OSCILLATORY MOTIONS IN RESTRICTED N-BODY PROBLEMS

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ABSTRACT. We consider the planar restricted N-body problem where the $N-1$ primaries are assumed to be in a central configuration whereas the infinitesimal particle escapes to infinity in a parabolic orbit. We prove the existence of transversal intersections between the stable and unstable manifolds of the parabolic orbits at infinity which guarantee the existence of a Smale's horseshoe. This implies the occurrence of chaotic motions but also of oscillatory motions, that is, orbits for which the massless particle leaves every bounded region but it returns infinitely often to some fixed bounded region. Our achievement is based in an adequate scaling of the variables which allows us to write the Hamiltonian function as the Hamiltonian of the Kepler problem plus higher-order terms that depend on the chosen configuration. We compute the Melnikov function related to the first non-null perturbative term and characterize the cases where it has simple zeroes. Concretely, for some combinations of the configuration parameters, i.e. mass values and positions of the primaries, and for a specific value of a parameter related to the angular momentum vector, the Melnikov function vanishes, otherwise it has simple zeroes and the transversality condition is satisfied. When the Melnikov function corresponding to the principal part of the perturbation is zero we compute the next non-zero Melnikov function proving that it has simple zeroes. The theory is illustrated for various cases of restricted N-body problems, including the circular restricted three-body problem. No restrictions on the mass parameters are assumed.

1. Introduction

In restricted N-body problems there exists a class of unbounded orbits called *oscillatory*, where the motion of the $N-1$ primaries remain bounded while the motion of the infinitesimal mass is unbounded, but nevertheless it leads back near the primaries an infinite number of times. This type of motion was hypothesized by Chazy [4] for the 3-body problem, Sitnikov [21] was the first to construct an oscillatory motion in a restricted 3-body problem, he also described an initial condition set for such solutions. Later on, using the theory of quasi-random dynamical systems, Alekseev [1] studied the Sitnikov problem showing the existence of oscillatory motion for small but positive infinitesimal mass. By the same time Melnikov [15] and Arnold [3] developed a method to study the formation of transversal intersections of stable and unstable manifolds. This technique, now known as the Melnikov method, replaces variational equations by the computation of certain integrals.

An orbit of an infinitesimal particle in a restricted N-body problem is *parabolic* if the infinitesimal particle escapes to infinity with zero limit radial velocity. In [12] McGehee introduced a suitable set of coordinates that brings the infinity into the origin. He also proved that the set of parabolic orbits is formed by two real analytic manifolds, that can be regarded as the stable and unstable manifolds of an unstable periodic orbit, or a hyperbolic fixed point in a suitable Poincaré map at infinity. This result was used by Moser [19] to clarify the proof of the existence of oscillatory motion in the Sitnikov problem. The key mechanism is to show the existence of transversal homoclinic intersections of the parabolic manifolds, which leads to a Smale's horseshoe map that guarantees the existence of symbolic dynamics, giving rise to the existence of oscillatory orbits as a consequence.

Llibre and Simó [11] followed Moser's approach to prove the existence of oscillatory solutions in the planar circular restricted 3-body problem (RPC3BP). They achieved it by demonstrating the transversal intersection of the stable and unstable parabolic manifolds for a large Jacobi constant C and a sufficiently small mass ratio μ between the primary bodies. Xia [22] treated the RPC3BP by the Melnikov method, where μ is used again as a perturbation parameter. He proved the transversality of the homoclinic manifolds for sufficiently small μ, and C close to $\pm\sqrt{2}$. After that, he used analytic continuation to extend the transversality to almost any value of μ with C large enough.

Guardia et al. [7] demonstrated that oscillatory motions do occur in the RPC3BP for any $\mu \in (0, 1/2]$ and a sufficiently large Jacobi constant. Their result was achieved through an asymptotic formula of the distance between the stable and unstable manifolds of infinity in a level set of the Jacobi constant, and making C sufficiently large, they proved that these manifolds intersect transversally. More recently, Guardia et al. [5] followed the above method to prove the existence of oscillatory solutions in the planar elliptic restricted 3-body problem, for any mass ratio $\mu \in (0, 1/2]$ and small eccentricities of the Keplerian ellipses (the primaries perform nearly circular orbits). This is done by constructing an infinite transition chain of fixed points of the Poincaré map and then providing a lambda lemma which gives the existence of an orbit which shadows the chain.

As far as the authors are aware, there are few studies in the literature related with oscillatory motions in restricted N-body problems for N greater than or equal to four, see for instance [6].

Our goal is to analyze the planar restricted N-body problem where the $N - 1$ primaries form a given central configuration and the infinitesimal particle goes to infinity in a parabolic orbit. More specifically, we show the existence of the stable and unstable manifolds of the parabolic orbits at infinity by means of a Poincaré map and establish the transversal intersections between them. This is done by the application of McGehee's Theorem for the existence of manifolds associated to degenerate fixed points [12]. The transversal intersections allow us to prove the existence of a Smale's horseshoe [8, 20] and the subsequent occurrence of chaotic motions. Besides the appearance of oscillatory motions is ensured. These motions correspond to orbits such that the infinitesimal particle leaves every bounded region but it returns infinitely often to some fixed bounded region [19].

The approach carried out here does not require any restriction in the mass parameters of the primaries. Instead of using a small parameter related to the masses we make a symplectic scaling of the variables so that the infinitesimal particle is placed far away from the center of mass of the primaries. This is the so called *cometary problem* [16, 17]. The scaling is performed through the introduction of a small parameter that measures the distance from the infinitesimal particle to the origin. Thence, after expanding the Hamiltonian function in terms of it, the resulting system is given by the Kepler Hamiltonian plus higher-order terms that are easily obtained through Legendre polynomials. The perturbation depends on the specific configuration of the primaries.

The transversal intersection between the manifolds is proved by applying the Melnikov's method [9, 10, 8, 20] related to the first non-null perturbative term. Indeed we compute a general Melnikov function that works for all the configurations, determining when it has simple zeroes. There are some cases where the calculation of higher-order terms of the Melnikov's function are required. This concerns a certain value of the parameter related to the angular momentum, or two parameters which are given in terms of the masses and positions of the primaries.

In the context of oscillatory motions we generalize the analysis performed for the RPC3BP by considering any central configuration of the $N - 1$ primaries. A key point in our analysis is that, instead of using a small parameter related to the masses we apply a symplectic scaling. This allows us to treat all the problems together so that we can check whether the Melnikov function for a specific configuration has simple zeros after replacing the coefficients of the configuration. In this manner we have been able to deal with a large variety of restricted problems in a systematic and straightforward way.

This paper has been structured as follows. In Section 2 we formulate the Hamiltonian of the restricted N-body problem where the primaries are in central configuration in a rotating and an inertial frame. We also define the cometary case, introducing an appropriate small parameter so that the problem is expressed as a 2-body Kepler Hamiltonian plus a small perturbation. In Section 3, we introduce McGehee coordinates to study the behavior of the system in a vicinity of infinity as a Poincaré map near a homoclinic orbit of a degenerate periodic orbit with analytic stable and unstable manifolds. Section 4 is devoted to the unperturbed problem that can be expressed as a Duffing oscillator. The main results will be given in Section 5, where we establish the existence of transversal homoclinic intersections of stable and unstable manifolds for the perturbed Hamiltonian system. We analyze the Melnikov function up to perturbation orders 4 and 6, depending on the non vanishing order of the Melnikov function. This is the fundamental part in our study, since it gives a systematic way of concluding the transversality between the corresponding invariant manifolds which only depends on the mass parameters and the configuration of the primaries. Then, we use that the Melnikov function has simple zeroes to show the transverse homoclinic intersections in the perturbed problem. In general, this integral is hard to calculate, however, we manage to overcome this technical challenge. In Section 6 we illustrate the theory developed in the previous sections in some restricted N-body problems.

In the RPC3BP, we show that in general it is enough to calculate the terms of order 4 in ε, although sometimes higher orders of the Melnikov function are required. In addition to the above, other examples are considered, such as the restricted 4-body problem with primaries in equilateral (Lagrange) configuration, the 5-body problem with primaries in rhomboidal configuration, the collinear restricted N-body problem and some polygonal restricted N-body problems. Finally, the study of the Melnikov functions is addressed in the Appendix.

All the numeric and symbolic calculations have been performed with Mathematica. We have made the computations within 50 significant digits although we only show the first eight.

2. Problem statement

The planar restricted N-body problem is the study of the motion of an infinitesimal mass particle subject to the Newtonian gravitational attraction of $N - 1$ bodies called the primaries. It is assumed that the masses of the primaries are so big in relation to the mass of the infinitesimal particle that the latter does not exert any significant influence in the primaries, hence the motion of the primaries becomes an $(N - 1)$-body problem. Along this paper the primaries move in a central configuration rotating around their center of mass with a constant angular velocity. Without loss of generality we also set the total mass of the primaries to one and their angular velocity $\omega = 1$.

