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Research paper

A unified analytical disk cam profile generation methodology using the Instantaneous Center of Rotation for educational purpose

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ABSTRACT

Cam design is a fundamental part of the Mechanism and Machine Theory (MMT) and is included in the vast majority of MMT books. Cam profile design is usually determined with graphical and analytical methods. Graphical methods are didactically very successful to introduce the theory of cam profile generation in a simple way. In turn, analytical methods allow computer implementations of cam profile generation in order to reproduce it accurately. Most modern MMT books describe analytical methods using geometric equations and envelope theory. However, the analytical profile definition depends on the specific type of follower and there is a lack of a general formulation. This work presents a unified and general analytical formulation for the disk cam profile determination. Based on the Instantaneous Center of Rotation and the kinematic inversion, the formulation provides analytical expressions of the cam profile and is applicable to any type of follower. Thus, the unified formulation can be used in forthcoming books on this discipline.

1. Introduction

In Mechanism and Machine Theory (MMT) education, it is crucial to emphasize contemporary methodologies that can effectively leverage constantly evolving computer and software tools. Consequently, there is a necessity to update teaching techniques that incorporate simulations and computer-oriented approaches as well [1]. Thus, subjects in the field of MMT have undergone major changes in last decades. In [2], Rao reviews the evolution of MMT teaching in the 20th century, showing numerous examples of the application of computational techniques and describing various simulation methods applied to different mechanisms [3–5]. The literature on teaching methods in MMT is extensive, especially with regard to kinematics of mechanisms [6–8] and the use of multibody system dynamics for mechanical engineering [9–12]. Nevertheless, there are few papers concerning new teaching approaches for the topic of cam design.

The topic of cam design is a fundamental part of the MMT curriculum in any Mechanical Engineering school. Likewise, this topic is included in most of the classic and modern books on MMT and there is usually a problem on cams in the IFToMM Student Olympiad on MMS [13]. The chapter devoted to cams usually includes a description of the types of cams and followers, the definition of displacement diagrams of the follower trajectory, the concept of pressure angle and the synthesis of the cam profile. In general, for educational purpose, the approach given to these topics has undergone very few changes in the last 40 years. Regarding the cam profile synthesis, the vast majority of books explain graphical methods for obtaining the cam profile and also introduce different

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formulations to obtain the profile analytically. While graphical methods are pedagogically very valuable, the analytical methods used in the MMT books are mathematically involved and poorly related to the basic concepts of the MMT.

1.1. Review of MMT books

A state of the art review of the MMT books shows that their authors use different approaches to deal with the cam profile generation. For a given displacement diagram, most texts present a graphical approach based on envelope theory. This theory establishes that the successive positions of the follower generate an envelope that defines the geometry of the cam profile. The graphical implementation of the envelope theory is intuitive, easy to visualize and very convenient to introduce the profile determination through the kinematic inversion. However, if this method is used for manufacturing purposes, it does not have enough precision and, therefore, it can only be considered valid for generating low-speed cams [14,15]. For high-speed cams, an accurate cam profile is needed and the geometry must be determined analytically. While some classic texts only explain the determination of the cam profile using graphical methods [16–18], most of the existing modern books use geometric equations to obtain analytical expressions of the cam profile [14,15,19–30] and there are also books that come with cam synthesis programs [21,31,32].

Specifically, some books design the cam profile using Cartesian coordinates for roller and flat-faced translating followers using complex algebra [19,21]. Based on the work of Raven [33], Doughty explains thoroughly how to obtain the analytical profile in polar coordinates by using geometric equations [23]. This procedure is applied to translating and oscillating followers with flat-faced and roller contact surfaces. Same equations are also stated in polar coordinates in [14,24,27] and in Cartesian coordinates in [15,28,31]. In [22] it is explained how the analytical profile of the cam is obtained developing differential geometric equations. Likewise, in [20], the same cam geometric profile equations are stated for some translating and oscillating followers. Finally, in [25,34], envelope theory is used to obtain analytical expressions of the cam profile, applying this methodology to different follower types. Among the teaching books on MMT that have been consulted, only Cardona and Clos [35] make an analytical approach to the determination of the cam profile based on the Instantaneous Center of Rotation (ICR). However, this determination is not presented as a general approach but only for knife-edge and flat-faced followers.

At this point, it is worth noting that both ICR and analytical cam design are explained in the majority of MMT books [14,15,18–24,26–28,31]. However, almost none of them link the two concepts and there are only a few examples that use the ICR for cam profile generation, with the exception of said particular cases shown in [35]. For instance, in [19,21,24,28], the ICR is used for calculating the pressure angle in cam mechanism with roller or flat-faced followers, but authors do not use it for the profile generation. In [27] it is suggested the use of the ICR in the case of a roller translating follower with offset, but the developing of the necessary equations to define the cam profile equations is left for the reader. Other books [14,15,18,20,22,23,26,31] do not absolutely relate the ICR with the cam profile generation. Authors believe that the application of a previously studied concept such as ICR to the cam profile design can be pedagogically enriching, both for the understanding of cam profile design and for the review of the ICR.

In what concerns to books dealing exclusively with cams, none of the reviewed books approaches the profile generation making use of the ICR. For instance, in [36] the cam profile is obtained by means of graphical methods as a first step and then profile properties are established with numerical algorithms. Later, ICR is used to calculate the transmission loads. In [32] different approaches are used for the cam profile determination depending on the follower type but the ICR is only used in the case of oscillating flat-faced followers. In turn, Ref. [37] does not use the ICR for the profile generation.

Concerning pedagogically oriented research texts, [38] proposes a unified and general method for the design of the profile. It is based on conjugate geometry and shows a procedure for roller and flat-faced followers. However, the use of conjugate geometry leads to complex mathematical expressions and books on MMT have not incorporated it into their cam profile design methodologies.

In summary, having reviewed the most significant MMT books, it has not been found any that considered the use of ICR for the analytical definition of the cam profile.

1.2. Cam design and ICR in research articles

Out of the pedagogical approach, we must refer to relatively recent research articles [39–41] in which the method of analytical generation of the profile based on ICR is presented to subsequently make different studies on its accuracy. In addition, it is also worth mentioning an article [42] in which the analytical methods of envelope theory and the use of ICR for the definition of curves are compared, giving as an example of application the determination of the profile of disk cams.

