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Computing with uncertain truth degrees: a convolution-based approach

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Abstract

Fuzzy set theory can be seen as a body of mathematical tools exceptionally well-suited to deal with incomplete information, unsharpness and non-stochastic uncertainty. Indeed, as a tool for translating natural imprecise human language into mathematical objects, fuzzy sets are playing a crucial role in engineering for bridging the gap between man and computers. However, it is widely spread that the assignation of an exact membership degree is not an easy task. As a possible solution to this difficulty, several generalizations of fuzzy sets have been introduced and studied in the literature. Moreover, these generalizations have shown to be very useful in many applications leading to improved results when generalizations of fuzzy sets are considered.

Generalizations differ from fuzzy sets in the mathematical object used to model the imprecision and/or uncertainty. Specifically, fuzzy sets take elements in the unit interval $[0, 1]$ while the generalizations use more intricate mathematical objects such as intervals (interval-valued fuzzy sets), subsets of the unit interval (set-valued fuzzy sets or hesitant fuzzy sets), or functions (type-2 fuzzy sets), among others. Nevertheless, the use of the generalizations of fuzzy sets has a main drawback. Before applying any of the generalizations of fuzzy sets, it is necessary to adapt *ad-hoc* each theoretical notion to the corresponding mathematical object which represents the uncertainty in the considered application, i.e., it is necessary to redefine each theoretical notion from the unit interval $[0, 1]$ to more intricate mathematical objects.

Rather early in the history of fuzzy sets it became clear that the natural relationship between classical set theory and classical logic can be mimicked generating a natural relationship between fuzzy set theory and many-valued logic. This many-valued logic is nowadays called fuzzy logic. Similarly, each generalization of fuzzy sets constitutes a new fuzzy logical system. All these logical systems coincide in the sense that all of them model uncertainty, but they differ in the mathematical object which represents it.

It can be easily seen that the same problem of fuzzy sets and its generalizations is found in the different fuzzy logics, i.e., although all the logical systems are akin, every theoretical notion has to be redefined for each logic. This problem, as well as the large number of these logical systems that model uncertainty, has led us to study whether or not it is possible to find a system that can encompass these logics and it has motivated us to propose a logical system that can model the uncertainty in a malleable way. Especially focusing on those logical systems that turn up from fuzzy theory, in this dissertation we propound a new logical system which retrieves multiple logical systems in the literature. The main advantages of the proposed logical system are that:

- it will avoid the excessive repetitions of theoretical notions;
- it will allow to adapt the applications to the most suitable type of fuzzy set or fuzzy logic in a simpler way.

In this dissertation we present the semantics of the proposed logical system as well as a in-depth study of the convolution operations, which are applied to define the disjunction and conjunction connectives of the logical system.

Resumen

La teoría de los conjuntos difusos puede contemplarse como un conjunto de herramientas matemáticas excepcionalmente adaptadas para trabajar con información incompleta, falta de nitidez e incertidumbre no aleatoria. De hecho, como herramienta en ingeniería, para traducir el lenguaje natural humano impreciso en un objeto matemático, los conjuntos difusos juegan un papel decisivo para superar la brecha entre el hombre y los ordenadores. Sin embargo, es ampliamente conocido que la asignación de un valor preciso como pertenencia no es una tarea sencilla. En la literatura, se han propuesto y estudiado varias generalizaciones de los conjuntos difusos para resolver esta dificultad. Más aún, estas generalizaciones han demostrado ser una herramienta útil, al mejorar los resultados en diferentes aplicaciones.

Las generalizaciones difieren de los conjuntos difusos en el objeto matemático que se utiliza para modelar la imprecisión y/o incertidumbre. Específicamente, los conjuntos difusos toman elementos en el intervalo unidad $[0, 1]$ mientras que las generalizaciones toman objetos matemáticos más complejos como intervalos (conjuntos difusos intervalo-valorados), subconjuntos del intervalo unidad (conjuntos difusos "conjunto-valorados") o funciones (conjuntos difusos tipo-2), entre otros. No obstante, el uso de las generalizaciones de los conjuntos difusos tiene un gran inconveniente. Antes de aplicar las generalizaciones de los conjuntos difusos es necesario adaptar *ad hoc* cada noción teórica al correspondiente objeto matemático que modela la incertidumbre en la aplicación, es decir, es necesario redefinir cada noción teórica reemplazando el intervalo unidad $[0, 1]$ por objetos matemáticos más complejos.

En la historia de los conjuntos difusos quedó claro relativamente pronto que la relación natural entre la teoría de conjuntos y la lógica clásica podía ser imitada generando una relación entre la teoría de los conjuntos difusos y la lógica multi-valuada. Hoy en día esta lógica multi-valuada recibe el nombre de lógica difusa. Del mismo modo, cada generalización de los conjuntos difusos genera un nuevo sistema lógico. Todos estos sistemas lógicos coinciden en

que intentan modelar incertidumbre, pero difieren en el objeto matemático que representa esta incertidumbre.

Es fácil comprobar que el mismo problema entre conjuntos difusos y sus generalizaciones puede encontrarse en los distintos sistemas lógicos, es decir, aunque todos ellos son similares, cada noción teórica tiene que ser redefinida para cada lógica. Este problema, junto con el gran número de lógicas que modelan incertidumbre, nos ha llevado a estudiar si es o no posible encontrar un sistema que englobe estas lógicas y nos ha motivado a proponer un sistema lógico que permita modelar la incertidumbre de manera más flexible. Centrándonos especialmente en sistemas lógicos provenientes de la lógica difusa, en esta tesis doctoral proponemos un nuevo sistema lógico que recupera varias de las lógicas de la literatura. La principales ventajas de nuestra propuesta son:

- evitará la excesiva repetición de las nociones teóricas;
- permitirá adaptar la aplicación a la generalización de los conjuntos difusos más adecuada de una manera mucho más sencilla.

En esta tesis doctoral presentamos la semántica del modelo lógico propuesto junto con un estudio en profundidad de la operación de convolución que se utiliza para definir las conectivas disyunción y conjunción del sistema.

Contents

Chapter 1

Introduction

1.1 Motivation

Many-valued logical systems are logical calculi in which there are more than two possible truth values [54]. Their roots can be traced back to Aristotle (IVth century b.c.) when philosophers discovered that a wide category of sentences (especially those expressing future events) cannot be evaluated from the point of view of truth and falsity. Serious attempts to define a non-classical logical did not start until the XIXth century. In 1920, Jan Łukasiewicz proposed the first formal and well-founded studies on many-valued logical systems defining and studying the 3-valued logic which considers a third truth value of "indetermination". Later on, in 1975, the origins of fuzzy logic were established.

The history of fuzzy logic begins with the introduction of fuzzy sets [81] in 1965 by Lotfi Zadeh. In this paper, motivated by problems in information processing and in pattern classification, Zadeh proposed the idea of fuzzy sets as a generalization of classical set theory in which the elements can have intermediary membership grades. Specifically, the unit interval is taken as the range of these membership grades. Moreover, the influence of fuzzy set theory initiated in 1975 [84] the study of fuzzy logic as a class of many-valued logical systems that can model non-stochastic uncertainty.

Soon after the introduction of fuzzy sets, it was discovered that the attribution of membership degrees is an arduous task. As a consequence of this difficulty, many different types of fuzzy sets have appeared in the literature [12]. These generalizations of fuzzy sets have demonstrated to be considerably useful in several applications. However, they require to redefine *ad hoc* every theoretical notion for each particular type of fuzzy set.

The case of fuzzy logic is even more intricate. Each generalization of fuzzy sets constitutes a different logical system. The main difference of these logical systems is underpinned by the mathematical object they use to model uncertainty. For instance, in fuzzy logic as introduced by Zadeh, uncertainty is modeled with a value in the interval $[0, 1]$ while in the generalizations of fuzzy logic uncertainty is modeled with more complex mathematical objects such as intervals, subsets of the unit interval, or functions, among others. Furthermore, similarly to the case of set theory in fuzzy sets, it is necessary to redefine each theoretical notion for every particular logical system.

The existence of such a large number of similar logical systems that model uncertainty, each of them by means of a different mathematical object, has motivated us to study whether or not we can find a general framework which encompasses these logical systems into a single one. Specifically, we search for a logical system that models the uncertainty in a more flexible way. The main advantage of the new logical system is that it will avoid the excessive repetitions and difficult efforts of redefining each theoretical notion in both fuzzy theory and fuzzy logic. Moreover, it will be easy to adapt these notions to the different generalizations of fuzzy sets.

1.2 Description of the problem

The basis of a logical system is a formalized language that includes some well-formed formulas (in this dissertation, called sentences) whose logical values (truth values) are determined either semantically or syntactically [49, 62]. The syntactic determination deals with the analysis of the grammatical structure of the sentences and texts, for example understanding which part is the predicate. The semantic determination refers to the meaning of the sentence, i.e., the understanding of what is being said. The truth value, i.e., the degree of truth of a sentence, and more interestingly, its logical relation to other sentences, is then evaluated with respect to a model. In this dissertation we focus on propositional logic, which is determined semantically.

Propositional logic is a branch of logic concerned with the study of the truth values of composed sentences generated by simple sentences with the use of logical connectives [1, 28, 61]. This means that each simple sentence, i.e., each well-formed formula that only communicates a single idea, has some truth value due to its meaning. We can then determine the truth values of more complex sentences (composed sentences) by means of the truth values of the simple sentences and the connectives involved.

Classical propositional logic is based on two basic assumptions: the principle of bivalence and the principle of compositionality (also called principle of extensionality in the literature) [8, 34, 35]. The principle of bivalence assumes that a sentence is either true "or" false. It is important to mention that the preceding "or" has to be understood in an exclusive way, so that the bivalence principle implies that

- there are only two possible truth values: false ($\mathbb{F}$) and true ($\mathbb{T}$);
- a sentence cannot simultaneously be false and true.

The principle of compositionality assumes that the truth value of each composed sentence can be determined using the truth values of its simple sentences. The most important consequence of the exclusive "or" (that only allows for a single truth value) and the principle of compositionality is the fact that each of the connectives, such as negation, disjunction, conjunction, etc., is semantically determined by functions from the set of truth values (with appropriate arity) into itself. This is usually referred to in the literature as truth-functional logic, or, it is said that the connectives are truth-functional. As a consequence of truth-functionality, considering the connectives as algebraic operations, a crucial point in propositional logic consists not only in studying the set of truth values, but also in studying the universal algebras or algebraic structures that the connectives constitute in the logical system.

On the one hand, many-valued logic differs from classical logic in the fact that it rejects the bivalence principle [11, 55]. Specifically, based on the semantics of a sentence, many-valued logics allow for a variety of truth values. On the other hand, many-valued logics reinforce the compositionality principle. Hence, the study of which universal algebras are constituted still makes sense.

In this dissertation, we propound the first steps into the direction of building a logical system which encompasses several logical systems in the literature. In order to do so, we consider two bounded lattices $\mathbb{L}_1 = (L_1, \vee_1, \wedge_1, 0_1, 1_1)$ and $\mathbb{L}_2 = (L_2, \vee_2, \wedge_2, 0_2, 1_2)$, with corresponding order relations $\leq_1$ and $\leq_2$, and the set of functions $\mathcal{F} = \{f \mid f : L_1 \to L_2\}$ between them. The degree of truth of a sentence in the proposed logical propositional system is determined by an element of $\mathcal{F}$, where the bounded lattice $\mathbb{L}_1$ represents the truth values with respect to the semantic model and the bounded lattice $\mathbb{L}_2$ represents a possibility degree [26, 83]. In this way, for a function $f \in \mathcal{F}$ and for any truth value $x \in \mathbb{L}_1$, $f(x) \in \mathbb{L}_2$ represents the possibility of a sentence having the truth value $x \in \mathbb{L}_1$.

The choice of taking $\mathbb{L}_1$ and $\mathbb{L}_2$ as two bounded lattices comes from the idea that the set of truth values ($\mathbb{L}_1$) must at least contain a truth value that represents the falsehood of a sentence and another one which represents the truth of the sentence. Moreover, we consider these values ordered in a truth-degree *scale*[1], so that, false is the bottom of the truth values and true is the top. Since we do not impose a linearly ordered set, taking a bounded lattice to represent the truth values seems natural. Similarly, for the possibility degrees, we assume that some value, usually represented by 1, means "is possible" while another value, usually represented by 0, means "is not possible". We also consider the possibility degrees to be somehow ordered, but without imposing a linearly ordered set, so that the concept of a bounded lattice for $\mathbb{L}_2$ naturally arises again.

It is worth mentioning that in our proposal we reject two different aspects of the bivalence principle. On the one hand, the bounded lattice $\mathbb{L}_1$ allows to consider a variety of truth values that is more general than the classical true and false. On the other hand, more than one truth value can be considered as possible simultaneously.

Once we have fixed the logical values of the logical propositional system, the next step is to define and study the simplest connectives. For that purpose, we need operations whose domain and codomain lie in the set $\mathcal{F}$. Specifically, following the ideas of Zadeh's extension principle [48, 82], we consider the convolution operation, which allows to extend classical connectives to our logical system.

The mathematical operation of convolution and related operations play a pivotal role in science, engineering and mathematics [20, 59]. In the considered context, convolution takes two real functions as input and outputs a third real function that represents the integral of the pointwise multiplication of the two functions as a function of the amount that one of the original functions is translated. More formally, given two real functions f and g, their convolution[2] is the function $f * g$ defined by

$$(f * g)(t) = \int_D f(\lambda)g(t - \lambda)\mathrm{d}\lambda\,,$$

where D is the integration domain. The convolution operation has applications in probability and statistics, differential equations, signal processing, natural language processing, image

[1] We use the term scale without referring to a linearly ordered set.

[2] One of the earliest uses of the above expression of convolution can be found in [51].

processing and computer vision, and engineering. It satisfies many mathematical properties, including commutativity, associativity, distributivity w.r.t. pointwise addition of functions, and has an identity element (Dirac delta function) and an absorbing element (the constant function equal to 0) [67]. Obviously, convolution is not defined in general, and needs a restriction to an appropriate subset of functions. This notion of convolution has been generalized to various other types of object, such as distribution functions, probability measures, complex functions, functions defined on a group endowed with a measure, and so on.

Similar operations are at the basis of mathematical morphology, a particular direction in image processing [66]. Recall that for subsets A and B of $\mathbb{R}^n$, the Minkowski addition $A \oplus B$ and subtraction $A \ominus B$ are defined as

$$\begin{aligned} A \oplus B &= \{y \in \mathbb{R}^n \mid (\exists x \in B)(y - x \in A)\} \\ A \ominus B &= \{y \in \mathbb{R}^n \mid (\forall x \in B)(y - x \in A)\}\,. \end{aligned}$$

Defining $-B = \{x \in \mathbb{R}^n \mid -x \in B\}$, the dilation $D(A, B)$ and erosion $E(A, B)$ of A by B are defined as $D(A, B) = A \oplus (-B)$ and $E(A, B) = A \ominus (-B)$. Identifying sets with their characteristic mapping, we can also write

$$\begin{aligned} (A \oplus B)(y) &= \sup_{x\in\mathbb{R}^n} \min(B(x), A(y - x)) \\ (A \ominus B)(y) &= \inf_{x\in\mathbb{R}^n} \max(1 - B(x), A(y - x))\,. \end{aligned}$$

The subsets A and B represent the sets of black pixels in a black-and-white image. Treating gray-scale images as fuzzy subsets of $\mathbb{R}^n$ through rescaling, the above expressions are immediately applicable to gray-scale images, which is at the basis of first approaches to fuzzy mathematical morphology [16, 17] and further generalizations [10, 15]. Clearly, they bear a striking similarity with the traditional convolution operation. An alternative approach to gray-scale morphology is based on functions taking values in $\overline{\mathbb{R}} = [-\infty, +\infty]$, leading to expressions that are even more similar to the traditional convolution operation [41]. Interestingly, the theory of mathematical morphology has been further generalized to the lattice-theoretic setting [42, 63, 66].

In fuzzy set theory [47], a similar convolutional spirit can be recognized in Zadeh's seminal extension principle [48, 82]. This principle allows to extend any function $f : X \to Y$ between two universes X and Y to a function between the fuzzy subsets of X and the fuzzy subsets of Y in the following natural way:

$$f(A)(y) = \sup_{f(x)=y} A(x)\,;$$

and, similarly, for a composite universe $X = X_1 \times X_2$:

$$f(A_1, A_2)(z) = \sup_{f(x,y)=z} \min(A_1(x), A_2(y))\,.$$

In particular, this principle is invoked to extend Moore's interval calculus to the computation with fuzzy intervals, leading to fuzzy interval arithmetic [26, 58]. For instance, the maximum, minimum and addition of two fuzzy intervals A and B are defined by

$$\begin{aligned} \max(A,B)(z) &= \sup_{\max(x,y)=z} \min(A(x), B(y)) \\ \min(A,B)(z) &= \sup_{\min(x,y)=z} \min(A(x), B(y)) \\ (A+B)(z) &= \sup_{x} \min(A(x), B(z-x))\,. \end{aligned}$$

The convolutional spirit is most easily recognized in the latter expression of the addition. Further generalizations of the extension principle, replacing $\min(A(x), B(y))$ by $T(A(x), B(y))$, with T a more general triangular norm, have been developed as well [25]. Additionally, settings in which the fuzzy intervals correspond to more general interactive fuzzy variables have also been explored [65].

In particular, the extension principle can be used to extend Boolean operations [18, 26]. The Boolean operations "or" and "and" in classical logic, two binary operations on the set of truth values $\{\mathbb{F}, \mathbb{T}\}$, can be extended to binary operations on the fuzzy subsets of $\{\mathbb{F}, \mathbb{T}\}$ as follows:

$$\begin{aligned} (f \text{ "or" } g)(\mathbb{F}) &= \min(f(\mathbb{F}), g(\mathbb{F})) \\ (f \text{ "or" } g)(\mathbb{T}) &= \max(\min(f(\mathbb{F}), g(\mathbb{T})), \min(f(\mathbb{T}), g(\mathbb{F})), \min(f(\mathbb{T}), g(\mathbb{T}))) \end{aligned}$$

and

$$\begin{aligned} (f \text{ "and" } g)(\mathbb{F}) &= \max(\min(f(\mathbb{F}), g(\mathbb{F})), \min(f(\mathbb{F}), g(\mathbb{T})), \min(f(\mathbb{T}), g(\mathbb{F}))) \\ (f \text{ "and" } g)(\mathbb{T}) &= \min(f(\mathbb{T}), g(\mathbb{T}))\,. \end{aligned}$$

It is worth mentioning that if f and g are possibilistic truth values, i.e., $\max(f(\mathbb{F}), f(\mathbb{T})) = 1$ and $\max(g(\mathbb{F}), g(\mathbb{T})) = 1$, then $(f \text{ "or" } g)$ and $(f \text{ "and" } g)$ are possibilistic truth values as well.

This approach goes back to the early years of fuzzy set theory [18, 74], and has given rise to further generalizations [19].

Zadeh's extension principle has been used in an extensive series of contributions on type-2 fuzzy sets [43, 44, 56, 57, 68], where the elements of $\mathcal{F}([0,1])$, i.e., functions from [0,1] to [0,1], play the role of fuzzy truth values. In particular, the Boolean operations "or" and "and" have been extended to binary operations on $\mathcal{F}([0,1])$ as follows, which can be considered convolution operations as well

$$(f \sqcup g)(x) = \sup_{\max(u,v)=x} \min(f(u), g(v))$$

$$(f \sqcap g)(x) = \sup_{\min(u,v)=x} \min(f(u), g(v))\,.$$

In particular, C.Walker, E. Walker and coauthors have studied in depth the algebraic (lattice) theoretic properties of these convolution operations [38, 40, 76, 77]. A similar approach replacing the unit interval $[0,1]$ by a finite chain has been developed in [78, 75]. Notwithstanding these remarkable achievements, the results obtained cannot easily be extended to a non-linear framework[3], as the proof methods are heavily based on the distributivity and linearity (of the unit interval or a finite chain). This renders these results of little use for the proposed logical system.

The structure of this dissertation is as follows. In Chapter 2, in order to set the notations, we introduce some preliminary notions from both branches of mathematics that are required in this dissertation: algebra and logic. Chapters 3 and 4 are devoted to the study of a general lattice-theoretic framework in which it is meaningful to study the above convolution operations to define the disjunction and conjunction connectives in the proposed logical system. Specifically, in Chapter 3 we identify the main algebraic differences between the convolutions when the functions from the unit interval into itself are replaced by functions between two bounded lattices $\mathbb{L}_1$ and $\mathbb{L}_2$. Moreover, we analyze the possibility of modifying the "or" and "and" convolutions in order to overcome some negative aspects. Chapter 4 is devoted to identify whether or not it is possible to find a particular class of functions where the disjunction and conjunction connectives defined through convolution constitute a bounded lattice. Chapter 5 is devoted to show how several logical systems in the literature are encompassed in the proposed one. We end with some conclusions and future research in Chapter 6.

[3]We refer to a framework that does not depend on a linear order, but on other kind of ordering (lattice, etc).

1.3 Objectives

As we have mentioned, the general objective of this dissertation is to set the first steps into the direction of building a propositional logical system which encompasses several logical systems in the literature. In order to do so, after setting the logical functions as degrees of truth in the system, we require:

- To study the convolution as a suitable operation to define the disjunctive ("or") and conjunctive ("and") connectives of the logical system (Chapter 3).
- To study which universal algebras the disjunctive and conjunction connectives constitute in the proposed logical system (Chapter 4).
- To study which logical systems are encompassed by the proposed one (Chapter 5).

Chapter 2

Preliminaries

2.1 Abstract algebra

The term modern or abstract algebra is nowadays used to refer to a branch of mathematics that studies *universal algebras*, i.e., sets with one or more n-ary operations defined on them. Examples of universal algebras are groups, monoids, rings or the most important universal algebra for this dissertation: lattices. We recall here most of the definitions of abstract algebra required for the development of this dissertation.

Definition 2.1. Let $\mathbb{L} = (L, \star)$ be a universal algebra equipped with a binary operation $\star : L \times L \mapsto L$.

(i) The operation satisfies the idempotency law if it holds that $a \star a = a$, for any $a \in L$.

(ii) The operation satisfies the commutativity law if it holds that $a \star b = b \star a$, for any $a, b \in L$.

(iii) The operation satisfies the associativity law if it holds that $a \star (b \star c) = (a \star b) \star c$, for any $a, b, c \in L$.

(iv) The operation satisfies the annihilation law if there exists an element $\ell_0 \in L$ such that $\ell_0 \star a = a \star \ell_0 = \ell_0$ holds for any $a \in L$. The element ℓ_0 is said to be an annihilator or absorbing element of the operation $\star$.

(v) The operation satisfies the identity law if there exists an element $\ell_1 \in L$ (with $\ell_0 \neq \ell_1$ when there exists annihilator element) such that $\ell_1 \star a = a \star \ell_1 = a$ holds for any $a \in A$. The element ℓ_0 is said to be a neutral or identity element of the operation $\star$.

Note that in case the annihilator and/or identity element exist, they are unique.

Operation	Property	Formulae
$\star$	Idempotency law	$a \star a = a$ for any $a \in L$
	Commutativity law	$a \star b = a \star b$ for any $a, b \in L$
	Associativity law	$a \star (b \star c) = (a \star b) \star c$ for any $a, b, c \in L$
	Annihilation law	$\exists \ell_0 \in L$ s.t. $\ell_0 \star a = a \star \ell_0 = \ell_0$ for any $a \in L$
	Identity law	$\exists \ell_0 \in L$ s.t. $\ell_0 \star a = a \star \ell_0 = a$ for any $a \in L$
$\star_1$, $\star_2$	Distributivity laws	$a \star_1 (b \star_2 c) = (a \star_1 b) \star_2 (a \star_1 c)$ for any $a, b, c \in L$ $a \star_2 (b \star_1 c) = (a \star_2 b) \star_1 (a \star_2 c)$ for any $a, b, c \in L$
	Absorption laws	$a \star_1 (a \star_2 b) = a$ for any $a, b \in L$ $a \star_2 (a \star_1 b) = a$ for any $a, b \in L$
	the Birkhoff equation	$a \star_1 (a \star_2 b) = a \star_2 (a \star_1 b)$ for any $a, b \in L$

Table 2.1: Summary of some algebraic properties involving one or two binary operations: idempotency law, commutativity law, associativity law, annihilation law, identity law, distributivity laws, absorption laws and the Birkhoff equation.

Definition 2.2. Let $\mathbb{L} = (L, \star_1, \star_2)$ be a universal algebra equipped with two binary operations $\star_1, \star_2 : L \times L \mapsto L$, such that they satisfy the commutativity law.

(i) We say that the operation $\star_1$ distributes over $\star_2$ if it holds that $a \star_1 (b \star_2 c) = (a \star_1 b) \star_2 (a \star_1 c)$, for any $a, b, c \in L$. We refer to this property as the distributivity law.

(ii) We say that the operation $\star_1$ absorbs $\star_2$ if it holds that $a \star_1 (a \star_2 b) = a$, for any $a, b \in L$. We refer to this property as the absorption law.

(iii) We say that the operations $\star_1$ and $\star_2$ satisfy the Birkhoff equation if it holds that $a \star_1 (a \star_2 b) = a \star_2 (a \star_1 b)$, for any $a, b \in L$.

Note that in every algebra equipped with two commutative operations $\star_1$, $\star_2$ there are two distributivity laws: it can hold that the operation $\star_1$ distributes over $\star_2$ and/or that the operation $\star_2$ distributes over $\star_1$. Similarly, there are two absorption laws: it can hold that the operation $\star_1$ absorbs $\star_2$ and/or that the operation $\star_2$ absorbs $\star_1$. Table 2.1 summarizes all the properties of the operations we have introduced in Definitions 2.1 and 2.2.

We recall some classes of universal algebras that we will use in the subsequent development of this dissertation. For more information on these universal algebras, see [9, 14, 37, 46, 13]. We include a summary of the classes of universal algebras in Table 2.2.

Definition 2.3. Let $\mathbb{L} = (L, \star)$ be a universal algebra equipped with one binary operation.