The Hamiltonian that governs the motion of the infinitesimal particle in a rotating frame is

$$H(Q, P) = \frac{1}{2}|P|^2 - Q^T J P - U(Q), \tag{1}$$

where $Q, P \in \mathbb{R}^2$ are the position and momentum, $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right)$ and the potential is given by

$$U(Q) = \sum_{k=1}^{N-1} \frac{m_k}{|Q - a_k|}. \tag{2}$$

The terms $a_k = a_{k1} + a_{k2}\,i$ and m_k correspond to the position and mass respectively, of the k-th primary, $k = 1, \ldots, N - 1$. Since they are in central configuration the following equations are satisfied:

$$a_k = -\sum_{\substack{j=1 \\ j \neq k}}^{N-1} \frac{m_j(a_j - a_k)}{|a_j - a_k|^3}, \quad k = 1, \ldots, N - 1 \ \text{ and } \ \sum_{k=1}^{N-1} m_k a_k = 0, \tag{3}$$

with $m_1 + \cdots + m_{N-1} = 1$. We observe that the Hamiltonian H represents an autonomous system with two degrees of freedom. This quantity is preserved through the changes of coordinates in spite of the fact that some characteristics, like the time independence of the flow or the Hamiltonian structure, are lost. More details related to the restricted N-body problem can be seen in [16, 17].

In order to work in an inertial frame we define the change of coordinates $(Q, P) = e^{-it}(q, p)$. The Hamiltonian accounting for the motion of the infinitesimal particle in inertial coordinates yields

$$H(q, p, t) = \frac{1}{2}|p|^2 - U(q, t), \tag{4}$$

where the potential is given by

$$U(q, t) = \sum_{k=1}^{N-1} \frac{m_k}{|e^{-it}q - a_k|}. \tag{5}$$

Now the Hamiltonian is time dependent.

We are interested in the study of the motion of the infinitesimal particle near parabolic orbits, that is when the particle escapes to infinity with zero limit velocity. Thus, it is convenient to scale the Hamiltonian by introducing a small positive parameter ε through the change $q \to \varepsilon^{-2}q$, $p \to \varepsilon p$ and $t \to \varepsilon^3 t$. This is a symplectic transformation with multiplier ε. Hence $H \to \varepsilon H$. This problem now is the cometary regime of the restricted N-body problem, see [16, 17, 18].

Thus, Hamiltonian (4) becomes

$$H_\varepsilon(q, p, t) = \frac{1}{2}|p|^2 - \sum_{k=1}^{N-1} \frac{m_k}{|\varepsilon^{-2}e^{-it}q - a_k|}. \tag{6}$$

Its expansion in powers of ε yields

$$H_\varepsilon(q, p, t) = \frac{1}{2}|p|^2 - \frac{1}{|q|} - \sum_{j=2}^{\infty} \frac{\varepsilon^{2j}}{|q|^{j+1}} \sum_{k=1}^{N-1} m_k |a_k|^j P_j(\cos \gamma_k), \tag{7}$$

where P_j is the j-th term of the Legendre polynomial, and γ_k is the angle between the k-th primary's position and $e^{-it}q$. Let us note that the zero term of the sum is $-1/|q|$ and the next term is zero because we have placed the center of mass at the origin. Hence the problem is a Kepler problem plus a perturbation term of order ε^4. The series is convergent in the region of the configuration space where $\varepsilon^2|a_k| < |q|$.

Now we make the symplectic polar change of variables

$$q = re^{i\theta}, \qquad p = Re^{i\theta} + \frac{\Theta}{r}ie^{i\theta},$$

where r is the distance of the infinitesimal particle to the origin, θ is the argument of latitude, R stands for the radial velocity and Θ for the angular momentum. For our analysis we restrict to Θ to be non-zero and bounded, which is equivalent to consider the angular momentum before the scaling being a big quantity as long as ε is small.

Let α_k be the angle between the position of k-th primary and the horizontal axis of the inertial frame, thus $\gamma_k = \alpha_k - (\theta - t)$. Then, Hamiltonian (7) can be written as

$$
\begin{aligned}
H_\varepsilon(r, \theta, R, \Theta, t) = \ & \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{1}{r} \\
& - \sum_{j=2}^{\infty} \frac{\varepsilon^{2j}}{r^{j+1}} \sum_{k=1}^{N-1} m_k |a_k|^j P_j\left(\cos\left(\alpha_k - (\theta - t)\right)\right).
\end{aligned}
\tag{8}
$$

We observe that the argument of the Legendre polynomial P_j depends on a_k through the relations $\cos(\alpha_k) = a_{k1}/|a_k|$ and $\sin(\alpha_k) = a_{k2}/|a_k|$.

The associated Hamiltonian system is given by

$$
\begin{aligned}
\dot{r} &= R, \\
\dot{R} &= -\frac{1}{r^2} + \frac{\Theta^2}{r^3} - \sum_{j=2}^{\infty} \frac{(j+1)\varepsilon^{2j}}{r^{j+2}} \sum_{k=1}^{N-1} m_k |a_k|^j P_j\left(\cos\left(\alpha_k - (\theta - t)\right)\right), \\
\dot{\theta} &= \frac{\Theta}{r^2}, \\
\dot{\Theta} &= \sum_{j=2}^{\infty} \frac{\varepsilon^{2j}}{r^{j+1}} \sum_{k=1}^{N-1} m_k |a_k|^j Q_j\left(\cos\left(\alpha_k - (\theta - t)\right)\right)\sin\left(\alpha_k - (\theta - t)\right),
\end{aligned}
\tag{9}
$$

where $Q_j(w) = dP_j(w)/dw$.

3. Application of McGehee's Theorem

In order to study the motion of the infinitesimal mass near infinity, we make the change of coordinates introduced by McGehee [12]:

$$r = x^{-2}, \qquad R = -\sqrt{2}y, \qquad s = t - \theta,$$

where $s \in \mathbb{S}^1$. In these new coordinates the equations (9) become

$$
\begin{aligned}
\dot{x} &= \tfrac{1}{\sqrt{2}} x^3 y, \\
\dot{y} &= \tfrac{1}{\sqrt{2}} x^4 - \tfrac{1}{\sqrt{2}} \Theta^2 \, x^6 \\
&\quad - \tfrac{1}{\sqrt{2}} \sum_{j=2}^{\infty} (j+1)\varepsilon^{2j} x^{2j+4} \sum_{k=1}^{N-1} m_k \, |a_k|^j \, P_j \left(\cos\left(\alpha_k + s\right) \right), \\
\dot{s} &= 1 - x^4 \, \Theta, \\
\dot{\Theta} &= \sum_{j=2}^{\infty} \varepsilon^{2j} x^{2j+2} \sum_{k=1}^{N-1} m_k \, |a_k|^j \sin(\alpha_k + s) Q_j \left(\cos\left(\alpha_k + s\right) \right).
\end{aligned}
\tag{10}
$$

Similarly to the circular restricted 3-body problem, the above system can be smoothly extended to $x = 0$; in addition, it has a Jacobi-like first integral given by the equation (1). Let C be an arbitrary value of it. Thus

$$
C = H_\varepsilon(x, y, \theta, \Theta, t) - \Theta,
\tag{11}
$$

where $H_\varepsilon(x, y, \theta, \Theta, t)$ is given in (8) after replacing r and R for their values in terms of x and y. In order to verify the hypotheses of McGehee's Theorem [12], we observe that Θ can be written in terms of x, y, s and the value of the Jacobi-like integral C as follows:

$$
\Theta = \frac{1 \pm \sqrt{1 + 2x^4(C + x^2 - y^2)}}{x^4} + O(\varepsilon^4).
\tag{12}
$$

We take the negative sign in (12) because we are interested in small values of x. Therefore, the variable Θ depends smoothly on $x \geq 0$, $y \in \mathbb{R}$, $s \in \mathbb{S}^1$. In fact $\Theta = -C$ when $x = y = 0$, hence the equation referring to $\dot{\Theta}$ can be dropped in system (10). Expanding (12) in power series around $x = 0$, replacing its value in (10), the resulting system becomes:

$$
\begin{aligned}
\dot{x} &= \tfrac{1}{2} x^3 y, \\
\dot{y} &= \tfrac{1}{2} x^4 + x^6 f(x, y, \varepsilon^4, s, C), \\
\dot{s} &= 1 + C x^4 - x^6 + x^4 y^2 + \varepsilon^4 x^8 g(x, y, \varepsilon^4, s, C).
\end{aligned}
\tag{13}
$$

Observe that the functions $f(x, y, \varepsilon^4, s, C)$ and $g(x, y, \varepsilon^4, s, C)$ are real analytic.

Now, we are going to exploit the fact that the solution $\gamma(t)$ of system (13) with $x = 0$ and $y = 0$ is given by

$$
x(t) = 0, \quad y(t) = 0, \quad s(t) = s_0 + t \bmod 2\pi.
\tag{14}
$$

This solution does not depend on the Jacobi-like first integral C and is a 2π-periodic solution in t. Let $\Sigma = \{(x, y, s) : s = s_0\}$ be a transversal

section to this periodic orbit. This set is parametrized by the coordinates x and y. For the point $\kappa_0 = (0, 0, s_0) \in \gamma \cap \Sigma$, the Implicit Function Theorem shows that there exist an open set $V \subset \Sigma$ containing κ_0 and a smooth function $\sigma : V \to \mathbb{R}$, the return time, such that the trajectories starting in V come back to Σ in a time σ close to 2π.

Let $\varphi^t (x_0, y_0) = (x (t, x_0, y_0) , y (t, x_0, y_0) , s (t, x_0, y_0))$ be the flow solution of (13) with initial condition $(x(0), y(0), s(0)) = (x_0, y_0, s_0 \bmod 2\pi)$. This solution depends on ε, but it is usually omitted.

