]^n \to [0,1]$ is (T, Z)-increasing, then F is weakly increasing.

Proof. Let $F : [0,1]^n \to [0,1]$ be (T, Z)-increasing and let $\mathbf{x} \in [0,1]^n$ and $c > 0$ such that $\mathbf{x} + c\mathbf{1} \in [0,1]^n$. Thus, we have that $\min(x_i + c, 1) = x_i + c$ for all $i \in \{1, \dots, n\}$ and, then, if we set $\mathbf{z} = (c, \dots, c) \in Z$,

$$F(\mathbf{x} + c\mathbf{1}) = F\left(\min(x_1 + c, 1), \dots, \min(x_n + c, 1)\right) = F(T(\mathbf{x}, \mathbf{z})) \geq F(\mathbf{x}).$$

Therefore, F is weakly increasing. □

The following example shows that the type of monotonicity presented in Proposition 5.3 and weak monotonicity are not equivalent.

Example 5.4. Let $F : [0,1]^2 \to [0,1]$ be given by

$$F(x_1,x_2) = \begin{cases} 5x_2^3 + 5x_1^2x_2 - 10x_1x_2^2 + 4x_2^2 - 4x_1x_2 + x_2, & \text{if } x_1 \geq x_2 \\ 5x_1^3 + 5x_2^2x_1 - 10x_2x_1^2 + 4x_1^2 - 4x_2x_1 + x_1, & \text{otherwise.} \end{cases} \tag{6}$$

Clearly, F is a continuous semi-aggregation function. It is not an aggregation function because it is not increasing, as, for example, $(1,0.2) \leq (1,0.5)$, but

$$F(1,0.2) = 0.2 > 0.125 = F(1,0.5).$$

Fig. 2 shows the graph representation of F.

Although not increasing, F is weakly increasing. Indeed, let $(x_1,x_2) \in [0,1]^2$ and $c > 0$ such that $(x_1+c, x_2+c) \in [0,1]^2$. Without loss of generality, let us assume that $x_1 \geq x_2 > 0$. Thus, it can be checked that

$$F(x_1+c, x_2+c) = \frac{c+x_2}{x_2}F(x_1,x_2),$$

which means that F is weakly increasing.

However, let us see that F is not (T,Z)-increasing in the sense of Proposition 5.3. Considering $Z = \{(z,z) \mid z \in [0,1]\}$ and $T : [0,1]^2 \times Z \to \mathbb{R}^2$ as in (5) with $n = 2$, and taking $(x_1,x_2) = (1,0.2)$ and $z = 0.3$, we have that

$$\begin{aligned} F\left(T((x_1,x_2),(z,z))\right) &= F\left(\min(x_1+z,1), \min(x_2+z,1)\right) \\ &= F(1,0.5) \\ &< F(1,0.2) \\ &= F(x_1,x_2). \end{aligned}$$

Example 5.5. The function known as Lehmer mean, $L : [0,1]^2 \to [0,1]$, given by

$$L(x_1,x_2) = \begin{cases} 0, & \text{if } x_1 + x_2 = 0 \\ \dfrac{x_1^2 + x_2^2}{x_1 + x_2}, & \text{otherwise,} \end{cases} \tag{7}$$

is another example of a continuous semi-aggregation function that is not increasing, but is weakly increasing as proved in [11].

Let us see that L is not (T,Z)-increasing in the sense of Proposition 5.3. Considering $Z = \{(z,z) \mid z \in [0,1]\}$ and $T : [0,1]^2 \times Z \to \mathbb{R}^2$ as in (5) with $n = 2$, and taking $(x_1,x_2) = (1,0)$ and $z = 0.3$, we have that

$$\begin{aligned} L\left(T((x_1,x_2),(z,z))\right) &= L\left(\min(x_1+z,1), \min(x_2+z,1)\right) \\ &= L(1,0.3) \\ &< L(1,0) \\ &= L(x_1,x_2). \end{aligned}$$

The property that the functions in Examples 5.4 and 5.5 are lacking is that the upper marginal functions $f_1, f_2 : [0,1] \to [0,1]$ given by $f_1(t) = F(t,1)$ and $f_2(t) = F(1,t)$ are not increasing.

In dimension 3, i.e., for a function $F : [0,1]^3 \to [0,1]$, the upper marginal functions of dimension 1 would be:

- $f_1^{(1)}(t) = F(t,1,1)$; $f_2^{(1)}(t) = F(1,t,1)$; and $f_3^{(1)}(t) = F(1,1,t)$.

The upper marginal functions of dimension 2 would be:

- $f_1^{(2)}(t,s) = F(t,s,1)$; $f_2^{(2)}(t,s) = F(1,t,s)$; and $f_3^{(2)}(t,s) = F(s,1,t)$.

If F is weakly increasing, for F to be (T,Z)-increasing, we would need that the marginal functions of dimension 2 are weakly increasing and the marginal functions of dimension 1 are increasing (which is the same as being weakly increasing in dimension 1).

Thus, we can achieve (T,Z)-monotonicity if we require this extra property to a weakly increasing function.

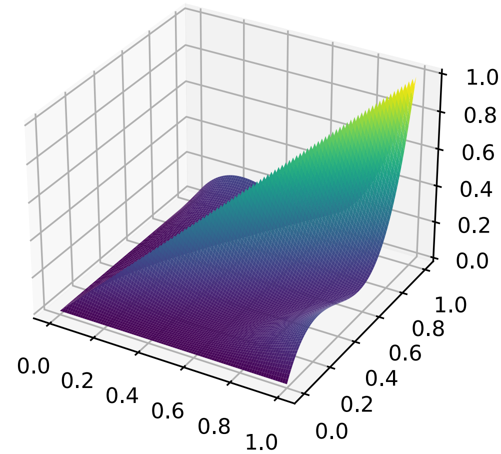

Fig. 2. Graph representation of the function given in Eq. (6) in Example 5.4.

Proposition 5.6. *If a function $F : [0,1]^n \to [0,1]$ is weakly increasing and all its upper marginal functions of dimensions $1,\dots,n-1$ are weakly increasing, then F is (T,Z)-increasing, where T and Z are as in Proposition 5.3.*

Proof. It is straightforward. ☐

Remark 5.7. Although weak increasingness and the property of (T,Z)-increasingness (as in Proposition 5.3) are essentially different, being the latter more restrictive, as far as we know, in all the applications in which a weakly increasing function has been needed, a (T,Z)-increasing function has been used. This fact leads to think that this property captures more precisely the actual requirements of a fusion function in order to fuse information when we think of weak increasingness.

In the literature, we can find numerous examples of such functions being applied in different settings. For example, $C_{\mathcal{T}}$-integrals are, in general, not increasing but they satisfy (T,Z)-increasingness and they have been used in fuzzy rule based classification systems [19,25] and in multimodal data fusion for enhancing the motor-imagery-based brain computer interface [26].