(i) We say that $\mathbb{L} = (L, \star)$ is a monoid if the operation $\star$ satisfies the associativity and identity laws. Moreover, if $\star$ satisfies the commutativity law, the monoid is called commutative or Abelian.

(ii) We say that $\mathbb{L} = (L, \star)$ is a semilattice if the operation $\star$ satisfies the idempotency, commutativity and associativity laws.

Definition 2.4. Let $\mathbb{L} = (L, \star_1, \star_2)$ be a universal algebra equipped with two binary operations.

(i) We say that $\mathbb{L} = (L, \star_1, \star_2)$ is a Birkhoff system if each of the operations $\star_1$ and $\star_2$ constitutes a semilattice and they satisfy the Birkhoff equation.

(ii) We say that $\mathbb{L} = (L, \star_1, \star_2)$ is a lattice if each of the operations $\star_1$ and $\star_2$ constitutes a semilattice and the two absorption laws hold. Moreover, the lattice is called bounded if each of the operations satisfies the identity law.

In a lattice the operations are usually called join and meet. The symbol $\vee$ is frequently used to refer to the join operation and the symbol $\wedge$ to refer to the meet operation. It can be proven that the fulfillment of the idempotency laws in a lattice is a consequence of the fulfillment of absorption laws. Moreover, in a bounded lattice it holds that the identity element of the join operation $\vee$ is the annihilator element of the meet operation $\wedge$ and the identity element of the meet operation $\wedge$ is the annihilator element of the join operation $\vee$. We use the notation $\mathbb{L} = (L, \vee, \wedge, 0_L, 1_L)$ to refer to a bounded lattice, where 0_L is the identity element of the operation $\vee$ and 1_L is the identity element of the operation $\wedge$.

2.2 Lattice theory

2.2.1 Basic notions of lattice theory

The notion of a lattice will be a central concept in this dissertation. In Section 2.1, we have defined it from an algebraic point of view. There is an equivalent definition starting from a partially ordered set that we recall here. Note that in this dissertation, we will switch back and forth between the algebraic and order-theoretic interpretation when it is convenient.

<table>
<tr><th>Set / Operation</th><th>Universal Algebra</th><th>Properties</th></tr>
<tr><td rowspan="2">$(\mathbb{L},\star)$</td><td>(Commutative) monoid</td><td>Associativity law
Identity law
(Commutativity law)</td></tr>
<tr><td>Semilattice</td><td>Idempotency law
Commutativity law
Associativity law</td></tr>
<tr><td rowspan="2">$(\mathbb{L},\star_1,\star_2)$</td><td>Birkhoff system</td><td>$(L,\star_1)$ semilattice
$(L,\star_2)$ semilattice
the Birkhoff equation</td></tr>
<tr><td>(Bounded) lattice</td><td>$(L,\star_1)$ semilattice
$(L,\star_2)$ semilattice
Absorption laws
(Identity laws)</td></tr>
</table>

Table 2.2: Summary of some algebraic structures involving one or two binary operations: (commutative) monoid, semilattice, Birkhoff system, (bounded) lattice.

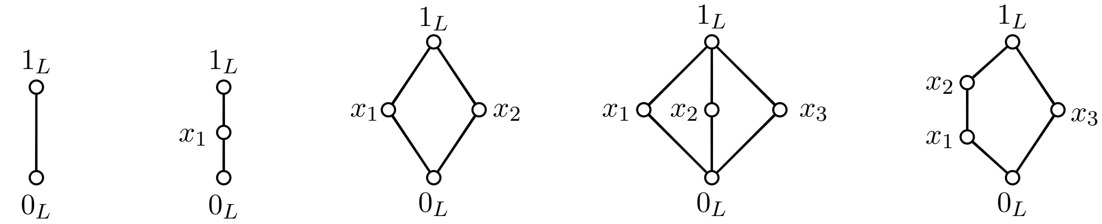
Definition 2.5. A partially ordered set (poset) is a set L equipped with a partial order relation $\leq$, i.e., a binary relation that satisfies the following conditions: for any $a, b, c \in L$, it holds that

(i) $a \leq a$ (reflexivity);

(ii) if $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry);

(iii) if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity).

Let $(L, \leq)$ be a poset and $B \subseteq L$. An element $u \in L$ is said to be an upper bound of B if $b \leq u$ for any $b \in B$. A set may have one, several or none at all upper bounds. An upper bound u^* of B is said to be a least upper bound of B if $u^* \leq u$ for any upper bound u of B. If a least upper bound of B exists, then it is unique, it is called the supremum of B and is denoted by $\sup B$. Analogously, $\ell \in L$ is said to be a lower bound of B if $\ell \leq b$ for any $b \in B$. A lower bound ℓ_* of B is said to be a greatest lower bound of B if $\ell \leq \ell_*$ for any lower bound ℓ of B. If a greatest lower bound of B exists, then it is unique, it is called the infimum of B and is denoted by $\inf B$. If any two-element subset $\{a, b\} \subseteq L$ has a supremum, denoted $a \vee b$, and an infimum, denoted $a \wedge b$, then the poset $(L, \leq)$ can be seen as a lattice $\mathbb{L} = (L, \vee, \wedge)$.



(a) $\mathbb{C}_2$ (b) $\mathbb{C}_3$ (c) $\mathbb{M}_2$ (d) $\mathbb{M}_3$ (e) $\mathbb{N}_5$

Figure 2.1: Hasse diagrams of finite lattices: (a) lattice $\mathbb{C}_2$, (b) lattice $\mathbb{C}_3$, (c) lattice $\mathbb{M}_2$, (d) lattice $\mathbb{M}_3$ (also called diamond), and (e) lattice $\mathbb{N}_5$ (also called pentagon).

Moreover, if the poset $(L, \leq)$ is bounded, i.e., there exist two elements 0_L and 1_L such that $0_L \leq a \leq 1_L$ for any $a \in L$, then both operations of the lattice satisfies the identity law.

Lemma 2.1. (The Connecting Lemma) [14] Let $\mathbb{L} = (L, \vee, \wedge)$ be a lattice and $a, b \in L$. The following statements are equivalent:

(i) $a \leq b$;

(ii) $a \vee b = b$;

(iii) $a \wedge b = a$.

As a consequence of the preceding lemma, given a lattice $\mathbb{L} = (L, \vee, \wedge)$, the relation $\leq$ defined by $a \leq b$ if $a \vee b = b$, and the relation $\leq'$ defined by $a \leq' b$ if $a \wedge b = a$, coincide and turn $\mathbb{L}$ into the same poset.

Definition 2.6. Let $(L, \leq)$ be a poset and $a, b \in L$. We say that the elements a and b are

(i) comparable: if it holds that $a \leq b$ or $b \leq a$;

(ii) incomparable: if neither $a \leq b$ nor $b \leq a$ holds. In this case we use the notation $a \parallel b$.

The Hasse diagram is a graphical tool for representing a finite poset $(L, \leq)$, or, for our dissertation, a method for representing a finite lattice $\mathbb{L}$, as a graph. The elements of the set L are represented by vertices in the graph. If $a \leq b$, then the vertex representing b is placed higher than the vertex representing a. Moreover, if there is no $c \in L$ such that $a \leq c \leq b$, then a straight segment from a to b is depicted. In Fig. 2.1 are depicted some Hasse diagrams of finite lattices that we will use in the development of this dissertation.

Definition 2.7. Let $\mathbb{L} = (L, \vee, \wedge, 0_L, 1_L)$ be a bounded lattice.

(i) The lattice is called distributive if it satisfies the distributivity laws. Moreover, in a lattice the fulfillment of one distributive law implies the fulfillment of the other [9, 14].

(ii) The lattice is called a chain if all the elements are comparable. Note that in this case the Hasse diagram of the lattice $\mathbb{L}$ is a vertical line from the bottom 0_L to the top 1_L composed of vertical segments from an element to another one. The Hasse diagrams of Figs. 2.1(a)–(b) represent chains with 2 and 3 elements, respectively. Generally, we use the notation $\mathbb{C}_n$, with $n \in \mathbb{N}$, to refer to the chain with n elements.

(iii) The lattice is said to be complete if $\sup B$ and $\inf B$ exist for any $B \subseteq L$. Note that any finite bounded lattice is complete. For the sake of convenience, in this paper, instead of $\sup B$, we will also use the more explicit notation $\bigvee_{b\in B} b$.

(iv) A complete lattice is called a frame [37] if it satisfies the meet-continuity property: for any $a \in L$ and any $\emptyset \subset B \subseteq L$, it holds that

$$a \wedge \left(\bigvee_{b\in B} b\right) = \bigvee_{b\in B} (a \wedge b)\,. \tag{2.1}$$

Note that any frame is distributive [9, 14]. Any finite distributive lattice satisfies the meet-continuity property and, consequently, any finite distributive lattice is a frame.

Depending on the source, frames are also called complete Heyting algebras or complete Brouwerian lattices [9], due to the fact that in complete lattices, meet-continuity and the residuation property [29] are equivalent (see, for example p. 128 in [9]). In this dissertation, we will only make use of the meet-continuity property, and we will therefore stick to the term frame.

Definition 2.8. A sublattice $\mathbb{M}$ of a lattice $\mathbb{L}$ is a subset M of L such that M is closed under join and meet operations, i.e., for any $a, b \in M$ it holds that $a \vee b \in M$ and $a \wedge b \in M$.

We recall the $\mathbb{M}_3 - \mathbb{N}_5$ theorem, a.k.a. diamond-pentagon theorem, about the distributivity of a lattice [14].

Theorem 2.1. A lattice $\mathbb{L}$ is not distributive if and only if it has a sublattice that is isomorphic to the lattice $\mathbb{M}_3$ or the lattice $\mathbb{N}_5$ (see Figs. 2.1(d)–(e)).

Note that the lattices depicted in Figs. 2.1(a)–(c) neither have a sublattice that is isomorphic to the lattice $\mathbb{M}_3$ nor to the lattice $\mathbb{N}_5$. Consequently, due to Theorem 2.1 the lattices depicted in Figs. 2.1(a)–(c) are finite distributive lattices, i.e., they are particular instances of frames. Also the unit interval equipped with the max and min operations is a frame with identity elements 0 and 1 respectively, i.e., the lattice $\mathbb{L} = ([0,1], \max, \min, 0, 1)$ is a frame. We will use the notation $[\![0,1]\!]$ to refer to the unit interval as a frame.

(a) $\mathbb{M}_2$ (b) Function f

Figure 2.2: (a) Hasse diagram of the lattice $\mathbb{M}_2$ and (b) graphical representation of a function $f : M_2 \to [0, 1]$.

2.2.2 Functions between lattices

In this dissertation we consider two bounded lattices $\mathbb{L}_1 = (L_1, \vee_1, \wedge_1, 0_1, 1_1)$ and $\mathbb{L}_2 = (L_2, \vee_2, \wedge_2, 0_2, 1_2)$, with corresponding order relations $\leq_1$ and $\leq_2$, and the set of functions $\mathcal{F}(\mathbb{L}_1, \mathbb{L}_2) = \{f \mid f : L_1 \to L_2\}$ between them. The elements of $\mathcal{F}(\mathbb{L}_1, \mathbb{L}_2)$ are called lattice functions. Note that the elements of $\mathcal{F}(\mathbb{L}_1, \mathbb{L}_2)$ depend on the selected $\mathbb{L}_1$ and $\mathbb{L}_2$. However, when no confusion can occur and for the sake of simplicity, we will refer to $\mathcal{F}$ without explicitly indicating the lattices $\mathbb{L}_1 = (L_1, \vee_1, \wedge_1, 0_1, 1_1)$ and $\mathbb{L}_2 = (L_2, \vee_2, \wedge_2, 0_2, 1_2)$.

When L_1 is finite, a function $f : L_1 \to L_2$ can be conveniently visualised by replacing the elements of L_1 in the Hasse diagram of $\mathbb{L}_1$ by the corresponding function values in L_2. For instance, the function f from the bounded lattice $\mathbb{M}_2$ in Fig. 2.2(a) to $[\![0, 1]\!]$ defined as:

$$f(x) = \begin{cases} 0.6 & , \text{ if } x = 0\,, \\ 0 & , \text{ if } x = x_1\,, \\ 0.8 & , \text{ if } x = x_2\,, \\ 0.1 & , \text{ if } x = 1\,, \end{cases}$$

is depicted in Fig. 2.2(b).

Given a lattice function $f \in \mathcal{F}$, the element $s_f \in L_2$ defined as $s_f := \bigvee_{x \in L_1} f(x)$ is called the supremum of f. We also consider the functions $\mathbf{0}_a$ and $\mathbf{1}_a$ (with $a \in L_2$) defined as:

$$\mathbf{0}_a(x) = \begin{cases} a & , \text{ if } x = 0\,, \\ 0 & , \text{ otherwise}\,; \end{cases} \qquad \mathbf{1}_a(x) = \begin{cases} a & , \text{ if } x = 1\,, \\ 0 & , \text{ otherwise}\,. \end{cases}$$

An obvious way to turn the set $\mathcal{F}$ into a lattice is by extending the lattice operations of $\mathbb{L}_2$ to operations on $\mathcal{F}$ in a pointwise manner.

Definition 2.9. For any $f, g \in \mathcal{F}$,

(i) the pointwise join of f and g is the lattice function $f \vee g$ defined by:

$$(f \vee g)(x) = f(x) \vee_2 g(x)\,;$$

(ii) the pointwise meet of f and g is the lattice function $f \wedge g$ defined by:

$$(f \wedge g)(x) = f(x) \wedge_2 g(x)\,.$$

Theorem 2.2. Let $\vee$ and $\wedge$ be the pointwise operations on $\mathcal{F}$ introduced in Definition 2.9. Then the universal algebra $\mathbb{F} = (\mathcal{F}, \vee, \wedge, \underline{\mathbf{0}}, \overline{\mathbf{1}})$ is a bounded lattice where $\underline{\mathbf{0}}$ and $\overline{\mathbf{1}}$ are defined by $\underline{\mathbf{0}}(x) = 0_2$ and $\overline{\mathbf{1}}(x) = 1_2$. Moreover, the corresponding order relation $\leq$ is given by $f \leq g$ if $f(x) \leq_2 g(x)$ for any $x \in L_1$, where $\leq_2$ is the order relation of the lattice $\mathbb{L}_2$.

2.3 Propositional logic

2.3.1 General concepts of logic

In this section we look at the basics of propositional logic [52]. We consider sentences or propositions as the most elementary linguistic terms that convey information. A composed sentence is generated by simple sentences joined by what is called a connective. The most frequent connectives are:

- the negation: (not P) referred as $\overline{P}$;
- the disjunction: (P or Q) referred as $P \vee Q$;
- the conjunction: (P and Q) referred as $P \wedge Q$;
- the implication: (P implies Q) referred as $P \Rightarrow Q$;
- the equivalence: (P is equivalent to Q) referred as $P \Leftrightarrow Q$.

Note that the first connective differs from the others in the fact that only one sentence is required. From an algebraic point of view, the connectives can be seen as operations on the set of sentences, i.e., if we consider

$$\mathbb{S} = \{S \mid S \text{ is a sentence}\}\,,$$

the connectives can be understood as algebraic unary or binary operations on $\mathbb{S}$. Hence, the set of sentences $\mathbb{S}$ equipped with some connectives constitutes some universal algebra.

2.3.2 Classical logic

As explained in the introduction of this dissertation, classical logic, also called bivalent logic in the literature is based on the principle of bivalence. This means that each sentence of the logical system can be either true or false, i.e., a sentence can only have one of two truth values: false ($\mathbb{F}$) and true ($\mathbb{T}$). Since the considered or of the possible truth values is exclusive, i.e., a sentence can not be both true and false simultaneously, and due to the principle of compositionality we can associate a function $v : \mathbb{S} \mapsto \{\mathbb{T}, \mathbb{F}\}$, called evaluation function, which assigns each sentence to its truth value (degree of truth).

The evaluation of the negation connective yields the contrary truth value, i.e., for any $P \in \mathbb{S}$

- if $v(P) = \mathbb{F}$, then $v(\overline{P}) = \mathbb{T}$;
- if $v(P) = \mathbb{T}$, then $v(\overline{P}) = \mathbb{F}$.

Note that since this connective is a unary operation, it does not generate a composed sentence, but it reverses the degree of truth of the sentence. All the presented connectives: disjunction, conjunction, implication and equivalence generate composed sentences.

The evaluation of the composed sentence generated by the disjunction connective only yields false if both sentences are false and it yields true in all the other cases, i.e., for any $P, Q \in \mathbb{S}$

- if $v(P) = \mathbb{F}$ and $v(Q) = \mathbb{F}$, then $v(P \vee Q) = \mathbb{F}$;
- if $v(P) = \mathbb{F}$ and $v(Q) = \mathbb{T}$, then $v(P \vee Q) = \mathbb{T}$;
- if $v(P) = \mathbb{T}$ and $v(Q) = \mathbb{F}$, then $v(P \vee Q) = \mathbb{T}$;
- if $v(P) = \mathbb{T}$ and $v(Q) = \mathbb{T}$, then $v(P \vee Q) = \mathbb{T}$.

The evaluation of the composed sentences generated by the conjunction connective yields true only if both sentences are true and it yields false in all the other cases, i.e., for any $P, Q \in \mathbb{S}$

- if $v(P) = \mathbb{F}$ and $v(Q) = \mathbb{F}$, then $v(P \wedge Q) = \mathbb{F}$;
- if $v(P) = \mathbb{F}$ and $v(Q) = \mathbb{T}$, then $v(P \wedge Q) = \mathbb{F}$;
- if $v(P) = \mathbb{T}$ and $v(Q) = \mathbb{F}$, then $v(P \wedge Q) = \mathbb{F}$;
- if $v(P) = \mathbb{T}$ and $v(Q) = \mathbb{T}$, then $v(P \wedge Q) = \mathbb{T}$.

The evaluation of the composed sentence generated by implication connective yields false only if P is true and Q is false and it yields true in all the other cases, i.e., for any $P, Q \in \mathbb{S}$

- if $v(P) = \mathbb{F}$ and $v(Q) = \mathbb{F}$, then $v(P \Rightarrow Q) = \mathbb{T}$;
- if $v(P) = \mathbb{F}$ and $v(Q) = \mathbb{T}$, then $v(P \Rightarrow Q) = \mathbb{T}$;
- if $v(P) = \mathbb{T}$ and $v(Q) = \mathbb{F}$, then $v(P \Rightarrow Q) = \mathbb{F}$;
- if $v(P) = \mathbb{T}$ and $v(Q) = \mathbb{T}$, then $v(P \Rightarrow Q) = \mathbb{T}$.

P	$\overline{P}$
$\mathbb{F}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{F}$

$\vee$	$\mathbb{F}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$

$\wedge$	$\mathbb{F}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$

P	Q	$P \Rightarrow Q$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$

P	Q	$P \Leftrightarrow Q$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$

Table 2.3: Truth values of the connectives on classical logic: negation, disjunction, conjunction, implication and equivalence.

The evaluation of the composed sentence generated by the equivalence connective yields true if the evaluation of P is equal to the evaluation of Q and it yields false in all the other cases, i.e., for any $P, Q \in \mathbb{S}$

- if $v(P) = \mathbb{F}$ and $v(Q) = \mathbb{F}$, then $v(P \Leftrightarrow Q) = \mathbb{T}$;
- if $v(P) = \mathbb{F}$ and $v(Q) = \mathbb{T}$, then $v(P \Leftrightarrow Q) = \mathbb{F}$;
- if $v(P) = \mathbb{T}$ and $v(Q) = \mathbb{F}$, then $v(P \Leftrightarrow Q) = \mathbb{F}$;
- if $v(P) = \mathbb{T}$ and $v(Q) = \mathbb{T}$, then $v(P \Leftrightarrow Q) = \mathbb{T}$.

Let $P \in \mathbb{S}$ be a sentence. In this dissertation we do not make use of P but only of its evaluation $v(P)$, i.e., we are only interested in the truth values of the sentences. Hence, since no confusion can occur and for the sake of simplicity, we will use P to refer to its evaluation $v(P)$.

The truth values of classical logic can be also reformulated in terms of standard numerical operations. Let 0 refer to the truth value $\mathbb{F}$ and 1 refer to the truth value $\mathbb{T}$. The evaluation of the sentence generated by the logical connectives are determined as follows:

- The evaluation of the negation $\overline{P} = 1 - P$, for any $P \in \mathcal{S}$.
- The evaluation of the disjunction $P \vee Q = \max(P, Q)$, for any $P, Q \in \mathcal{S}$.
- The evaluation of the conjunction $P \wedge Q = \min(P, Q)$, for any $P, Q \in \mathcal{S}$.
- The evaluation of the implication $P \Rightarrow Q = \max(\overline{(P)}, Q)$, for any $P, Q \in \mathcal{S}$.
- The evaluation of the equivalence $P \Leftrightarrow Q = \min(\max(\overline{P}, Q), \max(P, \overline{Q}))$, for any $P, Q \in \mathcal{S}$.

One easily verifies that in classical logic the universal algebra of the set of sentences $\mathbb{S}$ equipped with the connectives negation, disjunction and conjunction constitutes a Boolean algebra. This means that the disjunction and conjunction connectives constitute a bounded distributive lattice and the complementarity laws are satisfied, i.e., the universal algebra $(\mathbb{S}, \vee, \wedge)$ is a bounded distributive lattice and, it holds that $P \vee \overline{P} = 1$ and $P \wedge \overline{P} = 0$, for any $P \in \mathbb{S}$.

P	$\overline{P}$
$\mathbb{F}$	$\mathbb{T}$
$\mathbb{U}$	$\mathbb{U}$
$\mathbb{T}$	$\mathbb{F}$

$\vee$	$\mathbb{F}$	$\mathbb{U}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{U}$	$\mathbb{T}$
$\mathbb{U}$	$\mathbb{U}$	$\mathbb{U}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$

$\wedge$	$\mathbb{F}$	$\mathbb{U}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{U}$	$\mathbb{F}$	$\mathbb{U}$	$\mathbb{U}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{U}$	$\mathbb{T}$

Table 2.4: Truth values of the connectives in Łukasiewicz logic: negation, disjunction and conjunction.

2.3.3 Łukasiewicz logic

Łukasiewicz logic is one of the three-valued logics [11], i.e., a logic where three linearly ordered truth values are considered. The three truth values are: false ($\mathbb{F}$), undecided ($\mathbb{U}$) and true ($\mathbb{T}$).

In the preceding section we have listed the truth values of the sentences generated by the five most frequent connectives: negation, disjunction, conjunction, implication and equivalence. However, for the purpose of this dissertation, we will only need the truth values of the negation, disjunction and conjunction connectives. Moreover, for the sake of simplicity, we summarize their truth values in Table 2.4.

Note that since we do not recall the complete logical system, i.e., we do not recall the values of implication and equivalence, some other 3-valued logics, such as Kleene logic [11], coincide with the presented truth tables in Table 2.4. For simplicity, we will only use the name of Łukasiewicz logic.

Once again, if 0 refers to the truth value $\mathbb{F}$, $\frac{1}{2}$ to the truth value $\mathbb{U}$ and 1 to the truth value $\mathbb{T}$, we can retrieve the truth values of the evaluations of the composed sentences generated by the connectives as follows:

- The evaluation of the negation $\overline{P} = 1 - P$, for any $P \in \mathbb{S}$.
- The evaluation of the disjunction $P \vee Q = \max(P, Q)$, for any $P, Q \in \mathbb{S}$.
- The evaluation of the conjunction $P \wedge Q = \min(P, Q)$, for any $P, Q \in \mathbb{S}$.

One easily verifies that in Łukasiewicz logic the universal algebra of the set of sentences $\mathbb{S}$ equipped with the connectives negation, disjunction and conjunction constitutes a De Morgan algebra. This means that the disjunction and conjunction connectives constitute a bounded distributive lattice and the De Morgan laws are satisfied, i.e., the universal algebra $(\mathbb{S}, \vee, \wedge)$ is a bounded distributive lattice and it holds that $\overline{P \wedge Q} = \overline{P} \vee \overline{Q}$ and $\overline{P \vee Q} = \overline{P} \wedge \overline{Q}$ for any $P, Q \in \mathbb{S}$. Note that the complementarity laws do not hold.

P	$\overline{P}$
$\mathbb{F}$	$\mathbb{T}$
$\mathbb{B}$	$\mathbb{B}$
$\mathbb{N}$	$\mathbb{N}$
$\mathbb{T}$	$\mathbb{F}$

$\vee$	$\mathbb{F}$	$\mathbb{B}$	$\mathbb{N}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{B}$	$\mathbb{N}$	$\mathbb{T}$
$\mathbb{B}$	$\mathbb{B}$	$\mathbb{B}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{N}$	$\mathbb{N}$	$\mathbb{T}$	$\mathbb{N}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$

$\wedge$	$\mathbb{F}$	$\mathbb{B}$	$\mathbb{N}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{B}$	$\mathbb{F}$	$\mathbb{B}$	$\mathbb{F}$	$\mathbb{B}$
$\mathbb{N}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{N}$	$\mathbb{N}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{B}$	$\mathbb{N}$	$\mathbb{T}$

Table 2.5: Truth values of the connectives in Belnap logic: negation, disjunction and conjunction

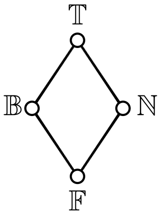


Figure 2.3: Lattice $\mathbb{L}$ representing the truth values of Belnap logic.

2.3.4 Belnap logic

Belnap logic is a four-valued logic [6, 7], i.e., a logic where four possible truth values are considered. The four truth values are: false ($\mathbb{F}$), true ($\mathbb{T}$), both ($\mathbb{B}$) and none ($\mathbb{N}$).

The design of Belnap logic was developed in order to generate a system which distinguishes between inconsistent and incomplete information and is frequently used in systems where some (more than one) computers/agents return some information to the system.[1] A sentence will be determined:

- false if one or more computers have returned false and none of them has returned true;
- true if one or more computers have returned true and none of them has returned false;
- both if at least one computer has returned true and at least one computer has returned false;
- none if none of the computers has returned neither true nor false.

The truth values of the negation, disjunction and conjunction connectives are summarized in Table 2.5.