The Poincaré map $\mathcal{P} : V \subset \Sigma \to \Sigma$ of the periodic orbit γ (or of the fixed point $(0, 0, s_0)$) is given by $\mathcal{P}(x, y) = \varphi^{\sigma(x,y)}(x, y)$ where

$$
\mathcal{P} : \begin{cases} x \to x + \sqrt{2}\,\pi x^3 \left(y + \varepsilon^4 r_1(x, y, \varepsilon)\right) \\[2mm] y \to y + \sqrt{2}\,\pi x^3 \left(x - C^2 x^3 + \varepsilon^4 r_2(x, y, \varepsilon)\right) \end{cases} \tag{15}
$$

and r_1 and r_2 are real analytic functions of fourth order in x, y and second order in ε.

A straightforward computation shows that if we make the transformation $x = u + v$, $y = v - u$, the map $\mathcal{P}$ takes the form

$$
\tilde{\mathcal{P}}(u, v) = (u + p_1(u, v, \varepsilon) + \cdots , v + p_2(u, v, \varepsilon) + \cdots),
$$

where p_1 and p_2 are homogeneous polynomials of fourth degree in u and v and of order six in ε. For $u > 0$ we have $p_1(u, 0, \varepsilon) = -\sqrt{2}\pi u^4 < 0$, $p_2(u, 0, \varepsilon) = 0$ and $\frac{\partial p_2}{\partial v}(u, 0, \varepsilon) = \sqrt{2}\pi u^3 > 0$. These are the hypotheses of McGehee's Theorem [12], which we proceed to state in our context now.

For $\delta > 0$ and $\beta > 0$, the sector centred on the line $y = -x$ is defined by $B_1(\delta, \beta) = \{(x, y) : 0 \leq x \leq \delta, (-1 - \beta)x \leq y \leq (-1 + \beta)x\}$. Similarly, the sector centred on the line $y = x$ is $B_2(\delta, \beta) = \{(x, y) : 0 \leq x \leq \delta, (1 - \beta)x \leq y \leq (1 + \beta)x\}$. The sets

$$
W_\varepsilon^s(0, 0) = \big\{(x, y) \in B_1(\delta, \beta) : \ \forall n \geq 0, \ \mathcal{P}^n(x, y) \in B_1(\delta, \beta),
$$
$$
\lim_{n \to \infty} \mathcal{P}^n(x, y) = (0, 0)\big\},
$$

$$
W_\varepsilon^u(0, 0) = \big\{(x, y) \in B_2(\delta, \beta) : \ \forall n \leq 0, \ \mathcal{P}^n(x, y) \in B_2(\delta, \beta),
$$
$$
\lim_{n \to -\infty} \mathcal{P}^n(x, y) = (0, 0)\big\},
$$

are called the stable and unstable manifolds of the fixed point $(0, 0)$.

Theorem 3.1 (McGehee, [12])**.** *For the map $\mathcal{P} : V \subset \Sigma \to \Sigma$ given by (15), there is a $\delta > 0$ and a $\beta > 0$ such that the manifolds $W_\varepsilon^s(0, 0) \subset$*

$B_1(\delta, \beta)$ and $W^u_\varepsilon(0,0) \subset B_2(\delta, \beta)$ correspond to the graphs of two functions ψ_s, $\psi_u : [0, \delta] \to \mathbb{R}$ that are smooth, real analytic in $(0, \delta]$, and $\psi_s(0) = \psi_u(0) = 0$, $\psi'_s(0) = -1$, $\psi'_u(0) = 1$. In addition to that, they vary smoothly with ε.

The sets $W^s_\varepsilon(0,0)$ and $W^u_\varepsilon(0,0)$ are curves, that is, one-dimensional manifolds. From here it follows that

$$W^s_\varepsilon(\gamma) = \left\{ \varphi^t(x_\varepsilon, y_\varepsilon) : t \geq 0, \ (x_\varepsilon, y_\varepsilon) \in W^s_\varepsilon(0,0) \right\},$$
$$W^u_\varepsilon(\gamma) = \left\{ \varphi^t(x_\varepsilon, y_\epsilon) : t \leq 0, \ (x_\varepsilon, y_\varepsilon) \in W^u_\varepsilon(0,0) \right\}$$

are smooth manifolds of dimension two. They are formed by the orbits that escape to infinity ($x = 0$) with zero velocity ($y = 0$). These are called the *parabolic manifolds*.

4. The unperturbed problem

For $\varepsilon = 0$ Hamiltonian (8) corresponds to the Kepler problem, and the related equations (10) take the form

$$\dot{x} = \tfrac{1}{\sqrt{2}} x^3 y, \qquad \dot{y} = \tfrac{1}{\sqrt{2}} x^4 - \tfrac{1}{\sqrt{2}} \Theta^2 x^6,$$
$$\dot{s} = 1 - x^4 \Theta, \qquad \dot{\Theta} = 0. \tag{16}$$

At this point it is convenient to introduce a new time through $d\tau/dt = x^3/\sqrt{2}$. Fixing $\Theta = \Theta_0$, the equations for x and y become

$$\frac{d\,x}{d\,\tau} = x' = y, \qquad \frac{d\,y}{d\,\tau} = y' = x - \Theta_0^2 x^3. \tag{17}$$

Thus (17) is a Θ_0-parametrized Duffing equation. The origin $(0,0)$ is a hyperbolic saddle point and its stable and unstable manifolds form a homoclinic orbit $\xi(\tau)$ given by

$$x(\tau) = \frac{\sqrt{2}}{|\Theta_0|} \operatorname{sech} \tau, \qquad y(\tau) = -\frac{\sqrt{2}}{|\Theta_0|} \tanh \tau \operatorname{sech} \tau, \tag{18}$$

connecting the fixed point with itself as shown in Figure 1.

For $\varepsilon = 0$, the Hamiltonian associated to (17) is written as

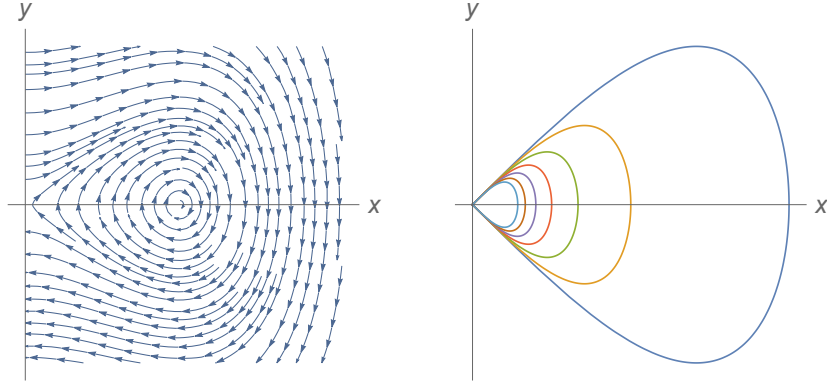$$H_0 = \tfrac{1}{2} y^2 - \tfrac{1}{2} x^2 + \tfrac{1}{4} x^4 \Theta_0^2. \tag{19}$$

FIGURE 1. From left to right, we show the flow of the unperturbed problem and the homoclinic loop for different values of Θ_0.

5. PERTURBED PROBLEM

For $\varepsilon = 0$ the manifolds $W_0^s(0,0) = W_0^u(0,0)$ are parametrized by the homoclinic orbit ξ. The smooth dependence on ε implies that $W_\varepsilon^s(\gamma)$ and $W_\varepsilon^u(\gamma)$ are parametrized by orbits of the form

$$\varphi_s\left(\tau, x_\varepsilon^s, y_\varepsilon^s\right) = \gamma(\tau + \tau_0) + \varepsilon \left.\frac{\partial \varphi_s}{\partial \varepsilon}\right|_{\varepsilon=0} + \cdots \qquad \text{for } \tau \geq 0,$$

$$\varphi_u\left(\tau, x_\varepsilon^u, y_\varepsilon^u\right) = \gamma(\tau + \tau_0) + \varepsilon \left.\frac{\partial \varphi_u}{\partial \varepsilon}\right|_{\varepsilon=0} + \cdots \qquad \text{for } \tau \leq 0,$$

where $\tau_0 = \tau(s_0)$. Our purpose now is to determine the speed of breaking up of $W_\varepsilon^s(0,0)$ and $W_\varepsilon^u(0,0)$ at $\varepsilon = 0$ under the perturbation. We point out that the normal direction to the level sets of H_0 is the only one to take care.

To measure the rate of separation between $W_\varepsilon^s(\gamma(\tau))$ and $W_\varepsilon^u(\gamma(\tau))$ with respect to ε, we take

$$H_0(\varphi_s\left(\tau_0, x_\varepsilon^s, y_\varepsilon^s\right)) - H_0(\varphi_u\left(\tau_0, x_\varepsilon^u, y_\varepsilon^u\right)) = \varepsilon^\nu \mathcal{M}_\nu(\tau_0) + \cdots,$$

where

$$\mathcal{M}_\nu(\tau_0) = \int_{-\infty}^{\infty} DH_\nu(\xi(\tau + \tau_0), \tau)\, \xi(\tau + \tau_0)\, d\tau$$

is the Melnikov function and H_ν is the first perturbation term of Hamiltonian (8) of order ν in ε whose function $\mathcal{M}_\nu$ is not identically zero. In fact, if $\mathcal{M}_\nu(\tau_0) = 0$ and $\mathcal{M}_\nu'(\tau_0) \neq 0$, then the stable and unstable manifolds intersect transversally for a small $\varepsilon > 0$ at a point close to $\xi(\tau_0)$, see [20] and references therein.

The consequence of the existence of transversal homoclinic manifolds is the appearance of a Smale's horseshoe, and thence the occurrence of chaotic motion of the infinitesimal particle. This includes the existence of oscillatory motion, see [19, 8].