On the other hand, if we modified (5) into $T(\mathbf{x},\mathbf{z}) = (x_1+z,\dots,x_n+z)$, we would recover the standard weak monotonicity.

Example 5.8. Given a fuzzy measure[1] $\mathfrak{m} : 2^N \to [0,1]$, where $N = \{1,\dots,n\}$, two specific examples of $C_{\mathcal{T}}$-integrals are the ones based on the minimum t-norm and the Hamacher t-norm (see [25]). These two functions are given by

$$C_{\mathfrak{m}}^{M}(\mathbf{x}) = \sum_{i=1}^{n} \min\left(x_{(i)} - x_{(i-1)}, \mathfrak{m}(A_{(i)})\right); \tag{8}$$

and

$$C_{\mathfrak{m}}^{H}(\mathbf{x}) = \sum_{i=1}^{n} \begin{cases} 0, & \text{if } x_{(i)} = x_{(i-1)},\ \mathfrak{m}(A_{(i)}) = 0 \\ \dfrac{(x_{(i)} - x_{(i-1)})\mathfrak{m}(A_{(i)})}{x_{(i)} - x_{(i-1)} + \mathfrak{m}(A_{(i)}) - (x_{(i)} - x_{(i-1)})\mathfrak{m}(A_{(i)})}, & \text{otherwise,} \end{cases} \tag{9}$$

where:

- $(x_{(1)},\dots,x_{(n)})$ is an increasing permutation of the input $\mathbf{x}$;

[1] A function $\mathfrak{m} : 2^N \to [0,1]$ is said to be a fuzzy measure if $\mathfrak{m}(\emptyset) = 0$, $\mathfrak{m}(N) = 1$ and, for all $X, Y \subseteq N$, if $X \subseteq Y$, then $\mathfrak{m}(X) \leq \mathfrak{m}(Y)$.

- $x_{(0)} = 0$; and
- $A_{(i)} = \{(i), \dots, (n)\}$ is the subset of the indices of the $n-i+1$ largest components of $\mathbf{x}$.

These two functions are (T, Z)-increasing, where T and Z are as in Proposition 5.3.

We can achieve a similar kind of relation between (T, Z)-monotonicity and the rest of the discussed types of monotonicity, except for $(\mathcal{F}, \mathcal{V})$-monotonicity due to the differences explained in Remark 5.2.

The next two results relate (T, Z)-monotonicity with directional and cone monotonicity, respectively.

Proposition 5.9. *Let $\mathbf{0} \neq \mathbf{r} \in [0,1]^n$ and $Z = \{(zr_1, \dots, zr_n) \in [0,1]^n \mid z \in [0,1]\}$ and $T : [0,1]^n \times Z \to \mathbb{R}^n$ be given by*

$$T(\mathbf{x}, \mathbf{z}) = \left(\min(x_1 + z_1, 1), \dots, \min(x_n + z_n, 1)\right).$$

If a function $F : [0,1]^n \to [0,1]$ is (T, Z)-increasing, then F is $\mathbf{r}$-increasing.

Proof. Let $F : [0,1]^n \to [0,1]$ be (T, Z)-increasing and let $\mathbf{x} \in [0,1]^n$ and $c > 0$ such that $\mathbf{x}+c\mathbf{r} \in [0,1]^n$. Thus, we have that $\min(x_i+cr_i, 1) = x_i+cr_i$ for all $i \in \{1, \dots, n\}$ and, then, if we set $\mathbf{z} = (cr_1, \dots, cr_n) \in Z$,

$$F(\mathbf{x} + c\mathbf{r}) = F\left(\min(x_1 + cr_1, 1), \dots, \min(x_n + cr_n, 1)\right) = F(T(\mathbf{x}, \mathbf{z})) \geq F(\mathbf{x}).$$

Therefore, F is $\mathbf{r}$-increasing. □

Proposition 5.10. *Let $C \subset [0,1]^n$ be a nonempty cone and $Z = \{z\mathbf{r} \in [0,1]^n \mid z \in [0,1],\ \mathbf{r} \in C\}$ and $T : [0,1]^n \times Z \to \mathbb{R}^n$ be given by*

$$T(\mathbf{x}, \mathbf{z}) = \left(\min(x_1 + z_1, 1), \dots, \min(x_n + z_n, 1)\right).$$

If a function $F : [0,1]^n \to [0,1]$ is (T, Z)-increasing, then F is cone increasing with respect to C.

Proof. The proof is similar to that of Proposition 5.9 taking into account all the vectors $\mathbf{r} \in C$. □

The following three results relate (T, Z)-monotonicity with dilative, reversed dilative and directional dilative monotonicity, respectively.

Proposition 5.11. *Let $Z = \{(z, \dots, z) \in [0,1]^n \mid z \in [0,1]\}$ and $T : [0,1]^n \times Z \to \mathbb{R}^n$ be given by*

$$T(\mathbf{x}, \mathbf{z}) = \left(\min(x_1/z, 1), \dots, \min(x_n/z, 1)\right),$$

with the conventions $\frac{0}{0} = 0$ and $\frac{a}{0} = \infty$ for $a > 0$. If a function $F : [0,1]^n \to [0,1]$ is (T, Z)-increasing, then F is dilative increasing.

Proof. Let $F : [0,1]^n \to [0,1]$ be (T, Z)-increasing and let $\mathbf{x} \in [0,1]^n$ and $d \geq 1$ such that $d\mathbf{x} \in [0,1]^n$. Thus, we have that $\min(dx_i, 1) = dx_i$ for all $i \in \{1, \dots, n\}$ and, then, if we set $\mathbf{z} = (\frac{1}{d}, \dots, \frac{1}{d}) \in Z$,

$$F(d\mathbf{x}) = F\left(\min(dx_1, 1), \dots, \min(dx_n, 1)\right) = F(T(\mathbf{x}, \mathbf{z})) \geq F(\mathbf{x}).$$

Therefore, F is dilative increasing. □

Proposition 5.12. *Let $Z = \{(z, \dots, z) \in [0,1]^n \mid z \in [0,1]\}$ and $T : [0,1]^n \times Z \to \mathbb{R}^n$ be given by*

$$T(\mathbf{x}, \mathbf{z}) = \left(\min(x_1 + z(1 - x_1), 1), \dots, \min(x_n + z(1 - x_n), 1)\right).$$

If a function $F : [0,1]^n \to [0,1]$ is (T, Z)-increasing, then F is reversed dilative increasing.

