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# Extensions of Fuzzy Sets in Image Processing: An Overview

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## 1. Introduction

Computer vision systems are made up of several stages in which they try to extract the information of the image. The stages and the purpose of its algorithms, depends on the application of the artificial vision system. Usually the systems are split in three main stages. The first stage is devoted to noise filtering, smoothing or contrast enhancement (it's also known as preprocessing or low level vision), the second stage is devoted to image segmentation, to split objects or regions according to their characteristics (intermediate level vision). Third stage involves the understanding of the scene (high level vision).

Uncertainty is present in every process of computer vision, therefore fuzzy techniques have been widely use in almost any of the processes. Extensions of fuzzy sets are not as specific as their counter-parts of fuzzy sets, but this lack of specificity makes them more realistic for some applications. Their advantage is that they allow us to express our uncertainty in identifying a particular membership function. This uncertainty is involved when extensions of fuzzy sets are processed, making results of the processing less specific but more reliable. Many authors based on this advantage proposed different image processing algorithms using extensions of fuzzy sets. This work presents a valuable review for the interested reader of the recent works using extensions of fuzzy sets in image processing. The chapter is divided as follows: first we recall the basics of the extensions of fuzzy sets, i.e. Type-2 fuzzy sets, Interval-valued fuzzy sets and Atanassov's Intuitionistic fuzzy sets. In sequent sections we review the methods proposed for noise removal (section 3), image enhancement (section 4), edge detection (section 5) and segmentation (section 6). There exist other image segmentation tasks such as video de-interlacing, stereo matching or object representation that are not described in this work.

## 2. Extensions of fuzzy sets

From the beginning it was clear that fuzzy set theory [48] was an extraordinary tool for representing human knowledge. Nevertheless, Zadeh himself established (see [49]) that sometimes, in decision-making processes, knowledge is better represented by means of some generalizations of fuzzy sets. A key problem of representing the knowledge by means of Fuzzy sets is to choose the membership function which best represents such knowledge.

Sometimes, it is appropriate to represent the membership degree of each element to the fuzzy set by means of an interval. From these considerations arises the extension of fuzzy sets called *theory of interval-valued fuzzy sets*, that is, fuzzy sets such that the membership degree of each element of the fuzzy set is given by a closed subinterval of the interval $[0, 1]$. Hence, not only vagueness (lack of sharp class boundaries), but also a feature of uncertainty (lack of information) can be addressed intuitively.

These sets were first introduced in the 1970s. In May 1975 Sambuc (see [37]) presented in his doctoral thesis the concept of an interval-valued fuzzy set named a $\Phi$-fuzzy set. That same year, Zadeh [49] discussed the representation of type 2 fuzzy sets and its potential in approximate reasoning.

The concept of a *type 2 fuzzy set* was introduced by Zadeh [49] as a generalization of an ordinary fuzzy set. Type 2 fuzzy sets are characterized by a fuzzy membership function, that is, the membership value for each element of the set is itself a fuzzy set in $[0, 1]$.

Formally, given the referential set $U$, a type 2 fuzzy set is defined as an object $\overline{\overline{A}}$ which has the following form:

$$\overline{\overline{A}} = \{(u, x, \mu_u(x)) | u \in U, x \in [0, 1]\},$$

where $x \in [0, 1]$ is the primary membership degree of $u$ and $\mu_u(x)$ is the secondary membership level, specific to a given pair $(u, x)$.

One year later, Grattan-Guinness [27] established a definition of an interval-valued membership function. In that decade interval-valued fuzzy sets appeared in the literature in various guises and it was not until the 1980s, that the importance of these sets, as well as their name, was definitely established.

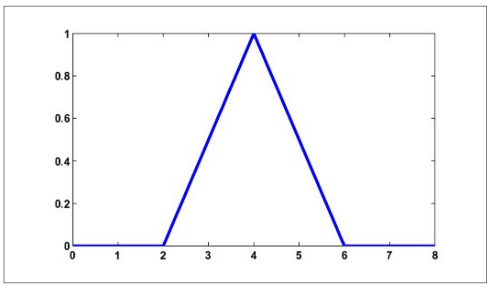


**Fig1. Fuzzy membership function.**

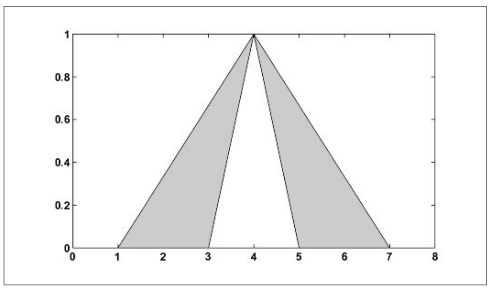


**Fig2. Interval valued fuzzy membership function. Lower bound and upper bound**

A particular case of a type 2 fuzzy set is an *interval type 2 fuzzy set* (see [31]–[32]). An interval type 2 fuzzy set $\overline{\overline{A}}$ in $U$ is defined by

$$\overline{\overline{A}} = \{(u, A(u), \mu_u(x)) | u \in U, A(u) \in L([0,1])\},$$

where $A(u)$ is a closed subinterval of $[0,1]$, and the function $\mu_u(x)$ represents the fuzzy set associated with the element $u \in U$ obtained when $x$ covers the interval $[0,1]$; $\mu_u(x)$ is given in the following way:

$$\mu_u(x) = \begin{cases} a \text{ if } \underline{A}(u) \leq x \leq \overline{A}(u) \\ 0 \text{ otherwise} \end{cases},$$

where $0 \leq a \leq 1$. It turns out that an interval type 2 fuzzy set is the same as an IVFS if we take $a = 1$.

Another important extension of fuzzy set theory is the theory of Atanassov's intuitionistic fuzzy sets ([1], [2]). Atanassov's intuitionistic fuzzy sets assign to each element of the universe not only a membership degree, but also a nonmembership degree, which is less than or equal to 1 minus the membership degree.

An Atanassov's intuitionistic fuzzy set (A-IFS) on $U$ is a set

$$\hat{A} = \{(u, \mu_{\hat{A}}(u), \nu_{\hat{A}}(u)) | u \in U\},$$

where $\mu_{\hat{A}}(u) \in [0,1]$ denotes the membership degree and $\nu_{\hat{A}}(u) \in [0,1]$ the nonmembership degree of $u$ in $\hat{A}$ and where, for all $u \in U$, $\mu_{\hat{A}}(u) + \nu_{\hat{A}}(u) \leq 1$.