In order to apply the results in the previous paragraphs we need to make some arrangements to Hamiltonian (8). Up to terms of order ε^4, it reads as

$$
\begin{aligned}
H_\varepsilon(r, \theta, R, \Theta, t) &= H_0 + \varepsilon^4 H_4 + O(\varepsilon^6) \\
&= \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{1}{r} \\
&\quad - \frac{\varepsilon^4}{4r^3}\left(c_1 + c_2 \cos 2(t - \theta) + c_3 \sin 2(t - \theta)\right) + O(\varepsilon^6),
\end{aligned}
\tag{20}
$$

where c_1, c_2 and c_3 are the following constant terms, that depend only on a_k and m_k:

$$
c_1 = \sum_{k=1}^{N-1} m_k(a_{k1}^2 + a_{k2}^2), \qquad c_2 = 3\sum_{k=1}^{N-1} m_k(a_{k1}^2 - a_{k2}^2),
$$

$$
c_3 = -6\sum_{k=1}^{N-1} m_k a_{k1} a_{k2}.
\tag{21}
$$

The next step consists in applying McGehee's transformation to the equations of motion associated to (20). After replacing $t - \theta$ by s, one gets

$$
\begin{aligned}
\dot{x} &= \tfrac{1}{\sqrt{2}} x^3 y, \\
\dot{y} &= \tfrac{1}{\sqrt{2}}(1 - \Theta^2 x^2)x^4 + \tfrac{3}{4\sqrt{2}}\varepsilon^4\left(c_1 + c_2 \cos 2s + c_3 \sin 2s\right) x^8 \\
&\quad + O(\varepsilon^6), \\
\dot{s} &= 1 - \Theta x^4, \\
\dot{\Theta} &= -\tfrac{1}{2}\varepsilon^4\left(c_3 \cos 2s - c_2 \sin 2s\right) x^6 + O(\varepsilon^6),
\end{aligned}
\tag{22}
$$

which corresponds to Hamiltonian equations (10).

With respect to the new time τ, Eq. (22) gets transformed into

$$
\begin{aligned}
x' &= y, \\
y' &= (1 - \Theta^2 x^2)x + \tfrac{3}{4}\varepsilon^4\left(c_1 + c_2 \cos 2s + c_3 \sin 2s\right) x^5 + O(\varepsilon^6), \\
s' &= \sqrt{2}(1 - \Theta x^4)x^{-3}, \\
\Theta' &= -\tfrac{1}{\sqrt{2}}\varepsilon^4\left(c_3 \cos 2s - c_2 \sin 2s\right) x^3 + O(\varepsilon^6).
\end{aligned}
\tag{23}
$$

To obtain the Melnikov function we use the corresponding integral of the zero order given by H_0 in (19), and compute the total derivative

$$\frac{dH_0}{d\tau} = \frac{\partial H_0}{\partial x} x' + \frac{\partial H_0}{\partial y} y',$$

where x', y' are given in (23). In this expression x and y are replaced by their explicit values obtained for the unperturbed problem and that were given in (18).

The next step consists in solving the differential equation for s' in (23) using the fact that Θ is assumed to be constant, i.e. $\Theta = \Theta_0$, and $x(\tau)$ is taken from the solution (18). The corresponding equation becomes

$$s'(\tau) = \pm \frac{1}{2} \Theta_0^3 \cosh^3 \tau \mp 2 \operatorname{sech} \tau.$$

We set $s(\tau_0) = s_0$ and for convenience we choose $\tau_0 = 0$. The solution yields

$$s(\tau) = s_0 \mp 4 \arctan(\tanh(\tau/2)) \pm \frac{1}{24} \Theta_0^3 \left(9 \sinh \tau + \sinh 3\tau\right). \quad (24)$$

The upper signs in s' and s are used when $\Theta_0 > 0$ and the lower ones when $\Theta_0 < 0$.

The resulting expression of the total derivative of H_0 with respect to τ becomes

$$\varepsilon^4 M_4 = \mp \frac{3\varepsilon^4 \operatorname{sech}^{10} \tau \tanh \tau}{4\Theta_0^6} \left(8c_1 \cosh^4 \tau + A \cos 2s_0 + B \sin 2s_0\right),$$

where

$$A = (c_2 c_\Theta + c_3 s_\Theta)(35 - 28 \cosh 2\tau + \cosh 4\tau)$$
$$-8(c_3 c_\Theta - c_2 s_\Theta)(7 \sinh \tau - \sinh 3\tau),$$

$$B = (c_3 c_\Theta - c_2 s_\Theta)(35 - 28 \cosh 2\tau + \cosh 4\tau)$$
$$+8(c_2 c_\Theta + c_3 s_\Theta)(7 \sinh \tau - \sinh 3\tau),$$

and

$$c_\Theta = \cos\left(\frac{\Theta_0^3}{12}(9 \sinh \tau + \sinh 3\tau)\right), \ \ s_\Theta = \sin\left(\frac{\Theta_0^3}{12}(9 \sinh \tau + \sinh 3\tau)\right).$$

The upper sign in M_4 is used when $\Theta_0 > 0$ and the lower one when $\Theta_0 < 0$.

Now we need to compute the integral of M_4 for τ between $-\infty$ and ∞. After performing the change of variable $z = \sinh \tau$ (note that $z' = \cosh \tau > 0$, hence the change is well-defined), simplifying the

resulting expressions and dropping the odd terms with respect to z their integrals are zero, we arrive at

$$\mathcal{M}_4(s_0; \Theta_0) = \int_{-\infty}^{\infty} M_4 \, dz = \pm \frac{6}{\Theta_0^6} \mathcal{F}_4(\Theta_0)(c_2 \sin 2s_0 - c_3 \cos 2s_0), \quad (25)$$

where

$$\mathcal{F}_4(\Theta_0) = \int_{-\infty}^{\infty} \frac{z}{(z^2 + 1)^6} \left(4z(z^2 - 1) \cos(\tfrac{\Theta_0^3}{3} z(z^2 + 3)) \right. \\ \left. + (z^4 - 6z^2 + 1) \sin(\tfrac{\Theta_0^3}{3} z(z^2 + 3)) \right) dz. \quad (26)$$

The upper sign applies for positive Θ_0 while the lower one for negative Θ_0.

The function $\mathcal{M}_4$ represents the fourth-order approximation in ε of the Melnikov function of the restricted N-body problem (6) related to parabolic motions of the infinitesimal particle near infinity. We state the following result.

Theorem 5.1. *When $\Theta_0 \neq \Theta_0^* = 1.44952926...$ and either c_2 or c_3 do not vanish, then for $\varepsilon > 0$ small enough the stable and unstable manifolds of the periodic orbit γ related to Hamiltonian (6) intersect transversally.*

Proof. On the one hand, in the Appendix we prove that the function $\mathcal{F}_4$ vanishes only at Θ_0^*. In this case (25) is zero and one has to study higher-order terms. On the other hand, when $\Theta_0 \neq \Theta_0^*$, we focus on the factor $f(s_0) = c_2 \sin 2s_0 - c_3 \cos 2s_0$ analyzing its possible zeros. Multiple roots of $f(s_0) = 0$ occur only when $c_2 = c_3 = 0$. Finally if the hypotheses of the Theorem hold, the equation $\mathcal{M}_4 = 0$ has only simple zeroes and Melnikov Theorem [8, 20] applies getting the transversality of the manifolds. $\square$

When $\Theta_0 = \Theta_0^*$ or $c_2 = c_3 = 0$ one has to resort to the next terms in the Legendre expansions of the Hamilton function (6). Concretely, the terms of order six in ε are given by

$$\varepsilon^6 H_6 = -\frac{\varepsilon^6}{8r^4} \left(d_1 \cos(t - \theta) + d_2 \sin(t - \theta) + d_3 \cos 3(t - \theta) \right. \\ \left. + d_4 \sin 3(t - \theta) \right),$$

where the d_j are constant terms depending on a_k and m_k given by

$$d_1 = 3 \sum_{k=1}^{N-1} m_k a_{k1}(a_{k1}^2 + a_{k2}^2), \quad d_2 = -3 \sum_{k=1}^{N-1} m_k a_{k2}(a_{k1}^2 + a_{k2}^2),$$

$$d_3 = 5 \sum_{k=1}^{N-1} m_k a_{k1}(a_{k1}^2 - 3a_{k2}^2), \quad d_4 = -5 \sum_{k=1}^{N-1} m_k a_{k2}(3a_{k1}^2 - a_{k2}^2). \tag{27}$$

The terms of order ε^6 that have to be added are those corresponding to $\dot{y}$ and $\dot{\Theta}$ in Eq. (22). They are, respectively,

$$\tfrac{\varepsilon^6}{2\sqrt{2}} (d_1 \cos s + d_2 \sin s + d_3 \cos 3s + d_4 \sin 3s) \, x^{10} + O(\varepsilon^8),$$

$$\tfrac{\varepsilon^6}{8} (d_1 \sin s - d_2 \cos s + 3d_3 \sin 3s - 3d_4 \cos 3s) \, x^8 + O(\varepsilon^8).$$

When transforming to the time τ these terms correspond to y' and Θ'. They are specified by

$$\tfrac{\varepsilon^6}{2} (d_1 \cos s + d_2 \sin s + d_3 \cos 3s + d_4 \sin 3s) \, x^7 + O(\varepsilon^8),$$

$$\tfrac{\varepsilon^6}{4\sqrt{2}} (d_1 \sin s - d_2 \cos s + 3d_3 \sin 3s - 3d_4 \cos 3s) \, x^5 + O(\varepsilon^8).$$