In parallel with the design of the cam profile, the same ICR concept can be used to obtain other design parameters that are decisive for the proper functioning of the cam. For instance, in [43] the authors use the ICR to analyze the sliding velocity at the contact point between the cam and the follower in constant-breadth cam mechanisms to predict their correct functioning. Also, the instantaneous radius of curvature of a disk cam profile can be determined by means of the joint locations of the equivalent linkage that can be obtained through the ICR [44]. These same equivalent linkage obtained by means of the ICR is used in Ref. [39] to obtain the mechanical errors of disk cam mechanism with flat-faced follower. The same authors present in [40] a computerized method for the analysis of the tolerances in disk cam mechanisms with a roller follower also based on the ICR. Thus, we can conclude that only few research papers scattered in the literature use the ICR for cam profile design or merely for the analysis as in [39].

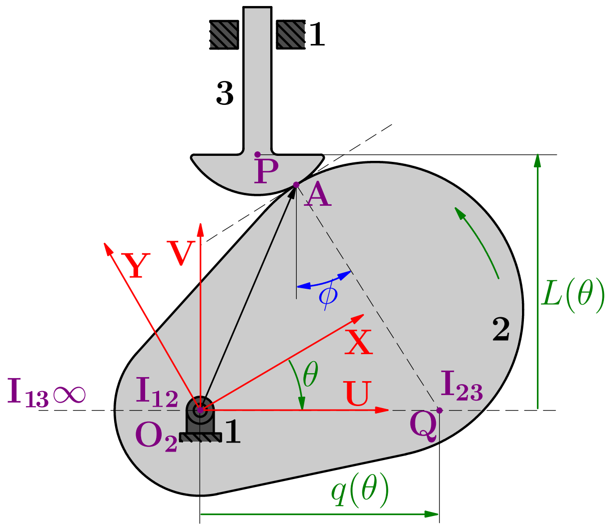

Fig. 1. Schematic of disk cam with translating arbitrary-geometry follower. Notice that the projection base XY is fixed to the cam (moving reference frame) while the projection base UV is fixed to the ground (fixed reference frame).

1.3. Proposed approach

The purpose of this paper is to present a unified and general methodology for the analytical determination of the disk cam profile based on the ICR. The vector formulation is compact and general, and being based on fundamental concepts of the MMT, it is appropriate for teaching cam profile design in the subjects related to MMT of Mechanical Engineering degrees. Special emphasis is placed on showing that the method is valid for any type of follower. As a consequence of this generality, a common notation can be used, making the vector equations common to all cases.

The paper shows the general formulation in Section 2, with which cam profile is calculated. Starting with a generic cam-follower system, the formulation is applied subsequently to knife-edge, flat-faced, roller and arbitrary-geometry followers. Section 3 shows an illustrative example detailing the equations of the profile and the pressure angle. The computational implementation of the method is also included, displaying the code in MATLAB for a couple of examples. Finally, Section 4 provides a summary and conclusions.

2. Determination of the cam profile based on the ICR

In this section a systematic description of the proposed method is given which determines the cam profile based on the ICR and the kinematic inversion. A single step of the procedure is particularized for knife-edge, flat-faced, roller and arbitrary-geometry followers, while in all cases the same notation is used to emphasize the generality of the methodology.

2.1. Velocity analysis and profile generation of a generic cam-follower system

Figs. 1 and 2 show two cam mechanisms with three bodies: ground (1), cam (2) and follower (3).

These three bodies share three ICRs. The ICR I_{12}, for both translating and oscillating followers, is immediately characterized and coincides with the point O_2. In the case of the translating follower, the ICR I_{13} is located at infinity, in the horizontal direction (orthogonal to the direction of translation of the follower with respect to the ground). In the case of the oscillating follower, the ICR I_{13} coincides with the point O_3. By the Aronhold-Kennedy Three-Center Theorem [45,46], for both cases, the ICR I_{23} must lie somewhere on the line defined by ICRs I_{12} and I_{13}. Point I_{23} is defined in terms of distance q, which is not yet known. Furthermore, since I_{23} is the point in the plane that has the same velocity as belonging to bodies 2 or 3, we can write the following identity:

$$\mathbf{v}_{_1}(\mathbf{Q}_2) = \mathbf{v}_{_1}(\mathbf{Q}_3) \tag{1}$$

where for brevity the ICR I_{23} has been denoted as $\mathbf{Q}$ and the velocity of the point $\mathbf{Q}$ as belonging to body i with respect to body j is denoted as $\mathbf{v}_{_j}(\mathbf{Q}_i)$.

In the case of the translating follower, Eq. (1) can be particularized to the following expression:

$$\mathbf{v}_{_1}(\mathbf{O}_2) + \boldsymbol{\omega}_{_1}(2) \times \mathbf{O}_2\mathbf{Q}_2 = \mathbf{v}_{_1}(\mathbf{P}) \tag{2}$$

where $\boldsymbol{\omega}_{_j}(i)$ represents the angular velocity of body i with respect to body j and $\mathbf{O}_2\mathbf{Q}_2$ represents the vector from $\mathbf{O}_2$ to $\mathbf{Q}_2$. Provided that the displacement diagram of the follower is known in terms of the rotation coordinate of the cam ($L = L(\theta)$), the previous equation simplifies to $q\,\dot{\theta} = \dot{L}$. Using the chain rule, the following expression for q is obtained:

$$q = L' \tag{3}$$

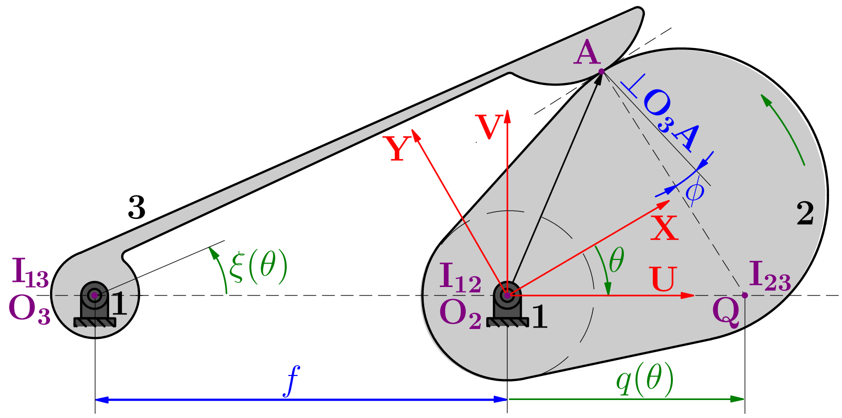

Fig. 2. Schematic of disk cam with oscillating arbitrary-geometry follower. Notice that the projection base XY is fixed to the cam (moving reference frame) while the projection base UV is fixed to the ground (fixed reference frame).

where the dot notation is used for derivatives with respect to time and the prime notation is used for partial derivatives with respect to the cam rotation coordinate θ.