In [1] Atanassov established that every Atanassov intuitionistic fuzzy set $\hat{A}$ on $U$ can be represented by an interval-valued fuzzy set $A$ given by

$$\begin{aligned} A: \quad & U \to L([0,1]) \\ & u \to [\mu_{\hat{A}}(u), 1 - \nu_{\hat{A}}(u)], \qquad \text{for all } u \in U. \end{aligned}$$

Using this representation, Atanassov proposed in 1983 that Atanassov's intuitionistic fuzzy set theory was equivalent to the theory of interval-valued fuzzy sets. This equivalence was proven in 2003 by Deschrijver and Kerre [20]. Therefore, from a mathematical point of view, the results that we obtain for IVFSs are easily adaptable to A-IFSs and vice versa. Nevertheless, we need to point out that, conceptually, the two types of sets are totally different. This is made clear when applications of these sets are constructed (see [43]).

In 1993, Gau and Buehrer introduced the concept of *vague sets* [26]. Later, in 1996, it was proven that vague sets are in fact A-IFSs [6].

A compilation of the sets that are equivalent (from a mathematical point of view) to interval-valued fuzzy sets can be found in [21]. Two conclusions are drawn from this study:

1.- Interval-valued fuzzy sets are equivalent to A-IFSs (and therefore vague sets), to grey sets (see [19]) and to $L$-fuzzy set in Goguen's sense with respect to a special lattice $L([0,1])$.

2.- IVFSs are a particular case of probabilistic sets (see [23]), of soft sets (see [3]), of Atanassov's interval-valued intuitionistic fuzzy sets and evidently of Type 2 fuzzy sets.

## 3. Noise Reduction

In this section we focus on gray scale images and the use of fuzzy logic theory extensions for noise removal.

Many applications of image processing perform image noise removal before any further processing such as segmentation, enhancement, edge detection or compression. Noise removal is a crucial task for most applications mainly for two reasons:

- Digital images are often corrupted by impulse noise during image acquisition, image transmission and image processing due to a number of imperfections present in these tasks environments.

- The noise can critically affect any image post-processing decisively compromising their performance.

For these reasons, noise removal is one of the most important steps for most image processing applications, regarding the overall performance of the applications and, is still a challenging problem in image processing.

When using fuzzy sets for noise removal, since their memberships are crisp values, it is not possible to know which one is the best membership function and, different membership functions will lead to different processing results. On the other hand, when using type-2 fuzzy sets, since their membership functions are also fuzzy, provides us with a more efficient way of dealing with uncertainty and, consequently, a more robust way of determining the membership functions.

In their work, Sun and Meng [39], make the assumption that, since impulse corruption is usually large compared with the strength of the signal, then the noise corrupted pixels have intensity values that are near the saturated values (lower and upper limits of the gray scale used). Based on this assumption, a inverted ladder membership function is used as initial membership function from which the type-2 fuzzy set is constructed (Fig. 3).

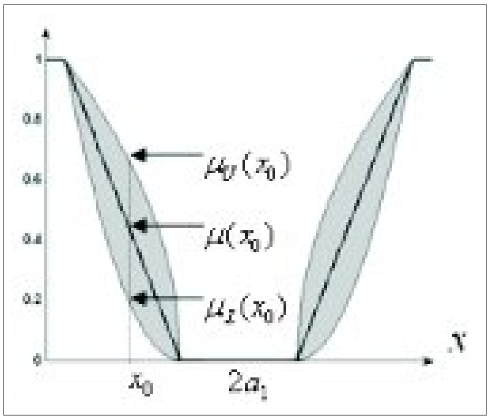


**Fig3. Type-2 Fuzzy Set constructed from the initial ladder membership function**

After defining the inverted ladder membership function based on parameters $a$, $a_1$ and $a_2$, the lower and upper limits of the type-2 fuzzy set are obtained using the following equations:

$$\mu_U(x) = [\mu(x)]^{\frac{1}{\alpha}}$$
$$\mu_L(x) = [\mu(x)]^{\alpha}$$

where $\alpha \in (1, 2]$.

After the construction of the type-2 fuzzy set, the probability of a pixel been corrupted by impulse noise (PPC) is represented by the centroid-type-reduction [46] of the set. This way, the larger PPC means the bigger probability of the pixel been corrupted and, the less PPC means the less probability. Finally, a threshold of corruption is established in such way that if the pixel's PPC is bigger than the threshold its restored value is the mean of the pixels intensities in the subimage ($n \times n$ filtering window centered at the pixel) and, if the pixel's PPC is less than the threshold it maintains its intensity.

Tulin, Basturk and Yuksel [42] proposed a type-2 fuzzy operator for detail preserving restoration of impulse noise corrupted images, that processes the pixels contained in a 3×3 filtering window and outputs the restored value of the window center pixel. The proposed operator is structure that combines four type-2 neuro-fuzzy filters, four defuzzifiers and a postprocessor. Each one of the type-2 NF filters are identical and accept the center pixel and two of its neighborhoods (processing the horizontal, vertical, diagonal and reverse diagonal pixel neighborhoods of the filtering window) and produces an output that represents the uncertainty interval (i.e., lower and upper bounds) for the center pixel restored value in the form of a type-1 interval. Each combination of the inputs of the filter and their associated type-2 interval Gaussian membership functions is represented by a rule in the form:

$$if(X_1 \in M_{i1})and(X_2 \in M_{i2})and(X_3 \in M_{i3}), then$$
$$R_i = k_{i1}X_1 + k_{i2}X_2 + k_{i3}X_3 + k_{i4}$$

There are $N$ fuzzy rules in the rulebase where, $R_i$ denotes the output of the $i$th rule ($N$ rules, $i = 1, 2, \cdots N$) and $M_{ij}$ denotes the $i$th Gaussian membership function of the $j$th input (3 inputs, $j = 1, 2, 3$) and is defined as follows:

$$M_{ij}(u) = \exp\left[-\frac{1}{2}(\frac{u - m_{ij}}{\sigma_{ij}})\right]$$

where $m_{ij}$ and $\sigma_{ij}$ are the mean and the standard deviation of the membership function, respectively.

Note that the membership functions $M_{ij}$ (Fig. 4) are interval membership functions with their boundaries characterized by upper and lower Gaussian membership functions (see [42]).