If we consider the truth values of Belnap logic represented by the lattice depicted in Fig. 2.3, then the evaluation of the composed sentences generated by the disjunction and conjunction

[1]It is worth mentioning that the semantics of this logic was criticised a lot in the literature [22, 23, 79]. However, we consider the discussion out of the scope of this dissertation.

connectives can be computed by the join and meet of the lattice as follows:

- The evaluation of the disjunction $P \vee Q = \sup(P, Q)$, for any $P, Q \in \mathbb{S}$.
- The evaluation of the conjunction $P \wedge Q = \inf(P, Q)$, for any $P, Q \in \mathbb{S}$.

Note that, unlike all the aforementioned examples, the set of truth values does not constitute a totally ordered set (i.e., a chain) but only a partially ordered set.

One easily verifies that in Belnap logic the universal algebra of the set of sentences $\mathbb{S}$ equipped with the connectives negation, disjunction and conjunction also constitutes a De Morgan algebra.

2.3.5 Fuzzy logic and its generalizations

The idea of fuzzy logic was first introduced by Lotfi Zadeh in 1965 [81]. Its design was developed in order to generate a logical system which deals with uncertainty and imprecise information in engineering applications.

On December 11, 2008, in the *bisc-group* mailing list Zadeh proposes the following definitions:

Definition 2.10. Fuzzy logic is a precise system of reasoning, deduction and computation in which the objects of discourse and analysis are associated with information which is, or is allowed to be, imperfect.

Definition 2.11. Imperfect information is defined as information which in one or more respects is imprecise, uncertain, vague, incomplete, partially true or partially possible.

As Zadeh explains in his paper on the occasion of the 50th anniversary of fuzzy sets [86]: One of the principal contributions of fuzzy logic is providing a basis for a progression from binarization to graduation, from binarism to pluralism, from black and white to shades of gray.

Zadeh also expressed in 1994 [85] that fuzzy logic can be understood in a narrow and a wide sense. In a wide sense, fuzzy logic is almost synonymous with fuzzy set theory, which includes tools such as fuzzy arithmetic, fuzzy topology or fuzzy data analysis, among others. In this dissertation, we are only interested in fuzzy logic in its narrow sense, i.e., we are only interested in fuzzy logic as a many-valued logic . Specifically, fuzzy logic in its narrow sense can be understood as a many-valued logic system where the set of truth values is the unit interval $[0, 1]$ [52, 53, 70]. Since the unit interval is an infinite set, we can not summarize the evaluation of the composed sentences generated by the connectives in a truth table (as in the preceding logical systems). However, mathematically, the evaluation of the connectives can be computed as follows:

- The evaluation of the negation $\overline{P} = 1 - P$, for any $P \in \mathbb{S}$.
- The evaluation of the disjunction $P \vee Q = \max(P, Q)$, for any $P, Q \in \mathbb{S}$.
- The evaluation of the conjunction $P \wedge Q = \min(P, Q)$, for any $P, Q \in \mathbb{S}$.

Once again, one easily verifies that in fuzzy logic the universal algebra of the set of sentences $\mathbb{S}$ equipped with the connectives negation, disjunction and conjunction also constitutes a De Morgan algebra.[2]

In 1971, and using the ideas given in [5], Zadeh stated in his work [82] that: *The problem of estimating the membership degrees of the elements to the fuzzy set is related to the abstraction, a problem that plays a central role in pattern recognition.* Therefore, the determination of the membership degree of each element to the set is the biggest problem for applying fuzzy set theory. Taking into account these considerations many different types of generalizations of fuzzy sets have been defined.

Some of these generalizations aim at solving the problem of constructing the membership degrees of the elements to the fuzzy set, and others focus on representing the uncertainty linked to the considered problem in a way different from the one proposed by Zadeh.

It is important to mention that each generalization of fuzzy set also generates a generalization of fuzzy logic, i.e., it generates a new many-valued logical system. The most important ones for this dissertation are:

- Interval-valued fuzzy logics: the considered truth values are closed subintervals of the unit interval $[0, 1]$.
- Hesitant fuzzy logics: the considered truth values are subsets of the unit interval $[0, 1]$.
- Type-2 fuzzy logics: the considered truth values are functions from the unit interval into itself.
- $\mathbb{L}$-fuzzy logics: the considered truth values are elements in a set $\mathbb{L}$. Moreover, as Goguen argues in [30], one of the most appropriate structures for $\mathbb{L}$ is complete lattices.

The presented generalizations of fuzzy logics will be crucial in Chapter 5. To avoid the repetition of the definitions of each generalization of fuzzy logic, we introduce them in Chapter 5.

[2]T-conorms and t-norms were later introduced as a generalization of disjunction and conjunction connectives in fuzzy logic but these notions are out of the scope of this dissertation [50, 60].

Chapter 3

Convolution operations

3.1 Join- and meet-convolution on $\mathcal{F}([0,1],[0,1])$

A first step into the direction of building the proposed logical system consists in defining the simplest connectives: disjunction and conjunction. For their definition, as we have explained in the introduction and following the ideas of Zadeh's extension principle [48, 82], we will use the mathematical operation of convolution.

The truth values in type-2 fuzzy logic are functions from the unit interval into itself. Following the notations introduced in Chapter 2, these truth values are elements of the set $\mathcal{F}(\mathbb{L}_1, \mathbb{L}_2)$, where $\mathbb{L}_1 = \mathbb{L}_2 = [\![0,1]\!]$. The disjunction and conjunction connectives in type-2 fuzzy logic, i.e., the Boolean operations "or" and "and", have been extended using Zadeh's extension principle as follows. For any $f, g \in \mathcal{F}([0,1],[0,1])$,

$$(f \sqcup g)(x) = \bigvee_{u \vee v = x} f(u) \wedge g(v) \tag{3.1}$$

$$(f \sqcap g)(x) = \bigvee_{u \wedge v = x} f(u) \wedge g(v)\,. \tag{3.2}$$

As a first approach to the definition of the "or" and "and" operations for the proposed logic system, we consider the straightforward generalization of these convolution operations. Specifically, we consider the same formulation as in Eqs. (3.1) and (3.2) where the functions belonging to the set $\mathcal{F}([0,1],[0,1])$ are replaced by functions belonging to the set $\mathcal{F}(\mathbb{L}_1, \mathbb{L}_2)$, where $\mathbb{L}_1 = (L_1, \vee_1, \wedge_1, 0_1, 1_1)$ and $\mathbb{L}_2 = (L_2, \vee_2, \wedge_2, 0_2, 1_2)$ are two bounded lattices. Since in classical logic the "or" and "and" operations constitute a bounded distributive lattice, our final goal consists in studying whether it is possible or not to constitute a bounded distributive lattice with the convolution operations defined on $\mathcal{F}(\mathbb{L}_1, \mathbb{L}_2)$. After identifying the main drawbacks of these convolution operations, we analyze two other definitions for disjunction and conjunction connectives for the proposed logical system. Although we will show these new definitions overcome some negative aspects of the firstly mentioned convolution operations, we will also show that the new definitions have other weaknesses. These weaknesses inspired us to analyze a different line of research in Chapter 4.

3.2 Join- and meet-convolution on $\mathcal{F}(\mathbb{L}_1, \mathbb{L}_2)$

The elements of $\mathcal{F}(\mathbb{L}_1, \mathbb{L}_2)$ depend on the bounded lattices $\mathbb{L}_1$ and $\mathbb{L}_2$. However, when no confusion can occur and for the sake of simplicity we will simply use $\mathcal{F}$ to refer to $\mathcal{F}(\mathbb{L}_1, \mathbb{L}_2)$.

Definition 3.1. For any $f, g \in \mathcal{F}$,

(i) the join-convolution of f and g is the lattice function $f \sqcup g$ defined by:

$$(f \sqcup g)(x) = \bigvee_{u \vee_1 v = x} f(u) \wedge_2 g(v) := \sup\{f(u) \wedge_2 g(v) \mid u \vee_1 v = x\}\,;$$

(ii) the meet-convolution of f and g is the lattice function $f \sqcap g$ defined by:

$$(f \sqcap g)(x) = \bigvee_{u \wedge_1 v = x} f(u) \wedge_2 g(v) := \sup\{f(u) \wedge_2 g(v) \mid u \wedge_1 v = x\}\,.$$

Obviously, the suprema in the above definition are taken in the lattice $\mathbb{L}_2$. The convolution operations are not well-defined unless we consider a complete lattice $\mathbb{L}_2$. Moreover, in the subsequent development of this dissertation, the meet-continuity of $\mathbb{L}_2$ will be determinant. We will therefore consider $\mathbb{L}_2$ to be a frame. Since the meet-continuity will be used extensively, we will invoke it without explicitly mentioning. The same applies to the distributivity of $\mathbb{L}_2$. In order not to overload the notations and since no confusion can occur, we will drop the subindices 1 and 2 from here on.

It is important to mention the strong analogy between the join- and meet-convolution. In particular, all the results in this dissertation display a duality between the join- and the meet-convolution. Even the proofs corresponding to each operation are almost identical. However, for the sake of completeness we have included them all.

We illustrate the computation of the join- and meet-convolution. The functions $f : L_1 \to L_2$ are visualized as explained in Section 2.2.2. Recall that $[\![0, 1]\!]$ refers to the unit interval.

x	u	v	$f(u)$	$g(v)$	$f(u) \wedge g(v)$	$\bigvee_{u \vee v = x} f(u) \wedge g(v)$
0	0	0	0.6	0	**0**	0
x_1	x_1	x_1	0	1	0	0.6
	0	x_1	0.6	1	**0.6**	
	x_1	0	0	0	0	
x_2	x_2	x_2	0.8	0.5	**0.5**	0.5
	0	x_2	0.6	0.5	**0.5**	
	x_2	0	0.8	0	0	
1	1	1	0.3	0.4	0.3	0.8
	1	0	0.3	0	0	
	0	1	0.6	0.4	0.4	
	1	x_1	0.3	1	0.3	
	x_1	1	0	0.4	0	
	1	x_2	0.3	0.5	0.3	
	x_2	1	0.8	0.4	0.4	
	x_2	x_1	0.8	1	**0.8**	
	x_1	x_2	0	0.5	0	

Table 3.1: Computation of the join-convolution $f \sqcup g$ in Example 3.1.

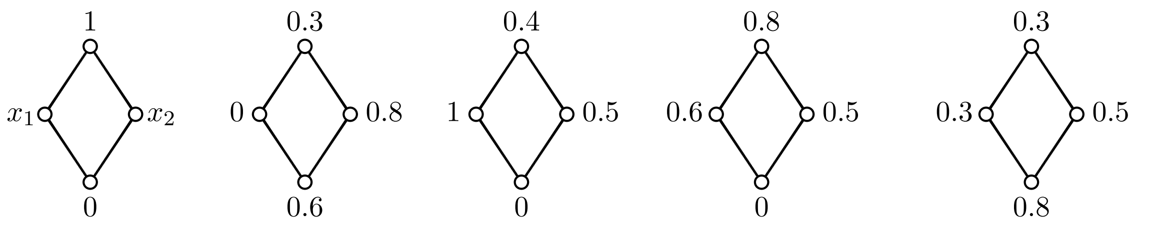
Example 3.1. Let $\mathbb{L}_1 = \mathbb{M}_2$ be the lattice whose Hasse diagram is depicted in Fig. 3.1(a) and $\mathbb{L}_2 = [\![0, 1]\!]$ (with the standard max and min operations). Consider the functions $f, g \in \mathcal{F}$ depicted in Figs. 3.1(b)–(c). Table 3.1 lists the computations of the corresponding join-convolution $f \sqcup g$ depicted in Fig. 3.1(d). Table 3.2 lists the computations of the corresponding meet-convolution $f \sqcap g$ depicted in Fig. 3.1(e).

One of the main drawbacks of the convolution operations when the functions of the set $\mathcal{F}([0,1],[0,1])$ are replaced by functions of the set $\mathcal{F}(\mathbb{L}_1, \mathbb{L}_2)$ is the non-fulfillment of the idempotency laws. In the following example, we show that the convolution operations on $\mathcal{F}$ are not idempotent, in general.

Example 3.2. Let $\mathbb{L}_1 = \mathbb{M}_2$ and $\mathbb{L}_2 = \mathbb{C}_2 = (\{0, 1\}, \max, \min)$. Consider the function $f \in \mathcal{F}$ depicted in Fig. 3.2(a). The join- and meet-convolution $f \sqcup f$ and $f \sqcap f$ are depicted in Figs. 3.2(b)–(c). One easily verifies that $f \sqcup f \neq f$ and $f \sqcap f \neq f$.

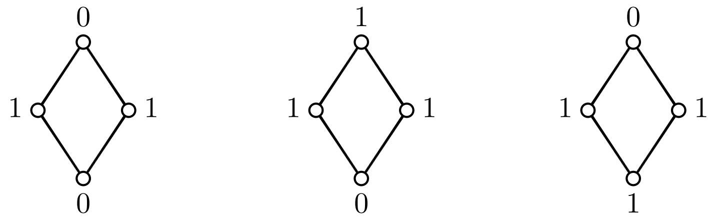
x	u	v	$f(u)$	$g(v)$	$f(u) \wedge g(v)$	$\bigvee_{u \wedge v = x} f(u) \wedge g(v)$
0	0	0	0.6	0	0	0.8
	1	0	0.3	0	0	
	0	1	0.6	0.4	0.4	
	x_1	0	0	0	0	
	0	x_1	0.6	1	0.6	
	0	x_2	0.6	0.5	0.5	
	x_2	0	0.8	0	0	
	x_2	x_1	0.8	1	**0.8**	
	x_1	x_2	0	0.5	0	
x_1	x_1	x_1	0	1	0	0.3
	x_1	1	0	0.4	0	
	1	x_1	0.3	1	**0.3**	
x_2	x_2	x_2	0.8	0.5	**0.5**	0.5
	x_2	1	0.8	0.4	0.4	
	1	x_2	0.3	0.5	0.3	
1	1	1	0.3	0.4	**0.3**	0.3

Table 3.2: Computation of the meet-convolution $f \sqcap g$ in Example 3.1.



(a) $\mathbb{L}_1$ (b) Function f (c) Function g (d) Function $f \sqcup g$ (e) Function $f \sqcap g$

Figure 3.1: Graphical representation of the functions in Example 3.1: (a) the Hasse diagram of the lattice $\mathbb{L}_1$, (b) the function f, (c) the function g, (d) the join-convolution $f \sqcup g$, and (e) the meet-convolution $f \sqcap g$.



(a) Function f (b) Function $f \sqcup f$ (c) Function $f \sqcap f$

Figure 3.2: Graphical representations of the functions in Example 3.2: (a) the function f, (b) the join-convolution $f \sqcup f$, and (c) the meet-convolution $f \sqcap f$.

In [56, 76], the authors show that the convolution operations on $\mathcal{F}([0,1],[0,1])$ are idempotent. This is due to the fact that the unit interval is a chain, as we prove in the following theorem.

Theorem 3.1. The following statements hold:

(i) the join-convolution satisfies the idempotency law if and only if $\mathbb{L}_1$ is a chain;

(ii) the meet-convolution satisfies the idempotency law if and only if $\mathbb{L}_1$ is a chain.

Proof. We first prove that the join-convolution satisfies the idempotency law if and only if $\mathbb{L}_1$ is a chain.

$\Rightarrow$ Suppose that $\sqcup$ satisfies the idempotency law, while $\mathbb{L}_1$ is not a chain. This means that there exist $x_1, x_2 \in L_1$ such that $x_1 \,||\, x_2$. We consider $f \in \mathcal{F}$ defined as

$$f(x) = \begin{cases} 1 & \text{, if } x \in \{x_1, x_2\}\,, \\ 0 & \text{, otherwise}\,. \end{cases} \tag{3.3}$$

Since the couple $(u, v) = (x_1, x_2)$ satisfies $u \vee v = x_1 \vee x_2$, it holds that

$$(f \sqcup f)(x_1 \vee x_2) \bigvee_{u \vee v = x_1 \vee x_2} f(u) \wedge g(v) \geq f(x_1) \wedge f(x_2) = 1 > 0 = f(x_1 \vee x_2),$$

which contradicts the idempotency law.

$\Leftarrow$ Suppose that $\mathbb{L}_1$ is a chain. For any $f \in \mathcal{F}$, since the couple $(u, v) = (x, x)$ satisfies $u \vee v = x$, it holds that

$$(f \sqcup f)(x) = \bigvee_{u \vee v = x} f(u) \wedge f(v) \geq f(x) \wedge f(x) = f(x)\,,$$

and, consequently, $f \leq f \sqcup f$.

Further, since $\mathbb{L}_1$ is a chain, it holds that

$$(f \sqcup f)(x) = \bigvee_{u \vee v = x} f(u) \wedge f(v) = \bigvee_{v \leq x} f(x) \wedge f(v) = f(x) \wedge \bigvee_{v \leq x} f(v) \leq f(x)\,,$$

and, consequently, $f \sqcup f \leq f$.

Taking into account both inequalities, it holds that $f \sqcup f = f$ for any $f \in \mathcal{F}$. Hence, the operation $\sqcup$ satisfies the idempotency law.

Dually, we now prove that the meet-convolution satisfies the idempotency law if and only if $\mathbb{L}_1$ is a chain.
$\Rightarrow$ Suppose that $\sqcap$ satisfies the idempotency law, while $\mathbb{L}_1$ is not a chain. This means that there exists $x_1, x_2 \in L_1$ such that $x_1 \,||\, x_2$. We consider $f \in \mathcal{F}$ defined as in Eq. (3.3). Since the couple $(u, v) = (x_1, x_2)$ satisfies $u \wedge v = x_1 \wedge x_2$, it holds that

$$(f \sqcap f)(x_1 \wedge x_2) \bigvee_{u \wedge v = x_1 \wedge x_2} f(u) \wedge g(v) \geq f(x_1) \wedge f(x_2) = 1 > 0 = f(x_1 \wedge x_2),$$

which contradicts the idempotency law.

$\Leftarrow$ Suppose that $\mathbb{L}_1$ is a chain. For any $f \in \mathcal{F}$, since the couple $(u, v) = (x, x)$ satisfies $u \wedge v = x$, it holds that

$$(f \sqcap f)(x) = \bigvee_{u \wedge v = x} f(u) \wedge f(v) > f(x) \wedge f(x) = f(x)\,,$$

and, consequently, $f \leq f \sqcap f$.

Further, since $\mathbb{L}_1$ is a chain, it holds that

$$(f \sqcap f)(x) = \bigvee_{u \wedge v = x} f(u) \wedge f(v) = \bigvee_{v \geq x} f(x) \wedge f(v) = f(x) \wedge \bigvee_{v \geq x} f(v) \leq f(x)\,,$$

and, consequently, $f \sqcap f \leq f$.

Taking into account both inequalities, it holds that $f \sqcap f = f$ for any $f \in \mathcal{F}$. Hence, the operation $\sqcap$ satisfies the idempotency law. $\square$

When $\mathbb{L}_1 = \mathbb{L}_2 = [\![0, 1]\!]$, the convolution operations can be equivalently formulated in terms of the cumulative functions introduced in the following definition.

Definition 3.2. For any $f \in \mathcal{F}$,

(i) the left-cumulative function f^L is the lattice function defined by:

$$f^L(x) = \bigvee_{y \leq x} f(y)\,;$$

(ii) the right-cumulative function f^R is the lattice function defined by:

$$f^R(x) = \bigvee_{y \geq x} f(y)\,.$$

Specifically, in [40, 76], it is proven that

$$\begin{aligned}(f\wedge g^L)\vee(f^L\wedge g) &= (f\vee g)\wedge f^L\wedge g^L = f\sqcup g\\ (f\wedge g^R)\vee(f^R\wedge g) &= (f\vee g)\wedge f^R\wedge g^R = f\sqcap g\,.\end{aligned} \tag{3.4}$$

Inspired by [40, 76], we analyze whether or not the join- and meet-convolution on $\mathcal{F}$ can be equivalently formulated in terms of the cumulative functions introduced in the preceding definition.

Proposition 3.1. Let $f, g \in \mathcal{F}$. The following equalities hold:

(i) $(f\wedge g^L)\vee(f^L\wedge g) = (f\vee g)\wedge f^L\wedge g^L$;

(ii) $(f\wedge g^R)\vee(f^R\wedge g) = (f\vee g)\wedge f^R\wedge g^R$.

Proof. We first prove statement (i). For any $x \in L_1$, it holds that

$$\begin{aligned}((f\wedge g^L)\vee(f^L\wedge g))(x) &= \Big(f(x)\wedge g^L(x)\Big)\vee\Big(f^L(x)\wedge g(x)\Big)\\ &= \bigg(\Big(f(x)\wedge g^L(x)\Big)\vee f^L(x)\bigg)\wedge\bigg(\Big(f(x)\wedge g^L(x)\Big)\vee g(x)\bigg)\\ &= \Big(f(x)\vee f^L(x)\Big)\wedge\Big(g^L(x)\vee f^L(x)\Big)\\ &\quad \wedge\Big(f(x)\vee g(x)\Big)\wedge\Big(g^L(x)\vee g(x)\Big)\,.\end{aligned}$$

Taking into account that $f(x) \le \bigvee_{y\le x} f(y) = f^L(x)$ and $g(x) \le \bigvee_{y\le x} g(y) = g^L(x)$, it follows that $f(x)\vee g(x) \le f^L(x)\vee g^L(x)$. Hence, it holds that

$$\begin{aligned}((f\wedge g^L)\vee(f^L\wedge g))(x) &= \Big(f(x)\vee f^L(x)\Big)\wedge\Big(g^L(x)\vee f^L(x)\Big)\\ &\quad \wedge\Big(f(x)\vee g(x)\Big)\wedge\Big(g^L(x)\vee g(x)\Big)\\ &= f^L(x)\wedge\Big(f(x)\vee g(x)\Big)\wedge g^L(x)\\ &= \Big(\Big(f\vee g\Big)\wedge f^L\wedge g^L\Big)(x)\,.\end{aligned}$$

Dually, we now prove statement (ii). For any $x \in L_1$, it holds that

$$\begin{aligned}((f \wedge g^R) \vee (f^R \wedge g))(x) &= \Big(f(x) \wedge g^R(x)\Big) \vee \Big(f^R(x) \wedge g(x)\Big) \\ &= \Big(\Big(f(x) \wedge g^R(x)\Big) \vee f^R(x)\Big) \wedge \Big(\Big(f(x) \wedge g^R(x)\Big) \vee g(x)\Big) \\ &= \Big(f(x) \vee f^R(x)\Big) \wedge \Big(g^R(x) \vee f^R(x)\Big) \\ &\quad \wedge \Big(f(x) \vee g(x)\Big) \wedge \Big(g^R(x) \vee g(x)\Big).\end{aligned}$$

Taking into account that $f(x) \leq \bigvee_{y \geq x} f(y) = f^R(x)$ and $g(x) \leq \bigvee_{y \geq x} g(y) = g^R(x)$, it follows that $f(x) \vee g(x) \leq f^R(x) \vee g^R(x)$. Hence, it holds that

$$\begin{aligned}((f \wedge g^R) \vee (f^R \wedge g))(x) &= \Big(f(x) \vee f^R(x)\Big) \wedge \Big(g^R(x) \vee f^R(x)\Big) \\ &\quad \wedge \Big(f(x) \vee g(x)\Big) \wedge \Big(g^R(x) \vee g(x)\Big) \\ &= f^R(x) \wedge \Big(f(x) \vee g(x)\Big) \wedge g^R(x) \\ &= \Big(\Big(f \vee g\Big) \wedge f^R \wedge g^R\Big)(x).\end{aligned}$$

□

However, when $\mathbb{L}_1$ is a bounded lattice and $\mathbb{L}_2$ is a frame, the convolution operations may fail to result in the aforementioned expressions as we show in the following example.

Example 3.3. Let $\mathbb{L}_1 = \mathbb{M}_2$ and $\mathbb{L}_2 = \mathbb{C}_2$. Consider the functions $f, g \in \mathcal{F}$ depicted in Figs. 3.3(a)–(b).

(i) The corresponding functions $f \sqcup g$, f^L, g^L and $(f \wedge g^L) \vee (f^L \wedge g)$ are depicted in Figs. 3.3(c),(e)–(g), respectively. One easily verifies that $(f \wedge g^L) \vee (f^L \wedge g) \neq f \sqcup g$.

(ii) The corresponding functions $f \sqcap g$, f^R, g^R and $(f \wedge g^R) \vee (f^R \wedge g)$ are depicted in Figs. 3.3(d),(h)–(j), respectively. One easily verifies that $(f \wedge g^R) \vee (f^R \wedge g) \neq f \sqcap g$.

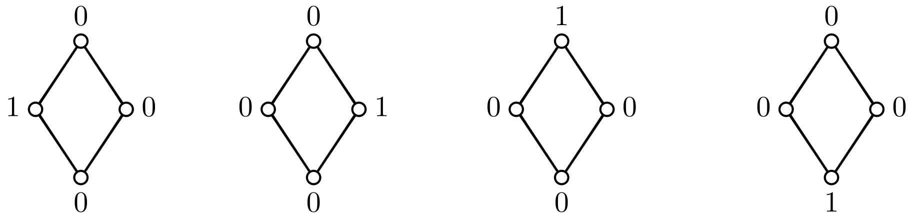
We find that although the identities

$$(f \wedge g^L) \vee (f^L \wedge g) = (f \vee g) \wedge f^L \wedge g^L = f \sqcup g \tag{3.5}$$

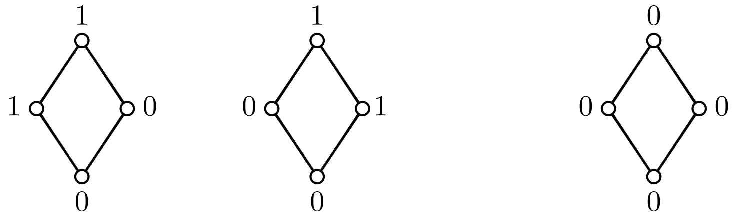
and

$$(f \wedge g^R) \vee (f^R \wedge g) = (f \vee g) \wedge f^R \wedge g^R = f \sqcap g \tag{3.6}$$

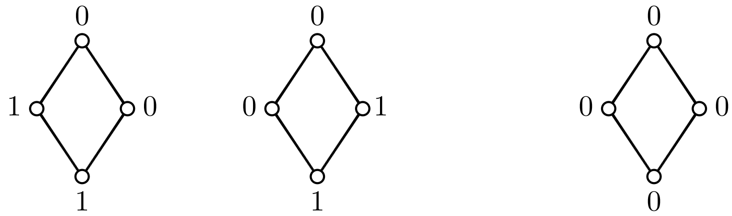
may not hold, the following inequalities are satisfied.