Proof. Let $F : [0,1]^n \to [0,1]$ be (T, Z)-increasing and let $\mathbf{x} \in [0,1]^n$ and $s \in [0,1]$ such that $\mathbf{x} + s(1 - \mathbf{x}) \in [0,1]^n$. Thus, we have that $\min(x_i + s(1 - x_i), 1) = x_i + s(1 - x_i)$ for all $i \in \{1, \dots, n\}$ and, then, if we set $\mathbf{z} = (s, \dots, s) \in Z$,

$$\begin{aligned} F(\mathbf{x} + s(1 - \mathbf{x})) &= F\left(\min(x_1 + s(1 - x_1), 1), \dots, \min(x_n + s(1 - x_n), 1)\right) \\ &= F(T(\mathbf{x}, \mathbf{z})) \\ &\geq F(\mathbf{x}). \end{aligned}$$

Therefore, F is reversed dilative increasing. □

Proposition 5.13. *Let $\mathbf{d} \in \mathbb{R}^n$ be such that $d_i \geq 1$ for all $i \in \{1, \dots, n\}$ and $Z = \{(z\frac{1}{d_1}, \dots, z\frac{1}{d_n}) \in [0,1]^n \mid z > 0\}$ and $T : [0,1]^n \times Z \to \mathbb{R}^n$ be given by*

$$T(\mathbf{x}, \mathbf{z}) = \left(\min(x_1/z_1, 1), \dots, \min(x_n/z_n, 1)\right),$$

with the conventions $\frac{0}{0} = 0$ and $\frac{a}{0} = \infty$ for $a > 0$. If a function $F : [0,1]^n \to [0,1]$ is (T, Z)-increasing, then F is dilative $\mathbf{d}$-increasing.

Proof. Let $F : [0,1]^n \to [0,1]$ be (T, Z)-increasing and let $\mathbf{x} \in [0,1]^n$ and $c > 0$ such that $c\mathbf{d} \cdot \mathbf{x} \in [0,1]^n$. Thus, we have that $\min(cd_i x_i, 1) = cd_i x_i$ for all $i \in \{1, \dots, n\}$ and, then, if we set $\mathbf{z} = (\frac{1}{c}\frac{1}{d_1}, \dots, \frac{1}{c}\frac{1}{d_n}) \in Z$,

$$F(c\mathbf{d} \cdot \mathbf{x}) = F\left(\min(cd_1 x_1, 1), \dots, \min(cd_n x_n, 1)\right) = F(T(\mathbf{x}, \mathbf{z})) \geq F(\mathbf{x}).$$

Therefore, F is dilative $\mathbf{d}$-increasing. □

In what follows, we study two particular cases of (T, Z)-monotonicity: the case in which T is formed by means of a disjunctive binary aggregation function and the case in which T is formed by means of a conjunctive binary aggregation function.

5.2. Special case: Monotonicity based on conjunctive and disjunctive binary aggregation functions

We pay special attention to two specific classes of (T, Z)-monotone functions, the ones in which T is constructed as a tuple of binary functions. Concretely, for a fixed binary function $H : [0,1]^2 \to [0,1]$, we consider function $T : [0,1]^n \times Z \to \mathbb{R}^n$ to be given by

$$T(\mathbf{x}, \mathbf{z}) = \left(H(x_1, z_1), \dots, H(x_n, z_n)\right),$$

i.e., the same function is applied to all the inputs and for a position j only the jth component of $\mathbf{x}$ and $\mathbf{z}$ play a role. More specifically, we focus on the case in which H is a disjunctive aggregation function or a conjunctive aggregation function.

Let us start with a disjunctive aggregation function.

Definition 5.14. Let $\emptyset \neq Z \subset [0,1]^n$ and $A : [0,1]^2 \to [0,1]$ be a binary disjunctive aggregation function. A function $F : [0,1]^n \to [0,1]$ is (A, Z)-increasing (resp. (A, Z)-decreasing) if it is $\mathcal{R}^D_{(A,Z)}$-increasing (resp. $\mathcal{R}^D_{(A,Z)}$-decreasing) for

$$\mathcal{R}^D_{(A,Z)} = \{(\mathbf{x}, \mathbf{y}) \in [0,1]^n \times [0,1]^n \mid \mathbf{y} = \left(A(x_1, z_1), \dots, A(x_n, z_n)\right) \text{ for some } \mathbf{z} \in Z\}. \quad (10)$$

Note that the fact that A is a disjunctive aggregation function makes it certain that the condition $\mathbf{x} \leq \mathbf{y}$ is satisfied.

Moreover, (A, Z)-monotonicity is of importance because it is the specific case of (T, Z)-monotonicity that allows to include all the other forms of monotonicity in the literature. Specifically, if we consider $A(x, y) = \min(x + y, 1)$, i.e., the Lukasiewicz t-conorm, we can relate to weak, directional and cone monotonicity (recall Propositions 5.3, 5.9 and 5.10). If we consider $A(x, y) = \min(x/y, 1)$, with the convention $0/0 = 0$ and $a/0 = \infty$ for $a > 0$, we can relate to dilative monotonicity (Proposition 5.11) and directional dilative monotonicity (Proposition 5.13). Finally, if we consider $A(x, y) = \min(x + y(1 - x), 1)$, we can relate to reversed dilative monotonicity (Proposition 5.12).

Similarly, we can consider a conjunctive aggregation function $B : [0,1]^2 \to [0,1]$. However, in this case, it is not possible to obtain a tuple

$$\mathbf{y} = \left(B(x_1, z_1), \dots, B(x_n, z_n)\right)$$

such that $\mathbf{x} \leq \mathbf{y}$, unless the equality is obtained. For that reason, we need to make an adjustment to define (B, Z)-increasingness with respect to a conjunctive operator.

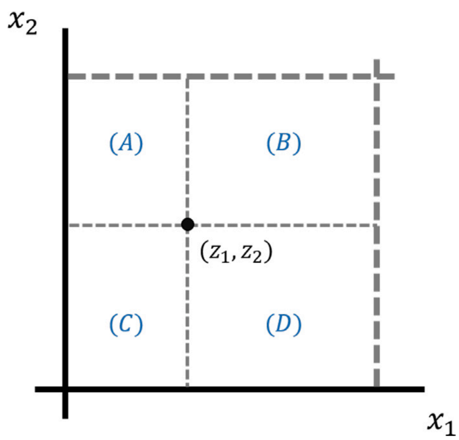

Fig. 3. Regions of the domain for the different conditions that (A,Z)- and (B,Z)-increasingness require to function F in Example 5.17.

Definition 5.15. Let $\emptyset \neq Z \subset [0,1]^n$ and $B : [0,1]^2 \to [0,1]$ be a binary conjunctive aggregation function. A function $F : [0,1]^n \to [0,1]$ is (B,Z)-increasing (resp. (B,Z)-decreasing) if it is $\mathcal{R}^C_{(B,Z)}$-decreasing (resp. $\mathcal{R}^C_{(B,Z)}$-increasing) for

$$\mathcal{R}^C_{(B,Z)} = \{(\mathbf{x},\mathbf{y}) \in [0,1]^n \times [0,1]^n \mid \mathbf{y} = \left(B(x_1,z_1),\dots,B(x_n,z_n)\right) \text{ for some } \mathbf{z} \in Z\}. \quad (11)$$

Remark 5.16. Note that in Definition 5.15 we have defined (B,Z)-increasingness in terms of $\mathcal{R}^C_{(B,Z)}$-decreasingness, and (B,Z)-decreasingness in terms of $\mathcal{R}^C_{(B,Z)}$-increasingness. This is due to the fact that we aim at defining an increasingness condition whenever the inputs increase and, when using a conjunctive aggregation function, we are making the inputs decrease.