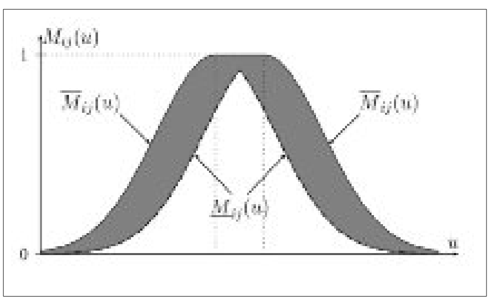


**Fig4. Type-2 interval Gaussian membership function**

The optimal values for the parameters of each one of the type-2 interval Gaussian membership functions are obtained by training using the least squares optimization algorithm. Finally, the output of each NF filter is the weighted average of the individual rules outputs.

The defuzzifier blocks convert the input fuzzy sets coming from the corresponding type-2 NF filters into a scalar value by performing centroid defuzzification (the centroid is the center of the type-1 interval fuzzy set).

The output of the defuzzifier blocks are four scalar values that are to be the candidates for the filtering window center pixel restored value. The postprocessor converts the four candidates into a single output value by discarding the lowest and the highest values and averaging the remaining two values.

In [42] Tulin, Basturk and Yuksel proposed an extension of this operator where the the original neighborhood topologies (horizontal, vertical, diagonal and reverse diagonal pixel neighborhoods) are extended to 28 possible neighborhood topologies corresponding to a filtering operator with 28 NF filters, 28 defuzzifiers and a postprocessor. However, authors emphasized that, for most filtering applications, one does not have to use all of these neighborhood topologies.

In this new operator, the NF filters and the defuzzifiers work in the same way as in the original operator. The way the new postprocessor produces the output value from the scalar values obtained at the outputs of the defuzzifiers is different from the original one. The postprocessor calculates the average of its inputs and truncates it to a 8-bit integer number, which is the output of the operator.

Wang, Chung, Hu and Wu [45] proposed a interval type-2 fuzzy filter for Gaussian noise suppression while maintaining the original structure of the image. Based on interval type-2 fuzzy sets is constructed a selective feedback fuzzy neural network (SFNN) suitable for image representation. In reality, this SFNN is a universal approximator that works as a filter for Gaussian noise suppression that preserves the fine structure of the image from the theoretical viewpoint.

First, the image is fuzzified in using linguistic concepts in such way that the gray scale interval is equally divided in $K_0$ partitions. Each one of these partitions is described with a Gaussian shape and its membership function obtained by drawing all the Gaussians having mean and standard deviation.

The proposed SFNN has five components: two neurons in the input layer (since, the input is a 2-D digital image), three hidden layers and the output layer. Since, a digital image can be viewed as a continuous 2-D function and, according to the structure of the SFNN (see [45]) it can be used to approximate a continuous function, the SFNN is used to express a digital image and, a Gaussian noise filter is designed based on it. For each one of the $K_0$ partitions a optimal gray level is computed in a small operating window (mean of the pixels in the window) and, the window gray level which is nearest to the optimal gray level is selected as the optimal gray level in the operating window.

After the operating window slides over the whole image, the mean absolute error of the input image and the output image (filtered image) are calculated and if the difference between these two values is less than a small positive number the process stops; otherwise the gray levels of the input image are replaced by the gray levels of the output image and the process is restarted.

In [4], Bigand and Colot, use IVFS entropy to take into account the uncertainty present in the image noise removal process. Their motivation is to assess, and ultimately remove, the uncertainty of the membership values using the length of the interval in an IVFSs (the longer the interval, the more uncertainty).

For each pixel, the uncertainty of a precise FS is modeled by the closed intervals delimited by the upper ($\mu_U(x)$) and lower ($\mu_L(x)$) membership functions defined as follows:

$$\mu_U(x) = [\mu(x; g, \sigma)]^{\frac{1}{2}}$$

$$\mu_L(x) = [\mu(x; g, \sigma)]^{2}$$

where $\mu(x; g, \sigma)$ is a Gaussian fuzzy number defined as:

$$\mu(x; g, \sigma) = \exp\left[-\frac{1}{2}(\frac{x-g}{\sigma})^2\right]$$

Then, in order to be used as a criterion to automatically find fuzzy region width and thresholds for segmentation of noisy images, the entropy (which they wrongly called index of ultrafuzziness) of each one of the constructed IVFSs is calculated as follows:

$$\Gamma(x) = \frac{1}{M.N}\sum_{g=0}^{G-1}[h(x).(\mu_U(x) - \mu_L(x))]$$

where $G$ is the number of gray levels of a $M \times N$ image.

This entropy is used to obtain a image homogram (in the same way as proposed by Cheng [18] using FSs) since, according to the authors, $\Gamma$ represents the homogeneity distribution across intensities of the considered image. This homogram is used to find all major homogeneous regions while filtering noise in such way that, if a pixel belongs to one of these regions then the pixel is noise-free else pixel is noisy.

Finally, using a $3\times3$ filtering window, the median filter is applied to all the pixels identified as noisy pixels.

**Discussion** Efficient noise removal in corrupted images is still a challenging problem in image processing mostly due to the imperfection/uncertainty inevitably present in noisy environments. The main idea of these works is to take into account the total amount of the imprecision/uncertainty present in the process of image noise filtering by means of the use of fuzzy sets extensions namely, interval-valued fuzzy sets and type-2 fuzzy sets.

All the presented filters inherit the advantages of fuzzy sets extensions theory, where an extra degree of fuzziness provides a more efficient way of dealing with uncertainty than with ordinary fuzzy sets.

Hence, using fuzzy sets extensions in noise filtering seems to bee a very promising idea that can lead to the design of efficient filtering operators.

## 4. Enhancement

In image enhancement, the main goal is to produce a new image that endows more accurate information for analysis than the original one. In this context, fuzzy logic extensions are used to represent and manipulate the uncertainty involved in the image enhancement process. Both type-2 fuzzy sets and A-IFSs approaches presented in this section, are able to model and minimize the effect that the uncertainty has in the image enhancement problem.

**Image Enhancement using Type-2 Fuzzy Sets** In [22], Ensafi and Tizhoosh proposed a type-2 fuzzy image enhancement method based on extension of the locally adaptive fuzzy histogram hyperbolization method [41], improving its performance by extending it from a type-1 to a type-2 method where the additional third dimension of the type-2 sets gives more degrees of freedom for better representation of the uncertainties associated with the image.

First, based on the value of homogeneity $\mu_{Homo}$, the image is divided into several sub-images.

$$\mu_{Homo} = \left(1 - \frac{g_{maxLocal} - g_{minLocal}}{g_{maxGlobal} - g_{minGlobal}}\right)^2$$

Setting the minimum and maximum local window sizes to 10 and 20 respectively and, using a fuzzy if-then-else rule, the local window size surrounding each supporting point is calculated.