Proceeding as in the previous paragraphs, the total derivative of H_0 with respect to τ corresponding to the terms of order six in ε is

$$\varepsilon^6 M_6 = \mp \frac{\varepsilon^6 \operatorname{sech}^{14}\tau \tanh\tau}{4\Theta_0^8} \, (C \sin s_0 + D \cos s_0 + E \sin 3s_0$$
$$+ F \cos 3s_0),$$

where

$$C = 16\big((d_4 c_\Theta + d_3 s_\Theta)(3 - \cosh 2\tau) - 4(d_3 c_\Theta - d_4 s_\Theta)\sinh\tau\big)\cosh^4\tau,$$

$$D = 16\big((d_3 c_\Theta - d_4 s_\Theta)(3 - \cosh 2\tau) + 4(d_4 c_\Theta + d_3 s_\Theta)\sinh\tau\big)\cosh^4\tau,$$

$$E = (d_2 \tilde{c}_\Theta + d_1 \tilde{s}_\Theta)(462 - 495\cosh 2\tau + 66\cosh 4\tau - \cosh 6\tau)$$
$$- 4(d_1 \tilde{c}_\Theta - d_2 \tilde{s}_\Theta)(198\sinh\tau - 55\sinh 3\tau + 3\sinh 5\tau),$$

$$F = (d_1 \tilde{c}_\Theta - d_2 \tilde{s}_\Theta)(462 - 495\cosh 2\tau + 66\cosh 4\tau - \cosh 6\tau)$$
$$+ 4(d_2 \tilde{c}_\Theta + d_1 \tilde{s}_\Theta)(198\sinh\tau - 55\sinh 3\tau + 3\sinh 5\tau),$$

and

$$\tilde{c}_\Theta = \cos\left(\tfrac{\Theta_0^3}{4}(9\sinh\tau + \sinh 3\tau)\right), \quad \tilde{s}_\Theta = \sin\left(\tfrac{\Theta_0^3}{4}(9\sinh\tau + \sinh 3\tau)\right).$$

The upper sign in M_6 applies for $\Theta_0 > 0$ while the lower sign is used for $\Theta_0 < 0$. Notice that $\tilde{c}_\Theta = c_\Theta(4c_\Theta^2 - 3)$ and $\tilde{s}_\Theta = s_\Theta(4c_\Theta^2 - 1)$.

In order to obtain the integral of M_6 with respect to τ we apply as before the change $z = \sinh\tau$, discard those terms with zero integral and simplify, ending up with

$$
\begin{aligned}
\mathcal{M}_6(s_0;\Theta_0) &= \int_{-\infty}^{\infty} M_6\, dz \\
&= \pm\frac{8}{\Theta_0^8}\big(\mathcal{F}_{6,1}(\Theta_0)(d_4\cos s_0 - d_3\sin s_0) \\
&\qquad +\mathcal{F}_{6,2}(\Theta_0)(d_2\cos 3s_0 - d_1\sin 3s_0)\big),
\end{aligned}
\tag{28}
$$

with

$$
\begin{aligned}
\mathcal{F}_{6,1}(\Theta_0) &= \int_{-\infty}^{\infty} \frac{z}{(z^2+1)^6}\Big(2z\cos(\tfrac{\Theta_0^3}{6}z(z^2+3)) \\
&\qquad\qquad +(z^2-1)\sin(\tfrac{\Theta_0^3}{6}z(z^2+3))\Big)\,dz, \\
\mathcal{F}_{6,2}(\Theta_0) &= \int_{-\infty}^{\infty} \frac{z}{(z^2+1)^8}\Big(2z(3z^4-10z^2+3)\cos(\tfrac{\Theta_0^3}{2}z(z^2+3)) \\
&\qquad +(z^6-15z^4+15z^2-1)\sin(\tfrac{\Theta_0^3}{2}z(z^2+3))\Big)\,dz.
\end{aligned}
\tag{29}
$$

The upper sign applies for positive Θ_0 while the lower one for negative Θ_0.

The function $\mathcal{M}_6$ stands for the sixth-order approximation in ε of the Melnikov function related to the restricted N-body problem (6). This is the Melnikov function related to parabolic motions of the infinitesimal particle when it is near infinity. We are ready to state our second theorem.

Theorem 5.2. *When $\Theta_0 = \Theta_0^*$ or $c_2 = c_3 = 0$, then for $\varepsilon > 0$ small enough the stable and unstable manifolds of the periodic orbit γ related to Hamiltonian (6) intersect transversally provided the expression*

$$
\begin{aligned}
&(d_3^2 + d_4^2)^2\mathcal{F}_{6,1}^4(\Theta_0) \\
&+8(d_1 d_3^3 + 3d_2 d_3^2 d_4 - 3d_1 d_3 d_4^2 - d_2 d_4^3)\mathcal{F}_{6,1}^3(\Theta_0)\mathcal{F}_{6,2}(\Theta_0) \\
&+18(d_1^2 + d_2^2)(d_3^2 + d_4^2)\mathcal{F}_{6,1}^2(\Theta_0)\mathcal{F}_{6,2}^2(\Theta_0) - 27(d_1^2 + d_2^2)^2\mathcal{F}_{6,2}^4(\Theta_0)
\end{aligned}
\tag{30}
$$

does not vanish.

Proof. Evaluating $\mathcal{F}_{6,1}$ and $\mathcal{F}_{6,2}$ at Θ_0^* we get $\mathcal{F}_{6,1}(\Theta_0^*) = 0.00144184...$ and $\mathcal{F}_{6,2}(\Theta_0^*) = 0.03009745...$, so $\mathcal{M}_6$ is not identically zero at Θ_0^*. In the Appendix it is proved that $\mathcal{F}_{6,1}$ vanishes only at $\Theta_0 = 1.45326624...$ whereas $\mathcal{F}_{6,2}$ does it at $\Theta_0 = 1.48295711...$, both being simple roots of the corresponding equations. Focusing on possible multiple zeroes of $\mathcal{M}_6 = 0$ with respect to s_0, this can occur only when there is a common root of $\mathcal{M}_6 = 0$ and $\mathcal{M}_6' = 0$. Hence, we eliminate s_0 from the system

$\mathcal{M}_6 = \mathcal{M}_6' = 0$, arriving at Formula (30). When the parameters d_i and Θ_0 make that the combination (30) becomes zero, one should calculate higher orders of the Melnikov function. However, when (30) does not vanish, $\mathcal{M}_6 = 0$ has only simple zeroes, Melnikov Theorem [8, 20] applies and one concludes the transversality of the manifolds. $\square$

In Section 6, more specifically when we will deal with the circular restricted 3-body problem and with the polygonal restricted N-body problem, we shall see how higher-order terms of the Melnikov function are needed to establish the transversality of the stable and unstable manifolds of the parabolic periodic orbits.

6. Applications

6.1. **Restricted circular 3-body problem.** We study the planar circular restricted 3-body problem. In this example the position and mass parameters can be chosen as:

$$a_{11} = 1 - \mu, \quad a_{12} = 0, \quad a_{21} = -\mu, \quad a_{22} = 0, \quad m_1 = \mu, \quad m_2 = 1 - \mu,$$

where $0 < \mu \le \frac{1}{2}$.

With these values, the perturbation parameters appearing in (21) read as:

$$c_1 = (1 - \mu)\mu, \quad c_2 = 3(1 - \mu)\mu, \quad c_3 = 0.$$

Then, according to Theorem 5.1, and taking into account that $c_2 \ne 0$, when $\Theta_0 \ne \Theta_0^*$ for $\varepsilon > 0$ small enough the stable and unstable manifolds of the periodic orbit γ related to Hamiltonian (6) intersect transversally.

For $\Theta_0 = \Theta_0^*$ we have to consider the terms of order six in ε. In this case the parameters d_i in (27) are given by

$$d_1 = 3(1 - \mu)\mu(1 - 2\mu), \quad d_2 = 0, \quad d_3 = 5(1 - \mu)\mu(1 - 2\mu), \quad d_4 = 0.$$

In order to apply Theorem 5.2 we compute the expression (30), leading to

$$(5\mathcal{F}_{6,1}(\Theta_0^*) - 3\mathcal{F}_{6,2}(\Theta_0^*))(5\mathcal{F}_{6,1}(\Theta_0^*) + 9\mathcal{F}_{6,2}(\Theta_0^*))^3(1 - 2\mu)^4(1 - \mu)^4\mu^4$$

$$= -0.00178670...(1 - 2\mu)^4(1 - \mu)^4\mu^4.$$

Thus, Theorem 5.2 applies provided $\mu \ne 1/2$ and the stable and unstable manifolds of the periodic orbit γ intersect transversally.