Analogously, for the oscillating follower of Fig. 2, provided that the displacement diagram of the follower is known in terms of the rotation coordinate of the cam ($\xi = \xi(\theta)$), for a known distance f between points O_2 and O_3, Eq. (1) can be particularized to:

$$\mathbf{v}_1(\mathbf{O}_2) + \boldsymbol{\omega}_1(2) \times \mathbf{O}_2\mathbf{Q}_2 = \mathbf{v}_1(\mathbf{O}_3) + \boldsymbol{\omega}_1(3) \times \mathbf{O}_3\mathbf{Q}_3 \tag{4}$$

which simplifies to: $q\,\dot{\theta} = (q + f)\,\dot{\xi}$. Using again the chain rule, the next expression for q is obtained:

$$q = f\,\frac{\xi'}{1 - \xi'} \tag{5}$$

At this step of the procedure, the calculation of the position of the cam-follower contact point (**A**) can be carried out in a simple way using the ICR, but this calculation will depend on the particular geometry of each follower. For this reason, the following sections will show how to use the ICR to determine the coordinates of vector $\mathbf{O_2A}$ in the projection base UV (fixed to the ground) for each of the followers.

Finally, assuming that the coordinates of $\mathbf{O_2A}$ are known in base UV, the analytical expression of the cam profile will be determined by the coordinates of this same vector expressed in base XY (fixed to the cam) which will be parameterized in terms of θ.

It is important to note that the *base change* of vector $\mathbf{O_2A}$ used in the analytical determination of the cam profile, is equivalent to the realization of a *kinematic inversion* in the graphical profile generation procedure. Therefore, by means of the parallelism that arises between these two tools that are fundamental in the kinematics of mechanisms, it is possible to relate the graphical and analytical methods [35], which is of great pedagogical interest.

At this point, it has been shown how useful it is to use the concept of the ICR to obtain analytical expressions of the geometry in terms of the displacement diagram and some geometric parameters, as the offset of the follower and the base radius of the cam. However, for the design of the cam, it is very useful to know other additional aspects of the kinematics of the cam, in order to evaluate the suitability of a preliminary design. Some of these variables are the pressure angle, the sliding velocity and the radius of curvature of the cam profile. Thus, it can be very useful to have analytical expressions of these magnitudes to be able to search for the combination of parameters that provides the best possible design. Fortunately, once the analytical expressions of the vectors used for the cam profile determination are available, it is almost immediate to obtain analytical expressions for pressure angle, the sliding velocity and the radius of curvature of the profile.

As an example, in order to calculate the pressure angle of the cam-follower system, it would suffice to calculate the angle between the vectors **V** and **AQ** for the case of translating follower, and between a vector perpendicular to $\mathbf{O_3A}$ and the vector **AQ** for the case of oscillating follower. As in previous steps we have obtained analytical expressions for these vectors, by straight forward calculations we can obtain an analytical expression for the pressure angle. Likewise, the calculation of the sliding velocity and radius of curvature of the cam profile can be derived directly from the same vectors.

2.2. The knife-edge follower

The simplest case of cam profile definition, regardless of the method used, is that of the knife-edge follower.

Fig. 3 shows a knife-edge translating follower. Making use of the expression of q provided by Eq. (3), it is trivial to obtain the components of vectors $\mathbf{O_2Q}$, **QP** and **PA** in base UV. Thus, the definition of the cam profile is simply given by the vector:

$$\mathbf{O_2A} = \mathbf{O_2Q} + \mathbf{QP} + \mathbf{PA} \tag{6}$$

written in base XY.

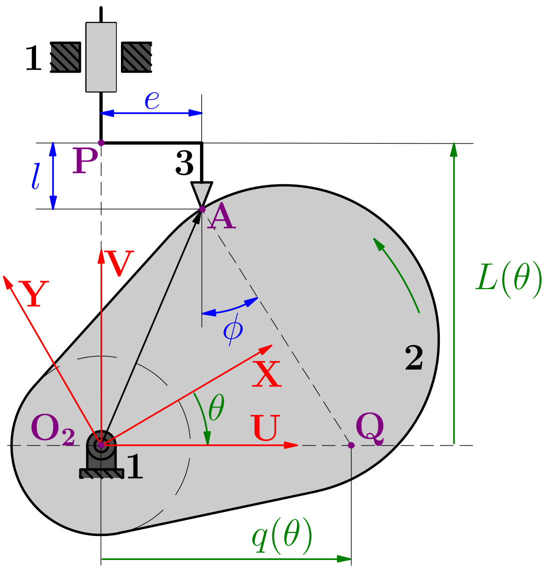

Fig. 3. Schematic of a disk cam with a knife-edge translating follower.

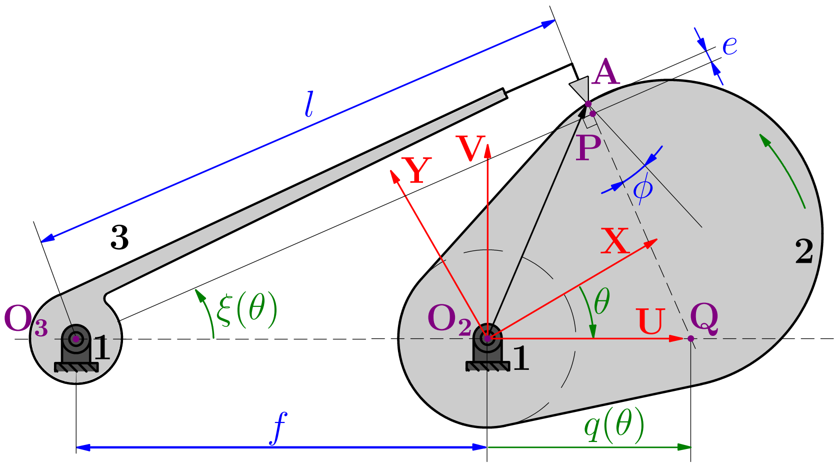

Fig. 4. Schematic of a disk cam with a knife-edge oscillating follower.