The type-2 fuzzy set is obtained by blurring the type-1 fuzzy set defined by the membership function $\mu(g_{mn})$ of each gray level.

$$\mu(g_{mn}) = \frac{g_{mn} - g_{min}}{g_{max} - g_{min}}$$

Where $g_{mn}$ represents the pixel gray level and, $g_{min}$ and $g_{max}$ represent the image minimum and maximum gray levels respectively.

The type-2 fuzzy set is constructed defining the upper and lower membership values using interval-valued fuzzy sets in the following way:

$$\mu_{UPPER}(x) = (\mu(x))^{0.5} \quad \mu_{LOWER}(x) = (\mu(x))^2$$

And, the proposed type-2 membership function is defined by:

$$\mu_{T2}(g_{mn}) = (\mu_{LOWER} \times \alpha) + (\mu_{UPPER} \times (1-\alpha))$$

with $\alpha = \frac{g_{Mean}}{L}$.

where, $L$ is the number of gray levels and $g_{Mean}$ the mean gray value of each sub-image.

Finally, using the $\mu_{T2}$ values, the new gray levels of the enhanced image are calculated using the following expression with $\beta = 1.1$:

$$g'_{mn} = \left(\frac{L \times 1}{e'^1 \times 1}\right) \times \left[e'^{\mu(g_{mn})^\beta} - 1\right]$$

**Image Enhancement using A-IFSs Entropy** In [44], Vlachos and Sergiadis studied the role of entropy in A-IFSs for contrast enhancement. Different concepts of entropy on A-IFss and their behavior are analyzed in the context of image enhancement.

In this work, image enhancement is regarded as a entropy optimization problem within an A-IFS image processing framework where, in the first stage the image is transferred into the fuzzy domain and sequentially into the A-IFS domain, where the proposed processing is performed. Finally, the inverse process is carried out in order to obtain the processed image in the gray-level domain.

Therefore, an intuitionistic fuzzification scheme for constructing an A-IFS for representing the image, based on entropy optimization, was proposed. First, the image ($N \times M$ pixels having $L$ gray-levels) is represented in the fuzzy domain by $A$.

$$A = \{\langle g_{ij}, \mu_A(g_{ij})\rangle | g_{ij} \in 0, \times, L-1\}$$

with, $i \in \{1, \times, M\}$ and $j \in \{1, \times, N\}$.

A optimal derivation of a combination of membership and non-membership functions that model the image gray-levels is achieved by maximizing the intuitionistic fuzzy entropy of the image (i.e., maximum intuitionistic fuzzy entropy principle).

First, the membership function $\mu_A(g)$ of the fuzzified image is calculated.

$$\mu_A(g) = \frac{g - g_{min}}{g_{max} - g_{min}}$$

The A-IFS for representing the image,

$$\hat{A} = \left\{\langle g, \mu_{\hat{A}}(g;\lambda), \nu_{\hat{A}}(g;\lambda)\rangle | g \in 0, \times, L-1\right\}$$

is then constructed, by means of $\mu_A(g)$, using the following expressions:

$$\mu_{\hat{A}}(g, \lambda) = 1 - (1 - \mu_A(g))^{\lambda}$$

and

$$\nu_{\hat{A}}(g, \lambda) = (1 - \mu_A(g))^{\lambda(\lambda+1)}$$

where $\lambda$ is obtained using an optimization criterion that is formulated as follows:

$$\lambda = \arg \max_{\lambda \geq 1} \{E(\hat{A};\lambda)\}$$

Where $E$ is an entropy measure. Entropies proposed by Burillo and Bustince [5] and by Szmidt and Kacprzyk [38] where used.

The defuzzification is made using the *maximum index of fuzziness intuitionistic defuzzification* [43] by selecting a parameter $\alpha$ in the following way:

$$\alpha = \begin{cases} 0, & \text{if } \alpha' < 0 \\ \alpha', & \text{if } 0 \leqslant \alpha' \leqslant 1 \\ 1, & \text{if } \alpha' > 1 \end{cases}$$

where

$$\alpha' = \frac{\sum_{g=0}^{L-1} h_A(g)\pi_{\hat{A}}(g;\lambda)(1 - 2\mu_{\hat{A}}(g;\lambda))}{2\sum_{g=0}^{L-1} h_A(g)\pi_{\hat{A}}^2(g;\lambda)}$$

with $h_A$ being the histogram of the fuzzified image $A$.

Finally, the new intensity gray levels $g'$ are obtained through the expression:

$$g' = (L-1)\mu_{D_\alpha(\hat{A})}(g)$$

where

$$\mu_{D_\alpha(\hat{A})}(g) = \alpha + (1-\alpha)\mu_{\hat{A}}(g;\lambda) - \alpha\nu_{\hat{A}}(g;\lambda)$$

**Discussion** In their work, Ensafi and Tizhoosh [22] demonstrated that using type-2 fuzzy logic a better image contrast enhancement is achieved than with its type-1 counterpart.

In Vlachos and Sergiadis approach [44], A-IFSs are used as a mathematical framework to deal with the vagueness present in a digital image by means of the A-IFS extra degree of freedom that allows a flexible modeling of imprecise and/or imperfect information present in images, better than classical fuzzy sets. They concluded that the different notions of intuitionistic fuzzy entropy used [5, 38] treat images in different ways, making the selection of the appropriate entropy measure to be application-dependent.

## 5. Edge Detection

In the ideal case, the result of applying an edge detector to an image may lead to a set of connected curves that indicate the boundaries of objects, the boundaries of surface markings as well curves that correspond to discontinuities in surface orientation. Thus, applying an edge detector to an image may significantly reduce the amount of data to be processed and may therefore filter out information that may be regarded as less relevant, while preserving the important structural properties of an image.

**Fig 6. Example image**

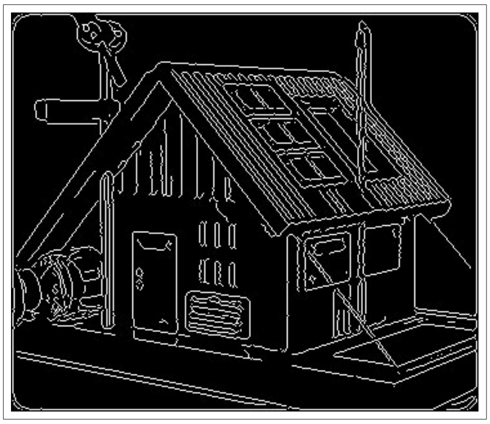

**Fig7. Example of edge detection**

One of the seminal works of edge detection was Canny's [15]. Its aim was to discover the optimal edge detection algorithm. In this situation, an optimal edge detector means:

- Good detection - the algorithm should mark as many real edges in the image as possible.
- Good localization - edges marked should be as close as possible to the edge in the real image.
- Minimal response - a given edge in the image should only be marked once, and where possible, image noise should not create false edges.