The remaining case is $\Theta_0 = \Theta_0^*$, $\mu = 1/2$. Then we have to go to order eight in ε. The relevant part of Hamiltonian (8) is

$$-\frac{1}{1024r^5}\varepsilon^8\big(9 + 20\cos\left(2(t - \theta)\right) + 35\cos\left(4(t - \theta)\right)\big).$$

Applying the same steps as in Section 5 we end up with

$$\varepsilon^8 M_8^* = 5\varepsilon^8 \frac{\operatorname{sech}^{10}\tau \tanh\tau}{32(\Theta_0^*)^{10}} \left(-9 + R_1\cos 2s_0 + R_2\sin 2s_0\right.$$
$$\left. + R_3\cos 4s_0 + R_4\sin 4s_0\right),$$

with

$$R_1 = -20c_\Theta + 40\operatorname{sech}^3\tau\left(4c_\Theta\sinh\tau + s_\Theta(\cosh 2\tau - 3)\right)\tanh\tau,$$

$$R_2 = 20s_\Theta + 40\operatorname{sech}^3\tau\left(c_\Theta(\cosh 2\tau - 3) - 4s_\Theta\sinh\tau\right)\tanh\tau,$$

$$R_3 = -35\bar{c}_\Theta\left(1 - 2\operatorname{sech}^8\tau(\sinh 3\tau - 7\sinh\tau)^2\right)$$
$$+ \tfrac{35}{8}\bar{s}_\Theta\operatorname{sech}^8\tau(\sinh 7\tau - 35\sinh 5\tau + 273\sinh 3\tau - 715\sinh\tau),$$

$$R_4 = \tfrac{35}{8}\bar{c}_\Theta\operatorname{sech}^8\tau(\sinh 7\tau - 35\sinh 5\tau + 273\sinh 3\tau - 715\sinh\tau)$$
$$+ 35\bar{s}_\Theta\left(1 - 2\operatorname{sech}^8\tau(\sinh 3\tau - 7\sinh\tau)^2\right),$$

and

$$\bar{c}_\Theta = \cos\left(\tfrac{\Theta_0^3}{6}(9\sinh\tau + \sinh 3\tau)\right), \quad \bar{s}_\Theta = \sin\left(\tfrac{\Theta_0^3}{6}(9\sinh\tau + \sinh 3\tau)\right).$$

Notice that $\bar{c}_\Theta = 2c_\Theta^2 - 1$, $\bar{s}_\Theta = 2c_\Theta s_\Theta$. In the formulae given the R_i it is assumed that c_Θ, s_Θ, $\bar{c}_\Theta$ and $\bar{s}_\Theta$ are evaluated at Θ_0^*.

After performing the change $z = \sinh\tau$ and simplifying the intermediate formulae we get

$$\mathcal{M}_8(s_0; \Theta_0^*) = \int_{-\infty}^{\infty} M_8^* dz$$
$$= \frac{25}{4096(\Theta_0^*)^{10}}\left(64\mathcal{F}_{8,1}^*(\Theta_0^*)\sin 2s_0 + 7\mathcal{F}_{8,2}^*(\Theta_0^*)\sin 4s_0\right). \tag{31}$$

The expressions $\mathcal{F}_{8,1}^*$ and $\mathcal{F}_{8,2}^*$ stand for the numerical integrals

$$\mathcal{F}_{8,1}^*(\Theta_0^*) = \int_{-\infty}^{\infty} \frac{8z}{(z^2+1)^8}\left(4z(z^2-1)\cos(\tfrac{(\Theta_0^*)^3}{3}z(z^2+3))\right.$$
$$\left. + (z^4 - 6z^2 + 1)\sin(\tfrac{(\Theta_0^*)^3}{3}z(z^2+3))\right) dz,$$

$$\mathcal{F}_{8,2}^*(\Theta_0^*) = \int_{-\infty}^{\infty} \frac{128z}{(z^2+1)^{10}} \times$$
$$\times \left(8z(z^6 - 7z^4 + 7z^2 - 1)\cos(\tfrac{2(\Theta_0^*)^3}{3}z(z^2+3))\right.$$
$$\left. + (z^8 - 28z^6 + 70z^4 - 28z^2 + 1)\sin(\tfrac{2(\Theta_0^*)^3}{3}z(z^2+3))\right) dz \tag{32}$$

We have evaluated $\mathcal{F}_{8,1}^*$ and $\mathcal{F}_{8,2}^*$ up to fifty significant digits, obtaining

$$\mathcal{F}_{8,1}^*(\Theta_0^*) = -0.15786278..., \quad \mathcal{F}_{8,2}^*(\Theta_0^*) = -6.06283763....$$

Thus we obtain

$$\mathcal{M}_8(s_0; \Theta_0^*) = -0.00150580...\sin 2s_0 - 0.00632534...\sin 4s_0,$$

and the equation $\mathcal{M}_8 = 0$ has eight single roots in $\mathbb{S}^1$ but no multiple roots. (Indeed, the equation $f(s_0) = A\sin 2s_0 + B\sin 4s_0 = 0$ has multiple roots if and only if $A = \pm 2B$, but in our case $A - 2B = 0.01114487...$, $A + 2B = -0.01415649....$) Hence, applying Melnikov Theorem as in Theorems 5.1, 5.2, the stable and unstable manifolds of γ have transversal intersections.

With the approach described above we have completed the analysis of the parabolic orbits for the circular restricted 3-body problem for any $\mu \in (0, 1/2]$. This problem is also treated in [7].

6.2. **Equilateral restricted 4-body problem.** In this example the three massive particles form an equilateral triangle, therefore a central configuration, thus (3) is satisfied. The parameter values are:

$$a_{11} = \tfrac{1}{2}(1 - m_1 - 2m_2), \quad a_{12} = \tfrac{\sqrt{3}}{2}(1 - m_1),$$
$$a_{21} = \tfrac{1}{2}(2 - m_1 - 2m_2), \quad a_{22} = -\tfrac{\sqrt{3}}{2}m_1,$$
$$a_{31} = -\tfrac{1}{2}(m_1 + 2m_2), \qquad a_{32} = -\tfrac{\sqrt{3}}{2}m_1,$$
$$m_3 = 1 - m_1 - m_2,$$

where $m_1, m_2 > 0$ and $m_1 + m_2 < 1$.

Then, the coefficients c_i of (21) take the following values:

$$c_1 = (m_1 + m_2)(1 - m_1) - m_2^2,$$
$$c_2 = -\tfrac{3}{2}\left((m_1 - 2m_2)(1 - m_1) + 2m_2^2\right),$$
$$c_3 = -\tfrac{3\sqrt{3}}{2}m_1(1 - m_1 - 2m_2).$$

Therefore the transversal intersection of the manifolds is established as Theorem 5.1 applies, except for the case $m_1 = m_2 = 1/3$ where $c_2 = c_3 = 0$, and the case $\Theta_0 = \Theta_0^*$.

When conditions of Theorem 5.1 are not fulfilled, Theorem 5.2 can be applied for some specific combinations of the parameters. On the one hand, when $m_1 = m_2 = 1/3$ we calculate the coefficients d_i getting $d_1 = d_3 = d_4 = 0$, $d_2 = 5/3^{3/2}$ and the expression (30) results in $-625\mathcal{F}_{6,2}^4(\Theta_0)/27$ which is non-zero excepting at $\Theta_0 = 1.48295711....$ On the other hand, for $\Theta_0 = \Theta_0^*$, we also compute the coefficients d_i and condition (30) yields a polynomial in m_1, m_2 of total degree 12 with numerical coefficients, that we call $g(m_1, m_2)$.

Thence when $\Theta_0 \neq 1.48295711...$ and $m_1 = m_2 = 1/3$ or when $\Theta_0 = \Theta_0^*$ and $g(m_1, m_2) \neq 0$, Theorem 5.2 is applied, accomplishing the intersection of the manifolds of the parabolic orbits. The remaining cases should be analyzed after taking into account terms of at least order eight in ε.

The case $m_1 = m_2 = 1/3$ is included as a particular situation of the polygonal restricted N-body problem that will be addressed in Subsection 6.5.

6.3. **Restricted rhomboidal 5-body problem.** We consider the case where masses m_1 to m_4 are equal by pairs and form a convex polygon, a rhombus, see [2] and references therein. The parameters that define the problem are as follows:

$$a_{11} = -x, \qquad a_{12} = 0, \qquad a_{21} = 0, \qquad a_{22} = y,$$
$$a_{31} = x, \qquad a_{32} = 0, \qquad a_{41} = 0, \qquad a_{42} = -y,$$
$$m_1 = m_3 = \mu, \quad m_2 = m_4 = \tfrac{1}{2} - \mu,$$

where $0 < \mu < 1/2$ and $x, y > 0$. To get a central configuration the parameters μ, x, y must be related. For this purpose we impose that Eqs. (3) are satisfied. It is convenient to introduce two parameters $a, b > 0$ such that

$$x = \frac{a}{2\sqrt{a^2 + b^2}} \left(\frac{64a^3b^3 - (a^2 + b^2)^3}{16a^3b^3 - (a^3 + b^3)(a^2 + b^2)^{3/2}} \right)^{1/3},$$

$$y = \frac{b}{2\sqrt{a^2 + b^2}} \left(\frac{64a^3b^3 - (a^2 + b^2)^3}{16a^3b^3 - (a^3 + b^3)(a^2 + b^2)^{3/2}} \right)^{1/3},$$

then a central configuration occurs provided μ is taken as

$$\mu = \frac{a^3 \left(8b^3 - (a^2 + b^2)^{3/2} \right)}{2 \left(16a^3b^3 - (a^3 + b^3)(a^2 + b^2)^{3/2} \right)}.$$

To ensure that $\mu \in (0, 1/2)$ we must restrict a, b so that $0 < b < \sqrt{3}a < 3b$.