Another way of interpreting the difference between Definitions 5.14 and 5.15 is by the sign of the inequality: for a function $F : [0,1]^n \to [0,1]$, we say that F is (A,Z)-increasing if

$$F(\mathbf{x}) \le F\left(A(x_1,z_1),\dots,A(x_n,z_n)\right),$$

and F is (B,Z)-increasing if

$$F(\mathbf{x}) \ge F\left(B(x_1,z_1),\dots,B(x_n,z_n)\right).$$

These notions are general and, as we later show, capable of generalizing various types of monotonicity. But, their generality covers the most basic forms of monotonicity. In particular, if the set Z is finite, (A,Z)- and (B,Z)-increasingness are reduced to a condition regarding the value of certain regions of the domain with respect to the points in Z. The next example illustrates this fact when Z is a singleton.

Example 5.17. Let $F : [0,1]^2 \to [0,1]$ be a fusion function and let $Z = \{(z_1,z_2)\} \subset [0,1]^2$ and let $A = \max$ and $B = \min$.

The conditions of $(\max,Z)$- and $(\min,Z)$-increasingness are equivalent to the following conditions for the points in the domain of F, according to the regions that are depicted in Fig. 3.

On the one hand, F is $(\max,Z)$-increasing whenever:

- $F(z_1,x_2) \ge F(x_1,x_2)$ for all (x_1,x_2) in (A);
- $F(x_1,x_2) \ge F(x_1,x_2)$ for all (x_1,x_2) in (B);
- $F(z_1,z_2) \ge F(x_1,x_2)$ for all (x_1,x_2) in (C);
- $F(x_1,z_2) \ge F(x_1,x_2)$ for all (x_1,x_2) in (D).

On the other hand, F is $(\min,Z)$-increasing whenever:

- $F(x_1,z_2) \le F(x_1,x_2)$ for all (x_1,x_2) in (A);
- $F(z_1,z_2) \le F(x_1,x_2)$ for all (x_1,x_2) in (B);
- $F(x_1,x_2) \le F(x_1,x_2)$ for all (x_1,x_2) in (C);
- $F(z_1,x_2) \le F(x_1,x_2)$ for all (x_1,x_2) in (D).

Note that, within each region, there are no further requirements; F could take any value satisfying the described conditions and it would still satisfy the monotonicity properties. In fact, $(\max,Z)$-increasingness does not require that any condition is fulfilled for the points in (B) and nor does $(\min,Z)$ for the points in (C).

The following is an example of a function that is $(\min,Z)$-increasing with Z a set with infinitely many elements, which makes the condition more restrictive.

Example 5.18. Let $Z = \{(z,z) \mid z \in [0,1]\}$. The function $F : [0,1]^2 \to [0,1]$, given by

$$F(x_1,x_2) = \begin{cases} 0.5, & \text{if } (x_1,x_2) = (1,0.5) \\ 1, & \text{if } x_1 = 1, x_2 \neq 0.5 \\ 0, & \text{otherwise,} \end{cases}$$

is $(\min,Z)$-increasing but not increasing. Indeed, F is not increasing as $(1,0.5) \ge (1,0.3)$ but

$$F(1,0.5) = 0.5 < 1 = F(1,0.3).$$

In order to show that F is $(\min,Z)$-increasing, let $\mathbf{x} \in [0,1]^2$ and $\mathbf{z} = (z,z) \in Z$. Then, let us break it into cases:

- If $(x_1,x_2) = (1,0.5)$, then clearly $F(x_1,x_2) = 0.5$. Now, let $z \in [0,1]$. If $z < 1$, then $F(\min(1,z),\min(0.5,z)) = 0$. If $z = 1$, then $F(\min(1,1),\min(0.5,1)) = 0.5$. Hence, it holds that $F(\min(x_1,z),\min(x_2,z)) \le F(x_1,x_2)$.
- If $x_1 = 1$ and $x_2 \neq 0.5$, then $F(x_1,x_2) = 1$ and, therefore, $F(\min(x_1,z),\min(x_2,z)) \le F(x_1,x_2)$.
- If (x_1,x_2) does not fall in any of the previous cases, then $F(x_1,x_2) = 0$ and notice that $x_1 < 1$. Therefore, $\min(x_1,z) < 1$ for any $z \in [0,1]$ and, consequently, $F(\min(x_1,z),\min(x_2,z)) = 0$.

The cases $(\min,Z)$- and $(\max,Z)$-increasingness with Z as in Example 5.18 are of particular interest as shown in the next results.

The next one shows that the boundary conditions of F makes it impossible for it to be $(\min,Z)$-decreasing or $(\max,Z)$-decreasing.

Proposition 5.19. *Let $Z = \{(z,\dots,z) \mid z \in [0,1]\}$ and $F : [0,1]^n \to [0,1]$ be a function satisfying $F(\mathbf{0}) = 0$ and $F(\mathbf{1}) = 1$. Then, the following hold:*

(a) F is not $(\min,Z)$-decreasing.
(b) F is not $(\max,Z)$-decreasing.

Proof.

(a) If F was $(\min,Z)$-decreasing, then it would hold that

$$F(\min(x_1,z),\dots,\min(x_n,z)) \ge F(\mathbf{x})$$

for any $\mathbf{x} \in [0,1]^n$ and $z \in [0,1]$. However, if we take $\mathbf{x} = \mathbf{1}$ and $z = 0$, we have that

$$F(\min(x_1,z),\dots,\min(x_n,z)) = F(\mathbf{0}) = 0 < 1 = F(\mathbf{x}).$$

(b) If F was $(\max,Z)$-decreasing, then it would hold that

$$F(\max(x_1,z),\dots,\max(x_n,z)) \le F(\mathbf{x}),$$

for any $\mathbf{x} \in [0,1]^n$ and $z \in [0,1]$. However, if we take $\mathbf{x} = \mathbf{0}$ and $z = 1$, we have that

$$F(\max(x_1,z),\dots,\max(x_n,z)) = 1 > 0 = F(\mathbf{x}). \quad \square$$

Moreover, to be both $(\min,Z)$- and $(\max,Z)$-increasing is a strong property of monotonicity. In fact, for two dimensional functions, it coincides with being increasing.

For the proof of the following Theorem, we make use of the concept of OD monotonicity [13].