Some fuzzy approaches to edge detection already exist and perform quite well (see [28]). Why then work with an extension of fuzzy sets? A classical definition of what an edge within an image should be is: *a significantly change in the intensity of adjacent neighboring pixels.* To state a specific threshold on how large the intensity change between two neighboring pixels must be, is not a simple task and obviously depends on the scene, illumination etc. So it's clear that extensions of fuzzy sets can be used to deal with this uncertain concept.

In the literature of extensions of fuzzy sets applied to edge detection there exist three different approaches. In the first one the main idea is to assign each pixel of the image with an interval, and then measuring its entropy decide if it's edge or not. In the second approach a interval type fuzzy system is used to classify pixels and, in the third approach AIFSs are used to deal with the uncertainty of edge pattern matching.

**First approach [12]:** Consider the fact that edge detection techniques attempt to find pixels whose intensity (gray level) is very different from those of its neighbors. An element of f (image) belongs to an edge if there is a big enough difference between its intensity and its neighbors' intensities. (Notice that this definition is intentionally fuzzy in its own right). The method begins assigning an IVFS to each matrix f and therefore each element has associated an interval as membership degree. The lower and upper bounds of this interval are determined by the concepts of tn-processing and sn-processing

**Definition 1** *([10]) Consider a matrix $f \in \mathcal{M}$, any two t-norms $T_1$ and $T_2$ in $[0,1]$, and a positive integer $n$ less than or equal to $\frac{N-1}{2}$ and $\frac{M-1}{2}$. We define the tn-processing of $f$ as follows:*

$$g^n_{T_1,T_2} : \mathcal{M} \to \mathcal{M}_n \text{ given by}$$
$$g^n_{T_1,T_2}(f(x,y)) = \overset{n}{\underset{\substack{i=-n\\ j=-n}}{T_1}} \left(T_2(f(x-i,y-j), f(x,y))\right)$$
$$\text{with } n \le x \le N-(n+1), n \le y \le M-(n+1)$$

In this case it is said that we use a submatrix of order $(2n+1)\times(2n+1)$.

**Definition 2** *([10]) Let $n$ be an integer number greater than zero. We define the <u>IVn matrix</u> associated with $f \in \mathcal{M}$ as the interval-valued fuzzy set $G^n$ given by*

$$G^n = \big\{((x,y), G^n(x,y) = \big[g^n_{T_1,T_2}(f(x,y)), g^n_{S_1,S_2}(f(x,y))\big] \in L([0,1]))|x \in X, y \in Y\big\}$$

*being $g^n_{T_1,T_2}$ and $g^n_{S_1,S_2}$ the tn-processing and the sn-processing given by Definition 1.*

Obviously, we can also associate with each matrix $f$ the following interval-valued fuzzy set $\mathbf{f} : \mathbf{f} = \{((x,y), \mathbf{f}(x,y) = [f(x,y), f(x,y)] \in L([0,1]))|x \in X, y \in Y\}$.

The following definition associates with the IVFS $G^n$ a fuzzy set whose membership function is the length of the intervals in $G^n$.

**Definition 3** *([10]) Given a matrix $f \in \mathcal{M}$ and its corresponding $G^n$. We call W-matrix of $f$, a new matrix obtained by assigning to each of its elements the corresponding interval length of $G^n$. It is denoted as $W(G^n)$. Therefore $W(G^n) = \big\{((x,y), W(G^n)(x,y) = g^n_{S_1,S_2}(f(x,y)) - g^n_{T_1,T_2}(f(x,y)))|(x,y) \in X \times Y\big\} \in FSs(X \times Y)$.*

Every matrix $f$ is thus associated with an interval-valued fuzzy set $G^n$ and a fuzzy set $W(G^n)$.

The normalized entropy of $G^n$ is given by the following expression:

$$\mathcal{E}_{\mathcal{N}}(G^n) = \frac{\sum\limits_{\substack{n \le x \le N-(n+1)\\ n \le y \le M-(n+1)}} g^n_{S_1,S_2}(f(x,y)) - g^n_{T_1,T_2}(f(x,y))}{(N-2\cdot n)\times(M-2\cdot n)} \quad (2)$$

It is logical to relate those elements of $G^n$ whose membership degree intervals are large to the location of the edge. This fact leads us to establish the following:

The normalized entropy of $G^n$ establishes the average length of the intervals that represent the membership function of the elements. A large entropy implies that the intervals are large, meaning that the difference between $g^n_{T_1,T_2}$ and $g^n_{S_1,S_2}$ tends to be large for each element. In such a matrix there are many changes in intensity, and a higher proportion of elements belong to an edge. Conversely, a small entropy implies that few elements of the matrix belong to an edge. The normalized entropy of $G^n$ therefore indicates the *proportion of elements* that are part of an edge.

**Definition 4** *We say that an element $(x,y)$ belongs to the bright zone edge if it satisfies the following two items:*

*(i)* $f(x,y) \geq \frac{g^n_{\vee,\vee}(f(x,y))+g^n_{\wedge,\wedge}(f(x,y))}{2} = K_{0.5}(G^n(x,y))$, *and*

*(ii) its interval has sufficient length.*

To identify elements of $G^n$ with a long enough interval to belong to an edge, we start by considering two intensity values $p$ and $q$ whose values are yet to be determined. For the moment, we simply require $p \leq q$ and $p, q \in [0,1]$ (in a binary image, we would have $p = 0$ and $q = 1$). From these two values we base on the following rule to obtain the edges:

(a) *If* $W(G^n(x,y)) \geq q$*, then the element belongs to an edge.*

(b) *If* $p \leq W(G^n(x,y)) < q$*, then we need a way of distinguishing elements that belong to the edge from those that do not belong.*

(c) *If* $W(G^n(x,y)) < p$*, then the element* $(x,y)$ *generally does not belong to the edge.* Such elements can be considered, however, if very few elements satisfy conditions (a) and (b).

From previous rule is proposed the following algorithm, in which relevant pixels are added to the edge by means of different actions (please see [12]).