(a) Function f (b) Function g (c) Function $f \sqcup g$ (d) Function $f \sqcap g$



(e) Function f^L (f) Function g^L (g) Function $(f \wedge g^L) \vee (f^L \wedge g)$



(h) Function f^R (i) Function g^R (j) Function $(f \wedge g^R) \vee (f^R \wedge g)$

Figure 3.3: Graphical representation of the functions in Example 3.3: (a) the function f, (b) the function g, (c) the join-convolution $f \sqcup g$, (d) the meet-convolution $f \sqcap g$, (e) the corresponding function f^L, (f) the corresponding function g^L, (g) the corresponding function $(f \wedge g^L) \vee (f^L \wedge g)$, (h) the corresponding function f^R, (i) the corresponding function g^R, and (j) the corresponding function $(f \wedge g^R) \vee (f^R \wedge g)$.

Proposition 3.2. Let $f, g \in \mathcal{F}$. The following statements hold:

(i) $(f \wedge g^L) \vee (f^L \wedge g) = (f \vee g) \wedge f^L \wedge g^L \leq f \sqcup g$;

(ii) $(f \wedge g^R) \vee (f^R \wedge g) = (f \vee g) \wedge f^R \wedge g^R \leq f \sqcap g$.

Proof. We first prove that $(f \wedge g^L) \vee (f^L \wedge g) \leq f \sqcup g$ holds. For any $x \in L_1$, it holds that

$$\begin{aligned}
((f \wedge g^L) \vee (f^L \wedge g))(x) &= \Big(f(x) \wedge g^L(x)\Big) \vee \Big(f^L(x) \wedge g(x)\Big) \\
&= \Big(f(x) \wedge \bigvee_{y_1 \leq x} g(y_1)\Big) \vee \Big(\bigvee_{y_2 \leq x} f(y_2) \wedge g(x)\Big) \\
&= \Big(\bigvee_{x \vee y_1 = x} f(x) \wedge g(y_1)\Big) \vee \Big(\bigvee_{y_2 \vee x = x} f(y_2) \wedge g(x)\Big) \\
&\leq \bigvee_{x_1 \vee x_2 = x} f(x_1) \wedge g(x_2) = (f \sqcup g)(x) \,.
\end{aligned}$$

Dually, we now prove that $(f \wedge g^R) \vee (f^R \wedge g) \leq f \sqcap g$ holds. For any $x \in L_1$, it holds that

$$\begin{aligned}
((f \wedge g^R) \vee (f^R \wedge g))(x) &= \Big(f(x) \wedge g^R(x)\Big) \vee \Big(f^R(x) \wedge g(x)\Big) \\
&= \Big(f(x) \wedge \bigvee_{y_1 \geq x} g(y_1)\Big) \vee \Big(\bigvee_{y_2 \geq x} f(y_2) \wedge g(x)\Big) \\
&= \Big(\bigvee_{x \wedge y_1 = x} f(x) \wedge g(y_1)\Big) \vee \Big(\bigvee_{y_2 \wedge x = x} f(y_2) \wedge g(x)\Big) \\
&\leq \bigvee_{x_1 \wedge x_2 = x} f(x_1) \wedge g(x_2) = (f \sqcap g)(x) \,.
\end{aligned}$$

□

3.3 Strict convolution operations

In [56, 76], it is shown that for appropriate functions in the set of functions $\mathcal{F}([0,1],[0,1])$, the convolution operations constitute a bounded distributive lattice. Hence, the convolution operations on $\mathcal{F}([0,1],[0,1])$ satisfy the idempotency laws. The fact that the convolution operations on $\mathcal{F}(\mathbb{L}_1, \mathbb{L}_2)$ do not satisfy the idempotency laws is a major drawback. In this section, we study another possible definition for the join and meet-convolution on $\mathcal{F}(\mathbb{L}_1, \mathbb{L}_2)$. However, for our goals an indispensable requirement of the new definition is that it should coincide with the convolution operations introduced in Definition 3.1 when $\mathbb{L}_1 = \mathbb{L}_2 = [\![0,1]\!]$. Specifically, we consider a modification of the convolution operations which turns out to be a reformulation of the preceding convolution operations when $\mathbb{L}_1$ is a chain.

Considering $x \in L_1$, the convolution operations in Definition 3.1 can be equivalently formulated as

$$(f \sqcup g)(x) = \bigvee_{(u,v)\in U_x} f(u) \wedge g(v)$$

$$(f \sqcap g)(x) = \bigvee_{(u,v)\in V_x} f(u) \wedge g(v)\,,$$

where $U_x = \{(u,v) \in {L_1}^2 \mid u \vee v = x\}$ and $V_x = \{(u,v) \in {L_1}^2 \mid u \wedge v = x\}$. It holds that

$$U_x = \{(x,a) \in {L_1}^2 \mid a \leq x\} \cup \{(a,x) \in {L_1}^2 \mid a \leq x\} \cup \{(a,b) \in {L_1}^2 \mid a \parallel b \text{ and } a \vee b = x\}\,;$$

$$V_x = \{(x,a) \in {L_1}^2 \mid a \geq x\} \cup \{(a,x) \in {L_1}^2 \mid a \geq x\} \cup \{(a,b) \in {L_1}^2 \mid a \parallel b \text{ and } a \wedge b = x\}\,.$$

When $\mathbb{L}_1$ is a chain, the sets $\{(a,b) \in {L_1}^2 \mid a \parallel b \text{ and } a \vee b = x\}$ and $\{(a,b) \in {L_1}^2 \mid a \parallel b \text{ and } a \wedge b = x\}$ are empty. However, this no longer holds when $\mathbb{L}_1$ is not a chain (see, for example, Tables 3.1 and 3.2, where the couple (x_1, x_2) belongs to the sets U_1 and V_0, respectively). We therefore consider the following modification of the convolution operations.

Definition 3.3. For any $f, g \in \mathcal{F}$,

(i) the strict join-convolution of f and g is the lattice function $f \sqcup^* g$ defined by:

$$(f \sqcup^* g)(x) = \bigvee_{(u,v)\in U_x^*} f(u) \wedge g(v)\,,$$

where $U_x^* = \{(x,a) \in {L_1}^2 \mid a \leq x\} \cup \{(a,x) \in {L_1}^2 \mid a \leq x\}$;

(ii) the strict meet-convolution of f and g is the lattice function $f \sqcap^* g$ defined by:

$$(f \sqcap^* g)(x) = \bigvee_{(u,v)\in V_x^*} f(u) \wedge g(v)\,,$$

where $V_x^* = \{(x,a) \in {L_1}^2 \mid a \geq x\} \cup \{(a,x) \in {L_1}^2 \mid a \geq x\}$.

Remark 3.1. We have opted for the name strict in the preceding definition since it holds that for any $x \in L_1$, $U_x^* \subseteq U_x$ and $V_x^* \subseteq V_x$, i.e., because the sets U_x^* and V_x^* are smaller than the sets U_x and V_x, respectively. Moreover, we find that

$$f \sqcup^* g \leq f \sqcup g \qquad \text{and} \qquad f \sqcap^* g \leq f \sqcap g\,.$$

In the following example, we illustrate the computation of the strict convolution operations. Note that the difference with the convolution operations introduced in Definition 3.1 is that the couple (x_1, x_2) is not considered for the computation.

x	u	v	$f(u)$	$g(v)$	$f(u) \wedge g(v)$	$\bigvee_{(u,v)\in U_x^*} f(u) \wedge g(v)$
0	0	0	0.6	0	**0**	0
x_1	x_1	x_1	0	1	0	0.6
	0	x_1	0.6	1	**0.6**	
	x_1	0	0	0	0	
x_2	x_2	x_2	0.8	0.5	**0.5**	0.5
	0	x_2	0.6	0.5	**0.5**	
	x_2	0	0.8	0	0	
1	1	1	0.3	0.4	0.3	0.4
	1	0	0.3	0	0	
	0	1	0.6	0.4	**0.4**	
	1	x_1	0.3	1	0.3	
	x_1	1	0	0.4	0	
	1	x_2	0.3	0.5	0.3	
	x_2	1	0.8	0.4	**0.4**	

Table 3.3: Computation of the strict join-convolution $f \sqcup^* g$ in Example 3.4.

Example 3.4. Let $\mathbb{L}_1 = \mathbb{M}_2$ be the lattice whose Hasse diagram is depicted in Fig. 3.4(a) and $\mathbb{L}_2 = [\![0,1]\!]$. Consider the functions $f, g \in \mathcal{F}$ depicted in Figs. 3.4(b)–(c). Table 3.3 lists the computations of the corresponding strict join-convolution $f \sqcup^* g$ depicted in Fig. 3.4(d). Table 3.4 lists the computations of the corresponding strict meet-convolution $f \sqcap^* g$ depicted in Fig. 3.4(e).

The main advantage of the strict convolution operations is that they satisfy the idempotency laws.

Proposition 3.3. The following statements hold:

(i) the strict join-convolution satisfies the idempotency law;

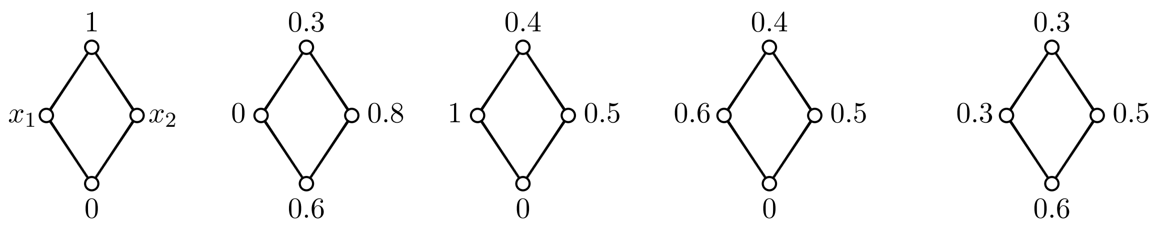
(ii) the strict meet-convolution satisfies the idempotency law.

Proof. We first prove that the strict join-convolution satisfies the idempotency law. For any $f \in \mathcal{F}$, since $(x,x) \in U_x^*$, it holds that

$$(f \sqcup^* f)(x) = \bigvee_{(u,v)\in U_x^*} f(u) \wedge f(v) \geq f(x) \wedge f(x) = f(x).$$

x	u	v	$f(u)$	$g(v)$	$f(u) \wedge g(v)$	$\bigvee_{(u,v)\in V_x^*} f(u) \wedge g(v)$
0	0	0	0.6	0	0	0.6
	1	0	0.3	0	0	
	0	1	0.6	0.4	0.4	
	x_1	0	0	0	0	
	0	x_1	0.6	1	**0.6**	
	0	x_2	0.6	0.5	0.5	
	x_2	0	0.8	0	0	
x_1	x_1	x_1	0	1	0	0.3
	x_1	1	0	0.4	0	
	1	x_1	0.3	1	**0.3**	
x_2	x_2	x_2	0.8	0.5	**0.5**	0.5
	0	x_2	0.6	0.5	**0.5**	
	x_2	0	0.8	0	0	
1	1	1	0.3	0.4	**0.3**	0.3

Table 3.4: Computation of the strict meet-convolution $f \sqcap^* g$ in Example 3.4.



(a) $\mathbb{L}_1$ (b) Function f (c) Function g (d) Function $f \sqcup^* g$ (e) Function $f \sqcap^* g$

Figure 3.4: Graphical representation of the functions in Example 3.4: (a) the Hasse diagram of the lattice $\mathbb{L}_1$, (b) the function f, (c) the function g, (d) the strict join-convolution $f \sqcup^* g$, and (e) the strict meet-convolution $f \sqcap^* g$.

Further, we find that

$$(f \sqcup^* f)(x) = \bigvee_{(u,v)\in U_x^*} f(u) \wedge f(v) = \bigvee_{u \leq x} f(x) \wedge f(u) \leq f(x).$$

Taking into account both inequalities, it holds that $f \sqcup^* f = f$ for any $f \in \mathcal{F}$. Hence, the strict join-convolution satisfies the idempotency law.

Dually, we now prove that the strict meet-convolution satisfies the idempotency law. For any $f \in \mathcal{F}$, since $(x, x) \in V_x^*$, it holds that

$$(f \sqcap^* f)(x) = \bigvee_{(u,v)\in V_x^*} f(u) \wedge f(v) \geq f(x) \wedge f(x) = f(x).$$

Further, we find that

$$(f \sqcap^* f)(x) = \bigvee_{(u,v)\in V_x^*} f(u) \wedge f(v) = \bigvee_{u \geq x} f(x) \wedge f(u) \leq f(x).$$

Taking into account both inequalities, it holds that $f \sqcap^* f = f$ for any $f \in \mathcal{F}$. Hence, the strict meet-convolution satisfies the idempotency law. $\square$

Moreover, with the strict convolution operations we also retrieve the identities with the cumulative functions.

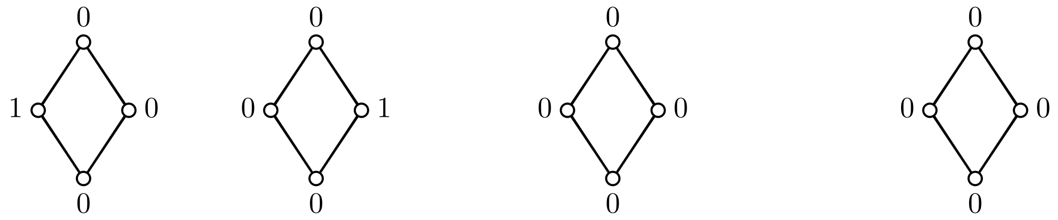
Proposition 3.4. Let $f, g \in \mathcal{F}$. The following statements hold:

(i) $f \sqcup^* g = (f \wedge g^L) \vee (f^L \wedge g) = (f \vee g) \wedge f^L \wedge g^L$;

(ii) $f \sqcap^* g = (f \wedge g^R) \vee (f^R \wedge g) = (f \vee g) \wedge f^R \wedge g^R$.

Proof. We first prove that $f \sqcup^* g = (f \wedge g^L) \vee (f^L \wedge g)$. For any $x \in L_1$, it holds that

$$\begin{aligned}
(f \sqcup^* g)(x) &= \bigvee_{(u,v)\in U_x^*} f(u) \wedge g(v) \\
&= \left(\bigvee_{v \leq x} f(x) \wedge g(v) \right) \vee \left(\bigvee_{u \leq x} f(u) \wedge g(x) \right) \\
&= \left(f(x) \wedge \bigvee_{v \leq x} g(v) \right) \vee \left(g(x) \wedge \bigvee_{u \leq x} f(u) \right) \\
&= \left(f(x) \wedge g^L(x) \right) \vee \left(f^L(x) \wedge g(x) \right).
\end{aligned}$$



(a) Function f (b) Function g (c) Function $f \sqcup^* (f \sqcap^* g)$ (d) Function $f \sqcap^* (f \sqcup^* g)$

Figure 3.5: Graphical representations of the functions in Example 3.5: (a) the function f, (b) the function g, (c) the corresponding $f \sqcup^* (f \sqcap^* g)$, and (d) the corresponding $f \sqcap^* (f \sqcup^* g)$.

Dually, we now prove dually that $f \sqcap^* g = (f \wedge g^R) \vee (f^R \wedge g)$. For any $x \in L_1$, it holds that

$$
\begin{aligned}
(f \sqcap^* g)(x) &= \bigvee_{(u,v) \in V_x^*} f(u) \wedge g(v) \\
&= \left(\bigvee_{v \geq x} f(x) \wedge g(v) \right) \vee \left(\bigvee_{u \geq x} f(u) \wedge g(x) \right) \\
&= \left(f(x) \wedge \bigvee_{v \geq x} g(v) \right) \vee \left(g(x) \wedge \bigvee_{u \geq x} f(u) \right) \\
&= (f(x) \wedge g^R(x)) \vee (f^R(x) \wedge g(x))
\end{aligned}
$$

□

However, the strict convolution operations do not satisfy the absorption laws, in general.

Example 3.5. Let $\mathbb{L}_1 = \mathbb{M}_2$ and $L_2 = \mathbb{C}_2$. Consider the functions $f, g \in \mathcal{F}$ depicted in Figs. 3.5(a)-(b). The corresponding $f \sqcup^* (f \sqcap^* g)$ and $f \sqcap^* (f \sqcup^* g)$ are depicted in Figs. 3.5(c)–(d). One easily verifies that $f \sqcup^* (f \sqcap^* g) \neq f$ and $f \sqcap^* (f \sqcup^* g) \neq f$.

Remark 3.2. The non-fulfillment of the absorption laws is not the only negative aspect of the strict convolution operations. Another negative aspect is that the corresponding convolution operations seem to be very restrictive and they take small values of the lattice $\mathbb{L}_2$. For instance, in Example 3.5 both $f \sqcup^* g = \underline{\mathbf{0}}$ and $f \sqcap^* g = \underline{\mathbf{0}}$ where $\underline{\mathbf{0}}$ is the function defined by $\underline{\mathbf{0}}(x) = 0$. Moreover, any other convolution with the function $\underline{\mathbf{0}}$ is again $\underline{\mathbf{0}}$. This is, up to a certain point, due to the fact that the couples $(x_1, x_2) \in {L_1}^2$ such that $x_1 \parallel x_2$ are never taken into account for the computation of the strict convolutions, i.e, when $x_1 \parallel x_2$, it holds that the couple (x_1, x_2) does not belong to any U_x, for $x \in L_1$. This is what we attempt to solve in the following section.

3.4 Extended convolution operations

Finally, we propose another modification of the convolution operations: the extended convolution operations. Once again, this definition turns out to be a reformulation of the convolution operations in Definition 3.1 when $\mathbb{L}_1$ is a chain. As we will show in this section, the extended convolution operations satisfy the idempotency law. Moreover, they solve the negative aspect of the strict convolution operation, where the corresponding convolution operations are very restrictive. The main idea consists in including the incomparable couples that are never considered for the computation of the strict convolutions. Note that these couples are also taken into account in the convolution operations introduced in Definition 3.1 but in the extended convolutions we use them for computing the values of the convolutions at different points.

Definition 3.4. For any $f, g \in \mathcal{F}$,

(i) the extended join-convolution of f and g is the lattice function $f \sqcup^{\circledast} g$ defined by:

$$(f \sqcup^{\circledast} g)(x) = \bigvee_{(u,v)\in U_x^{\circledast}} f(u) \wedge g(v)\,,$$

where $U_x^{\circledast} = \{(x,a) \in {L_1}^2 \mid a \leq x \text{ or } a \parallel x\} \cup \{(a,x) \in {L_1}^2 \mid a \leq x \text{ or } a \parallel x\}$;

(ii) the extended meet-convolution of f and g is the lattice function $f \sqcap^{\circledast} g$ defined by:

$$(f \sqcap^{\circledast} g)(x) = \bigvee_{(u,v)\in V_x^{\circledast}} f(u) \wedge g(v)\,,$$

where $V_x^{\circledast} = \{(x,a) \in {L_1}^2 \mid a \geq x \text{ or } a \parallel x\} \cup \{(a,x) \in {L_1}^2 \mid a \geq x \text{ or } a \parallel x\}$.

Remark 3.3. We have opted for the name extended in the preceding convolutions, since it holds that for any $x \in L_1$, $U_x^* \subseteq U_x^{\circledast}$ and $V_x^* \subseteq V_x^{\circledast}$, i.e., because the sets $U_x^{\circledast}$ and $V_x^{\circledast}$ are larger than the sets U_x^* and V_x^*, respectively. Hence, once again the corresponding strict and extended convolution operations satisfy the pointwise partial relations:

$$f \sqcup^* g \leq f \sqcup^{\circledast} g \qquad \text{and} \qquad f \sqcap^* g \leq f \sqcap^{\circledast} g\,.$$

It is also important to highlight the difference between the extended convolution operations and the convolution operations introduced in Definition 3.1. The main difference is that, in contrast with the operations in Definition 3.1, in the computation of the extended convolution at a point x, we only take into account couples in which at least one of the elements is exactly x. It is also important to highlight that there is no pointwise relation between these operations. One can easily verify this comparing the convolutions depicted in Figs. 3.1 and 3.6.

x	u	v	$f(u)$	$g(v)$	$f(u) \wedge g(v)$	$\bigvee_{(u,v)\in U_x^{\circledast}} f(u) \wedge g(v)$
0	0	0	0.6	0	**0**	0
x_1	x_1	x_1	0	1	0	0.8
	0	x_1	0.6	1	0.6	
	x_1	0	0	0	0	
	x_1	x_2	0	0.5	0	
	x_2	x_1	0.8	1	**0.8**	
x_2	x_2	x_2	0.8	0.5	0.5	0.8
	0	x_2	0.6	0.5	0.5	
	x_2	0	0.8	0	0	
	x_1	x_2	0	0.5	0	
	x_2	x_1	0.8	1	**0.8**	
	x_1	x_2	0	0.5	0	
1	1	1	0.3	0.4	0.3	0.4
	1	0	0.3	0	0	
	0	1	0.6	0.4	**0.4**	
	1	x_1	0.3	1	0.3	
	x_1	1	0	0.4	0	
	1	x_2	0.3	0.5	0.3	
	x_2	1	0.8	0.4	**0.4**	

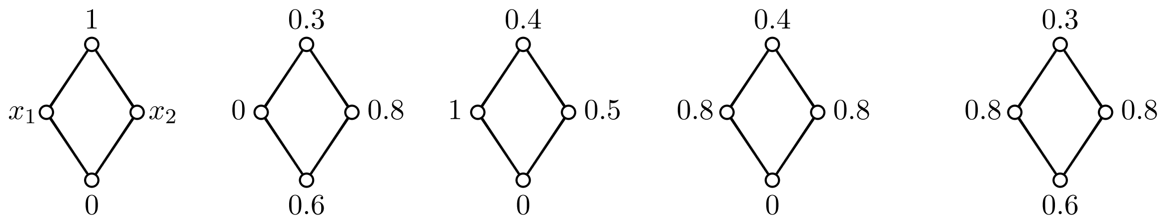
Table 3.5: Computation of the extended join-convolution $f \sqcup^{\circledast} g$ in Example 3.6.

In the following example, we illustrate the computation of the extended convolution operations.

Example 3.6. Let $\mathbb{L}_1 = \mathbb{M}_2$ be the lattice whose Hasse diagram is depicted in Fig. 3.6(a) and $\mathbb{L}_2 = [\![0, 1]\!]$. Consider the functions $f, g \in \mathcal{F}$ depicted in Figs. 3.6(b)–(c). Table 3.5 lists the computations of the corresponding extended join-convolution $f \sqcup^{\circledast} g$ depicted in Fig. 3.6(d). Table 3.6 lists the computations of the corresponding extended meet-convolution $f \sqcap^{\circledast} g$ depicted in Fig. 3.6(e). Note that the difference between the strict and extended convolution operations is that the couple (x_1, x_2) is considered for the computation of the value of the convolutions at x_1 and x_2.

x	u	v	$f(u)$	$g(v)$	$f(u) \wedge g(v)$	$\bigvee_{(u,v)\in V_x^{\circledast}} f(u) \wedge g(v)$
0	0	0	0.6	0	0	0.6
	1	0	0.3	0	0	
	0	1	0.6	0.4	0.4	
	x_1	0	0	0	0	
	0	x_1	0.6	1	**0.6**	
	0	x_2	0.6	0.5	0.5	
	x_2	0	0.8	0	0	
x_1	x_1	x_1	0	1	0	0.8
	x_1	1	0	0.4	0	
	1	x_1	0.3	1	0.3	
	x_1	x_2	0	0.5	0	
	x_2	x_1	0.8	1	**0.8**	
x_2	x_2	x_2	0.8	0.5	0.5	0.8
	0	x_2	0.6	0.5	0.5	
	x_2	0	0.8	0	0	
	x_2	x_1	0.8	1	**0.8**	
	x_1	x_2	0	0.5	0	
1	1	1	0.3	0.4	**0.3**	0.3

Table 3.6: Computation of the extended meet-convolution $f \sqcap^{\circledast} g$ in Example 3.6.



(a) $\mathbb{L}_1$ (b) Function f (c) Function g (d) Function $f \sqcup^{\circledast} g$ (e) Function $f \sqcap^{\circledast} g$

Figure 3.6: Graphical representation of the functions in Example 3.4: (a) the Hasse diagram of the lattice $\mathbb{L}_1$, (b) the function f, (c) the function g, (d) the extended join-convolution $f \sqcup^{\circledast} g$, and (e) the extended meet-convolution $f \sqcap^{\circledast} g$.

The main advantage of the extended convolution operations is that they satisfy the idempotency laws.

Proposition 3.5. The following statements hold:

(i) the extended join-convolution satisfies the idempotency law;

(ii) the extended meet-convolution satisfies the idempotency law.

Proof. We first prove that the extended join-convolution satisfies the idempotency law. For any $f \in \mathcal{F}$, since $(x, x) \in U_x^{\circledast}$, it holds that

$$(f \sqcup^{\circledast} f)(x) = \bigvee_{(u,v)\in U_x^{\circledast}} f(u) \wedge f(v) \geq f(x) \wedge f(x) = f(x).$$

Further, we find that

$$(f \sqcup^{\circledast} f)(x) = \bigvee_{(u,v)\in U_x^{\circledast}} f(u) \wedge f(v) = \bigvee_{u\leq x \text{ or } u\|x} f(x) \wedge f(u) \leq f(x).$$

Taking into account both inequalities, it holds that $f \sqcup^{\circledast} f = f$ for any $f \in \mathcal{F}$. Hence, the operation $\sqcup^{\circledast}$ satisfies the idempotency law.