In terms of x, y, the perturbation parameters in (21) are expressed by:

$$c_1 = y^2 + 2\mu(x^2 - y^2),$$
$$c_2 = -3y^2 + 6\mu(x^2 + y^2),$$
$$c_3 = 0.$$

When c_2 is non-zero and $\Theta_0 \neq \Theta_0^*$ Theorem 5.1 applies. For $\Theta_0 = \Theta_0^*$ or $c_2 = 0$ we should go to higher orders in ε. The coefficient c_2 vanishes in

three cases: (i) If $a = b$, then $\mu = 1/4$, which corresponds to the square configuration, i.e. the polygonal restricted 4-body problem, that will be treated in the next subsection; (ii) when $a = 1.32018439...b$; and (iii) the reverse case to (ii), i.e. $a = 0.75746994...b$. These values come as the only (three) real roots of the 14-degree homogeneous polynomial equation given by

$$a^{14} + 2a^{13}b + 6a^{12}b^2 + 10a^{11}b^3 + 17a^{10}b^4 + 22a^9b^5 - 36a^8b^6 - 100a^7b^7$$

$$-36a^6b^8 + 22a^5b^9 + 17a^4b^{10} + 10a^3b^{11} + 6a^2b^{12} + 2ab^{13} + b^{14} = 0.$$

We have checked that the parameters d_i in (27) are identically zero for all possible choices of a, b. Therefore, when $\Theta_0 = \Theta_0^*$ or when $a = b$, $a = 1.32018439...b$ or $a = 0.75746994...b$, higher orders should be analyzed, at least, those of order eight in ε. The case $a = b$ corresponds to a particular situation of the polygonal restricted N-body problem dealt with in Subsection 6.5.

6.4. **Collinear restricted N-body problem.** In the following we present two examples where the $N - 1$ massive particles are collinear in a planar central configuration.

Note that in all collinear problems $a_{k2} = 0$ for $k = 1, \ldots, N - 1$. Then, the parameter c_3 of (21) always vanishes.

6.4.1. *Collinear restricted 8-body problem.* Let us consider a collinear configuration of seven bodies with equal masses that are placed in a symmetric configuration with respect to the origin on the horizontal axis. In order to calculate the positions we impose that the conditions (3) are satisfied and so, the seven bodies form a central configuration. Then, by means of the determination of the resultant of two polynomials we find that a collinear configuration occurs if the parameters are chosen as follows:

$$\begin{aligned}
a_{11} &= -1.17858061..., \quad a_{21} = -0.73861375..., \\
a_{31} &= -0.35910513..., \quad a_{41} = 0, \\
a_{51} &= -a_{31}, \qquad a_{61} = -a_{21}, \qquad a_{71} = -a_{11}, \\
a_{k2} &= 0, \quad m_k = \tfrac{1}{7}, \quad \text{for } k = 1, \ldots, 7.
\end{aligned}$$

Then, the perturbation parameters appearing in (21) are:

$$c_1 = 0.58958829..., \quad c_2 = 1.76876487..., \quad c_3 = 0.$$

Thus, $c_2 \neq 0$ and the conditions of Theorem 5.1 are satisfied except for the value $\Theta_0 = \Theta_0^*$. Thence, for $\varepsilon > 0$ small enough and $\Theta_0 \neq \Theta_0^*$, the stable and unstable manifolds of the periodic orbit γ related to

Hamiltonian (6) intersect transversally. When $\Theta_0 = \Theta_0^*$ we consider the Melnikov function $\mathcal{M}_6$ but then $d_i = 0$ and higher-order terms should be computed to decide on the transversality of the manifolds.

6.4.2. *Collinear restricted* 11-*body problem.* Here we consider ten particles in collinear configuration placed at equidistant positions. Then, we calculate the masses so as to obtain a central configuration. For that, we impose that Eqs. (3) are satisfied and solve the resulting system of linear equations to get

$$a_{11} = -1.44194062..., \qquad a_{21} = -1.12150937...,$$

$$a_{31} = -0.80107812..., \qquad a_{41} = -0.48064687...,$$

$$a_{51} = -0.16021562...,$$

$$a_{61} = -a_{51}, \ a_{71} = -a_{41}, \ a_{81} = -a_{31}, \ a_{91} = -a_{21}, \ a_{101} = -a_{11},$$

$$a_{k2} = 0 \quad \text{for } k = 1, \ldots, 10,$$

$$m_1 = m_{10} = 0.05585772..., \qquad m_2 = m_9 = 0.08684056...,$$

$$m_3 = m_8 = 0.10794726..., \qquad m_4 = m_7 = 0.121390422...,$$

$$m_5 = m_6 = 0.12796403....$$

Note that, although here we put an approximation of the values correct to eight decimal digits, we have obtained them using integer arithmetic.

We calculate the coefficients (21) and get their values also exactly, an approximation of them accurate up to eight decimal places being:

$$c_1 = 0.65193332..., \quad c_2 = 1.95579995..., \quad c_3 = 0.$$

Then, as $c_2 \neq 0$, Theorem 5.1 is satisfied except for the value $\Theta_0 = \Theta_0^*$. Regarding the terms of order six in ε, using (27) we have obtained $d_i = 0$, thus Theorem 5.2 cannot be applied and we should compute higher orders to conclude the transversality condition of the manifolds of γ.

6.5. **Polygonal restricted N-body problem.** In this case particles 1 to $N-1$ form a regular $(N-1)$-gon determined by the following constants:

$$a_{k1} = \text{Re}(\mathrm{e}^{2\pi i \frac{k-1}{N-1}}), \qquad a_{k2} = \text{Im}(\mathrm{e}^{2\pi i \frac{k-1}{N-1}}), \qquad m_k = \frac{1}{N-1},$$

for $k = 1$ to $N-1$ where $N \geq 4$.

We want to write down Hamiltonian (8) for the specific cases of the polygonal restricted problem. After some manipulations and simplifications that include an induction over the integer N, we notice that the Hamiltonian function can be written in terms of symplectic polar coordinates in a rather compact way by

$$
\begin{aligned}
H_\varepsilon(r,\theta,R,\Theta,t) = & \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{1}{r} - \sum_{j=1}^{2N-3}\frac{\varepsilon^j}{r^{j/2+1}}U_j \\
& -\frac{\varepsilon^{2N-2}}{r^N}\Big(V_{N-1} + W_{N-1}\cos((N-1)(t-\theta))\Big) \\
& +O(\varepsilon^{2N-1}),
\end{aligned}
\tag{33}
$$

where

$$
U_j = \frac{(1+(-1)^j + 2\cos(j\pi/2))\,(\Gamma(j/4+1/2))^2}{4\pi(\Gamma(j/4+1))^2},
$$

$$
V_{N-1} = \frac{(1+(-1)^{N-1})(\Gamma(N/2))^2}{2\pi(\Gamma(N/2+1/2))^2},
$$

$$
W_{N-1} = \frac{2\Gamma(N-1/2)}{\sqrt{\pi}\,\Gamma(N)},
$$

and Γ stands for the gamma function.

To obtain the Melnikov function we emphasize the convenience of developing H_ε to order $2N-2$ because the first appearance of θ occurs at this order and the previous orders would yield zero.

At this point we apply the same steps as in the previous sections, arriving at the total derivative

$$
\begin{aligned}
\frac{dH_0}{d\tau} = & -\tfrac{1}{4}\Theta_0^2 \sinh 2\tau \sum_{j=1}^{2N-3}\varepsilon^j U_j \frac{2^{j/2+1}(j+2)}{(|\Theta_0|\cosh\tau)^{j+4}} \\
& -\varepsilon^{2N-2}\left(V_{N-1} + W_{N-1}\cos q(s_0,\tau)\right)\frac{2^N N \sinh\tau}{\Theta_0^{2N}\cosh^{2N+1}\tau},
\end{aligned}
$$

where

$$
q(s_0,\tau) = (N-1)\Big(s_0 \mp 4\arctan(\tanh(\tau/2)) \pm \tfrac{\Theta_0^3}{24}(9\sinh\tau + \sinh 3\tau)\Big),
$$

where the upper signs apply when $\Theta_0 > 0$ and the lower ones when $\Theta_0 < 0$.

Now, we observe that by the parity of the derivative with respect to τ, the only term with no zero integral in the derivative is that factorized

by W_{N-1}, thus we introduce

$$\varepsilon^{2N-2} M_{2N-2} = -\varepsilon^{2N-2} \frac{2^N N W_{N-1}}{\Theta_0^{2N}} \frac{\sinh \tau \cos q(s_0, \tau)}{\cosh^{2N+1} \tau}.$$

We introduce the change $z = \sinh \tau$ and define the Melnikov function as

$$\mathcal{M}_{2N-2}(s_0, \Theta_0) = -\frac{2^N N W_{N-1}}{\Theta_0^{2N}} \int_{-\infty}^{\infty} \frac{z}{(z^2 + 1)^{N+1}} \cos \tilde{q}(s_0, z) \, dz, \quad (34)$$

with

$$\tilde{q}(s_0, z) = (N - 1)\big(s_0 \pm \tfrac{\Theta_0^3}{6} z(z^2 + 3) \mp 4 \arctan(\tanh(\tfrac{1}{2}\mathrm{arcsinh} z)))\big).$$

In order to illustrate how the theory of this paper applies, we particularize the calculations for two specific cases, namely $N = 7, 8$.