**(BZ1)** *Calculate the IVn matrix* $G^n$ *and its associated W-matrix (by means of* $g^n_{\wedge,\wedge}$ *and* $g^n_{\vee,\vee}$*).*

**(BZ2)** *Calculate* $p$ *and* $q$*, then construct the sets* $G^n_p$*,* $G^n_{p,q}$ *and* $G^n_q$*.*

**(BZ3)** *Calculate the entropies* $\mathcal{E}_\mathcal{N}(G^n_p)$*,* $\mathcal{E}_\mathcal{N}(G^n_{p,q})$ *and* $\mathcal{E}_\mathcal{N}(G^n_q)$*.*

**(BZ4)** *Calculate* $\mathbf{T}_{\wedge,\wedge}(\mathbf{f}, G^n)$*.*

**(BZ5)** *Calculate* $\mathcal{E}_\mathcal{N}(\mathbf{T}_{\wedge,\wedge}(\mathbf{f}, G^n)^n_p)$*,* $\mathcal{E}_\mathcal{N}(\mathbf{T}_{\wedge,\wedge}(\mathbf{f}, G^n)^n_{p,q})$ *and* $\mathcal{E}_\mathcal{N}(\mathbf{T}_{\wedge,\wedge}(\mathbf{f}, G^n)^n_q)$*.*

**(BZ6)** *Execute* ***(Action1)****,* ***(Action2)*** *or* ***(Action3)****.*

**(BZ7)** *Sum the binary images obtained in (BZ6).*

**(BZ8)** *Clean and thin the lines.*

**Fig 8. Example image**

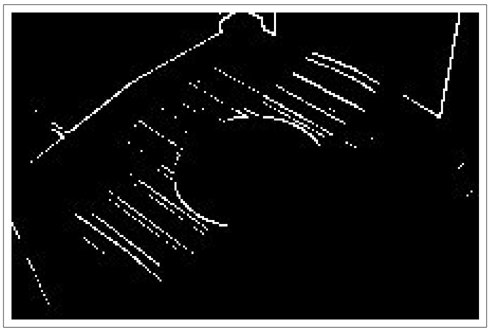

**Fig 9. Edge image after Action 1**

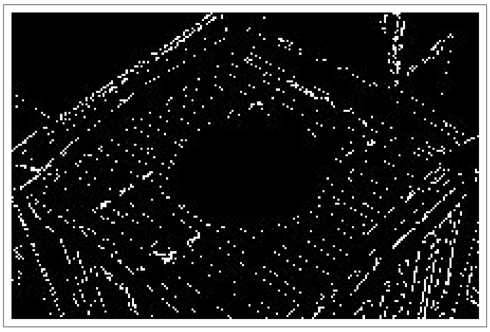

**Fig 10. Edge image after Action 2**

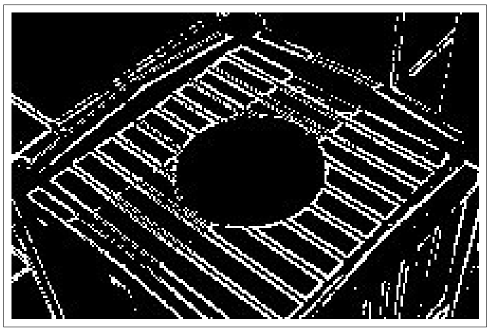

**Fig 11. Edge image after Action 3**

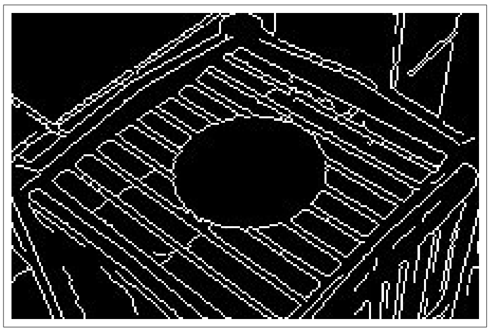

**Fig 12. Edge image after BZ8**

**Second approach [33, 34]**: the goal is to design a system which makes it easier to include edges in low contrast regions, but which does not favor false edges by effect of noise. Because of this specifications the authors design an Interval type 2 fuzzy system.

The system has 4 inputs and one output that is the degree of edginess of each pixel.

The input variables are the gradients with respect to x-axis and y-axis, to which they call DH and DV respectively. The other two inputs are the pixels filtered when convolute two masks to the original image. One is a high-pass filter and the other a low-pass filter. The high-pass filter HP detects the contrast of the image to guarantee the border detection in relative low contrast regions. The low-pass filter M allow to detect image pixels belonging to regions of the input were the mean gray level is lower. These regions are proportionally more affected by noise, supposed uniformly distributed over the whole image.

Seven interval valued fuzzy rules allow to evaluate the input variables, so that the obtained image displays the edges of the image in color near white (HIGH tone), whereas the background was in tones near black (tone LOW).

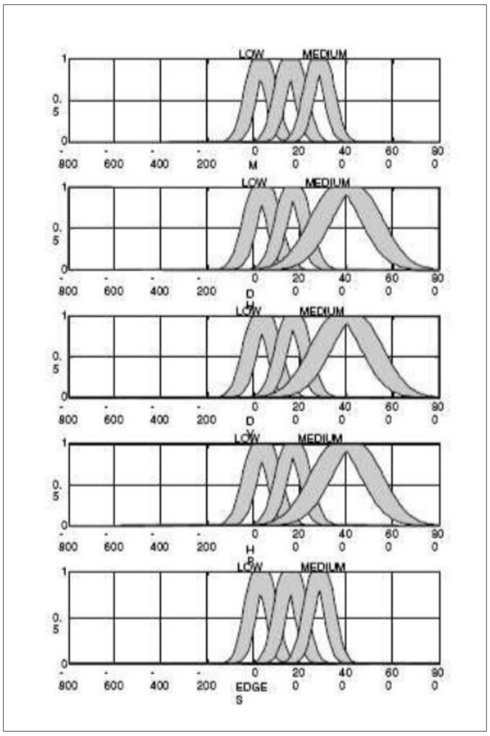


**Fig 13. Interval valued membership functions designed for the input variables and the output variable: Degree of "edginess".**

1. If (DH is LOW) and (DV is LOW) then (EDGES is LOW)
2. If (DH is MEDIUM) and (DV is MEDIUM) then (EDGES is HIGH)
3. If (DH is HIGH) and (DV is HIGH) then (EDGES is HIGH)
4. If (DH is MEDIUM) and (HP is LOW) then (EDGES is HIGH)
5. If (DV is MEDIUM) and (HP is LOW) then (EDGES is HIGH)
6. If (M is LOW) and (DV is MEDIUM) then (EDGES is LOW)
7. If (M is LOW) and (DH is MEDIUM) then (EDGES is LOW)

**Third Approach [16]:**the authors propose an intuitionistic fuzzy divergence to deal with the uncertainty in an edge pattern matching scheme. The intuitionistic fuzzy set takes into account the uncertainty in assignment of membership degree known as hesitation degree.