Fig. 4, on the other hand, shows the schematic corresponding to a knife-edge oscillating follower. For this case, we proceed in an analogous manner. Using the expression for the distance q provided by Eq. (5), it is trivial to write the components of the vectors $\mathbf{O_2Q}$ and $\mathbf{QO_3}$ in base UV and $\mathbf{O_3P}$ and $\mathbf{PA}$ in a base fixed to the follower. Thus, the definition of the cam profile is now given by the vector $\mathbf{O_2A}$ obtained by means of:

$$\mathbf{O_2A} = \mathbf{O_2Q} + \mathbf{QO_3} + \mathbf{O_3P} + \mathbf{PA} \tag{7}$$

written in base XY.

It should be noted that for the case of knife-edge followers, both translating and oscillating, it is not strictly necessary to use the Instantaneous Center of Rotation to obtain the cam profile. The reason is that, for these cases, the contact point between the cam and the follower (point **A**) can be achieved by means of a constant (and known) vector from the point that defines the translation or rotation of the follower (point **P**). However, for the other cases, the contact point is not known directly and it has to be calculated, for example, by means of the ICR.

2.3. Cam profile determination for flat-faced followers

For the determination of the cam profile, in the case of flat-faced, roller and arbitrary-geometry followers, the ICR concept will be used. The procedure described below is used in [39] for analyzing mechanical errors of disk cam mechanism.

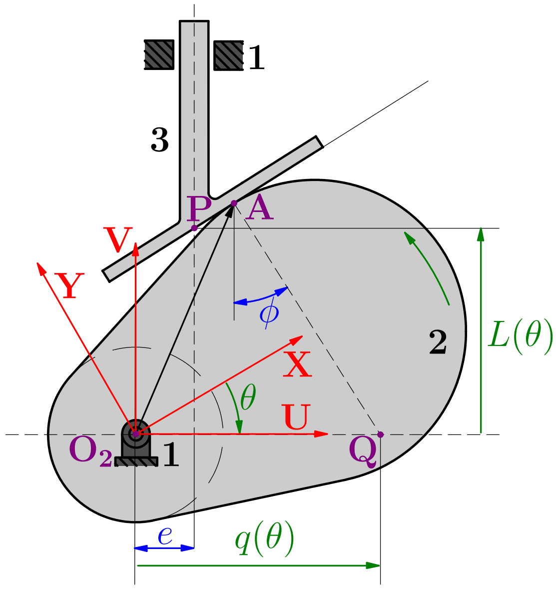

Fig. 5. Schematic of a disk cam with a slanted translating flat-faced follower.

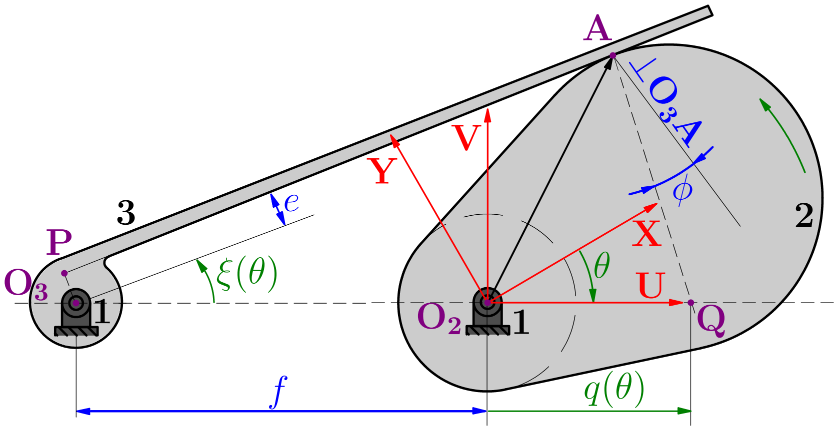

Fig. 6. Schematic of a disk cam with an oscillating flat-faced follower.

For the translating follower of Fig. 5, using the expression for the distance q of Eq. (3), it is trivial to write the components of the vector $\mathbf{QP}$ in base UV. As the joint between the cam and follower is a higher pair (commonly named as *cam joint*) the ICR between cam and follower (point $\mathbf{Q}$) is constrained to be in the line that contains the contact point and is orthogonal to both profiles at the contact point. Thus, scalar multiplying vector $\mathbf{QP}$ by a unit vector in the direction orthogonal to the flat face of the slanted follower (whose orientation is given by ϕ) the distance $\|\mathbf{QA}\|$ is obtained. Furthermore, multiplying this distance by the mentioned unit vector, $\mathbf{QA}$ is obtained. Finally, vector $\mathbf{O_2A}$ is calculated as in Eq. (6) and the cam profile is defined writing it in base XY.

Fig. 6, on the other hand, shows the schematic corresponding to a flat-faced oscillating follower. For this case, the procedure is analogous to the previous one. Using the expression for the distance q provided by Eq. (5), it is trivial to write the components of the vector $\mathbf{QO_3}$ in base UV and the components of vector $\mathbf{O_3P}$ in a base fixed to the follower. Multiplying the sum of these vectors ($\mathbf{QP}$) by a unit vector in the direction orthogonal to the flat face of the follower (whose orientation is given by $\xi(\theta)$), the distance $\|\mathbf{QA}\|$ is obtained. Furthermore, multiplying this distance by the mentioned unit vector, $\mathbf{QA}$ is obtained. Finally, vector $\mathbf{O_2A}$ is calculated as in Eq. (7) and the cam profile is defined writing it in base XY.

2.4. Roller followers

The determination of the cam profile when using roller followers such as those shown in Figs. 7 and 8 is quite similar to that of flat-faced followers. The procedure explained below is previously used in [40] for analyzing the tolerances in disk cam mechanisms with roller followers.