Definition 5.20 (*[13]*)**.** Let $\mathbf{0} \neq \mathbf{r} \in \mathbb{R}^n$. A function $F : [0,1]^n \to [0,1]$ is said to be Ordered Directionally (OD) $\mathbf{r}$-increasing if for each $\mathbf{x} \in [0,1]^n$, and any permutation σ with $x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)}$ and any $c > 0$ such that

$$1 \geq x_{\sigma(1)} + cr_1 \geq \cdots \geq x_{\sigma(n)} + cr_n \geq 0, \tag{12}$$

it holds that

$$F(\mathbf{x} + c\mathbf{r}_{\sigma^{-1}}) \geq F(\mathbf{x}),$$

where $\mathbf{r}_{\sigma^{-1}} = (r_{\sigma^{-1}(1)}, \ldots, r_{\sigma^{-1}(n)})$.

Theorem 5.21. *Let $Z = \{(z,z) \mid z \in [0,1]\}$ and $F : [0,1]^2 \to [0,1]$. Then, F is both $(\min, Z)$- and $(\max, Z)$-increasing if and only if F is increasing.*

Proof. If F is increasing, then it is easy to check that F is also $(\min, Z)$- and $(\max, Z)$-increasing.

For the converse, let us assume that F is both $(\min, Z)$- and $(\max, Z)$-increasing and let $\mathbf{x} = (x_1, x_2) \in [0,1]^2$. By Proposition 6.1 in [14], we know that F is increasing if and only if it is OD $(0,1)$-increasing and OD $(1,0)$-increasing. Let us start with OD $(0,1)$-increasingness.

- If $x_1 \geq x_2$, let $c > 0$ such that $x_1 \geq x_2 + c$. Now, take $z = x_2 + c$. Thus, since F is $(\max, Z)$-increasing,

  $$F(x_1, x_2 + c) = F(\max(x_1, z), \max(x_2, z)) \geq F(\mathbf{x}).$$
- If $x_1 < x_2$, let $c > 0$ such that $x_1 + c \leq x_2$. If we take $z = x_1 + c$, by $(\max, Z)$-increasingness,

  $$F(x_1 + c, x_2) = F(\max(x_1, z), \max(x_2, z)) \geq F(\mathbf{x}).$$

Therefore, F is OD $(0,1)$-increasing. Let us know show that it is also OD $(1,0)$-increasing.

- If $x_1 \geq x_2$, let $c > 0$ such that $x_1 + c \geq x_2$. Now, take $z = x_1$. Thus, since F is $(\min, Z)$-increasing,

  $$F(x_1 + c, x_2) \geq F(\min(x_1 + c, z), \min(x_2, z)) = F(z, x_2) = F(x_1, x_2).$$
- If $x_1 < x_2$, let $c > 0$ such that $x_1 \leq x_2 + c$. If we take $z = x_2$, by $(\min, Z)$-increasingness,

  $$F(x_1, x_2 + c) \geq F(\min(x_1, z), \min(x_2 + c, z)) = F(x_1, z) = F(x_1, x_2).$$

Hence, F is both OD $(0,1)$- and OD $(1,0)$-increasing and, consequently, increasing. □

However, when $n \geq 3$, the two conditions are not equivalent to standard increasingness. If a function $F : [0,1]^n \to [0,1]$ with $n \geq 3$ is increasing, then it is also $(\min, Z)$- and $(\max, Z)$-increasing. But a function that is both $(\min, Z)$- and $(\max, Z)$-increasing needs not be increasing in the standard sense for $n \geq 3$.

Example 5.22. Let $n \geq 3$ and $F : [0,1]^n \to [0,1]$ be given by

$$F(\mathbf{x}) = x_{(1)} - x_{(2)} + x_{(3)}, \tag{13}$$

where $(\cdot)$ is a permutation of $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ such that $x_{(1)} \geq x_{(2)} \geq \cdots \geq x_{(n)}$.

Function F satisfies the boundary conditions $F(\mathbf{0}) = 0$ and $F(\mathbf{1}) = 1$ and F is not increasing. Indeed, if we consider $\mathbf{x} = (0.7, 0.5, 0.2, \ldots, 0.2) \leq \mathbf{y} = (0.7, 0.6, 0.2, \ldots, 0.2)$, it holds that $F(\mathbf{x}) = 0.4 > 0.3 = F(\mathbf{y})$. Let us show that F is both $(\min, Z)$- and $(\max, Z)$-increasing. Firstly, we show that

$$x_{(1)} - x_{(2)} + x_{(3)} \leq \max(x_{(1)}, z) - \max(x_{(2)}, z) + \max(x_{(3)}, z) \tag{14}$$

for any z. Note that $x_{(1)} - x_{(2)} + x_{(3)} \leq \max(x_{(1)}, z) + x_{(3)} - x_{(2)}$. It is easy to show that $x_{(3)} - x_{(2)} \leq \max(x_{(3)}, z) - \max(x_{(2)}, z)$, so inequality (14) is valid for any z.

We now show that

$$\min(x_{(1)}, z) - \min(x_{(2)}, z) + \min(x_{(3)}, z) \leq x_{(1)} - x_{(2)} + x_{(3)} \tag{15}$$

for any z. Note that $x_{(1)} - x_{(2)} + x_{(3)} \geq x_{(1)} - x_{(2)} + \min(x_{(3)}, z)$. It is easy to show that $x_{(1)} - x_{(2)} \geq \min(x_{(1)}, z) - \min(x_{(2)}, z)$, so inequality (15) is satisfied for any z. Summing up, function (13) is both $(\min, Z)$-increasing and $(\max, Z)$-increasing, but not increasing.

Moreover, any function of the form

$$F(\mathbf{x}) = x_{(1)} - x_{(2)} + x_{(3)} - x_{(4)} + \cdots + x_{(2k-1)} - x_{(2k)} + x_{(2k+1)}$$

has also these properties for any k such that $2k + 1 \leq n$.

6. Application to text classification

In this section we present an example of a possible application of the proposed theoretical developments that illustrates their usability. Concretely, we make use of functions that satisfy the discussed types of monotonicity in an information fusion process in a text classification problem regarding sentiment analysis.

Sentiment analysis falls within the context of natural language processing and consists in the identification and extraction of subjective opinions or attitudes expressed in text. A typical form of sentiment analysis is a classification process in which the goal is to classify a text as expressing a positive or negative sentiment.

Several techniques have been proposed to tackle the problem of sentiment analysis, including supervised and unsupervised Machine Learning techniques [27]. Nowadays, the majority of methods that come on top of the benchmark datasets are based on Deep Learning [28, 29]. However, our proposal does not aim at overcoming the state of the art techniques for sentiment analysis, but we feel it serves to illustrate the appropriateness of the studied types of fusion function properties in certain information fusion processes.

We base our proposal on the fusion of feature vectors that represent each word in the text to classify. These vectors are commonly known as word embeddings [30], which are numeric vectors that encode the meaning of words in a way that words that share a similar meaning are closer in the feature space. An schema of the whole document classification process can be seen in Fig. 4, which shows the following steps:

1. The words of the document are extracted and mapped to their feature vector representation.
2. The word-level feature vectors are fused component-wise by means of a fusion function to generate a document-level feature vector.
3. The document-level feature vector is fed as input to a Machine Learning classifier, which outputs the class, i.e., the polarity, of the document.