When $N = 7$ we get

$$\mathcal{M}_{12}(s_0, \Theta_0) = \pm \frac{1617}{4\Theta_0^{14}} \mathcal{F}_{12}(\Theta_0) \sin 6s_0$$

where

$$\mathcal{F}_{12}(\Theta_0) = \int_{-\infty}^{\infty} \frac{z}{(z^2 + 1)^{14}} \big(p_1(z) \cos(\Theta_0^3 z(z^2 + 3)) \\ + p_2(z) \sin(\Theta_0^3 z(z^2 + 3))\big) \, dz,$$

with

$$p_1(z) = 4z(3z^{10} - 55z^8 + 198z^6 - 198z^4 + 55z^2 - 3),$$
$$p_2(z) = z^{12} - 66z^{10} + 495z^8 - 924z^6 + 495z^4 - 66z^2 + 1.$$

For $N = 8$ the corresponding Melnikov function reads as

$$\mathcal{M}_{14}(s_0, \Theta_0) = \pm \frac{858}{\Theta_0^{16}} \mathcal{F}_{14}(\Theta_0) \sin 7s_0$$

where

$$\mathcal{F}_{14}(\Theta_0) = \int_{-\infty}^{\infty} \frac{-z}{(z^2 + 1)^{16}} \big(p_3(z) \cos(\tfrac{7\Theta_0^3}{6} z(z^2 + 3)) \\ + p_4(z) \sin(\tfrac{7\Theta_0^3}{6} z(z^2 + 3))\big) \, dz,$$

with

$$p_3(z) = 2z(z^6 - 21z^4 + 35z^2 - 7)(7z^6 - 35z^4 + 21z^2 - 1),$$
$$p_4(z) = z^{14} - 91z^{12} + 1001z^{10} - 3003z^8 + 3003z^6 - 1001z^4 \\ + 91z^2 - 1.$$

From the expressions of both Melnikov functions it is clearly deduced that the equations $\mathcal{M}_{12} = 0$ and $\mathcal{M}_{14} = 0$ have simple roots provided,
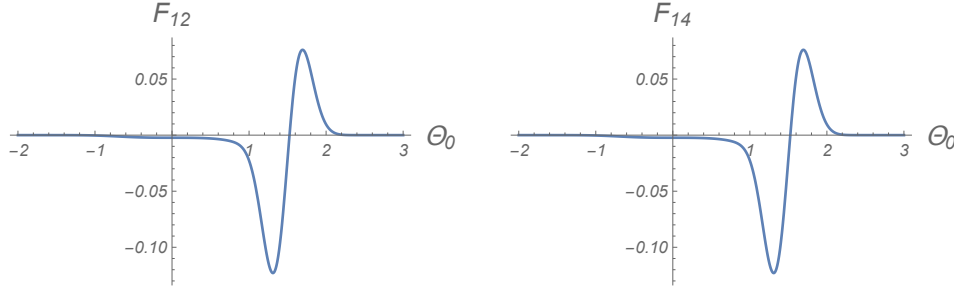
FIGURE 2. On the left: graph of $\mathcal{F}_{12}$; on the right, graph of $\mathcal{F}_{14}$.

respectively, $\mathcal{F}_{12}$ and $\mathcal{F}_{14}$ do not vanish. However $\mathcal{F}_{12}$, $\mathcal{F}_{14}$ are of the same type as the functions $\mathcal{F}_4$, $\mathcal{F}_{6,1}$, $\mathcal{F}_{6,2}$ of Section 5 analyzed in the Appendix. Then they have a global maximum and a global minimum, their graphs cut the horizontal axis at a unique point and tend asymptotically to zero as long as Θ_0 tends to $\pm\infty$, see also Figure 2.

More specifically, $\mathcal{F}_{12}$ becomes zero for $\Theta_0 = 1.52516447...$ while $\mathcal{F}_{14}$ vanishes at $\Theta_0 = 1.53230637....$ Thus, for the polygonal restricted 7-body problem, the stable and unstable manifolds of the parabolic orbits γ intersect transversally excepting at $\Theta_0 = 1.52516447...$ while for the polygonal restricted 8-body problem the corresponding stable and unstable manifolds of γ intersect transversally, if we ignore the case $\Theta_0 = 1.53230637....$

Regarding the case $N = 4$, corresponding to the equilateral restricted 4-body problem of Subsection 6.2 when $m_1 = m_2 = m_3 = 1/3$, the Melnikov function obtained from (34) is a particular case of the function $\mathcal{M}_6$ in (28), thus the degeneracy achieved in Subsection 6.2 for $\Theta_0 = 1.48295711...$, persists. When $N = 5$, the square configuration studied in Subsection 6.3 is analyzed using the Melnikov function (34) which is of order eight in ε. Proceeding similarly to what we did for $N = 7, 8$ we conclude that $\mathcal{M}_8$ has simple roots provided $\Theta_0 \neq 1.50262022...$, thus extending the analysis performed in Subsection 6.3 when $a = b$.

7. APPENDIX: QUALITATIVE STUDY OF FUNCTIONS $\mathcal{F}_4(\Theta_0)$, $\mathcal{F}_{6,1}(\Theta_0)$ AND $\mathcal{F}_{6,2}(\Theta_0)$

The function $\mathcal{F}_4$ has been defined in (26) and its graph is given in Figure 3. Considered as a function in z, the integrand is smooth in $\mathbb{R}$ and bounded by above in the intervals in $(-\infty, -1] \cup [1, \infty)$ by the improperly integrable function $16/|z|^7$. Now the comparison test

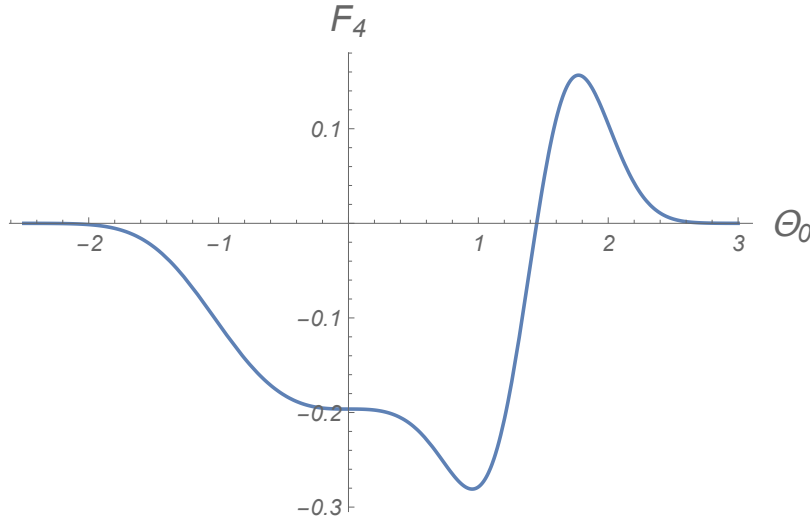FIGURE 3. The graph of the function $\mathcal{F}_4(\Theta_0)$.

for improper integrals implies that $\mathcal{F}_4(\Theta_0)$ is a well defined smooth function for all $\Theta_0 \in \mathbb{R}$. In addition it has only one zero, namely $\Theta_0^* = 1.44952926...$, and $\mathcal{F}_4(\Theta_0) < 0$ for $\Theta_0 < \Theta_0^*$, while $\mathcal{F}_4(\Theta_0) > 0$ for $\Theta_0 > \Theta_0^*$. The function $\mathcal{F}_4$ takes only one maximum and one minimum values. Furthermore

$$\lim_{\Theta_0 \to \pm\infty} \mathcal{F}_4(\Theta_0) = 0.$$

The functions $\mathcal{F}_{6,1}$, $\mathcal{F}_{6,2}$ were introduced in (29) and present an analogous behavior to the function $\mathcal{F}_4$, as it can be seen in Figure 4. The corresponding improper integrals are absolutely convergent. Specifically, $\mathcal{F}_{6,1} = 0$ has its unique root at $\Theta_0 = 1.45326624...$ whereas the root of $\mathcal{F}_{6,2} = 0$ occurs at $\Theta_0 = 1.48295711....$ Besides, $\mathcal{F}_{6,1} > 0$ when $\Theta_0 < 1.45326624...$, $\mathcal{F}_{6,1} < 0$ when $\Theta_0 > 1.45326624...$ and $\mathcal{F}_{6,2} > 0$ when $\Theta_0 < 1.48295711...$, $\mathcal{F}_{6,2} < 0$ when $\Theta_0 > 1.48295711....$ As $\mathcal{F}_4$, the functions $\mathcal{F}_{6,1}$, $\mathcal{F}_{6,2}$ take a global maximum as well as a global minimum. Finally,

$$\lim_{\Theta_0 \to \pm\infty} \mathcal{F}_{6,1}(\Theta_0) = \lim_{\Theta_0 \to \pm\infty} \mathcal{F}_{6,2}(\Theta_0) = 0.$$

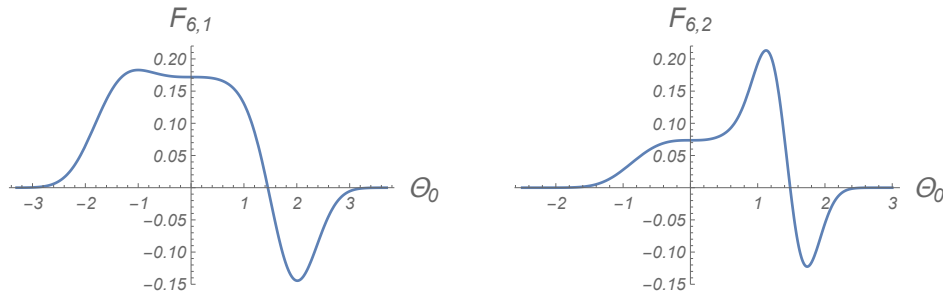Finally we remark that similar integrals have been analyzed and can be found in [13, 14].

FIGURE 4. The graphs of the functions $\mathcal{F}_{6,1}(\Theta_0)$ and $\mathcal{F}_{6,2}(\Theta_0)$.

Acknowledgements

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