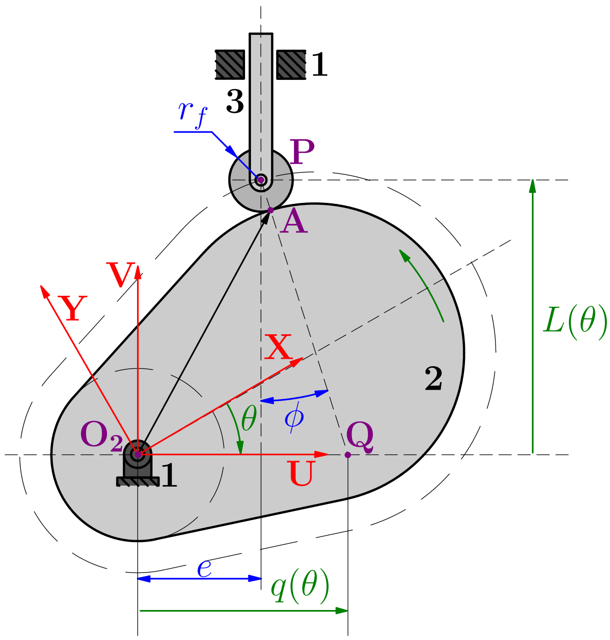

Fig. 7. Schematic of a disk cam with a translating roller follower.

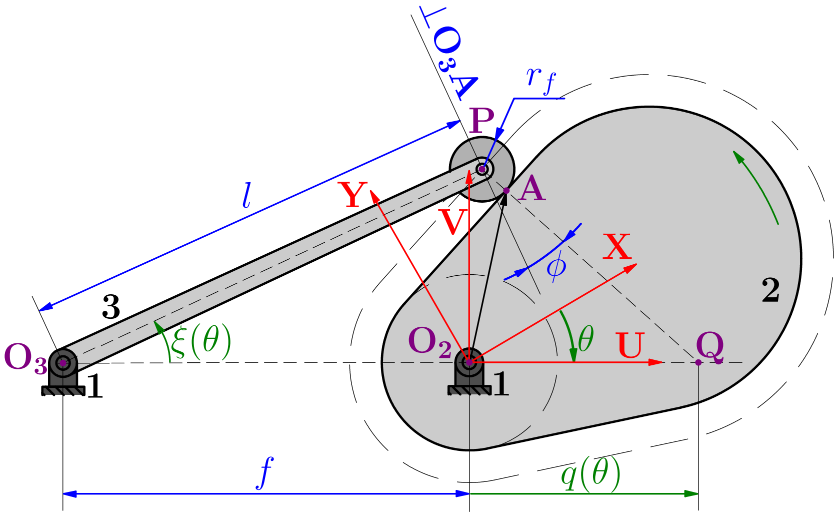

Fig. 8. Schematic of a disk cam with an oscillating roller follower.

For the translating roller follower of Fig. 7, as in previous cases, the first step is to find the ICR (point **Q**) that lies at a distance q from the point $\mathbf{O_2}$. As in previous cases q takes the value given in Eq. (3). Knowing the location of **Q**, vector **QP** can be easily obtained in base UV in terms of L and q. Since the follower is a roller, it is also known that the point of contact between cam and roller (point **A**) will lay on the line defined by points **Q** and **P**. Thus, vector **PA** can be written as a vector with direction **PQ** and modulus r_f (radius of the roller) as:

$$\mathbf{PA} = r_f \frac{\mathbf{PQ}}{\|\mathbf{PQ}\|} \tag{8}$$

As a result, it is once more possible to write the position vector that provides the cam profile by means of Eq. (6), finally writing it in base XY which is fixed to the cam.

For the case of an oscillating roller follower depicted in Fig. 8, the procedure is analogous to the previous cases. The distance q at which point **Q** is located is calculated by means of Eq. (5). Now it is trivial to write the components of vector $\mathbf{QO_3}$ in base UV and the components of vector $\mathbf{O_3P}$ in a base fixed to the follower. At this moment, knowing that the point of contact between cam

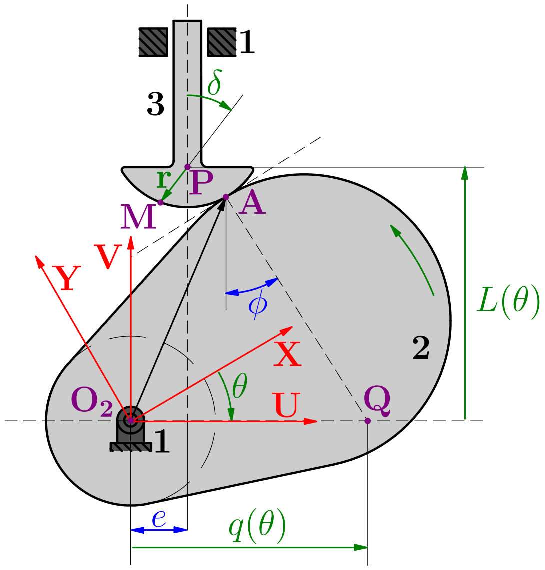

Fig. 9. Schematic of a disk cam with a translating arbitrary-geometry follower.

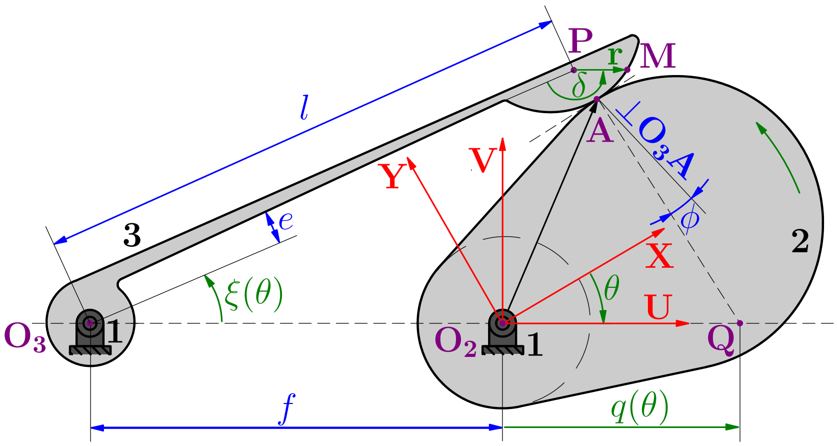

Fig. 10. Schematic of a disk cam with an oscillating arbitrary-geometry follower.

and roller (point **A**) will always be located on the line joining points **Q** and **P**, vector **PA** can be obtained by Eq. (8). Finally, the cam profile is defined by writing vector $\mathbf{O_2A}$ by means of Eq. (7) in base XY.