For our experimental framework, we set two methods of obtaining the word embeddings in Step 1; thirteen fusion functions to fuse the word embeddings into document feature vectors in Step 2 and a single classifier in Step 3.

Concretely, we use GloVe [31] and word2vec [32,33] as word embedding techniques. The former is based on word co-occurrence from a corpus and the latter is based on representing the words as the activations of a hidden layer of a specific kind of neural network. There exist available repositories with pre-trained word embeddings, both for the GloVe word embeddings[2] and for the word2vec word embeddings.[3] We test our proposal with different dimension of word embeddings: 50, 100, 200 and 300 for GloVe and 300 for word2vec. In order to be consistent with the input domain of the fusion functions that we study in this work, we translate the values of the pre-trained word embedding to $[0,1]$.

As fusion functions, we use the arithmetic mean (AM), the minimum (Min), the maximum (Max), the geometric mean (GM), the harmonic

[2] https://nlp.stanford.edu/projects/glove/.
[3] https://code.google.com/archive/p/word2vec/.

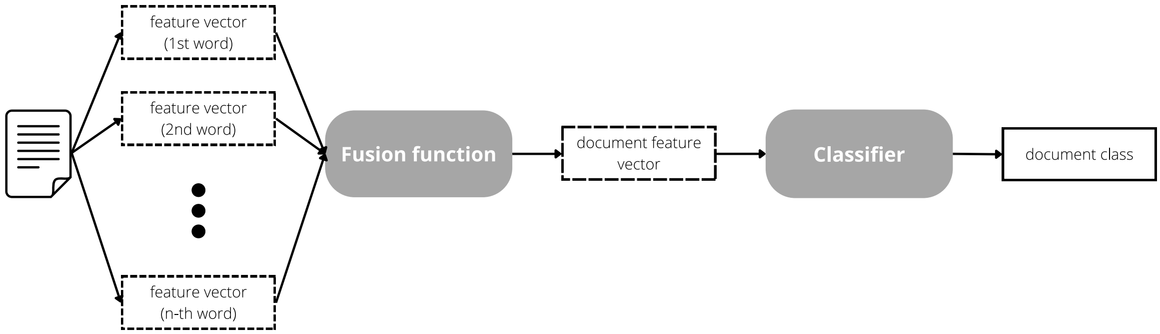

Fig. 4. Document classification process involving the fusion of word-level feature vectors into a document-level feature vector.

mean (HM), the product (P) and the probabilistic sum (PS) representing aggregation functions, i.e., functions that are increasing; the function given in Example 4.12 (DF), for $n \geq 2$, as a dilative increasing function; the dual function of the one given in Example 4.12 (RDF), for $n \geq 2$, as a reversed dilative increasing function; the Lehmer mean (LM) – see Example 5.5 – and the function given in Example 5.4 (F-5.4), for $n \geq 2$, as weakly increasing functions that are not (T, Z)-increasing as in Proposition 5.3; and the functions C^M_{m} and C^H_{m} in Eqs. (8) and (9), respectively, as functions that are both weakly increasing and (T, Z)-increasing as in Proposition 5.3.

As classifier, we use a feedforward neural network, also known as multilayer perceptron, with a single hidden layer with 100 neurons and ReLU as activation. The rationale of not using a excessively complex classifier is that we aim at illustrating the effect of the different fusion functions when generating the document feature vectors. Additionally, for each type of word embedding, we fix the same initialization weights for the different fusion functions so that we are able to compare the results.

In order to check the results of our proposal, we carry out experiments in a dataset that is often used as benchmark for sentiment analysis: the IMDb dataset[4] [34], which is a binary sentiment analysis dataset which has 50,000 reviews from the Internet Movie Database (IMDb). This dataset is balanced in terms of polarity, i.e., there are 25,000 positive and 25,000 negative reviews. Additionally, the data are evenly split into a training set and a test set. For our experiments, we carry out a stop-word removal process, which is a common preprocessing step in text classification [35].

For evaluation, we set as reference the test results of the process using the arithmetic mean as fusion function. We train each model 100 times, using different weight initializations, and compare the top 5 test accuracy values for each fusion function. To test whether there exist statistical significance, we use the nonparametric Mann–Whitney U test [36] setting as null hypothesis the fact that the distribution underlying the top 5 accuracy values of the process using the arithmetic mean is the same as the distribution underlying the top 5 accuracy values using each of the other fusion functions. We require a significance level of 0.05.

In Table 1 we show the Test results of the experiments in terms of the accuracy of the classification. First, we can see that the results progressively improve as we increase the dimension of the feature vectors, achieving the highest overall accuracy rates by word2vec. Additionally, we observe that the average values of the accuracy rates for each fusion function are quite similar, which suggest that all the established fusion techniques are valid.

With respect to the influence of each fusion function, we derive that, computing the average performance of each fusion function, the function C^H_{m}, satisfying (T, Z)-increasingness, comes on top. Focusing in lower dimensions (embeddings of 50, 100 or 200 components), the function C^H_{m}, satisfying (T, Z)-increasingness, yields the overall best results; it gets the best accuracy rate in the case of GloVe-50, it lies among the best performers in GloVe-100 and it is the runner-up in GloVe-200. Moreover, for all the embedding extraction techniques except word2vec, one of the functions satisfying (T, Z)-increasingness (C^M_{m} and C^H_{m}) overcome both the functions that are only weakly increasing (LM and F-5.4). For higher dimensions (embeddings of 300 components), the dilative increasing function (DF) yields the best results.

Another interesting outcome of the experimentation is that, for all the feature extraction settings except GloVe-100, none of the best fusion functions satisfies the condition of increasingness in the standard sense; they satisfy either dilative or (T, Z)-increasingness. In fact, focusing in the average performance of each fusion function across feature extraction methods, the functions DF, F-5.4, C^M_{m} and C^H_{m} overcome every aggregation function (AM, Min, Max, GM, HM, P and PS).

7. Conclusions

We have proposed a framework for general monotonicity for fusion functions based on a binary relation $\mathcal{R}$. This framework encompasses all the different forms of monotonicity of fusion functions in the literature and enables the introduction of additional forms of monotonicity.

In that sense, we have presented the concept of dilative monotonicity, which deals with the increasingness of a function when the inputs are multiplied by the same, greater than one, constant. In the same way that weak monotonicity is linked with shift-invariance, dilative monotonicity is linked with positive homogeneity. Moreover, we have studied the relation between the set of weakly increasing functions, dilative increasing functions and increasing functions, concluding that the latter is within the intersection of the former two, while there also exist functions that are both weakly and dilative increasing and not increasing. We have also proposed a construction method for dilative increasing functions and have studied its properties. Besides, dilative monotonicity has motivated the proposal of reversed dilative monotonicity and directional dilative monotonicity, whose properties and links are also studied in this work.