Dually, we now prove that the extended meet-convolution satisfies the idempotency law. For any $f \in \mathcal{F}$, since $(x, x) \in V_x^{\circledast}$, it holds that

$$(f \sqcap^{\circledast} f)(x) = \bigvee_{(u,v)\in V_x^{\circledast}} f(u) \wedge f(v) \geq f(x) \wedge f(x) = f(x).$$

Further, we find that

$$(f \sqcap^{\circledast} f)(x) = \bigvee_{(u,v)\in V_x^{\circledast}} f(u) \wedge f(v) = \bigvee_{u\geq x \text{ or } u\|x} f(x) \wedge f(u) \leq f(x).$$

Taking into account both inequalities, it holds that $f \sqcap^{\circledast} f = f$ for any $f \in \mathcal{F}$. Hence, the operation $\sqcap^{\circledast}$ satisfies the idempotency law. □

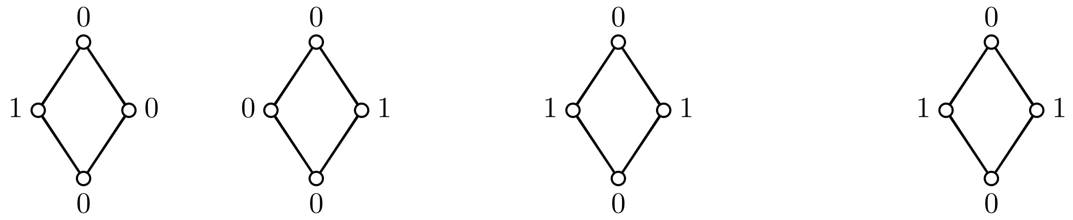
Once again, we study the relation between the extended convolution operations and the cumulative functions.

Corollary 3.1. Let $f, g \in \mathcal{F}$. The following statements hold:

(i) $(f \wedge g^L) \vee (f^L \wedge g) = (f \vee g) \wedge f^L \wedge g^L \leq f \sqcup^{\circledast} g$;

(ii) $(f \wedge g^R) \vee (f^R \wedge g) = (f \vee g) \wedge f^R \wedge g^R \leq f \sqcap^{\circledast} g$.

Proof. Recall that $(f \wedge g^L) \vee (f^L \wedge g) = (f \vee g) \wedge f^L \wedge g^L = f \sqcup^* g$ and $f \sqcup^* g \leq f \sqcup^{\circledast} g$. Similarly, $(f \wedge g^R) \vee (f^R \wedge g) = (f \vee g) \wedge f^R \wedge g^R = f \sqcap^* g$ and $f \sqcap^* g \leq f \sqcap^{\circledast} g$. □



(a) Function f (b) Function g (c) Function $f \sqcup^{\circledast} (f \sqcap^{\circledast} g)$ (d) Function $f \sqcap^{\circledast} (f \sqcup^{\circledast} g)$

Figure 3.7: Graphical representations of the functions in Example 3.7: (a) the function f, (b) the function g, (c) the corresponding $f \sqcup^{\circledast} (f \sqcap^{\circledast} g)$, and (d) the corresponding $f \sqcap^{\circledast} (f \sqcup^{\circledast} g)$.

Although the extended convolution operations satisfy the idempotency laws, in general they do not satisfy the absorption laws, as we show in the following example.

Example 3.7. Let $\mathbb{L}_1 = \mathbb{M}_2$ and $L_2 = \mathbb{C}_2$. Consider the functions $f, g \in \mathcal{F}$ depicted in Figs. 3.7(a)-(b). The corresponding $f \sqcup^{\circledast} (f \sqcap^{\circledast} g)$ and $f \sqcap^{\circledast} (f \sqcup^{\circledast} g)$ are depicted in Figs. 3.7(c)–(d). One easily verifies that $f < f \sqcup^{\circledast} (f \sqcap^{\circledast} g)$ and $f < f \sqcap^{\circledast} (f \sqcup^{\circledast} g)$.

Remark 3.4. The non-fulfillment of the absorption laws is not the only negative aspect of the extended convolution operations. Another negative aspect is that the corresponding convolution operations seem to be very lax and they can take the same value of $\mathbb{L}_2$ repeatedly. For instance, in Example 3.6 both $(f \sqcup^{\circledast} g)(x_1) = (f \sqcup^{\circledast} g)(x_2) = 0.8$. Similarly, it holds that $(f \sqcap^{\circledast} g)(x_1) = (f \sqcap^{\circledast} g)(x_2) = 0.8$. This is, up to a certain point, due to the fact that the couples $(x_1, x_2) \in {L_1}^2$ such that $x_1 \parallel x_2$ are taken into account for the computation of the extended convolutions at both x_1 and x_2, i.e. when $x_1 \parallel x_2$ it holds that the couple (x_1, x_2) belongs to both U_{x_1} and U_{x_2}.

3.5 Conclusions of Chapter 3

In this chapter, we have evaluated the suitability of the notion of convolution to define the disjunction connective, "or", and the conjunction connective, "and", for the proposed logical system.

Firstly, we have shown that when we replace the set of functions $\mathcal{F}([0,1],[0,1])$ with the set of functions $\mathcal{F}(\mathbb{L}_1, \mathbb{L}_2)$ in the "or" and "and" convolution operations, the idempotency laws do not longer hold. Moreover, these generalized convolution operations cannot be reformulated in terms of cumulative functions.

Secondly, two possible modifications of the convolutions called the strict and the extended convolution operations have been studied. Both strict and extended convolutions turn out to be a reformulation of the operations in Definition 3.1 when $\mathbb{L}_1$ is a chain. Although these convolution operations overcome the drawback of the non-fulfillment of the idempotency laws, none of these definitions satisfies the absorption laws. Moreover, the strict and extended convolution operations come along with some other negative aspects.

In [56, 76], it is proven that the convolution operations only constitute a lattice when the set of functions is restricted to some suitable subsets, i.e., when the considered functions satisfy some additional properties. In the following chapter, we will consider the original definition of convolution operations introduced in Definition 3.1 and we will study whether or not there are some subsets where the convolution operations constitute some interesting algebras. Specifically, we will focus on lattices and their algebraic laws.

Chapter 4

Algebraic structures of convolution operations

4.1 Introduction

For the constitution of a lattice with the convolution operations in type-2 fuzzy logic, some particular classes of functions have been considered, i.e., the functions are required to fulfill some additional properties [56, 76]. This means that the convolution operations do not constitute a lattice on the set $\mathcal{F}([0,1],[0,1])$, but on a subset $\mathcal{G}([0,1],[0,1]) \subset \mathcal{F}([0,1],[0,1])$.

In Chapter 3, we have analyzed the main drawbacks when the set $\mathcal{F}([0,1],[0,1])$ is replaced by the set $\mathcal{F}(\mathbb{L}_1,\mathbb{L}_2)$, where $\mathbb{L}_1$ is a bounded lattice and $\mathbb{L}_2$ is a frame. We have already proven that the straightforward generalization of the convolution operations introduced in Definition 3.1 does not satisfy the idempotency laws. Hence, the join- and meet-convolution operations do not constitute a lattice. As a first attempt, we have tried to modify the definition of the convolution operations. However, we have shown negative aspects for each of the possible modifications. Since we cannot achieve an adequate modification of the convolution operations, in this chapter we return to the convolution operations introduced in Definition 3.1. Similarly to some other studies on type-2 fuzzy sets [56, 76], we will analyze whether or not there exists a subset $\mathcal{G}(\mathbb{L}_1,\mathbb{L}_2) \subseteq \mathcal{F}(\mathbb{L}_1,\mathbb{L}_2)$ such that the convolution operations constitute a lattice on $\mathcal{G}(\mathbb{L}_1,\mathbb{L}_2)$.

Summarizing, we consider the set $\mathcal{F}$ (note that, once again, we refer to $\mathcal{F}(\mathbb{L}_1, \mathbb{L}_2)$ without explicitly indicating the lattices) and the join- and meet-convolution operations on $\mathcal{F}$ defined as:

$$(f \sqcup g)(x) = \bigvee_{u \vee v = x} f(u) \wedge g(v)$$

$$(f \sqcap g)(x) = \bigvee_{u \wedge v = x} f(u) \wedge g(v)\,.$$

In this chapter, we will study if there exists $\mathcal{G} \subseteq \mathcal{F}$ such that for any $f, g, h \in \mathcal{G}$, it holds that

(i) $f \sqcup g = g \sqcup f$ and $f \sqcap g = g \sqcap f$ (commutativity laws);

(ii) $f \sqcup (g \sqcup h) = (f \sqcup g) \sqcup h$ and $f \sqcap (g \sqcap h) = (f \sqcap g) \sqcap h$ (associativity laws);

(iii) $f \sqcup (f \sqcup g) = f$ and $f \sqcap (f \sqcap g) = f$ (absorption laws);

(iv) $f \sqcup 0_{\mathcal{F}} = f$ and $f \sqcap 1_{\mathcal{F}} = f$ (identity laws).

Remark 4.1. In several studies on type-2 fuzzy sets, for functions belonging to the set $\mathcal{F}([0,1],[0,1])$, the identities in Eqs. (3.5) and (3.6) are determinant in proving the lattice properties of the convolution operations [40, 76]. For the general setting considered here, we will be forced to base our proofs on the explicit expressions of the operations.

4.2 First properties of the convolution operations

4.2.1 Binary relations

Let $\mathbb{L} = (L, \vee, \wedge)$ be a lattice. The relation $\leq$ defined by $a \leq b$ if $a \vee b = b$, and the relation $\leq'$ defined by $a \leq' b$ if $a \wedge b = a$, coincide (see Section 2.2.1, for more information). A first step in the study of the join- and meet-convolution operations on $\mathcal{F}$ consists in exploring the connection between the convolution operations and their corresponding binary relations.

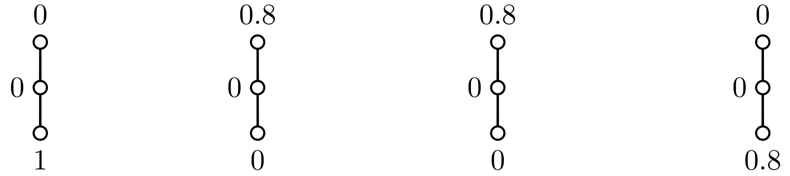
Definition 4.1. We consider the following relations.

(i) With the join-convolution operation $\sqcup$ on $\mathcal{F}$, we associate the binary relation $\sqsubseteq_{\sqcup}$ on $\mathcal{F}$ defined by:

$$f \sqsubseteq_{\sqcup} g \text{ if } f \sqcup g = g\,.$$

(ii) With the meet-convolution operation $\sqcap$ on $\mathcal{F}$, we associate the binary relation $\sqsubseteq_{\sqcap}$ on $\mathcal{F}$ defined by:

$$f \sqsubseteq_{\sqcap} g \text{ if } f \sqcap g = f\,.$$



(a) Function f (b) Function g (c) Function $f \sqcup g$ (d) Function $f \sqcap g$

Figure 4.1: Graphical representation of the functions in Example 4.1: (a) the function f, (b) the function g, (c) the join-convolution $f \sqcup g$, and (d) the meet-convolution $f \sqcap g$.

Note that since the convolution operations do not constitute a lattice on $\mathcal{F}$, the relations $\sqsubseteq_{\sqcup}$ and $\sqsubseteq_{\sqcap}$ do not coincide in general. We illustrate this in Example 4.1.

Example 4.1. Let $\mathbb{L}_1 = \mathbb{C}_3$ and $\mathbb{L}_2 = [\![0, 1]\!]$. Consider the functions $f, g \in \mathcal{F}$ depicted in Figs. 4.1(a)–(b). The join- and meet-convolution $f \sqcup g$ and $f \sqcap g$ are depicted in Figs. 4.1(c)–(d). One easily verifies that $f \sqcup g = g$, while $f \sqcap g \neq f$. Consequently, $f \sqsubseteq_{\sqcup} g$, while $f \not\sqsubseteq_{\sqcap} g$.

4.2.2 Monotone functions

There is no reason to expect the pointwise operations and the convolution operations introduced in Definition 2.9 to coincide. However, as we will show next, there exist subsets of $\mathcal{F}$ for which this effectively holds. In particular, we consider the set of increasing functions

$$\mathcal{M}_I = \{f \in \mathcal{F} \mid (\forall (x_1, x_2) \in {L_1}^2)(x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2))\}$$

and the set of decreasing functions

$$\mathcal{M}_D = \{f \in \mathcal{F} \mid (\forall (x_1, x_2) \in {L_1}^2)(x_1 \leq x_2 \Rightarrow f(x_2) \leq f(x_1))\}\,.$$

Proposition 4.1. The following statements hold:

(i) if $f, g \in \mathcal{M}_I$, then $f \sqcup g = f \wedge g$;

(ii) if $f, g \in \mathcal{M}_D$, then $f \sqcap g = f \wedge g$.

Proof. We first prove statement (i). Let $f, g \in \mathcal{M}_I$, then for any $x \in L_1$, it holds that

$$\begin{aligned}(f \sqcup g)(x) &= \bigvee_{u \vee v = x} f(u) \wedge g(v) \\ &= \bigvee_{\substack{u \vee v = x \\ u \leq x, v \leq x}} f(u) \wedge g(v) \\ &\leq \bigvee_{\substack{u \vee v = x \\ u \leq x, v \leq x}} f(x) \wedge g(x) \\ &= f(x) \wedge g(x)\,.\end{aligned}$$

Further, since the couple $(u, v) = (x, x)$ satisfies $u \vee v = x$, it holds that

$$(f \sqcup g)(x) \geq f(x) \wedge g(x)\,.$$

Taking into account both inequalities, it holds that $(f \sqcup g)(x) = f(x) \wedge g(x)$ for any $x \in L_1$.

Dually, we now prove statement (ii). Let $f, g \in \mathcal{M}_D$, then for any $x \in L_1$, it holds that

$$\begin{aligned}(f \sqcap g)(x) &= \bigvee_{u \wedge v = x} f(u) \wedge g(v) \\ &= \bigvee_{\substack{u \wedge v = x \\ u \geq x, v \geq x}} f(u) \wedge g(v) \\ &\leq \bigvee_{\substack{u \wedge v = x \\ u \geq x, v \geq x}} f(x) \wedge g(x) \\ &= f(x) \wedge g(x)\,.\end{aligned}$$

Further, since the couple $(u, v) = (x, x)$ satisfies $u \wedge v = x$, it holds that

$$(f \sqcap g)(x) \geq f(x) \wedge g(x)\,.$$

Taking into account both inequalities, it holds that $(f \sqcap g)(x) = f(x) \wedge g(x)$ for any $x \in L_1$.

□

Corollary 4.1. The following statements hold:

(i) if $f, g \in \mathcal{M}_I$, then $f \sqsubseteq_\sqcup g$ if and only if $g \leq f$;

(ii) if $f, g \in \mathcal{M}_D$, then $f \sqsubseteq_\sqcap g$ if and only if $f \leq g$.

Proof. We first prove statement (i). Due to Proposition 4.1, for any $f, g \in \mathcal{M}_I$ and for any $x \in L_1$, it holds that $(f \sqcup g)(x) = f(x) \wedge g(x)$. Hence, $(f \sqcup g)(x) = g(x)$ if and only if $g(x) \leq f(x)$ for any $x \in L_1$, i.e., $f \sqsubseteq_\sqcup g$ if and only if $g \leq f$.

Dually, we now prove statement (ii). Due to Proposition 4.1, for any $f, g \in \mathcal{M}_D$ and for any $x \in L_1$, it holds that $(f \sqcap g)(x) = f(x) \wedge g(x)$. Hence, $(f \sqcap g)(x) = f(x)$ if and only if $f(x) \leq g(x)$ for any $x \in L_1$, i.e., $f \sqsubseteq_\sqcup g$ if and only if $f \leq g$. □

Consequently, the pointwise operations coincide with the convolution operations when restricted to the subset $\mathcal{M}_I \cap \mathcal{M}_D$ of $\mathcal{F}$. However, this subset consists of constant functions only, and is of no further interest in this dissertation.

4.3 Lattice laws of the convolution operations

4.3.1 General properties

In this section, we study whether or not the convolution operations satisfy the algebraic laws of lattice operations. For some properties, we will be forced to pay special attention to an appropriate subset of the set of lattice functions. An important question that then pops up is whether or not such subset is effectively closed under the convolution operations. As this question is not trivial at all, Section 4.4 is devoted to it.

Recall that, considering a lattice function $f \in \mathcal{F}$, the element $s_f \in L_2$ refers to the supremum of f, i.e., $s_f := \bigvee_{x \in L_1} f(x)$. The function $\underline{\mathbf{0}}$ refers to the constant function defined by $\underline{\mathbf{0}}(x) = 0$. Moreover, the functions $\mathbf{0}_a$ and $\mathbf{1}_a$ (with $a \in L_2$) refer to the functions defined as:

$$\mathbf{0}_a(x) = \begin{cases} a & , \text{ if } x = 0\,, \\ 0 & , \text{ otherwise}\,; \end{cases} \qquad \mathbf{1}_a(x) = \begin{cases} a & , \text{ if } x = 1\,, \\ 0 & , \text{ otherwise}\,. \end{cases}$$

Theorem 4.1. Let $f, g, h \in \mathcal{F}$. The following statements hold:

(i) $f \sqcup g = g \sqcup f$ and $f \sqcap g = g \sqcap f$;

(ii) $f \sqcup (g \sqcup h) = (f \sqcup g) \sqcup h$ and $f \sqcap (g \sqcap h) = (f \sqcap g) \sqcap h$;

(iii) $f \sqcup \mathbf{0}_a = f$ if and only if $s_f \leq a$;

(iv) $f \sqcap \mathbf{1}_a = f$ if and only if $s_f \leq a$;

(v) $f \sqcup \underline{\mathbf{0}} = \underline{\mathbf{0}}$ and $f \sqcap \underline{\mathbf{0}} = \underline{\mathbf{0}}$.

Proof. Statement (i) is a direct consequence of the commutativity of the operations $\vee$ and $\wedge$ on L_1. Similarly, statement (v) is a direct consequence of the definition of join- and meet-convolution.

For statement (ii), we first prove $f \sqcup (g \sqcup h) = (f \sqcup g) \sqcup h$. For any $x \in L_1$, it holds that

$$
\begin{aligned}
((f \sqcup g) \sqcup h)(x) &= \bigvee_{q_1 \vee w = x} \Big((f \sqcup g)(q_1) \wedge h(w)\Big) \\
&= \bigvee_{q_1 \vee w = x} \left(\left(\bigvee_{u \vee v = q_1} f(u) \wedge g(v)\right) \wedge h(w)\right) \\
&= \bigvee_{u \vee v \vee w = x} f(u) \wedge g(v) \wedge h(w) \\
&= \bigvee_{u \vee q_2 = x} \left(f(u) \wedge \left(\bigvee_{v \vee w = q_2} g(v) \wedge h(w)\right)\right) \\
&= \bigvee_{u \vee q_2 = x} \Big(f(u) \wedge (g \sqcup h)(q_2)\Big) = (f \sqcup (g \sqcup h))(x)\,.
\end{aligned}
$$

Dually, we now prove that $f \sqcap (g \sqcap h) = (f \sqcap g) \sqcap h$. For any $x \in L_1$, it holds that

$$
\begin{aligned}
((f \sqcap g) \sqcap h)(x) &= \bigvee_{q_1 \wedge w = x} \Big((f \sqcap g)(q_1) \wedge h(w)\Big) \\
&= \bigvee_{q_1 \wedge w = x} \left(\left(\bigvee_{u \wedge v = q_1} f(u) \wedge g(v)\right) \wedge h(w)\right) \\
&= \bigvee_{u \wedge v \wedge w = x} f(u) \wedge g(v) \wedge h(w) \\
&= \bigvee_{u \wedge q_2 = x} \left(f(u) \wedge \left(\bigvee_{v \wedge w = q_2} g(v) \wedge h(w)\right)\right) \\
&= \bigvee_{u \wedge q_2 = x} \Big(f(u) \wedge (g \sqcap h)(q_2)\Big) = (f \sqcap (g \sqcap h))(x)\,.
\end{aligned}
$$

Next, we prove statement (iii). For any $x \in L_1$, it holds that

$$
\begin{aligned}
(f \sqcup \mathbf{0}_a)(x) &= \bigvee_{u \vee v = x} f(u) \wedge \mathbf{0}_a(v) \\
&= \left(\bigvee_{\substack{u \vee v = x \\ v = 0}} f(u) \wedge \mathbf{0}_a(v)\right) \vee \left(\bigvee_{\substack{u \vee v = x \\ v \neq 0}} f(u) \wedge \mathbf{0}_a(v)\right) \\
&= (f(x) \wedge a) \vee \left(\bigvee_{\substack{u \vee v = x \\ v \neq 0}} f(u) \wedge 0\right) \\
&= f(x) \wedge a\,.
\end{aligned}
$$

Hence, $f \sqcup \mathbf{0}_a = f$ if and only if it holds that $f(x) \leq a$ for any $x \in L_1$, i.e., $s_f \leq a$.

Dually, we now prove statement (iv). For any $x \in L_1$, it holds that

$$\begin{aligned}(f \sqcap \mathbf{1}_a)(x) &= \bigvee_{u \wedge v = x} f(u) \wedge \mathbf{1}_a(v) \\ &= \left(\bigvee_{\substack{u \wedge v = x \\ v = 1}} f(u) \wedge \mathbf{1}_a(v) \right) \vee \left(\bigvee_{\substack{u \wedge v = x \\ v \neq 1}} f(u) \wedge \mathbf{1}_a(v) \right) \\ &= (f(x) \wedge a) \vee \left(\bigvee_{\substack{u \wedge v = x \\ v \neq 1}} f(u) \wedge 0 \right) \\ &= f(x) \wedge a\,.\end{aligned}$$

Hence, $f \sqcap \mathbf{1}_a = f$ if and only if it holds that $f(x) \leq a$ for any $x \in L_1$, i.e., $s_f \leq a$. □

The following corollary is a direct consequence of Theorem 4.1.

Corollary 4.2. The following hold:

(i) the relations $\sqsubseteq_\sqcup$ and $\sqsubseteq_\sqcap$ are antisymmetric;

(ii) the relations $\sqsubseteq_\sqcup$ and $\sqsubseteq_\sqcap$ are transitive;

(iii) it holds that $\mathbf{0}_1 \sqsubseteq_\sqcup f$ and $f \sqsubseteq_\sqcap \mathbf{1}_1$ for any $f \in \mathcal{F}$.

Remark 4.2. Note that as a consequence of statements (iii) and (iv) of Theorem 4.1, we have the following equivalence: the lattice function $\mathbf{0}_a$ is the neutral element of the join-convolution if and only if the lattice function $\mathbf{1}_a$ is the neutral element of the meet-convolution.

As mentioned in Chapter 2, in a bounded lattice the identity element of the join operation is the absorbing element of the meet operation, while the identity element of the meet operation is the absorbing element of the join operation. However, the join- and meet-convolution on $\mathcal{F}$ have the same absorbing element $\underline{\mathbf{0}}$. Since an element cannot be identity and absorbing element at the same time (unless the lattice consists of a single element, i.e., $\mathcal{F} = \{\underline{\mathbf{0}}\}$), we will need to study when the lattice function $\mathbf{1}_a$ is the absorbing element of the join-convolution as well as when the lattice function $\mathbf{0}_a$ is the absorbing element of the meet-convolution.

Proposition 4.2. Let $f \in \mathcal{F}$. The following statements hold:

(i) $f \sqcup \mathbf{1}_a = \mathbf{1}_a$ if and only if $a \leq s_f$;

(ii) $f \sqcap \mathbf{0}_a = \mathbf{0}_a$ if and only if $a \leq s_f$.

Proof. We first prove statement (i). If $x \neq 1$, then

$$(f \sqcup \mathbf{1}_a)(x) = \bigvee_{u \vee v = x} f(u) \wedge \mathbf{1}_a(v) = \bigvee_{\substack{u \vee v = x \\ v \neq 1}} f(u) \wedge \mathbf{1}_a(v) = \bigvee_{\substack{u \vee v = x \\ v \neq 1}} f(u) \wedge 0 = 0\,.$$

If $x = 1$, then

$$\begin{aligned}(f \sqcup \mathbf{1}_a)(1) &= \bigvee_{u \vee v = 1} f(u) \wedge \mathbf{1}_a(v) \\ &= \left(\bigvee_{\substack{u \vee v = x \\ v = 1}} f(u) \wedge \mathbf{1}_a(v) \right) \vee \left(\bigvee_{\substack{u \vee v = x \\ v \neq 1}} f(u) \wedge \mathbf{1}_a(v) \right) \\ &= \left(\bigvee_{u \in L_1} f(u) \wedge a \right) \vee \left(\bigvee_{\substack{u \vee v = x \\ v \neq 1}} f(u) \wedge 0 \right) = s_f \wedge a\,.\end{aligned}$$

Consequently, $f \sqcup \mathbf{1}_a = \mathbf{1}_a$ if and only if $a \leq s_f$.

Dually, we now prove statement (ii). If $x \neq 0$, then

$$(f \sqcap \mathbf{0}_a)(x) = \bigvee_{u \wedge v = x} f(u) \wedge \mathbf{0}_a(v) = \bigvee_{\substack{u \wedge v = x \\ v \neq 0}} f(u) \wedge \mathbf{0}_a(v) = \bigvee_{\substack{u \wedge v = x \\ v \neq 0}} f(u) \wedge 0 = 0\,.$$

If $x = 0$, then

$$\begin{aligned}(f \sqcap \mathbf{0}_a)(0) &= \bigvee_{u \wedge v = 0} f(u) \wedge \mathbf{0}_a(v) \\ &= \left(\bigvee_{\substack{u \wedge v = x \\ v = 0}} f(u) \wedge \mathbf{0}_a(v) \right) \vee \left(\bigvee_{\substack{u \wedge v = x \\ v \neq 0}} f(u) \wedge \mathbf{0}_a(v) \right) \\ &= \left(\bigvee_{u \in L_1} f(u) \wedge a \right) \vee \left(\bigvee_{\substack{u \wedge v = x \\ v \neq 0}} f(u) \wedge 0 \right) = s_f \wedge a\,.\end{aligned}$$

Consequently, $f \sqcap \mathbf{0}_a = \mathbf{0}_a$ if and only if $a \leq s_f$. □

Remark 4.3. Note that as a consequence of Proposition 4.2, we have the following equivalence: $f \sqcup \mathbf{1}_a = \mathbf{1}_a$ if and only if $f \sqcap \mathbf{0}_a = \mathbf{0}_a$.