The main idea behind template edge matching is to detect tipical intensity distributions that are usually in edges. The authors propose the following algorithm to deal with uncertainty present in the matching process by means of an intuitionistic divergence.

- Step 1. Form 16 edge-detected templates.
- Step 2. Apply the edge templates over the image by placing the center of each template at each point (i,j) over the normalized image.
- Step 3. Calculate the intuitionistic fuzzy divergence (IFD) between each elements of each template and the image window and choose the minimum IFD value.
- Step 4. Choose the maximum of all the 16 minimum intuitionistic fuzzy divergence values.
- Step 5. Position the maximum value at the point where the template was centered over the image.
- Step 6. For all the pixel positions, the max-min value has been selected and positioned.
- Step 7. A new intuitionistic divergence matrix has been formed.
- Step 8. Threshold the intuitionistic divergence matrix and thin.
- Step 9. An edge-detected image is obtained.

Experimental studies reveal that, for edge detection the result is completely dependent on the selection of hesitation constant and thereby by the hesitation degree (also called the intuitionistic fuzzy index). The intuitionistic method detects the dominant edges clearly, while removing the unwanted edges.

## 6. Segmentation

Image segmentation is a critical and essential component of image analysis and/or pattern recognition system and is one of the most difficult tasks in image processing, that can determine the quality of the final result of the system.

The goal of image segmentation is the partition of an image in different areas or regions.

**Definition 5** *Segmentation is grouping pixels into regions such that*

1. $\cup_{i=1}^{k} P_i$ = *Entire image* ($\{P_i\}$ *is an exhaustive partitioning).*
2. $P_i \cap P_j = 0, i \neq j$ ($\{P_i\}$ *is an exclusive partitioning).*
3. *Each region* $P_i$ *satisfies a predicate; that is, all points of the partition have some common property.*
4. *Pixels belonging to adjacent regions, when taken jointly, do not satisfy the predicate.*

There exist three different approaches using fuzzy methods:

- Histogram thresholding.
- Feature space clustering.
- Rule based systems.

There exist different works that use extensions of fuzzy sets within the three approaches. The most commonly studied method is thresholding with extensions of fuzzy sets. The first work on this topic was made by Tizhoosh [40].

One of the earlier papers of fuzzy thresholding is [36] in 1983. The idea behind the fuzzy thresholding is to first transfer the selected image feature into a fuzzy subset by means of a proper membership function and then select and optimize a global or local fuzzy measure to attain the goal of image segmentation.

In [36] the authors used the S-function to fuzzify the image. They minimize the entropy in such a way that the final segmented image is the one which has less doubtful pixels.

Said membership functions represent the brightness set within the image. The basic idea of using this membership function is that, if we take the value of a parameter as the threshold value, the dark pixels should have low membership degrees, and on the contrary, brighter pixels should have high membership degrees. The pixels with membership function near 0.5 should be the ones that are not clearly classified. Therefore the set with less entropy is the set with less amount of pixels with uncertain membership (around 0.5).

In 2005 Tizhoosh [40] presented a paper that uses Interval type 2 fuzzy sets in image thresholding (we must point out that he tries to use type 2 fuzzy sets, however in the paper he only uses Interval Type 2 fuzzy sets). His study is based on the modification of the classical fuzzy algorithm of Huang and Wang [29], so that he applies an $\alpha$ factor as interval generator to the membership function. Starting from a membership function, Tizhoosh obtains an interval valued fuzzy set that "contains" different membership functions and is useful for finding the threshold of an image.

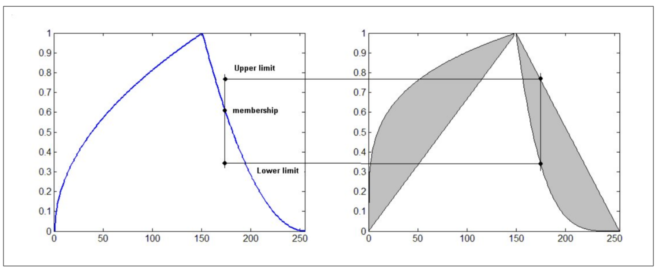


**Fig.14 Threshold as a fuzzy number used by Tizhoosh. Transformation of a FS into a IVFS.**

He assigns each intensity with the following interval of membership, $\mu_U$ and $\mu_L$ being the upper and lower membership degrees:

$$\mu_L(g) = \mu(g)^{\alpha} \qquad (3)$$
$$\mu_U(g) = \mu(g)^{1/\alpha}$$

With $\alpha \in (1, \infty)$, and therefore $0 \leq \mu_L(q) \leq \mu_U(q) \leq 1$. Sometimes parameter $\alpha$ can be interpreted as linguistic edges.

Tizhoosh's idea for proposing his algorithm is to *"remove the uncertainty of membership values by using type II fuzzy sets"*.

Vlachos and Sergiaidis [43] also propose a modification of Tizhoosh's algorithm, using Atanassov's intuitionistic fuzzy sets (see [1]). Their basis are membership functions similar to Huang's but, instead of minimizing the entropy, the algorithm minimizes the divergence with set $\hat{\mathbf{1}}$ (see [17]). The structure of the intuitionistic algorithm is the same as Tizhoosh's. The construction of the intuitionistic fuzzy sets is done in the following way:

$$\mu_{\hat{A}}(g,t) = \lambda\mu_A(g,t) \qquad (4)$$
$$\nu_{\hat{A}}(g,t) = (\hat{1} - \lambda\mu_A(g,t))^{\lambda}$$

With $\lambda \in [0, 1]$, being $\hat{A}$ an intuitionistic fuzzy set, and the divergence:

$$D_{IFS}(\hat{A}, \hat{1}, t) = \sum_{g=0}^{L-1} h_A(g) \left( \mu_{\hat{A}(g,t)} ln \frac{2\mu_{\hat{A}}(g,t)}{1 + \mu_{\hat{A}}(g,t)} + \nu_{\hat{A}}(g,t) ln2 + ln \frac{2}{1 + \mu_{\hat{A}}(g,t)} \right) \qquad (5)$$

Tizhooh's algorithm is applied directly to color segmentation using RGB in [35]. Moreover in [47] it's used to segment color image skin lesions.