Additionally, we have presented a new class of monotonicity that is based on a subset Z of the domain and a function $T : [0,1]^n \times Z \to \mathbb{R}^n$, which makes this type of monotonicity is more general than the previous ones. We have studied its properties and focused on the specific case that T is defined in terms of binary disjunctive and conjunctive aggregation functions. The link between this type of monotonicity and the rest shows interesting results. In particular, the case in which (T, Z) monotonicity recovers a more restrictive property than weak monotonicity is interesting because we have checked that in the applications that data were fused according to a weak monotone function, that function satisfied (T, Z) monotonicity in all the cases [19, 25,26].

[4] The IMDb dataset is available at: https://ai.stanford.edu/~amaas/data/sentiment/.

Table 1
Average classification accuracy (%) of the top 5 test result for each fusion method in the IMDb dataset. The best accuracy rate for each feature extraction method is highlighted.

	GloVe-50	GloVe-100	GloVe-200	GloVe-300	word2vec	AVG
AM	0.7010 ± 0.0003	0.7494 ± 0.0007	0.7864 ± 0.0001	0.7901 ± 0.0001	0.7983 ± 0.0002	0.7650 ± 0.0003
Min	0.6909 ± 0.0212	0.7640 ± 0.0010*	0.7866 ± 0.0001	0.7891 ± 0.0001*	0.7926 ± 0.0001*	0.7647 ± 0.0045
Max	0.6845 ± 0.0126	0.7665 ± 0.0004*	0.7856 ± 0.0001*	0.7892 ± 0.0002*	0.7922 ± 0.0002*	0.7636 ± 0.0027
GM	0.6723 ± 0.0003*	**0.7669 ± 0.0002***	0.7858 ± 0.0001*	0.7885 ± 0.0001*	0.7932 ± 0.0001*	0.7613 ± 0.0002
HM	0.6762 ± 0.0007*	0.7660 ± 0.0005*	0.7858 ± 0.0002*	0.7887 ± 0.0002*	0.7933 ± 0.0001*	0.7620 ± 0.0003
P	0.6933 ± 0.0237	0.7660 ± 0.0002*	0.7856 ± 0.0002*	0.7886 ± 0.0002*	0.7924 ± 0.0002*	0.7652 ± 0.0049
PS	0.6935 ± 0.0131	0.7607 ± 0.0026*	0.7854 ± 0.0003*	0.7880 ± 0.0003*	0.7922 ± 0.0001*	0.7640 ± 0.0033
DF	0.7047 ± 0.0080	0.7551 ± 0.0016*	0.7862 ± 0.0002	**0.7905 ± 0.0001***	**0.7989 ± 0.0002***	0.7671 ± 0.0020
RDF	0.6886 ± 0.0005*	0.7571 ± 0.0043*	0.7859 ± 0.0002*	0.7902 ± 0.0002	0.7957 ± 0.0001*	0.7635 ± 0.0011
LM	0.6847 ± 0.0005*	0.7556 ± 0.0006*	0.7860 ± 0.0001*	0.7892 ± 0.0001*	0.7952 ± 0.0001*	0.7621 ± 0.0003
F-5.4	0.7143 ± 0.0050*	0.7560 ± 0.0077	0.7864 ± 0.0002	0.7897 ± 0.0001*	0.7958 ± 0.0003*	0.7685 ± 0.0026
C_m^M	0.7011 ± 0.0162	0.7653 ± 0.0001*	**0.7867 ± 0.0002***	0.7899 ± 0.0001*	0.7939 ± 0.0001*	0.7674 ± 0.0033
C_m^H	**0.7210 ± 0.0042***	0.7641 ± 0.0004*	0.7866 ± 0.0004	0.7890 ± 0.0001*	0.7936 ± 0.0001*	**0.7709 ± 0.0010**

*Results indicate a p-value lesser than 0.05 in the comparison with AM.

Finally, we have presented a possible application of the proposed notions of general monotonicity. We have utilized functions satisfying the studied types of monotonicity in an information fusion process of a sentiment analysis (text classification) problem. The results show that, for most of the tested settings, the best function to carry out the fusion is not increasing, but rather dilative or (T, Z)-increasing.

For a future work, we think it would be interesting to carry out an experimental study comparing the performance of purely weakly increasing functions with the stronger (T, Z)-increasing functions to determine whether this stronger property is indeed desirable in a fuzzy rule based classification system.

CRediT authorship contribution statement

Mikel Sesma-Sara: Conceptualization, Formal analysis, Writing – original draft, Writing – review & editing, Software. **Adam Šeliga:** Formal analysis. **Michał Boczek:** Conceptualization, Formal analysis, Writing – original draft, Writing – review & editing. **LeSheng Jin:** Conceptualization, Formal analysis, Writing – original draft. **Marek Kaluszka:** Conceptualization, Formal analysis, Writing – original draft, Writing – review & editing. **Martin Kalina:** Formal analysis. **Humberto Bustince:** Supervision, Funding acquisition. **Radko Mesiar:** Conceptualization, Formal analysis, Supervision, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The link to the data is in the text

Acknowledgements

This work has been funded by the Spanish ministry MCIN, with the project PID2019-108392GB-I00/AEI/10.13039/501100011033, by the Public University of Navarra under the project PJUPNA25-2022 and by the grant APVV-18-0052 by the Slovak Research and Development Agency. Open access funding provided by Universidad Pública de Navarra.