If we want to ensure that $\mathbf{0}_a$ is both the neutral element of the join-convolution and the absorbing element of the meet-convolution, it follows from Theorem 4.1 and Proposition 4.2 that it should hold $s_f = a$. Similarly, if we want to ensure that $\mathbf{1}_a$ is both the neutral element of the meet-convolution and the absorbing element of the join-convolution, it should hold $s_f = a$ as well. Consequently, we are forced to consider a subset of functions that share the same supremum, i.e., we are forced to consider a set $\mathcal{N}_a = \{f \in \mathcal{F} \mid s_f = a\}$ for some $a \in L_2$.

4.3.2 Idempotency laws

The only lattice laws not studied so far are the absorption laws. However, as mentioned in Chapter 2, in a lattice the fulfillment of the absorption laws imply that idempotency laws are also satisfied. In Chapter 3 we have shown that the idempotency laws do not hold on $\mathcal{F}$. We are therefore forced to study whether or not there is a subset of $\mathcal{F}$ such that the convolution operations satisfy the idempotency laws on it. Although in the preceding subsection we have shown that for the constitution of a bounded lattice we will be forced to restrict our attention to a subset $\mathcal{N}_a$ for some $a \in L_2$, we make an independent study of the idempotency laws of the convolution operations. In general, we find the following inequalities.

Proposition 4.3. Let $f \in \mathcal{F}$. The following statements hold:

(i) $f \leq f \sqcup f$;

(ii) $f \leq f \sqcap f$.

Proof. We first prove statement (i). For any $x \in L_1$, since the couple $(u, v) = (x, x)$ satisfies $u \vee v = x$, it holds that

$$(f \sqcup f)(x) = \bigvee_{u \vee v = x} f(u) \wedge f(v) \geq f(x) \wedge f(x) = f(x)\,.$$

Hence, $f \leq f \sqcup f$.

Dually, we now prove statement (ii). For any $x \in L_1$, since the couple $(u, v) = (x, x)$ satisfies $u \wedge v = x$, it holds that

$$(f \sqcap f)(x) = \bigvee_{u \wedge v = x} f(u) \wedge f(v) \geq f(x) \wedge f(x) = f(x)\,.$$

Hence, $f \leq f \sqcap f$. $\square$

In order to have the idempotency laws satisfied, we consider the following subsets of $\mathcal{F}$:

$$\mathcal{I}_{\sqcup} = \left\{ f \in \mathcal{F} \mid (\forall (x, y) \in {L_1}^2)(f(x) \wedge f(y) \leq f(x \vee y)) \right\}$$

and

$$\mathcal{I}_{\sqcap} = \left\{ f \in \mathcal{F} \mid (\forall (x, y) \in {L_1}^2)(f(x) \wedge f(y) \leq f(x \wedge y)) \right\}\,.$$

We also use the notation $\mathcal{I} := \mathcal{I}_{\sqcup} \cap \mathcal{I}_{\sqcap}$. As is justified by the following theorem, we refer to the functions of the sets $\mathcal{I}_{\sqcup}$, $\mathcal{I}_{\sqcap}$ and $\mathcal{I}$ as join-idempotent, meet-idempotent and idempotent functions, respectively.

Theorem 4.2. Let $f \in \mathcal{F}$. The following statements hold:

(i) $f \sqcup f = f$ if and only if $f \in \mathcal{I}_{\sqcup}$;

(ii) $f \sqcap f = f$ if and only if $f \in \mathcal{I}_{\sqcap}$;

(iii) $f \sqcup f = f$ and $f \sqcap f = f$ if and only if $f \in \mathcal{I}$.

Proof. We first prove statement (i).
$\Rightarrow$ Suppose that $f \sqcup f = f$, while $f \notin \mathcal{I}_{\sqcup}$. Then there exist $x, y \in L_1$ such that $f(x) \wedge f(y) \not\leq f(x \vee y)$. Note that this means that $(f(x) \wedge f(y)) \vee f(x \vee y) > f(x \vee y)$.

Further, since both couples $(u, v) = (x, y)$ and $(u, v) = (x \vee y, x \vee y)$ satisfy $u \vee v = x \vee y$, it holds that

$$
\begin{aligned}
(f \sqcup f)(x \vee y) &= \bigvee_{u \vee v = x} f(u) \wedge g(v) \\
&\geq (f(x) \wedge f(y)) \vee (f(x \vee y) \wedge f(x \vee y)) \\
&= (f(x) \wedge f(y)) \vee f(x \vee y) > f(x \vee y)\,,
\end{aligned}
$$

which contradicts $f \sqcup f = f$.

$\Leftarrow$ Due to Proposition 4.3, it holds that $f \leq f \sqcup f$ and, hence, it only remains to prove that $f \sqcup f \leq f$.

If $f \in \mathcal{I}_{\sqcup}$, then it holds that

$$
(f \sqcup f)(x) = \bigvee_{u \vee v = x} f(u) \wedge f(v) \leq \bigvee_{u \vee v = x} f(u \vee v) = \bigvee_{u \vee v = x} f(x) = f(x).
$$

Hence, $f \sqcup f = f$.

Dually, we now prove statement (ii).
$\Rightarrow$ Suppose that $f \sqcap f = f$, while $f \notin \mathcal{I}_{\sqcap}$. Then there exist $x, y \in L_1$ such that $f(x) \wedge f(y) \not\leq f(x \wedge y)$. Note that this means that $(f(x) \wedge f(y)) \vee f(x \wedge y) > f(x \wedge y)$.
Further, since both couples $(u, v) = (x, y)$ and $(u, v) = (x \wedge y, x \wedge y)$ satisfy $u \wedge v = x \wedge y$, it holds that

$$
\begin{aligned}
(f \sqcap f)(x \wedge y) &= \bigvee_{u \wedge v = x} f(u) \wedge g(v) \\
&\geq (f(x) \wedge f(y)) \vee (f(x \wedge y) \wedge f(x \wedge y)) \\
&= (f(x) \wedge f(y)) \vee f(x \wedge y) > f(x \wedge y)\,,
\end{aligned}
$$

which contradicts $f \sqcap f = f$.
$\Leftarrow$ Due to Proposition 4.3, it holds that $f \leq f \sqcap f$ and, hence, it only remains to prove that $f \sqcap f \leq f$.

If $f \in \mathcal{I}_{\sqcap}$, then it holds that

$$(f \sqcap f)(x) = \bigvee_{u \wedge v = x} f(u) \wedge f(v) \leq \bigvee_{u \wedge v = x} f(u \wedge v) = \bigvee_{u \wedge v = x} f(x) = f(x).$$

Hence, $f \sqcap f = f$.

Statement (iii) is a direct consequence of (i) and (ii). □

Consequently, if we want to ensure that the convolution operations satisfy the idempotency laws, we are forced to consider the set of idempotent functions $\mathcal{I}$ (or a certain subset of it). Note that in case $\mathbb{L}_1$ is a bounded chain, it holds that $\mathcal{I}_{\sqcup} = \mathcal{I}_{\sqcap} = \mathcal{I} = \mathcal{F}$, and the convolution operations satisfy the idempotency laws (as it is proven in Theorem 3.1).

Corollary 4.3. The following hold:

(i) the relation $\sqsubseteq_{\sqcup}$ is reflexive on $\mathcal{I}_{\sqcup}$;

(ii) the relation $\sqsubseteq_{\sqcap}$ is reflexive on $\mathcal{I}_{\sqcap}$.

Hence, due to Corollaries 4.2 and 4.3, the relation $\sqsubseteq_{\sqcup}$ constitutes a partial order on $\mathcal{I}_{\sqcup}$, while the relation $\sqsubseteq_{\sqcap}$ constitutes a partial order on $\mathcal{I}_{\sqcap}$.

4.3.3 Absorption laws

Since the idempotency laws do not hold, the absorption laws surely do not hold in general either. However, the following result holds.

Proposition 4.4. Let $f, g \in \mathcal{F}$. The following statements are equivalent:

(i) $f \leq f \sqcup (f \sqcap g)$;

(ii) $f \leq f \sqcap (f \sqcup g)$;

(iii) $s_f \leq s_g$.

Proof. We first prove the equivalence of statements (i) and (iii).

$\Rightarrow$ Suppose that $f \leq f \sqcup (f \sqcap g)$. For any $x \in L_1$, it holds that

$$(f \sqcup (f \sqcap g))\,(x) = \bigvee_{u_1 \vee (u_2 \wedge v) = x} f(u_1) \wedge f(u_2) \wedge g(v) \leq \bigvee_{u_1 \vee (u_2 \wedge v) = x} g(v) \leq \bigvee_{v \in L_1} g(v) = s_g\,.$$

Consequently, $f(x) \leq (f \sqcup (f \sqcap g))\,(x) \leq s_g$ for any $x \in L_1$ and $s_f \leq s_g$.

$\Leftarrow$ Suppose that $s_f \leq s_g$, then it holds that $f(x) \leq \bigvee_{y \in L_1} g(y)$, for any $x \in L_1$. Due to the absorption laws in $\mathbb{L}_1$, for any $x, y \in L_1$, it holds that $x \vee (x \wedge y) = x$. Hence, for any $x \in L_1$, it holds that

$$\begin{aligned}(f \sqcup (f \sqcap g))(x) &= \bigvee_{u_1 \vee (u_2 \wedge v) = x} f(u_1) \wedge f(u_2) \wedge g(v) \\ &\geq \bigvee_{\substack{u_1 \vee (u_2 \wedge v) = x \\ u_1 = x,\, u_2 = x}} f(x) \wedge f(x) \wedge g(v) \\ &= f(x) \wedge \left(\bigvee_{v \in L_1} g(v) \right) = f(x)\,.\end{aligned}$$

Dually, we now prove the equivalence of statements (ii) and (iii).

$\Rightarrow$ Suppose that $f \leq f \sqcap (f \sqcup g)$. For any $x \in L_1$, it holds that

$$(f \sqcap (f \sqcup g))(x) = \bigvee_{u_1 \wedge (u_2 \vee v) = x} f(u_1) \wedge f(u_2) \wedge g(v) \leq \bigvee_{u_1 \wedge (u_2 \vee v) = x} g(v) \leq \bigvee_{v \in L_1} g(v) = s_g\,.$$

Consequently, $f(x) \leq (f \sqcap (f \sqcup g))(x) \leq s_g$ for any $x \in L_1$, and $s_f \leq s_g$.

$\Leftarrow$ Suppose that $s_f \leq s_g$, then it holds that $f(x) \leq \bigvee_{y \in L_1} g(y)$, for any $x \in L_1$. Due to the absorption laws in $\mathbb{L}_1$, for any $x, y \in L_1$, it holds that $x \wedge (x \vee y) = x$. Hence, for any $x \in L_1$, it holds that

$$\begin{aligned}(f \sqcap (f \sqcup g))(x) &= \bigvee_{u_1 \wedge (u_2 \vee v) = x} f(u_1) \wedge f(u_2) \wedge g(v) \\ &\geq \bigvee_{\substack{u_1 \wedge (u_2 \vee v) = x \\ u_1 = x,\, u_2 = x}} f(x) \wedge f(x) \wedge g(v) \\ &= f(x) \wedge \left(\bigvee_{v \in L_1} g(v) \right) = f(x)\,.\end{aligned}$$

□

Corollary 4.4. Let $f, g \in \mathcal{F}$. The following statements hold:

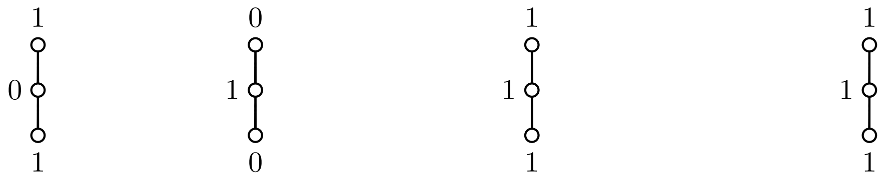
(i) if $f \sqcup (f \sqcap g) = f$ and $g \sqcup (g \sqcap f) = g$, then $s_f = s_g$;

(ii) if $f \sqcap (f \sqcup g) = f$ and $g \sqcap (g \sqcup f) = g$, then $s_f = s_g$.

Proof. We first prove statement (i). Suppose that $f \sqcup (f \sqcap g) = f$ and $g \sqcup (g \sqcap f) = g$. Due to Proposition 4.4, it then holds that $s_f \leq s_g$ and $s_g \leq s_f$, i.e., $s_f = s_g$.

Dually, we now prove statement (ii). Suppose that $f \sqcap (f \sqcup g) = f$ and $g \sqcap (g \sqcup f) = g$. Due to Proposition 4.4, it then holds that $s_f \leq s_g$ and $s_g \leq s_f$, i.e., $s_f = s_g$. □

Consequently, if we want to ensure that the absorption laws hold for the convolution operations, we are again forced to consider a subset $\mathcal{N}_a = \{f \in \mathcal{F} \mid s_f = a\}$ for some $a \in L_2$.



(a) Function f (b) Function g (c) Function $f \sqcup (f \sqcap g)$ (d) Function $f \sqcap (f \sqcup g)$

Figure 4.2: Graphical representation of the functions in Example 4.2: (a) the function f, (b) the function g, (c) the corresponding function $f \sqcup (f \sqcap g)$, and (d) the corresponding function $f \sqcap (f \sqcup g)$.

Moreover, since the idempotency laws only hold when restricting to the set of idempotent functions $\mathcal{I}$, the same restriction is additionally required for the absorption laws. In the following example we show that a further restriction will be necessary.

Example 4.2. Let $\mathbb{L}_1 = \mathbb{C}_3$ and $\mathbb{L}_2 = \mathbb{C}_2$. Consider the functions $f, g \in \mathcal{N}_1 \cap \mathcal{I}$ depicted in Figs. 4.2(a)–(b). The convolutions $f \sqcup (f \sqcap g)$ and $f \sqcap (f \sqcup g)$ are depicted in Figs. 4.2(c)–(d). One easily verifies that $f \neq f \sqcup (f \sqcap g)$ and $f \neq f \sqcap (f \sqcup g)$.

In order to have the absorption laws satisfied we consider, as the authors in [40, 76], the set of (order-)convex functions

$$\mathcal{C} = \{f \in \mathcal{F} \mid (\forall (x_1, x_2, x_3) \in {L_1}^3)(x_1 \leq x_2 \leq x_3 \Rightarrow f(x_1) \wedge f(x_3) \leq f(x_2))\} .$$

Note that the term convex function does not refer to the traditional convex real-valued functions, but it stems from the concept of convex fuzzy set introduced by Zadeh in [81].

Theorem 4.3. Let $f \in \mathcal{F}$. The following statements hold:

(i) $f \sqcup (f \sqcap g) = f$, for any $g \in \mathcal{N}_{s_f}$, if and only if $f \in \mathcal{I}_\sqcup \cap \mathcal{C}$;

(ii) $f \sqcap (f \sqcup g) = f$, for any $g \in \mathcal{N}_{s_f}$, if and only $f \in \mathcal{I}_\sqcap \cap \mathcal{C}$;

(iii) $f \sqcup (f \sqcap g) = f$ and $f \sqcap (f \sqcup g) = f$, for any $g \in \mathcal{N}_{s_f}$, if and only $f \in \mathcal{I} \cap \mathcal{C}$.

Proof. We first prove statement (i).

$\Rightarrow$ Suppose that, for any $g \in \mathcal{N}_{s_f}$, it holds that $f \sqcup (f \sqcap g) = f$, while $f \notin \mathcal{I}_\sqcup \cap \mathcal{C}$. We distinguish two different cases.

(a) Case 1: $f \notin \mathcal{I}_\sqcup$. Let $g = \mathbf{1}_{s_f} \in \mathcal{N}_{s_f}$. Due to Theorem 4.1 (iv), it holds that $f \sqcap g = f$. Hence, it holds that

$$f \sqcup (f \sqcap g) = f \sqcup f .$$

Since $f \sqcup f \neq f$, it follows that $f \sqcup (f \sqcap g) \neq f$, a contradiction.

(b) Case 2: $f \notin \mathcal{C}$. This means that there exist $x_1, x_2, x_3 \in L_1$ such that $x_1 \leq x_2 \leq x_3$ and $f(x_1) \wedge f(x_3) \nleq f(x_2)$. Consequently, $f(x_2) < f(x_2) \vee (f(x_1) \wedge f(x_3))$. Let $g \in \mathcal{N}_{s_f}$ be the function

$$g(x) = \begin{cases} s_f & , \text{ if } x = x_2 \,, \\ 0 & , \text{ otherwise } \,, \end{cases} \tag{4.1}$$

Since the triplets $(u_1, u_2, v) = (x_2, x_2, x_2)$ and $(u_1, u_2, v) = (x_1, x_3, x_2)$ satisfy $u_1 \vee (u_2 \wedge v) = x_2$, it holds that

$$\begin{aligned}
(f \sqcup (f \sqcap g))(x_2) &= \bigvee_{u_1 \vee (u_2 \wedge v) = x_2} f(u_1) \wedge f(u_2) \wedge g(v) \\
&\geq (f(x_2) \wedge f(x_2) \wedge g(x_2)) \vee (f(x_1) \wedge f(x_3) \wedge g(x_2)) \\
&= (f(x_2) \wedge f(x_2) \wedge s_f) \vee (f(x_1) \wedge f(x_3) \wedge s_f) \\
&= s_f \wedge (f(x_2) \vee (f(x_1) \wedge f(x_3))) \\
&= f(x_2) \vee (f(x_1) \wedge f(x_3)) > f(x_2) \,,
\end{aligned}$$

a contradiction.

$\Leftarrow$ Suppose that $f \in \mathcal{I}_{\sqcup} \cap \mathcal{C}$. Due to Proposition 4.4, for any $g \in \mathcal{N}_{s_f}$, it holds that $f \leq f \sqcup (f \sqcap g)$. Hence, it only remains to prove that $f \sqcup (f \sqcap g) \leq f$, i.e., we need to verify that for any $x \in L_1$, it holds that

$$(f \sqcup (f \sqcap g))(x) = \bigvee_{u_1 \vee (u_2 \wedge v) = x} f(u_1) \wedge f(u_2) \wedge g(v) \leq f(x) \,.$$

If $u_1 \vee (u_2 \wedge v) = x$, then $u_1 \leq u_1 \vee (u_2 \wedge v) = x$ and $u_1 \leq x$. Similarly, it holds that $x = u_1 \vee (u_2 \wedge v) \leq u_1 \vee u_2$ and we find that $u_1 \leq x \leq u_1 \vee u_2$. Due to $f \in \mathcal{I}_{\sqcup} \cap \mathcal{C}$, for any $x \in L_1$, it holds that

$$\begin{aligned}
(f \sqcup (f \sqcap g))(x) &= \bigvee_{u_1 \vee (u_2 \wedge v) = x} f(u_1) \wedge f(u_2) \wedge g(v) \\
&= \bigvee_{u_1 \vee (u_2 \wedge v) = x} f(u_1) \wedge f(u_1) \wedge f(u_2) \wedge g(v) \\
&\leq \bigvee_{u_1 \vee (u_2 \wedge v) = x} f(u_1) \wedge f(u_1 \vee u_2) \wedge g(v) \\
&\leq \bigvee_{u_1 \vee (u_2 \wedge v) = x} f(x) \wedge g(v) \\
&\leq \bigvee_{u_1 \vee (u_2 \wedge v) = x} f(x) \\
&= f(x) \,.
\end{aligned}$$

Dually, we now prove statement (ii).

$\Rightarrow$ Suppose that, for any $g \in \mathcal{N}_{s_f}$, it holds that $f \sqcap (f \sqcup g) = f$, while $f \notin \mathcal{I}_\sqcap \cap \mathcal{C}$. We distinguish two different cases.

(a) Case 1: $f \notin \mathcal{I}_\sqcap$. Let $g = \mathbf{0}_{s_f} \in \mathcal{N}_{s_f}$. Due to Theorem 4.1 (iii), it holds that $f \sqcup g = f$. Hence, it holds that

$$f \sqcap (f \sqcup g) = f \sqcap f\,.$$

Since $f \sqcap f \neq f$, it follows that $f \sqcap (f \sqcup g) \neq f$, a contradiction.

(b) Case 2: $f \notin \mathcal{C}$. This means that there exist $x_1, x_2, x_3 \in L_1$ such that $x_1 \leq x_2 \leq x_3$ and $f(x_1) \wedge f(x_3) \nleq f(x_2)$. Consequently, $f(x_2) < f(x_2) \vee (f(x_1) \wedge f(x_3))$. Consider $g \in \mathcal{N}_{s_f}$ as in Eq. (4.1). Since the triplets $(u_1, u_2, v) = (x_2, x_2, x_2)$ and $(u_1, u_2, v) = (x_3, x_1, x_2)$ satisfy $u_1 \wedge (u_2 \vee v) = x_2$, it holds that

$$\begin{aligned}
(f \sqcap (f \sqcup g))(x_2) &= \bigvee_{u_1 \wedge (u_2 \vee v) = x_2} f(u_1) \wedge f(u_2) \wedge g(v) \\
&\geq (f(x_2) \wedge f(x_2) \wedge g(x_2)) \vee (f(x_3) \wedge f(x_1) \wedge g(x_2)) \\
&= (f(x_2) \wedge f(x_2) \wedge s_f) \vee (f(x_3) \wedge f(x_1) \wedge s_f) \\
&= s_f \wedge (f(x_2) \vee (f(x_1) \wedge f(x_3))) \\
&= f(x_2) \vee (f(x_1) \wedge f(x_3)) > f(x_2)\,,
\end{aligned}$$

a contradiction.

$\Leftarrow$ Suppose that $f \in \mathcal{I}_\sqcup \cap \mathcal{C}$. Due to Proposition 4.4, for any $g \in \mathcal{N}_{s_f}$, it holds that $f \leq f \sqcap (f \sqcup g)$. Hence, it only remains to prove that $f \sqcap (f \sqcup g) \leq f$, i.e., we need to verify that for any $x \in L_1$, it holds that

$$(f \sqcap (f \sqcup g))(x) = \bigvee_{u_1 \wedge (u_2 \vee v) = x} f(u_1) \wedge f(u_2) \wedge g(v) \leq f(x)\,.$$

If $u_1 \wedge (u_2 \vee v) = x$, then $u_1 \wedge u_2 \leq u_1 \wedge (u_2 \vee v) = x$ and $u_1 \wedge u_2 \leq x$. Similarly, it holds that $x = u_1 \wedge (u_2 \vee v) \leq u_1$ and we find that $u_1 \wedge u_2 \leq x \leq u_1$.

Due to $f \in \mathcal{I}_\sqcap \cap \mathcal{C}$, for any $x \in L_1$, it holds that

$$\begin{aligned}
(f \sqcap (f \sqcup g))(x) &= \bigvee_{u_1 \wedge (u_2 \vee v) = x} f(u_1) \wedge f(u_2) \wedge g(v) \\
&= \bigvee_{u_1 \wedge (u_2 \vee v) = x} f(u_1) \wedge f(u_1) \wedge f(u_2) \wedge g(v) \\
&\leq \bigvee_{u_1 \wedge (u_2 \vee v) = x} f(u_1) \wedge f(u_1 \wedge u_2) \wedge g(v) \\
&\leq \bigvee_{u_1 \wedge (u_2 \vee v) = x} f(x) \wedge g(v) \\
&\leq \bigvee_{u_1 \wedge (u_2 \vee v) = x} f(x) \\
&= f(x)\,.
\end{aligned}$$

Statement (iii) is a direct consequence of statements (i) and (ii). □

Consequently, if we want to ensure the absorption laws for the convolution operations, we are forced to consider the set $\mathcal{N}_a \cap \mathcal{I} \cap \mathcal{C}$ for some $a \in L_2$ (or a certain subset of it).

4.4 Universal algebras

4.4.1 Closedness of the sets of functions considered

In the study of the lattice laws of the convolution operations, some restrictions on the set of lattice functions have appeared naturally. More specifically, the subsets $\mathcal{N}_a$ (with $a \in L_2$), $\mathcal{I}_\sqcup$, $\mathcal{I}_\sqcap$, $\mathcal{I}$ and $\mathcal{C}$ have been considered, as well as the subsets $\mathcal{M}_I$ and $\mathcal{M}_D$. However, we have not yet verified whether or not these subsets of $\mathcal{F}$ are closed under the convolution operations. In other words, we have not yet verified whether or not the convolution operations are internal on these subsets. This issue is addressed in this subsection.

Proposition 4.5. The sets $\mathcal{N}_a$ (with $a \in L_2$) are closed under join- and meet-convolution.

Proof. Let $f, g \in \mathcal{N}_a$ for some $a \in L_2$. We first prove that $\mathcal{N}_a$ is closed under join-convolution.

For any $x \in L_1$, it holds that

$$
\begin{aligned}
\bigvee_{x\in L_1} (f \sqcup g)(x) &= \bigvee_{x\in L_1} \bigvee_{u\vee v=x} f(u) \wedge g(v) \\
&= \bigvee_{(u\vee v)\in L_1} f(u) \wedge g(v) \\
&= \bigvee_{\substack{u\in L_1 \\ v\in L_1}} f(u) \wedge g(v) \\
&= \left(\bigvee_{u\in L_1} f(u) \right) \wedge \left(\bigvee_{v\in L_1} g(v) \right) \\
&= a \wedge a = a\,.
\end{aligned}
$$

Dually, we now prove that $\mathcal{N}_a$ is closed under meet-convolution. For any $x \in L_1$, it holds that

$$
\begin{aligned}
\bigvee_{x\in L_1} (f \sqcap g)(x) &= \bigvee_{x\in L_1} \bigvee_{u\wedge v=x} f(u) \wedge g(v) \\
&= \bigvee_{(u\wedge v)\in L_1} f(u) \wedge g(v) \\
&= \bigvee_{\substack{u\in L_1 \\ v\in L_1}} f(u) \wedge g(v) \\
&= \left(\bigvee_{u\in L_1} f(u) \right) \wedge \left(\bigvee_{v\in L_1} g(v) \right) \\
&= a \wedge a = a\,.
\end{aligned}
$$

□

Proposition 4.6. The following statements hold:

(i) the set $\mathcal{I}_\sqcup$ is closed under join-convolution;

(ii) the set $\mathcal{I}_\sqcap$ is closed under meet-convolution.