But there exist an improvement of Tizhoosh algorithm, that arises from the selection of the membership functions. It was proved, that the membership functions that best represent the image are the ones used in [25, 29], that represent how similar the intensity of each pixel is to the mean of the intensities of the object or to the mean of the intensities of the background. By defining the functions in this way, the set with lowest entropy is the set that contains the greatest number of pixels around the mean of the intensities of the background and the mean of the intensities of the object.

Starting form the idea of obtaining the uncertainty from the information given by the user, we have proposed several approximations using A-IFS [8, 9] (where the key point is to calculate the intuitionistic index) and also using interval valued fuzzy sets [7] (where the key point is to calculate the lengths of the intervals). These works lead us to introduce the concept of an ignorance function to try to model the lack of knowledge from which experts may suffer when determining the membership degrees of some pixels of an image Q to the fuzzy set representing the background of the image, $Q_B$, and to the fuzzy set representing the object in the image, $Q_O$.

The classical fuzzy thresholding algorithm is modified due to the user should pick two functions, one to represent the background and another one to represent the object. We have chosen this representation since, in this way, the expert is able to get a better representation of the pixels for which he is not sure of their membership to the object or the background. In Fig. 15 we show two membership functions, one to represent the background and the other to represent the object.

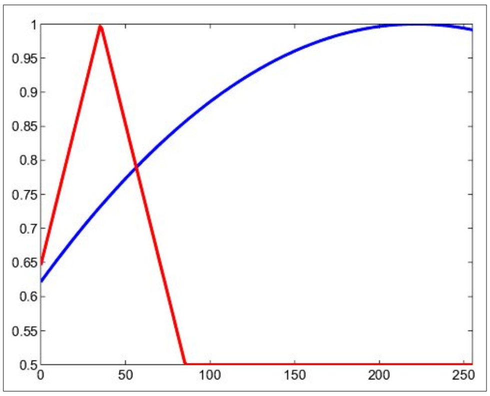

**Fig 15. Two different membership functions to represent the background and the object with t=150**

As we have already said in the previous paragraph, we are going to represent the images by means of two different fuzzy sets. For this reason, in our proposed algorithm we introduce the concept of *ignorance function* $G_u$. Such functions are a way to represent the user's ignorance for choosing the two membership functions used to represent the image (object and background). Therefore, in our algorithm we will associate to each pixel three numerical values:

- A value for representing its membership to the background, which we will interpret as the expert's knowledge of the membership of the pixel to the background.
- A value for representing its belongingness to the object, which we will interpret as the expert's knowledge of the membership of the pixel to the object.
- A value for representing the expert's ignorance of the membership of the pixels to the background or to the object. This ignorance hinders the expert from making an exact construction of the membership functions described in the first two items and therefore it also hinders the proper construction of step ($a$) of the fuzzy algorithm. The lower the value of ignorance is, the better the membership function chosen to represent the membership of that pixel to the background and the one chosen to represent the membership to the object will be. Evidently, there will be pixels of the image for which the expert will know exactly their membership to the background or to the object but there will also be pixels for which the expert is not able to determine if they belong to the background or to the object.

Under these conditions, if the value of the function of ignorance ($G_u$) for a certain pixel is zero, it means that the expert is positively sure about the belongingness of the pixel to the background or to the object. However, if the expert does not know at all whether the pixel belongs to the background or to the object he must represent its membership to both with the value 0.5, and under these conditions we can say that the expert has *total* ignorance regarding the membership of the pixel to the background and the membership of the same pixel to the object.

In [13] a methodology is proposed to construct Ignorance functions.

Ignorance functions can be constructed from t-norms as the miimum or te product or other functions like the geometric mean that are not t-norms. We have proposed several ways to construct IVFSs from a fuzzy membership function and an ignorance function.

In [13] it's proved that solution provided by the IVFS algorithm is better than the solution provided by the fuzzy algorithm when wrong membership functions are chosen and for special type of images (ultrasound images) the ignorance functions are useful for fast segmentation.

**Clustering:** A very common method for segmenting images is clustering. The most studied algorithm for this purpose is the Fuzzy Cluster Means (FCM), which aims to find the most characteristic point of each cluster, considered its centroid, and the membership degree of every object to each cluster. Some authors have adapted this algorithm to type-2 fuzzy sets. In [30] Hwang et al. try to define and manage the uncertainty of fuzzifier $m$ in FCM. They define the lower and upper interval memberships using two different values of $m$. To manage appropriately the uncertainty defined in an interval type-2 fuzzy set through all steps of FCM, they update cluster centers employing type reduction and defuzzification methods using type-2 fuzzy operations.

In [24] Jurio et al. transform the original image into an interval valued fuzzy set and adapt the FCM to it. In order to do that, we calculate the distance between each pattern and each cluster by the interval-valued restricted equivalence function.

**Fuzzy rule based systems:**

Data driven methods have also used in image segmentation. Starting from man made segmentations as ground truth data, machine learning algorithms have been used to create intellingent systems devoted to segment similar images as the trainig ones. Neural networks are the most common, and also fuzzy rule based systems have provided good results.

In [14] we introduce an application of interval-valued systems to the segmentation of prostate ultrasound images. The system classifies each pixel as prostate or background. The input variables are the values of each pixel in different processed images as proximity, edginess and enhanced image. The system has 20 rules and is trained with ideal images segmented by an expert.

## 7. Final remarks and future trends

As a brief resume of the conclusions of all the works reviewed using extensions of fuzzy sets in image processing, authors claim that the motivation of using extensions is their capability to deal with uncertainty present in all image processing steps. Looking to the results presented we can say that this research line is really promising. However we have found two points that we must notice:

1. All works that have been revised use interval type 2 fuzzy sets, although in the title are presented as

type-2 fuzzy sets. This occurs maybe due to the complexity of dealing with real type-2 fuzzy sets or due to the fact that it's really difficult to define a type-2 fuzzy set.

2. Works dealing with intuitionistic fuzzy sets, do not use the complete information given by the intuitionistic fuzzy sets. The non-membership does not represent anything. Authors use the hesitation index $\pi$ but as we have seen in the introduction it is equivalent to the length of an interval in IVFS. A really important conclusion is that all works can be done using Interval valued fuzzy sets. Therefore a problem of notation and work visibility is derived.

Hence future research, form the extensions point of view, must focus on problems that definition and computation of general type 2 fuzzy sets are tractable. Also, finding applications or image representations in which membership and non-membership can be generated independently in order to use all of the power of A-IFSs, should be researched.

From the point of view of image processing algorithms point of view, extensions of fuzzy sets can also be used in video summarization or content image retrieval.

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