References

[1] A.N. Kolmogorov, Sur la notion de la moyenne, Atti Accad. Naz. Lincei 12 (1930) 388–391.
[2] T. Calvo, A. Kolesárová, M. Komorníková, R. Mesiar, A review of aggregation operators, University of Alcala Press, Alcala de Henares, 2001.
[3] M. Grabisch, J. Marichal, R. Mesiar, E. Pap, Aggregation Functions, Cambridge University Press, 2009.
[4] G. Beliakov, H. Bustince, T. Calvo, A Practical Guide to Averaging Functions, in: Studies in Fuzziness and Soft Computing, Springer International Publishing, 2016.
[5] M. Boczek, A. Hovana, O. Hutník, M. Kaluszka, New monotone measure-based integrals inspired by scientific impact problem, European J. Oper. Res. 290 (1) (2021) 346–357.
[6] D. Li, Q. Zeng, Approximate reasoning with aggregation functions satisfying GMP rules, Artif. Intell. Rev. (2022) 1–21.
[7] R. Yang, X. Pu, D. Paternain, R. Yager, R. Mesiar, H. Bustince, L. Jin, Some methods for Yager preference involved aggregations in multi-criteria and multi-sources evaluation, Int. J. Uncertain. Fuzziness Knowl.-Based Syst. 29 (04) (2021) 587–602.
[8] R. Mesiar, A. Kolesárová, A. Stupňanová, Quo vadis aggregation? Int. J. Gen. Syst. 47 (2) (2018) 97–117, http://dx.doi.org/10.1080/03081079.2017.1402893.
[9] P.S. Bullen, Handbook of Means and their Inequalities, vol. 560, Springer Science & Business Media, 2013.
[10] T. Wilkin, G. Beliakov, Weakly monotonic averaging functions, Int. J. Intell. Syst. 30 (2) (2015) 144–169, http://dx.doi.org/10.1002/int.21692.
[11] H. Bustince, J. Fernandez, A. Kolesárová, R. Mesiar, Directional monotonicity of fusion functions, European J. Oper. Res. 244 (1) (2015) 300–308, http://dx.doi.org/10.1016/j.ejor.2015.01.018.
[12] G. Beliakov, T. Calvo, T. Wilkin, Three types of monotonicity of averaging functions, Knowl.-Based Syst. 72 (2014) 114–122, http://dx.doi.org/10.1016/j.knosys.2014.08.028.
[13] H. Bustince, E. Barrenechea, M. Sesma-Sara, J. Lafuente, G.P. Dimuro, R. Mesiar, A. Kolesárová, Ordered directionally monotone functions. Justification and application, IEEE Trans. Fuzzy Syst. 26 (4) (2018) 2237–2250, http://dx.doi.org/10.1109/TFUZZ.2017.2769486.
[14] M. Sesma-Sara, J. Lafuente, A. Roldán, R. Mesiar, H. Bustince, Strenghtened ordered directionally monotone functions. Links between the different notions of monotonicity, Fuzzy Sets and Systems 357 (2019) 151–172, http://dx.doi.org/10.1016/j.fss.2018.07.007.
[15] M. Sesma-Sara, L. De Miguel, A.F. Roldán López de Hierro, J. Lafuente, R. Mesiar, H. Bustince, Pointwise directional increasingness and geometric interpretation of directionally monotone functions, Inform. Sci. 501 (2019) 236–247, http://dx.doi.org/10.1016/j.ins.2019.06.002.
[16] R. Santiago, M. Sesma-Sara, J. Fernandez, Z. Takac, R. Mesiar, H. Bustince, F-homogeneous functions and a generalization of directional monotonicity, Int. J. Intell. Syst. In press (2022) http://dx.doi.org/10.1002/int.22823.
[17] M. Sesma-Sara, H. Bustince, E. Barrenechea, J. Lafuente, A. Kolesárová, R. Mesiar, Edge detection based on ordered directionally monotone functions, in: Advances in Fuzzy Logic and Technology 2017, Springer, 2017, pp. 301–307.
[18] G. Lucca, J. Sanz, G. Dimuro, B. Bedregal, M.J. Asiain, M. Elkano, H. Bustince, CC-integrals: Choquet-like copula-based aggregation functions and its application in fuzzy rule-based classification systems, Knowl.-Based Syst. 119 (2017) 32–43, http://dx.doi.org/10.1016/j.knosys.2016.12.004.
[19] G. Lucca, J.A. Sanz, G.P. Dimuro, B. Bedregal, H. Bustince, R. Mesiar, CF-integrals: A new family of pre-aggregation functions with application to fuzzy rule-based classification systems, Inform. Sci. 435 (2018) 94–110.

[20] R. Pérez-Fernández, B. De Baets, M. Gagolewski, A taxonomy of monotonicity properties for the aggregation of multidimensional data, Inf. Fusion 52 (2019) 322–334.

[21] B. De Baets, H. De Meyer, Maximal directions of monotonicity of an aggregation function, Fuzzy Sets and Systems 433 (2022) 54–78.

[22] M. Sesma-Sara, R. Mesiar, H. Bustince, Weak and directional monotonicity of functions on Riesz spaces to fuse uncertain data, Fuzzy Sets and Systems 386 (2020) 145–160, http://dx.doi.org/10.1016/j.fss.2019.01.019.

[23] G. Beliakov, S. James, A. Kolesárová, R. Mesiar, Cardinality-limiting extended pre-aggregation functions, Inf. Fusion 76 (2021) 66–74.

[24] A. Stupňanová, Y. Su, R. Mesiar, End-point linear functions, Iran. J. Fuzzy Syst. 18 (6) (2021) 1–12.

[25] G. Lucca, J.A. Sanz, G.P. Dimuro, B. Bedregal, R. Mesiar, A. Kolesárová, H. Bustince, Preaggregation functions: Construction and an application, IEEE Trans. Fuzzy Syst. 24 (2) (2016) 260–272, http://dx.doi.org/10.1109/TFUZZ.2015.2453020.

[26] L.-W. Ko, Y.-C. Lu, H. Bustince, Y.-C. Chang, Y. Chang, J. Ferandez, Y.-K. Wang, J.A. Sanz, G.P. Dimuro, C.-T. Lin, Multimodal fuzzy fusion for enhancing the motor-imagery-based brain computer interface, IEEE Comput. Intell. Mag. 14 (1) (2019) 96–106.

[27] B. Liu, Sentiment Analysis: Mining Opinions, Sentiments, and Emotions, Cambridge University Press, 2020.

[28] C. Sun, X. Qiu, Y. Xu, X. Huang, How to fine-tune BERT for text classification? in: China National Conference on Chinese Computational Linguistics, Springer, 2019, pp. 194–206.

[29] Z. Yang, Z. Dai, Y. Yang, J. Carbonell, R.R. Salakhutdinov, Q.V. Le, Xlnet: Generalized autoregressive pretraining for language understanding, Adv. Neural Inf. Process. Syst. 32 (2019).

[30] Y. Li, T. Yang, Word embedding for understanding natural language: a survey, in: Guide To Big Data Applications, Springer, 2018, pp. 83–104.

[31] J. Pennington, R. Socher, C.D. Manning, Glove: Global vectors for word representation, in: Empirical Methods in Natural Language Processing, EMNLP, 2014, pp. 1532–1543, URL http://www.aclweb.org/anthology/D14-1162.

[32] T. Mikolov, K. Chen, G. Corrado, J. Dean, Efficient estimation of word representations in vector space, 2013, arXiv preprint arXiv:1301.3781.

[33] T. Mikolov, I. Sutskever, K. Chen, G.S. Corrado, J. Dean, Distributed representations of words and phrases and their compositionality, Adv. Neural Inf. Process. Syst. 26 (2013).

[34] A.L. Maas, R.E. Daly, P.T. Pham, D. Huang, A.Y. Ng, C. Potts, Learning word vectors for sentiment analysis, in: Proceedings of the 49th Annual Meeting of the Association for Computational Linguistics: Human Language Technologies, Association for Computational Linguistics, Portland, Oregon, USA, 2011, pp. 142–150, URL http://www.aclweb.org/anthology/P11-1015.

[35] C. Silva, B. Ribeiro, The importance of stop word removal on recall values in text categorization, in: Proceedings of the International Joint Conference on Neural Networks, 2003., 3, IEEE, 2003, pp. 1661–1666.

[36] H.B. Mann, D.R. Whitney, On a test of whether one of two random variables is stochastically larger than the other, Ann. Math. Stat. (1947) 50–60.