Proof. Let $f, g \in \mathcal{I}_\sqcup$. We first prove that $\mathcal{I}_\sqcup$ is closed under join-convolution. Due to $f, g \in \mathcal{I}_\sqcup$, for any $x_1, x_2 \in L_1$, it holds that

$$
\begin{aligned}
(f \sqcup g)(x_1) \wedge (f \sqcup g)(x_2) &= \left(\bigvee_{u_1\vee v_1=x_1} f(u_1) \wedge g(v_1) \right) \wedge \left(\bigvee_{u_2\vee v_2=x_2} f(u_2) \wedge g(v_2) \right) \\
&= \bigvee_{\substack{u_1\vee v_1=x_1 \\ u_2\vee v_2=x_2}} f(u_1) \wedge g(v_1) \wedge f(u_2) \wedge g(v_2) \\
&\leq \bigvee_{\substack{u_1\vee v_1=x_1 \\ u_2\vee v_2=x_2}} f(u_1 \vee u_2) \wedge g(v_1 \vee v_2)\,.
\end{aligned}
$$

Further, since $(u_1 \vee u_2) \vee (v_1 \vee v_2) = (u_1 \vee v_1) \vee (u_2 \vee v_2) = x_1 \vee x_2$, by denoting $u = u_1 \vee u_2$ and $v = v_1 \vee v_2$, we find that

$$\begin{aligned}(f \sqcup g)(x_1) \wedge (f \sqcup g)(x_2) &\leq \bigvee_{\substack{u_1 \vee v_1 = x_1 \\ u_2 \vee v_2 = x_2}} f(u_1 \vee u_2) \wedge g(v_1 \vee v_2) \\ &\leq \bigvee_{u \vee v = x_1 \vee x_2} f(u) \wedge g(v) \\ &= (f \sqcup g)(x_1 \vee x_2)\,.\end{aligned}$$

Dually, we now prove that $\mathcal{I}_\sqcup$ is closed under meet-convolution. Due to $f, g \in \mathcal{I}_\sqcap$, for any $x_1, x_2 \in L_1$, it holds that

$$\begin{aligned}(f \sqcap g)(x_1) \wedge (f \sqcap g)(x_2) &= \left(\bigvee_{u_1 \wedge v_1 = x_1} f(u_1) \wedge g(v_1)\right) \wedge \left(\bigvee_{u_2 \wedge v_2 = x_2} f(u_2) \wedge g(v_2)\right) \\ &= \bigvee_{\substack{u_1 \wedge v_1 = x_1 \\ u_2 \wedge v_2 = x_2}} f(u_1) \wedge g(v_1) \wedge f(u_2) \wedge g(v_2) \\ &\leq \bigvee_{\substack{u_1 \wedge v_1 = x_1 \\ u_2 \wedge v_2 = x_2}} f(u_1 \wedge u_2) \wedge g(v_1 \wedge v_2)\,.\end{aligned}$$

Further, since $(u_1 \wedge u_2) \wedge (v_1 \wedge v_2) = (u_1 \wedge v_1) \wedge (u_2 \wedge v_2) = x_1 \wedge x_2$, by denoting $u = u_1 \wedge u_2$ and $v = v_1 \wedge v_2$, we find that

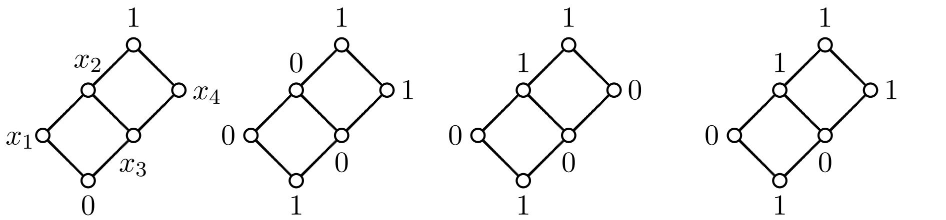
$$\begin{aligned}(f \sqcap g)(x_1) \wedge (f \sqcap g)(x_2) &\leq \bigvee_{\substack{u_1 \wedge v_1 = x_1 \\ u_2 \wedge v_2 = x_2}} f(u_1 \wedge u_2) \wedge g(v_1 \wedge v_2) \\ &\leq \bigvee_{u \wedge v = x_1 \wedge x_2} f(u) \wedge g(v) \\ &= (f \sqcap g)(x_1 \wedge x_2)\,.\end{aligned}$$

□

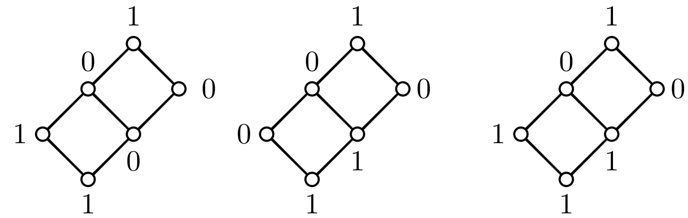
In the following example we show that neither the set $\mathcal{I}_\sqcap$ is closed under the join-convolution nor the set $\mathcal{I}_\sqcup$ is closed under the meet-convolution.

Example 4.3. Let $\mathbb{L}_1$ be the distributive lattice whose Hasse diagram is depicted in Fig. 4.3(a) and $\mathbb{L}_2 = \mathbb{C}_2$.

(i) Consider the functions $f_1, g_1 \in \mathcal{I}_\sqcap$ depicted in Figs. 4.3(b)–(c). The join-convolution $f_1 \sqcup g_1$ is depicted in Fig. 4.3(d). One easily verifies that $x_2 \wedge x_4 = x_3$, while $(f_1 \sqcup g_1)(x_2) \wedge (f_1 \sqcup g_1)(x_4) = 1 > (f_1 \sqcup g_1)(x_3) = 0$. Hence, $f_1 \sqcup g_1 \notin \mathcal{I}_\sqcap$.



(a) $\mathbb{L}_1$ (b) Function f_1 (c) Function g_1 (d) Function $f_1 \sqcup g_1$



(e) Function f_2 (f) Function g_2 (g) Function $f_2 \sqcap g_2$

Figure 4.3: Graphical representation of the functions in Example 4.3: (a) Hasse diagram of the lattice $\mathbb{L}_1$, (b) the function f_1, (c) the function g_1, (d) the join-convolution $f_1 \sqcup g_1$, (e) the function f_2, (f) the function g_2, and (g) the meet-convolution $f_2 \sqcap g_2$.

(ii) Consider the functions $f_2, g_2 \in \mathcal{I}_\sqcup$ depicted in Figs. 4.3(e)–(f). The meet-convolution $f_2 \sqcap g_2$ is depicted in Fig. 4.3(g). One easily verifies that $x_1 \vee x_3 = x_2$, while $(f_2 \sqcap g_2)(x_1) \wedge (f_2 \sqcap g_2)(x_3) = 1 > (f_1 \sqcap g_1)(x_2) = 0$. Hence, $f_2 \sqcap g_2 \notin \mathcal{I}_\sqcup$.

Proposition 4.7. The following statements hold:

(i) the set $\mathcal{M}_I$ is closed under join-convolution;

(ii) the set $\mathcal{M}_D$ is closed under meet-convolution.

Proof. We first prove that the set $\mathcal{M}_I$ is closed under join-convolution. Let $f, g \in \mathcal{M}_I$. Due to Proposition 4.1, for any $x_1, x_2 \in L_1$ such that $x_1 \leq x_2$, it holds that

$$(f \sqcup g)(x_1) = f(x_1) \wedge g(x_1) \leq f(x_2) \wedge g(x_2) = (f \sqcup g)(x_2)\,.$$

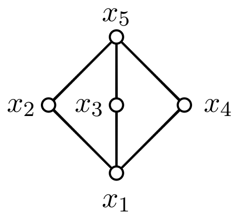
Hence, the set $\mathcal{M}_I$ is closed under join-convolution.

Dually, we now prove that the set $\mathcal{M}_D$ is closed under meet-convolution. Let $f, g \in \mathcal{M}_D$. Due to Proposition 4.1, for any $x_1, x_2 \in L_1$ such that $x_1 \leq x_2$, it holds that

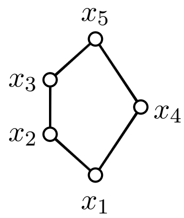
$$(f \sqcap g)(x_2) = f(x_2) \wedge g(x_2) \leq f(x_1) \wedge g(x_1) = (f \sqcup g)(x_1)\,.$$

Hence, the set $\mathcal{M}_D$ is closed under meet-convolution.

□



(a) $\mathbb{M}_3$



(b) $\mathbb{N}_5$

Figure 4.4: Hasse diagram of: (a) the sublattice $\mathbb{M}_3$, and (b) the sublattice $\mathbb{N}_5$.

We will prove the closedness of the set $\mathcal{M}_D$ under join-convolution and of the set $\mathcal{M}_I$ under meet-convolution under the additional assumption that $\mathbb{L}_1$ is a distributive lattice; the latter will turn out to be a necessary and sufficient condition. Our proofs make extensive use of the famous $\mathbb{M}_3$–$\mathbb{N}_5$ theorem that we have recalled in Theorem 2.1.

Proposition 4.8. The following statements hold:

(i) the set $\mathcal{M}_D$ is closed under join-convolution if and only if $\mathbb{L}_1$ is a distributive lattice;

(ii) the set $\mathcal{M}_I$ is closed under meet-convolution if and only if $\mathbb{L}_1$ is a distributive lattice.

Proof. We first prove that $\mathcal{M}_D$ is closed under join-convolution if and only if $\mathbb{L}_1$ is a distributive lattice.

$\Rightarrow$ Suppose that $\mathcal{M}_D$ is closed under join-convolution, while $\mathbb{L}_1$ is not distributive. Due to the $\mathbb{M}_3$–$\mathbb{N}_5$ theorem, $\mathbb{L}_1$ has a sublattice that is isomorphic to $\mathbb{M}_3$ or to $\mathbb{N}_5$. We distinguish two cases.

(a) Case 1: $\mathbb{L}_1$ has a sublattice isomorphic to $\mathbb{M}_3$. We refer to the elements of this sublattice as in Fig. 4.4(a). We consider the functions $f, g \in \mathcal{M}_D$ defined as:

$$f(x) = \begin{cases} 1 & , \text{ if } x \leq x_2 \,, \\ 0 & , \text{ otherwise} \,; \end{cases} \qquad g(x) = \begin{cases} 1 & , \text{ if } x \leq x_3 \,, \\ 0 & , \text{ otherwise} \,. \end{cases}$$

It holds that $(f \sqcup g)(x_5) \geq f(x_2) \wedge g(x_3) = 1$. Moreover,

$$\begin{aligned}(f \sqcup g)(x_4) &= \bigvee_{u \vee v = x_4} f(u) \wedge g(v) \\ &= \left(\bigvee_{\substack{u \vee v = x_4 \\ u \not\leq x_2 \text{ or } v \not\leq x_3}} f(u) \wedge g(v) \right) \vee \left(\bigvee_{\substack{u \vee v = x_4 \\ u \leq x_2 \text{ and } v \leq x_3}} f(u) \wedge g(v) \right) \\ &= 0 \vee \left(\bigvee_{\substack{u \vee v = x_4 \\ u \leq x_2 \text{ and } v \leq x_3}} 1 \right),\end{aligned}$$

which equals 1 unless the set

$$U = \{(u,v) \in {L_1}^2 \mid u \vee v = x_4 \,, u \leq x_2 \text{ and } v \leq x_3\}$$

is empty.

For any $u \in L_1$ such that $u \vee v = x_4$ and $u \leq x_2$, it follows that $u \leq x_2 \wedge x_4 = x_1$. Analogously, for any $v \in L_1$ such that $u \vee v = x_4$ and $v \leq x_3$, it follows that $v \leq x_3 \wedge x_4 = x_1$. It then follows that $x_2 = u \vee v \leq x_1 \vee x_1 = x_1$, a contradiction. Consequently, $U = \emptyset$ and $(f \sqcup g)(x_4) = 0$. We conclude that $(f \sqcup g)(x_4) = 0 < 1 = (f \sqcup g)(x_5)$, and, hence, $f \sqcup g \notin \mathcal{M}_D$.

(b) Case 2: $\mathbb{L}_1$ has a sublattice isomorphic to $\mathbb{N}_5$. We refer to the elements of this sublattice as in Fig. 4.4(b). We consider the functions $f, g \in \mathcal{M}_D$ defined as:

$$f(x) = \begin{cases} 1 & , \text{ if } x \leq x_2 \,, \\ 0 & , \text{ otherwise} \,; \end{cases} \qquad g(x) = \begin{cases} 1 & , \text{ if } x \leq x_4 \,, \\ 0 & , \text{ otherwise} \,. \end{cases}$$

It holds that $(f \sqcup g)(x_5) \geq f(x_2) \wedge g(x_4) = 1$. Moreover,

$$\begin{aligned}(f \sqcup g)(x_3) &= \bigvee_{u \vee v = x_3} f(u) \wedge g(v) \\ &= \left(\bigvee_{\substack{u \vee v = x_3 \\ u \not\leq x_2 \text{ or } v \not\leq x_4}} f(u) \wedge g(v) \right) \vee \left(\bigvee_{\substack{u \vee v = x_3 \\ u \leq x_2 \text{ and } v \leq x_4}} f(u) \wedge g(v) \right) \\ &= 0 \vee \left(\bigvee_{\substack{u \vee v = x_3 \\ u \leq x_2 \text{ and } v \leq x_4}} 1 \right),\end{aligned}$$

which equals 1 unless the set

$$U = \{(u,v) \in {L_1}^2 \mid u \vee v = x_3 \,, u \leq x_2 \text{ and } v \leq x_4\}$$

is empty.

For any $u \in L_1$ such that $u \vee v = x_3$ and $u \leq x_2$, it follows that $u \leq x_2 \wedge x_3 = x_2$. Analogously, for any $v \in L_1$ such that $u \vee v = x_3$ and $v \leq x_4$, it follows that $v \leq x_3 \wedge x_4 = x_1$. It then follows that $x_3 = u \vee v \leq x_2 \vee x_1 = x_2$, a contradiction. Consequently, $U = \emptyset$ and $(f \sqcup g)(x_3) = 0$. We conclude that $(f \sqcup g)(x_3) = 0 < 1 = (f \sqcup g)(x_5)$, and, hence, $f \sqcup g \notin \mathcal{M}_D$.

$\Leftarrow$ Let $\mathbb{L}_1$ be a distributive lattice and $f, g \in \mathcal{M}_D$. For any $x_1, x_2 \in L_1$ such that $x_1 \leq x_2$ and for any couple (u_2, v_2) such that $u_2 \vee v_2 = x_2$, it holds that $u_2 \wedge x_1 \leq u_2$ and $v_2 \wedge x_1 \leq v_2$. Hence, since $f, g \in \mathcal{M}_D$, it holds that

$$\begin{aligned}(f \sqcup g)(x_2) &= \bigvee_{u_2 \vee v_2 = x_2} f(u_2) \wedge g(v_2) \\ &\leq \bigvee_{u_2 \vee v_2 = x_2} f(u_2 \wedge x_1) \wedge g(v_2 \wedge x_1)\,.\end{aligned}$$

Further, since $\mathbb{L}_1$ is distributive, it follows that $(u_2 \wedge x_1) \vee (v_2 \wedge x_1) = (u_2 \vee v_2) \wedge x_1 = x_2 \wedge x_1 = x_1$. Hence, we find that

$$\begin{aligned}(f \sqcup g)(x_2) &\leq \bigvee_{u_2 \vee v_2 = x_2} f(u_2 \wedge x_1) \wedge g(v_2 \wedge x_1) \\ &\leq \bigvee_{u \vee v = x_1} f(u) \wedge g(v) \\ &= (f \sqcup g)(x_1)\,.\end{aligned}$$

Dually, we now prove that $\mathcal{M}_I$ is closed under meet-convolution if and only if $\mathbb{L}_1$ is a distributive lattice.

$\Rightarrow$ Suppose that $\mathcal{M}_I$ is closed under meet-convolution, while $\mathbb{L}_1$ is not distributive. Due to the $\mathbb{M}_3$–$\mathbb{N}_5$ theorem, $\mathbb{L}_1$ has a sublattice that is isomorphic to $\mathbb{M}_3$ or to $\mathbb{N}_5$. We distinguish two cases.

(a) Case 1: $\mathbb{L}_1$ has a sublattice isomorphic to $\mathbb{M}_3$. We refer to the elements of this sublattice as in Fig. 4.4(a). We consider the functions $f, g \in \mathcal{M}_D$ defined as:

$$f(x) = \begin{cases} 1 & \text{, if } x \geq x_2\,, \\ 0 & \text{, otherwise}\,; \end{cases} \qquad g(x) = \begin{cases} 1 & \text{, if } x \geq x_3\,, \\ 0 & \text{, otherwise}\,. \end{cases}$$

It holds that $(f \sqcap g)(x_1) \geq f(x_2) \wedge g(x_3) = 1$. Moreover,

$$
\begin{aligned}
(f \sqcap g)(x_4) &= \bigvee_{u \wedge v = x_4} f(u) \wedge g(v) \\
&= \left(\bigvee_{\substack{u \wedge v = x_4 \\ u \not\geq x_2 \text{ or } v \not\geq x_3}} f(u) \wedge g(v) \right) \vee \left(\bigvee_{\substack{u \wedge v = x_4 \\ u \geq x_2 \text{ and } v \geq x_3}} f(u) \wedge g(v) \right) \\
&= 0 \vee \left(\bigvee_{\substack{u \wedge v = x_4 \\ u \geq x_2 \text{ and } v \geq x_3}} 1 \right),
\end{aligned}
$$

which equals 1 unless the set

$$U = \{(u, v) \in {L_1}^2 \mid u \wedge v = x_4 \,, u \geq x_2 \text{ and } v \geq x_3\}$$

is empty.

For any $u \in L_1$ such that $u \wedge v = x_4$ and $u \geq x_2$, it follows that $u \geq x_2 \vee x_4 = x_5$. Analogously, for any $v \in L_1$ such that $u \wedge v = x_4$ and $v \geq x_3$, it follows that $v \geq x_3 \vee x_4 = x_5$. It then follows that $x_4 = u \wedge v \geq x_5 \wedge x_5 = x_5$, a contradiction. Consequently, $U = \emptyset$ and $(f \sqcap g)(x_4) = 0$. We conclude that $(f \sqcap g)(x_4) = 0 < 1 = (f \sqcap g)(x_1)$, and, hence, $f \sqcap g \notin \mathcal{M}_I$.

(b) Case 2: $\mathbb{L}_1$ has a sublattice isomorphic to $\mathbb{N}_5$. We refer to the elements of this sublattice as in Fig. 4.4(b). We consider the functions $f, g \in \mathcal{M}_D$ defined as:

$$
f(x) = \begin{cases} 1 & , \text{ if } x \geq x_3 \,, \\ 0 & , \text{ otherwise} \,; \end{cases} \qquad g(x) = \begin{cases} 1 & , \text{ if } x \geq x_4 \,, \\ 0 & , \text{ otherwise} \,. \end{cases}
$$

It holds that $(f \sqcap g)(x_1) \geq f(x_3) \wedge g(x_4) = 1$. Moreover,

$$
\begin{aligned}
(f \sqcap g)(x_2) &= \bigvee_{u \wedge v = x_2} f(u) \wedge g(v) \\
&= \left(\bigvee_{\substack{u \wedge v = x_2 \\ u \not\geq x_3 \text{ or } v \not\geq x_4}} f(u) \wedge g(v) \right) \vee \left(\bigvee_{\substack{u \wedge v = x_2 \\ u \geq x_3 \text{ and } v \geq x_4}} f(u) \wedge g(v) \right) \\
&= 0 \vee \left(\bigvee_{\substack{u \wedge v = x_2 \\ u \geq x_3 \text{ and } v \geq x_4}} 1 \right),
\end{aligned}
$$

which equals 1 unless the set

$$U = \{(u, v) \in {L_1}^2 \mid u \wedge v = x_2 \,, u \geq x_3 \text{ and } v \geq x_4\}$$

is empty.

For any $u \in L_1$ such that $u \wedge v = x_2$ and $u \geq x_3$, it follows that $u \geq x_2 \vee x_3 = x_3$. Analogously, for any $v \in L_1$ such that $u \wedge v = x_2$ and $v \geq x_4$, it follows that $v \geq x_4 \vee x_2 = x_5$. It then follows that $x_2 = u \wedge v \geq x_3 \wedge x_5 = x_3$, a contradiction. Consequently, $U = \emptyset$ and $(f \sqcap g)(x_2) = 0$. We conclude that $(f \sqcap g)(x_2) = 0 < 1 = (f \sqcap g)(x_1)$, and, hence, $f \sqcap g \notin \mathcal{M}_I$.

$\Leftarrow$ Let $\mathbb{L}_1$ be a distributive lattice and $f, g \in \mathcal{M}_I$. For any $x_1, x_2 \in L_1$ such that $x_1 \leq x_2$ and for any couple (u_1, v_1) such that $u_1 \wedge v_1 = x_1$, it holds that $u_1 \leq u_1 \vee x_2$ and $v_1 \leq v_1 \vee x_2$. Hence, since $f, g \in \mathcal{M}_I$, it holds that

$$\begin{aligned}(f \sqcap g)(x_1) &= \bigvee_{u_1 \wedge v_1 = x_1} f(u_1) \wedge g(v_1) \\ &\leq \bigvee_{u_1 \wedge v_1 = x_1} f(u_1 \vee x_2) \wedge g(v_1 \vee x_2)\,.\end{aligned}$$

Further, since $\mathbb{L}_1$ is distributive, it follows that $(u_1 \vee x_2) \wedge (v_1 \vee x_2) = (u_1 \wedge v_1) \vee x_2 = x_1 \vee x_2 = x_2$. Hence, we find that

$$\begin{aligned}(f \sqcap g)(x_1) &\leq \bigvee_{u_1 \wedge v_1 = x_1} f(u_1 \vee x_2) \wedge g(v_1 \vee x_2) \\ &\leq \bigvee_{u \wedge v = x_2} f(u) \wedge g(v) \\ &= (f \sqcap g)(x_2)\,.\end{aligned}$$

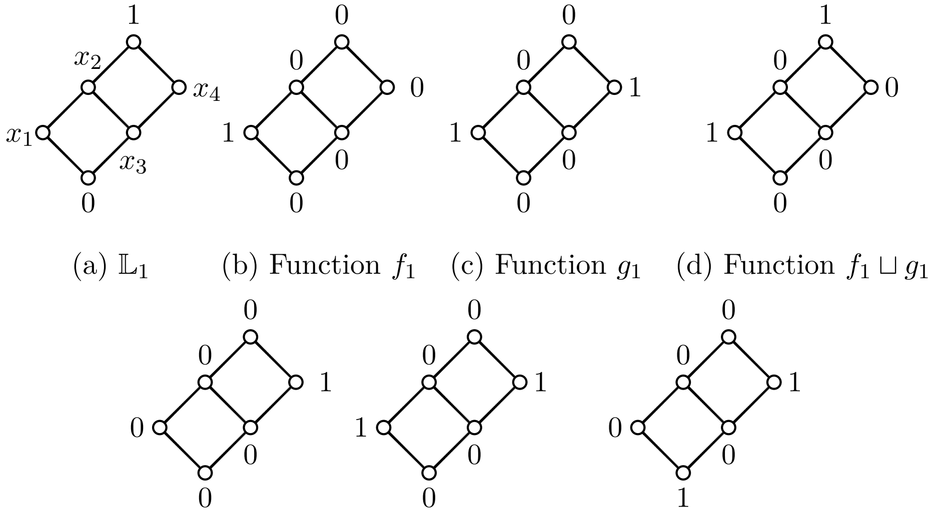
□

The only set of which we have not yet studied the closedness is $\mathcal{C}$. As we will show in the following example, $\mathcal{C}$ is not closed under the convolution operations.

Example 4.4. Let $\mathbb{L}_1$ be the distributive lattice whose Hasse diagram is depicted in Fig. 4.5(a) and $\mathbb{L}_2 = \mathbb{C}_2$.

(i) Consider the functions $f_1, g_1 \in \mathcal{C}$ depicted in Figs. 4.5(b)–(c). The join-convolution $f_1 \sqcup g_1$ is depicted in Fig. 4.5(d). One easily verifies that $x_1 \leq x_2 \leq 1$, while $(f_1 \sqcup g_1)(x_1) \wedge (f_1 \sqcup g_1)(1) = 1 > 0 = (f_1 \sqcup g_1)(x_2)$. Hence, $f_1 \sqcup g_1 \notin \mathcal{C}$.

(ii) Consider the functions $f_2, g_2 \in \mathcal{C}$ depicted in Figs. 4.5(e)–(f). The join-convolution $f_2 \sqcap g_2$ is depicted in Fig. 4.5(g). One easily verifies that $0 \leq x_3 \leq x_4$, while $(f_2 \sqcup g_2)(0) \wedge (f_2 \sqcup g_2)(x_4) = 1 > 0 = (f_2 \sqcup g_2)(x_3)$. Hence, $f_2 \sqcup g_2 \notin \mathcal{C}$.



(a) $\mathbb{L}_1$ (b) Function f_1 (c) Function g_1 (d) Function $f_1 \sqcup g_1$

(e) Function f_2 (f) Function g_2 (g) Function $f_2 \sqcap g_2$

Figure 4.5: Graphical representation of the functions in Example 4.4: (a) Hasse diagram of the lattice $\mathbb{L}_1$, (b) the function f_1, (c) the function g_1, (d) the join-convolution $f_1 \sqcup g_1$, (e) the function f_2, (f) the function g_2, and (g) the meet-convolution $f_2 \sqcap g_2$.