2.5. Arbitrary-geometry followers

Finally, we are going to deal with the analytical determination of the cam profile using arbitrary-geometry followers such as those schematized in Figs. 9 and 10. The geometry of the follower is parameterized by a set of equations. If these equations can be solved analytically, the profile definition will be obtained as a function of the cam angle of rotation. On the other hand, if they can only be solved numerically, the profile will be defined by a set of points.

The first steps of the procedure, again, are analogous to the previous cases. For the cases of translating and oscillating follower, first of all the location of point **Q** is determined through Eqs. (3) and (5), respectively.

At this point it is necessary to proceed differently than when the follower was a knife-edge, flat-faced or roller follower. Since its geometry is now arbitrary, we cannot know *a priori* where the contact point is located. To find it in the more general case, the geometry of the follower must be first parameterized. An arbitrary point **M** of the surface of the follower (defined from point **P**) is positioned by vector **r** whose modulus and direction are functions of coordinate δ, as shown in Figs. 9 and 10. The parameterization of the point **M** could also be done with two coordinates (x, y), in a base fixed to the follower, defining the vector **r** and an implicit function relating them $(F(x, y) = 0)$. For the parameterization used in Figs. 9 and 10, the equations to be considered are the following:

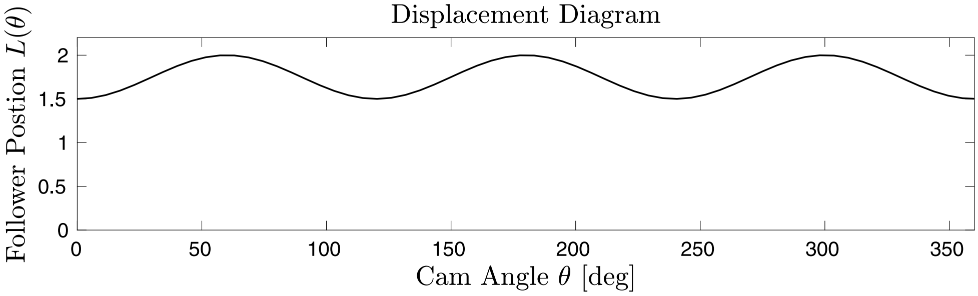

Fig. 11. Displacement diagram of the roller follower.

$$\frac{d\mathbf{r}}{d\delta} \cdot \mathbf{QA} = 0 \tag{9}$$

and

$$\mathbf{O_2Q} + \mathbf{QA} = \mathbf{O_2P} + \mathbf{r} \tag{10}$$

where the dot $(\cdot)$ denotes scalar product between two vectors. Thus, Eq. (9) imposes that the surface of the follower at point $\mathbf{M}$ be orthogonal to the vector $\mathbf{QA}$, while Eq. (10) imposes that points $\mathbf{A}$ and $\mathbf{M}$ coincide in the plane. As mentioned above, if these three scalar equations can be solved by giving an explicit expression of the vector $\mathbf{QA}$ in terms of q, f, e, l, θ, L and ξ, an explicit expression of the cam geometry can be obtained. Finally, we can define the cam profile by writing the vector $\mathbf{O_2A}$ by means of Eqs. (6) and (7) in the XY base for the translating and oscillating follower, respectively.

3. Illustrative example

This section details step by step two illustrative examples of cam profile synthesis by applying the proposed methodology, both for roller and arbitrary-geometry translating followers. After the exhaustive description of the theoretical procedure, and due to the generality of the method used, illustrating the theory with examples reduces to writing the components of several vectors in the corresponding bases.

3.1. Cam profile synthesis for translating roller follower

As mentioned above, Fig. 7 shows a schematic of the mechanism to be constructed where r_f is the roller radius and e is the offset. In this example, the displacement diagram of Fig. 11 defined by the Eq. (11) has been chosen:

$$L(\theta) = R_0 + H\frac{1-\cos 3\theta}{2} \tag{11}$$

where R_0 represents the base circle radius and H the amplitude of the displacement diagram. In practical applications the displacement diagrams will be commonly defined by piecewise functions. However, in order to keep the example as simple as possible, a single function is used for the whole cam rotation.

The chosen values for the parameters of Eq. (11) are $H = 0.5$ and $R_0 = 1.5$. Note that, although Figs. 11 and 12 are drawn for this specific displacement diagram, the analytic expressions are written in terms of a generic $L(\theta)$. Moreover, for the sake of simplicity, the dependence on the angle rotated by the cam will be omitted, i.e. $L(\theta) \equiv L$.

The first step is to obtain the position of the ICR, point $\mathbf{Q}$, according to Eq. (3):

$$q = L' = \frac{3}{2}H\sin(3\theta); \tag{12}$$

By means of this equation the components of the vector $\mathbf{QO_2}$ expressed in base UV can be obtained:

$$\{\mathbf{QO_2}\}_{UV} = \begin{Bmatrix} -q \\ 0 \end{Bmatrix}_{UV} \tag{13}$$

The rest of the necessary vectors can be easily obtained from Fig. 7. Vector $\mathbf{O_2P}$ looks as follows:

$$\{\mathbf{O_2P}\}_{UV} = \begin{Bmatrix} e \\ L \end{Bmatrix}_{UV} \tag{14}$$

and by means of Eq. (8) the components of $\mathbf{PA}$ are obtained:

$$\{\mathbf{PA}\}_{UV} = \frac{-r_f}{\sqrt{(e-q)^2 + L^2}} \begin{Bmatrix} e-q \\ L \end{Bmatrix}_{UV} \tag{15}$$

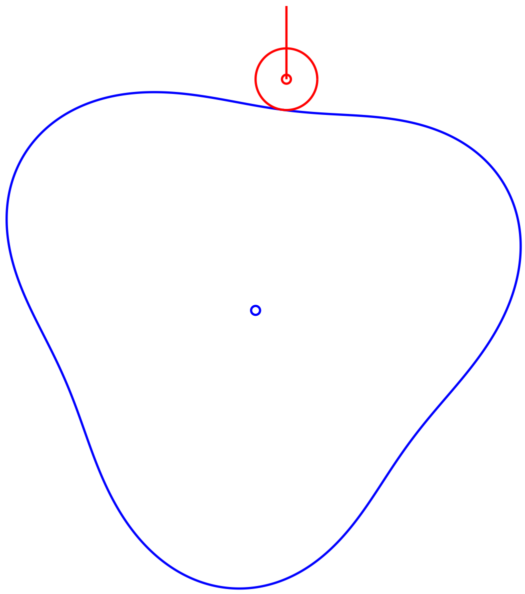

Fig. 12. Final solution for the cam profile for example 1.

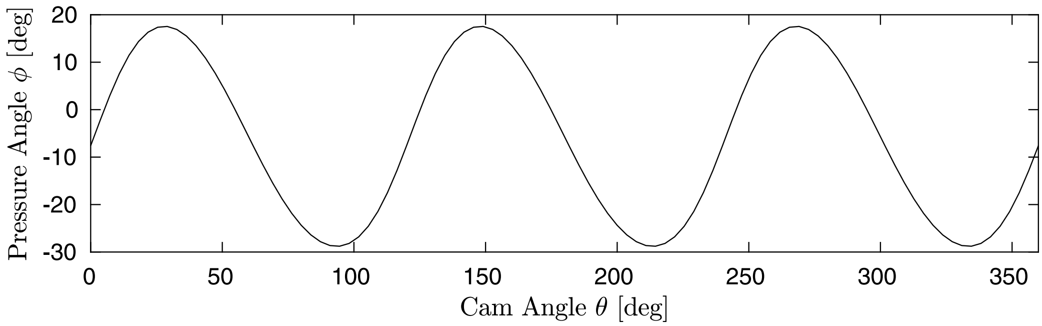

Fig. 13. Pressure angle in terms of the cam rotation angle for example 1.

The components of vector $\mathbf{O_2A}$ expressed in base UV can be obtained by Eq. (6), resulting in:

$$\left\{\mathbf{O_2A}\right\}_{UV} = \left\{\begin{matrix} e \\ L \end{matrix}\right\}_{UV} + d\left\{\begin{matrix} e-q \\ L \end{matrix}\right\}_{UV} = \left\{\begin{matrix} e+d(e-q) \\ L(1+d) \end{matrix}\right\}_{UV} \tag{16}$$

where $d = \dfrac{-r_f}{\sqrt{(e-q)^2+L^2}}$.

Finally, changing these components to the base XY fixed to the cam the expression that provides the cam profile is obtained:

$$\left\{\mathbf{O_2A}\right\}_{XY} = \left\{\begin{matrix} (e+d(e-q))\cos\theta + L(1+d)\sin\theta \\ -(e+d(e-q))\sin\theta + L(1+d)\cos\theta \end{matrix}\right\}_{XY} \tag{17}$$

Additionally, the pressure angle expression can be calculated as:

$$\phi = atan\left(\frac{q-e}{L}\right) \tag{18}$$

For the specific values of $e = 0.2$ and $r_f = 0.2$, the profile obtained can be seen in Fig. 12 and the pressure angle is shown in Fig. 13.

To show the simplicity of this method, the implementation in MATLAB of the cam profile design is shown in Listing 1.

```
clear all; close all; clc

% Angle to draw the cam profile
theta = linspace(0,2*pi,100);

R_0 = 1.5; % Base circle radius
```

```
e   = 0.2; % Off-set
H   = 0.5; % Displacement diagram max height
r_f = 0.2; % Roller radius

% Displacement diagram: L(theta)
L =H*(1-cos(3*theta))/2+R_0;

% IRC calculation: q(theta)
q = (3*H*sin(3*theta))/2;

% Kinematic inversion
for i=1:length(theta)
    O2_P = [ e;L(i)];
    Q_O2 = [-q(i);0];
    Q_P  = Q_O2 + O2_P;
    P_A  = -r_f*Q_P/norm(Q_P);
    O2_A = O2_P + P_A;

    % Change of base matrix for kinematic inversion
    II_xy2uv = [+cos(theta(i)),+sin(theta(i));
                -sin(theta(i)),+cos(theta(i))];

    profile(:,i) = II_xy2uv*O2_A;
    pressureAngle(:,i) = atan2(q(i) - e,L(i));
end

% Draw the profile
plot(profile(1,:),profile(2,:),'b');
axis equal

% Draw the pressure angle
figure()
plot((180/pi)*theta,(180/pi)*pressureAngle,'b');
axis([0 360 -30 20])
```

Listing 1: MATLAB implementation of the cam profile design for example 1.

3.2. Cam profile synthesis for translating arbitrary-geometry follower

Let us now solve the previous example changing only the geometry of the follower, from a simple roller to an elliptic follower. Thus, Fig. 9 corresponds with this example and the geometry of the follower is defined in terms of vector $\mathbf{r} = \mathbf{r}(\delta)$. For this particular example, let us define $\mathbf{r}$ for certain parameters a and b, as:

$$\{\mathbf{r}\}_{UV} = \begin{Bmatrix} -a\,\sin\delta \\ -b\,\cos\delta \end{Bmatrix}_{UV} \tag{19}$$

For the definition of the displacement diagram in the previous case, the expressions for $L(\theta)$, $\mathbf{QO_2}$ and $\mathbf{O_2P}$ remain unchanged, as defined in Eqs. (11), (13) and (14), respectively. Parameterizing the unknown vector $\mathbf{QA}$ as:

$$\{\mathbf{QA}\}_{UV} = \begin{Bmatrix} u \\ v \end{Bmatrix}_{UV} \tag{20}$$

vectors $\mathbf{r}$ and $\mathbf{QA}$ would have to be calculated solving for δ, u and v in the system of nonlinear Eqs. (9) and (10), which would take the form:

$$\begin{aligned} v\,b\,\sin\delta - u\,a\,\cos\delta &= 0 \\ u - e + q + a\,\sin\delta &= 0 \\ v - L + b\,\cos\delta &= 0 \end{aligned} \tag{21}$$

Once the values for δ, u and v are calculated for discrete values of θ, it only remains to calculate the cam profile defined by $\mathbf{OA}$ with either sides of Eq. (10).

For this example, the pressure angle would be calculated as:

$$\phi = atan\left(\frac{-u}{v}\right) \tag{22}$$

For the specific values of $e = 0.2$, $a = 0.4$ and $b = 0.2$, the profile obtained can be seen in Fig. 14 and the pressure angle is shown in Fig. 15.

For the sake of completeness, the MATLAB implementation of this problem is shown in Listing 2.

```
clear all; close all; clc

% Angle to draw the cam profile
theta = linspace(0,2*pi,100);

```

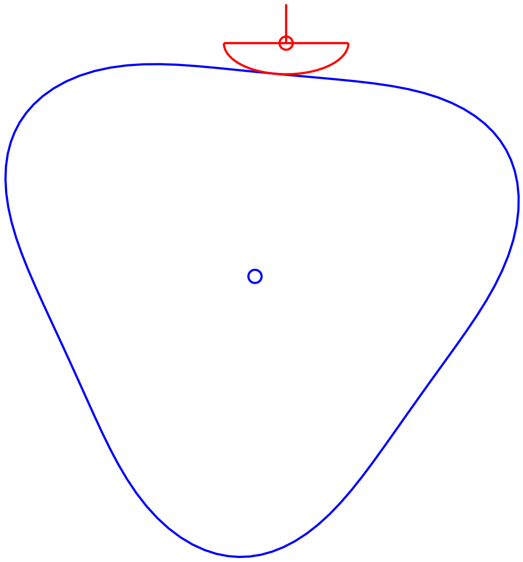

Fig. 14. Final solution for the cam profile for example 2.

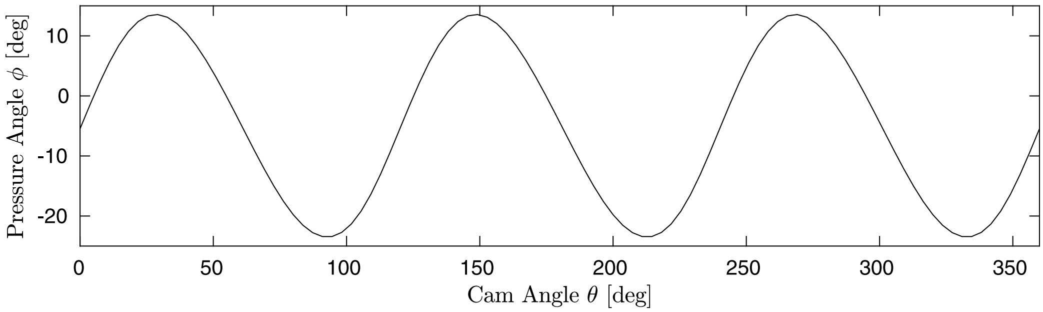

Fig. 15. Pressure angle in terms of the cam rotation angle for example 2.

```
R_0 = 1.5;   % Base circle radius
e   = 0.2; % Off-set
H   = 0.5; % Displacement diagram max height
a   = 0.4; % Horizontal semi-axis of the ellipse
b   = 0.2; % Vertical semi-axis of the ellipse

% Displacement diagram: L(theta)
L =H*(1-cos(3*theta))/2+R_0;

% O2_P calculation
O2_P=[e*ones(1,length(L));L];

% IRC calculation: q(theta)
q = (3*H*sin(3*theta))/2;

% Guessed value for the unknowns: unk_0=[delta,u,v];
unk_0=[0;-q(1)+e;L(1)];

% Kinematic inversion
for i=1:length(theta)
    % Parameters of the nonlinear equations
    params = [q(i);e;L(i);a;b];

    % Solve for the unknows [delta,u,v]
    unk_sol=fsolve(@(unk)nonlinear_system(unk,params),unk_0);
    unk_0=unk_sol;

    % Calculate vector O2_A = O2_Q + r
    O2_A = [q(i);0] + [unk_sol(2);unk_sol(3)];
```

```

    % Change of base matrix for kinematic inversion
    II_xyz2uvw = [+cos(theta(i)),+sin(theta(i));
                  -sin(theta(i)),+cos(theta(i))];

    profile(:,i) = II_xyz2uvw*O2_A;
    pressureAngle(:,i) = atan2(-unk_sol(2),unk_sol(3));
end

% Draw the profile
figure()
plot(profile(1,:),profile(2,:),'b');
axis equal

% Draw the pressure angle
figure()
plot(theta,pressureAngle,'b');

function res=nonlinear_system(unk,params)
    delta=unk(1);   u=unk(2); v=unk(3);
    q=params(1);  e=params(2); L=params(3); a=params(4); b=params(5);

    res=[ v*b*sin(delta) - u*a*cos(delta);
          u - e + q + a*sin(delta);
          v - L + b*cos(delta)];
end
```

Listing 2: MATLAB implementation of the cam profile design for example 2.

4. Conclusions

A review of books on Mechanism Machine Theory has been carried out to evaluate the approach of cam profile generation based on previously calculated displacement diagrams. It has been found that different approaches are used and they must be formulated individually according to the specific type of follower. There is therefore no general method in MMT books that can be applied to any type of follower. With the aim of having a unified and general methodology for teaching cam profile design, a compact analytical formulation has been proposed. The formulation is relatively simple and is based on two fundamental concepts of the Mechanism and Machine Theory: the Instantaneous Center of Rotation and the Kinematic Inversion.

The presented formulation is valid for any kind of follower, regardless its output movement (translating or oscillating) and its shape (knife-edge, flat-faced, roller or arbitrary-geometry). It has been shown that it can be easily applied to said different followers. Further, from a pedagogical point of view, it is based on previously studied concepts which allows to acquire a solid foundation and to advance firmly in the knowledge of the Mechanisms and Machines Theory.

Due to the generality, simplicity and the compactness of its formulation, the methodology can be easily implemented in a programming language. This implementation has been shown by means of two illustrative examples in MATLAB code.

Because of these qualities, the proposed formulation is considered to be a general method for the analytical determination of the cam profile in the teaching of the subject of Mechanism and Machine Theory in the degree of Mechanical Engineering. Authors believe that the proposed methodology can be used in forthcoming MMT books.

CRediT authorship contribution statement

Xabier Iriarte: Conceptualization, Formal analysis, Methodology, Software, Supervision, Writing – original draft, Writing – review & editing. **Julen Bacaicoa:** Data curation, Visualization, Writing – original draft, Writing – review & editing. **Aitor Plaza:** Conceptualization, Formal analysis, Methodology, Software, Supervision, Visualization, Writing – original draft, Writing – review & editing. **Jokin Aginaga:** Conceptualization, Methodology, Supervision, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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