Note that due to Theorem 4.3 the absorption laws surely do not hold outside the subset of lattice functions $\mathcal{I} \cap \mathcal{C}$. Hence, the closedness of the set $\mathcal{C}$ (under the convolution operations) and the closedness of the sets $\mathcal{I}_{\sqcap}$ (under join-convolution) and $\mathcal{I}_{\sqcup}$ (under meet-convolution) are crucial. Moreover, in both Examples 4.3 and 4.4 the considered lattice $\mathbb{L}_1$ is distributive. Hence, the sets are not closed under the additional assumption of $\mathbb{L}_1$ being distributive. However, one easily verifies that the functions f_1, g_1, f_2 and g_2 in Examples 4.3(i)-(ii) are not convex, while the functions g_1 and g_2 in Examples 4.4(i)-(ii) are not idempotent. We could therefore investigate the closedness of $\mathcal{I} \cap \mathcal{C}$. We will prove the closedness of this subset under the additional assumption that $\mathbb{L}_1$ is a distributive lattice; the latter will turn out to be a necessary and sufficient condition once again.

Theorem 4.4. The following statements hold:

(i) the set $\mathcal{I} \cap \mathcal{C}$ is closed under join-convolution if and only if $\mathbb{L}_1$ is a distributive lattice;

(ii) the set $\mathcal{I} \cap \mathcal{C}$ is closed under meet-convolution if and only if $\mathbb{L}_1$ is a distributive lattice.

Proof. We first prove that $\mathcal{I} \cap \mathcal{C}$ is closed under join-convolution if and only if $\mathbb{L}_1$ is a distributive lattice.

$\Rightarrow$ Suppose that $\mathcal{I} \cap \mathcal{C}$ is closed under join-convolution, while $\mathbb{L}_1$ is not distributive. Due to the $\mathbb{M}_3$–$\mathbb{N}_5$ theorem, $\mathbb{L}_1$ has a sublattice that is isomorphic to $\mathbb{M}_3$ or to $\mathbb{N}_5$. We distinguish two cases.

(a) Case 1: $\mathbb{L}_1$ has a sublattice isomorphic to $\mathbb{M}_3$. We refer to the elements of this sublattice as in Fig. 4.4(a). We consider the functions $f, g \in \mathcal{I} \cap \mathcal{C}$ defined as:

$$f(x) = \begin{cases} 1 & , \text{ if } x \in \{x_1, x_2\}\,, \\ 0 & , \text{ otherwise}\,; \end{cases} \qquad g(x) = \begin{cases} 1 & , \text{ if } x \in \{x_1, x_3\}\,, \\ 0 & , \text{ otherwise}\,. \end{cases}$$

It holds that $(f \sqcup g)(x) = 0$ for any $x \in L_1$ unless $x \in \{x_1, x_2, x_3, x_5\}$, where $f \sqcup g$ takes the value 1. Since $x_1 \leq x_4 \leq x_5$ and $(f \sqcup g)(x_4) = 0 < 1 = (f \sqcup g)(x_1) \wedge (f \sqcup g)(x_5)$, we conclude that $f \sqcup g \notin \mathcal{C}$, a contradiction.

(b) Case 2: $\mathbb{L}_1$ has a sublattice isomorphic to $\mathbb{N}_5$. We refer to the elements of this sublattice as in Fig. 4.4(b). We consider the functions $f, g \in \mathcal{I} \cap \mathcal{C}$ defined as:

$$f(x) = \begin{cases} 1 & , \text{ if } x = x_2\,, \\ 0 & , \text{ otherwise}\,; \end{cases} \qquad g(x) = \begin{cases} 1 & , \text{ if } x \in \{x_1, x_4\}\,, \\ 0 & , \text{ otherwise}\,. \end{cases}$$

It holds that $(f \sqcup g)(x) = 0$ for any $x \in L_1$ unless $x \in \{x_2, x_5\}$, where $f \sqcup g$ takes the value 1. Since $x_2 \leq x_3 \leq x_5$ and $(f \sqcup g)(x_3) = 0 < 1 = (f \sqcup g)(x_2) \wedge (f \sqcup g)(x_5)$, we conclude that $f \sqcup g \notin \mathcal{C}$, a contradiction.

$\Leftarrow$ Let $\mathbb{L}_1$ be a distributive lattice and $f, g \in \mathcal{I} \cap \mathcal{C}$. Since $\mathcal{I}_\sqcup$ is closed under join-convolution, it holds that $f \sqcup g \in \mathcal{I}_\sqcup$ and we only need to prove that $f \sqcup g \in \mathcal{I}_\sqcap \cap \mathcal{C}$.

First, we prove that $f \sqcup g \in \mathcal{I}_\sqcap$. For any $x_1, x_2 \in L_1$, it holds that

$$\begin{aligned}(f \sqcup g)(x_1) \wedge (f \sqcup g)(x_2) &= \left(\bigvee_{u_1 \vee v_1 = x_1} f(u_1) \wedge g(v_1) \right) \wedge \left(\bigvee_{u_2 \vee v_2 = x_2} f(u_2) \wedge g(v_2) \right) \\ &= \bigvee_{\substack{u_1 \vee v_1 = x_1 \\ u_2 \vee v_2 = x_2}} f(u_1) \wedge f(u_2) \wedge g(v_1) \wedge g(v_2)\,.\end{aligned}$$

Since $f \in \mathcal{I}$, it holds that $f(u_1) \wedge f(u_2) \leq f(u_1 \vee u_2)$ and $f(u_1) \wedge f(u_2) \leq f(u_1 \wedge u_2)$. Hence, $f(u_1) \wedge f(u_2) \leq f(u_1 \wedge u_2) \wedge f(u_1 \vee u_2)$. Similarly, since $g \in \mathcal{I}$, it holds that $g(v_1) \wedge g(v_2) \leq g(v_1 \wedge v_2) \wedge g(v_1 \vee v_2)$. This leads to

$$\begin{aligned}(f \sqcup g)(x_1) \wedge (f \sqcup g)(x_2) &= \bigvee_{\substack{u_1 \vee v_1 = x_1 \\ u_2 \vee v_2 = x_2}} f(u_1) \wedge f(u_2) \wedge g(v_1) \wedge g(v_2) \\ &\leq \bigvee_{\substack{u_1 \vee v_1 = x_1 \\ u_2 \vee v_2 = x_2}} f(u_1 \wedge u_2) \wedge f(u_1 \vee u_2) \wedge g(v_1 \wedge v_2) \wedge g(v_1 \vee v_2)\,.\end{aligned}$$

Taking into account that $u_1 \leq u_1 \vee v_1 = x_1$ and $u_2 \leq u_2 \vee v_2 = x_2$, it holds that $u_1 \wedge u_2 \leq x_1 \wedge x_2$. Moreover, since $u_1 \wedge u_2 \leq u_1 \vee u_2$, we find that

$$u_1 \wedge u_2 \leq (x_1 \wedge x_2) \wedge (u_1 \vee u_2) \leq (u_1 \vee u_2)\,.$$

Analogously, it follows that

$$v_1 \wedge v_2 \leq (x_1 \wedge x_2) \wedge (v_1 \vee v_2) \leq (v_1 \vee v_2)\,.$$

Since $f, g \in \mathcal{C}$, it holds that

$$\begin{aligned}(f \sqcup g)(x_1) \wedge (f \sqcup g)(x_2) &\leq \bigvee_{\substack{u_1 \vee v_1 = x_1 \\ u_2 \vee v_2 = x_2}} f(u_1 \wedge u_2) \wedge f(u_1 \vee u_2) \wedge g(v_1 \wedge v_2) \wedge g(v_1 \vee v_2) \\ &\leq \bigvee_{\substack{u_1 \vee v_1 = x_1 \\ u_2 \vee v_2 = x_2}} f((x_1 \wedge x_2) \wedge (u_1 \vee u_2)) \wedge g((x_1 \wedge x_2) \wedge (v_1 \vee v_2))\,.\end{aligned}$$

Further, since $\mathbb{L}_1$ is a distributive lattice, it holds that

$$\begin{aligned}&((x_1 \wedge x_2) \wedge (u_1 \vee u_2)) \vee ((x_1 \wedge x_2) \wedge (v_1 \vee v_2)) \\ &\qquad = (x_1 \wedge x_2) \wedge ((u_1 \vee u_2) \vee (v_1 \vee v_2)) \\ &\qquad = (x_1 \wedge x_2) \wedge ((u_1 \vee v_1) \vee (u_2 \vee v_2)) \\ &\qquad = (x_1 \wedge x_2) \wedge (x_1 \vee x_2) = x_1 \wedge x_2\,.\end{aligned}$$

By denoting $u = (x_1 \wedge x_2) \wedge (u_1 \vee u_2)$ and $v = (x_1 \wedge x_2) \wedge (v_1 \vee v_2)$, it holds that

$$\begin{aligned}(f \sqcup g)(x_1) \wedge (f \sqcup g)(x_2) &\leq \bigvee_{\substack{u_1 \vee v_1 = x_1 \\ u_2 \vee v_2 = x_2}} f((x_1 \wedge x_2) \wedge (u_1 \vee u_2)) \wedge g((x_1 \wedge x_2) \wedge (v_1 \vee v_2)) \\ &\leq \bigvee_{u \vee v = x_1 \wedge x_2} f(u) \wedge g(v) \\ &= (f \sqcup g)(x_1 \wedge x_2)\,.\end{aligned}$$

Consequently, $f \sqcup g \in \mathcal{I}_{\sqcap}$.

Secondly, we prove that $f \sqcup g \in \mathcal{C}$. For any $x_1, x_2, x_3 \in L_1$ such that $x_1 \leq x_2 \leq x_3$, it holds that

$$\begin{aligned}(f \sqcup g)(x_1) \wedge (f \sqcup g)(x_3) &= \left(\bigvee_{u_1 \vee v_1 = x_1} f(u_1) \wedge g(v_1)\right) \wedge \left(\bigvee_{u_3 \vee v_3 = x_3} f(u_3) \wedge g(v_3)\right) \\ &= \bigvee_{\substack{u_1 \vee v_1 = x_1 \\ u_3 \vee v_3 = x_3}} f(u_1) \wedge f(u_3) \wedge g(v_1) \wedge g(v_3)\,.\end{aligned}$$

Analogously to the case $\mathcal{I}_\sqcap$, since $f, g \in \mathcal{I}$, it holds that

$$
\begin{aligned}
(f \sqcup g)(x_1) \wedge (f \sqcup g)(x_3) &= \bigvee_{\substack{u_1 \vee v_1 = x_1 \\ u_3 \vee v_3 = x_3}} f(u_1) \wedge f(u_3) \wedge g(v_1) \wedge g(v_3) \\
&\leq \bigvee_{\substack{u_1 \vee v_1 = x_1 \\ u_3 \vee v_3 = x_3}} f(u_1 \wedge u_3) \wedge f(u_1 \vee u_3) \wedge g(v_1 \wedge v_3) \wedge g(v_1 \vee v_3)\,.
\end{aligned}
$$

Taking into account that $u_1 \wedge u_3 \leq u_1 \leq u_1 \vee v_1 = x_1 \leq x_2$ and $u_1 \wedge u_3 \leq u_1 \vee u_3$, it holds that

$$
u_1 \wedge u_3 \leq x_2 \wedge (u_1 \vee u_3) \leq u_1 \vee u_3\,.
$$

Analogously, it follows that

$$
v_1 \wedge v_3 \leq x_2 \wedge (v_1 \vee v_3) \leq v_1 \vee v_3\,.
$$

Since $f, g \in \mathcal{C}$, it holds that

$$
\begin{aligned}
(f \sqcup g)(x_1) \wedge (f \sqcup g)(x_3) &\leq \bigvee_{\substack{u_1 \vee v_1 = x_1 \\ u_3 \vee v_3 = x_3}} f(u_1 \wedge u_3) \wedge f(u_1 \vee u_3) \wedge g(v_1 \wedge v_3) \wedge g(v_1 \vee v_3) \\
&\leq \bigvee_{\substack{u_1 \vee v_1 = x_1 \\ u_3 \vee v_3 = x_3}} f(x_2 \wedge (u_1 \vee u_3)) \wedge g(x_2 \wedge (v_1 \vee v_3))\,.
\end{aligned}
$$

Further, since $\mathbb{L}_1$ is a distributive lattice, it holds that

$$
\begin{aligned}
(x_2 \wedge (u_1 \vee u_3)) \vee (x_2 \wedge (v_1 \vee v_3)) &= x_2 \wedge ((u_1 \vee u_3) \vee (v_1 \vee v_3)) \\
&= x_2 \wedge ((u_1 \vee v_1) \vee (u_3 \vee v_3)) \\
&= x_2 \wedge (x_1 \vee x_3) = x_2 \wedge x_3 = x_2\,.
\end{aligned}
$$

By denoting $u_2 = x_2 \wedge (u_1 \vee u_3)$ and $v_2 = x_2 \wedge (v_1 \vee v_3)$, it holds that

$$
\begin{aligned}
(f \sqcup g)(x_1) \wedge (f \sqcup g)(x_3) &\leq \bigvee_{\substack{u_1 \vee v_1 = x_1 \\ u_3 \vee v_3 = x_3}} f(x_2 \wedge (u_1 \vee u_3)) \wedge g(x_2 \wedge (v_1 \vee v_3)) \\
&\leq \bigvee_{u_2 \vee v_2 = x_2} f(u_2) \wedge g(v_2) \\
&= (f \sqcup g)(x_2)\,.
\end{aligned}
$$

Consequently, $f \sqcup g \in \mathcal{C}$.

Dually, we now prove that $\mathcal{I} \cap \mathcal{C}$ is closed under meet-convolution if and only if $\mathbb{L}_1$ is a distributive lattice.

$\Rightarrow$ Suppose that $\mathcal{I} \cap \mathcal{C}$ is closed under meet-convolution, while $\mathbb{L}_1$ is not distributive. Due to the $\mathbb{M}_3$–$\mathbb{N}_5$ theorem, $\mathbb{L}_1$ has a sublattice that is isomorphic to $\mathbb{M}_3$ or to $\mathbb{N}_5$. We distinguish two cases.

(a) Case 1: $\mathbb{L}_1$ has a sublattice isomorphic to $\mathbb{M}_3$. We refer to the elements of this sublattice as in Fig. 4.4(a). We consider the functions $f, g \in \mathcal{I} \cap \mathcal{C}$ defined as:

$$f(x) = \begin{cases} 1 & , \text{ if } x \in \{x_2, x_5\}\,, \\ 0 & , \text{ otherwise}\,; \end{cases} \qquad g(x) = \begin{cases} 1 & , \text{ if } x \in \{x_3, x_5\}\,, \\ 0 & , \text{ otherwise}\,. \end{cases}$$

It holds that $(f \sqcap g)(x) = 0$ for any $x \in L_1$ unless $x \in \{x_1, x_2, x_3, x_5\}$, where $f \sqcap g$ takes the value 1. Since $x_1 \leq x_4 \leq x_5$ and $(f \sqcup g)(x_4) = 0 < 1 = (f \sqcup g)(x_1) \wedge (f \sqcup g)(x_5)$, we conclude that $f \sqcup g \notin \mathcal{C}$, a contradiction.

(b) Case 2: $\mathbb{L}_1$ has a sublattice isomorphic to $\mathbb{N}_5$. We refer to the elements of this sublattice as in Fig. 4.4(b). We consider the functions $f, g \in \mathcal{I} \cap \mathcal{C}$ defined as:

$$f(x) = \begin{cases} 1 & , \text{ if } x = x_3\,, \\ 0 & , \text{ otherwise}\,; \end{cases} \qquad g(x) = \begin{cases} 1 & , \text{ if } x \in \{x_4, x_5\}\,, \\ 0 & , \text{ otherwise}\,. \end{cases}$$

It holds that $(f \sqcap g)(x) = 0$ for any $x \in L_1$ unless $x \in \{x_1, x_3\}$, where $f \sqcap g$ takes the value 1. Since $x_1 \leq x_2 \leq x_3$ and $(f \sqcap g)(x_2) = 0 < 1 = (f \sqcap g)(x_1) \wedge (f \sqcap g)(x_3)$, we conclude that $f \sqcap g \notin \mathcal{C}$, a contradiction.

$\Leftarrow$ Let $\mathbb{L}_1$ be a distributive lattice and $f, g \in \mathcal{I} \cap \mathcal{C}$. Since $\mathcal{I}_\sqcap$ is closed under meet-convolution, it holds that $f \sqcap g \in \mathcal{I}_\sqcap$ and we only need to show that $f \sqcap g \in \mathcal{I}_\sqcup \cap \mathcal{C}$.

First, we prove that $f \sqcap g \in \mathcal{I}_\sqcup$. For any $x_1, x_2 \in L_1$, it holds that

$$\begin{aligned} (f \sqcap g)(x_1) \wedge (f \sqcap g)(x_2) &= \left(\bigvee_{u_1 \wedge v_1 = x_1} f(u_1) \wedge g(v_1) \right) \wedge \left(\bigvee_{u_2 \wedge v_2 = x_2} f(u_2) \wedge g(v_2) \right) \\ &= \bigvee_{\substack{u_1 \wedge v_1 = x_1 \\ u_2 \wedge v_2 = x_2}} f(u_1) \wedge f(u_2) \wedge g(v_1) \wedge g(v_2)\,. \end{aligned}$$

Analogously to the proof of statement (i), since $f \in \mathcal{I}$, it holds that

$$\begin{aligned} (f \sqcap g)(x_1) \wedge (f \sqcap g)(x_2) &= \bigvee_{\substack{u_1 \wedge v_1 = x_1 \\ u_2 \wedge v_2 = x_2}} f(u_1) \wedge f(u_2) \wedge g(v_1) \wedge g(v_2) \\ &\leq \bigvee_{\substack{u_1 \wedge v_1 = x_1 \\ u_2 \wedge v_2 = x_2}} f(u_1 \wedge u_2) \wedge f(u_1 \vee u_2) \wedge g(v_1 \wedge v_2) \wedge g(v_1 \vee v_2)\,. \end{aligned}$$

Taking into account that $x_1 = u_1 \wedge v_1 \leq u_1$ and $x_2 = u_2 \wedge v_2 \leq u_2$, it holds that $x_1 \vee x_2 \leq u_1 \vee x_2$. Moreover, since $u_1 \wedge u_2 \leq u_1 \vee u_2$, we find that

$$u_1 \wedge u_2 \leq (x_1 \vee x_2) \vee (u_1 \wedge u_2) \leq (u_1 \vee u_2)\,.$$

Analogously, it follows that

$$v_1 \wedge v_2 \leq (x_1 \vee x_2) \vee (v_1 \wedge v_2) \leq (v_1 \vee v_2)\,.$$

Since $f, g \in \mathcal{C}$, it holds that

$$\begin{aligned}(f \sqcap g)(x_1) \wedge (f \sqcap g)(x_2) &\leq \bigvee_{\substack{u_1 \wedge v_1 = x_1 \\ u_2 \wedge v_2 = x_2}} f(u_1 \wedge u_2) \wedge f(u_1 \vee u_2) \wedge g(v_1 \wedge v_2) \wedge g(v_1 \vee v_2) \\ &\leq \bigvee_{\substack{u_1 \wedge v_1 = x_1 \\ u_2 \wedge v_2 = x_2}} f((x_1 \vee x_2) \vee (u_1 \wedge u_2)) \wedge g((x_1 \vee x_2) \vee (v_1 \wedge v_2))\,.\end{aligned}$$

Further, since $\mathbb{L}_1$ is a distributive lattice, it holds that

$$\begin{aligned}&((x_1 \vee x_2) \vee (u_1 \wedge u_2)) \wedge ((x_1 \vee x_2) \vee (v_1 \wedge v_2)) \\ &\qquad = (x_1 \vee x_2) \vee ((u_1 \wedge u_2) \wedge (v_1 \wedge v_2)) \\ &\qquad = (x_1 \vee x_2) \vee ((u_1 \wedge v_1) \wedge (u_2 \wedge v_2)) \\ &\qquad = (x_1 \vee x_2) \vee (x_1 \wedge x_2) = x_1 \vee x_2\,.\end{aligned}$$

By denoting $u = (x_1 \vee x_2) \vee (u_1 \wedge u_2)$ and $v = (x_1 \vee x_2) \vee (v_1 \wedge v_2)$, it holds that

$$\begin{aligned}(f \sqcap g)(x_1) \wedge (f \sqcap g)(x_2) &\leq \bigvee_{\substack{u_1 \wedge v_1 = x_1 \\ u_2 \wedge v_2 = x_2}} f((x_1 \vee x_2) \vee (u_1 \wedge u_2)) \wedge g((x_1 \vee x_2) \vee (v_1 \wedge v_2)) \\ &\leq \bigvee_{u \wedge v = x_1 \vee x_2} f(u) \wedge g(v) \\ &= (f \sqcap g)(x_1 \vee x_2)\,.\end{aligned}$$

Consequently, $f \sqcap g \in \mathcal{I}_{\sqcup}$.

Secondly, we prove that $f \sqcap g \in \mathcal{C}$. For any $x_1, x_2, x_3 \in L_1$ such that $x_1 \leq x_2 \leq x_3$, it holds that

$$\begin{aligned}(f \sqcap g)(x_1) \wedge (f \sqcap g)(x_3) &= \left(\bigvee_{u_1 \wedge v_1 = x_1} f(u_1) \wedge g(v_1)\right) \wedge \left(\bigvee_{u_3 \wedge v_3 = x_3} f(u_3) \wedge g(v_3)\right) \\ &= \bigvee_{\substack{u_1 \wedge v_1 = x_1 \\ u_3 \wedge v_3 = x_3}} f(u_1) \wedge f(u_3) \wedge g(v_1) \wedge g(v_3)\,.\end{aligned}$$

Analogously to the case of statement (i), since $f, g \in \mathcal{I}$, it holds that

$$\begin{aligned}(f \sqcap g)(x_1) \wedge (f \sqcap g)(x_3) &= \bigvee_{\substack{u_1 \wedge v_1 = x_1 \\ u_3 \wedge v_3 = x_3}} f(u_1) \wedge f(u_3) \wedge g(v_1) \wedge g(v_3) \\ &\leq \bigvee_{\substack{u_1 \wedge v_1 = x_1 \\ u_3 \wedge v_3 = x_3}} (f(u_1 \wedge u_3) \wedge f(u_1 \vee u_3)) \wedge (g(v_1 \wedge v_3) \wedge g(v_1 \vee v_3))\,.\end{aligned}$$

Taking into account that $x_2 \leq x_3 = u_3 \wedge v_3 \leq u_3 \leq u_3 \vee u_1$ and $u_1 \wedge u_3 \leq u_1 \vee u_3$, it holds that

$$u_1 \wedge u_3 \leq x_2 \vee (u_1 \wedge u_3) \leq u_1 \vee u_3 \,.$$

Analogously, it follows that

$$v_1 \wedge v_3 \leq x_2 \vee (v_1 \wedge v_3) \ \leq v_1 \vee v_3 \,.$$

Since $f, g \in \mathcal{C}$, it holds that

$$\begin{aligned}(f \sqcap g)(x_1) \wedge (f \sqcap g)(x_3) &\leq \bigvee_{\substack{u_1 \wedge v_1 = x_1 \\ u_3 \wedge v_3 = x_3}} (f(u_1 \wedge u_3) \wedge f(u_1 \vee u_3)) \wedge (g(v_1 \wedge v_3) \wedge g(v_1 \vee v_3)) \\ &\leq \bigvee_{\substack{u_1 \wedge v_1 = x_1 \\ u_3 \wedge v_3 = x_3}} (f(x_2 \vee (u_1 \wedge u_3))) \wedge (g(x_2 \vee (v_1 \wedge v_3))) \,.\end{aligned}$$

Further, since $\mathbb{L}_1$ is a distributive lattice, it holds that

$$\begin{aligned}(x_2 \vee (u_1 \wedge u_3)) \wedge (x_2 \vee (v_1 \wedge v_3)) &= x_2 \vee ((u_1 \wedge u_3) \wedge (v_1 \wedge v_3)) \\ &= x_2 \vee ((u_1 \wedge v_1) \wedge (u_3 \wedge v_3)) \\ &= x_2 \vee (x_1 \wedge x_3) = x_2 \vee x_1 = x_2 \,.\end{aligned}$$

By denoting $u_2 = x_2 \vee (u_1 \wedge u_3)$ and $v_2 = x_2 \vee (v_1 \wedge v_3)$, it holds that

$$\begin{aligned}(f \sqcap g)(x_1) \wedge (f \sqcap g)(x_3) &\leq \bigvee_{\substack{u_1 \wedge v_1 = x_1 \\ u_3 \wedge v_3 = x_3}} f(x_2 \vee (u_1 \wedge u_3)) \wedge g(x_2 \vee (v_1 \wedge v_3)) \\ &\leq \bigvee_{u_2 \wedge v_2 = x_2} f(u_2) \wedge g(v_2) \\ &= (f \sqcap g)(x_2) \,.\end{aligned}$$

Consequently, $f \sqcap g \in \mathcal{C}$.

□

4.4.2 Algebraic structures

Finally, in this subsection we conclude which types of universal algebras the convolution operations constitute on the different subsets of lattice functions considered. The following results are direct consequences of Section 4.3 and Subsection 4.4.1. For the sake of simplicity we recall here the required algebras but a well-structured summary is found in Section 2.1.

A monoid is a set equipped with a binary operation which satisfies the associativity and identity laws [37]. Moreover, if this operation satisfies the commutativity law, then the monoid is called commutative as well. In general, due to Theorem 4.1, the following proposition holds.

Proposition 4.9. The universal algebras $(\mathcal{F}, \sqcup, \mathbf{0}_1)$ and $(\mathcal{F}, \sqcap, \mathbf{1}_1)$ are commutative monoids.

Propositions 4.2, 4.5 and Theorem 4.1 lead to the following observations.

Proposition 4.10. Let $a \in L_2$.

(i) The universal algebra $(\mathcal{N}_a, \sqcup, \mathbf{0}_a)$ is a commutative monoid with absorbing element $\mathbf{1}_a$.

(ii) The universal algebra $(\mathcal{N}_a, \sqcap, \mathbf{1}_a)$ is a commutative monoid with absorbing element $\mathbf{0}_a$.

A semilattice is a set equipped with a binary operation that satisfies the idempotency, commutativity and associativity laws. Theorems 4.1 and 4.2 and Proposition 4.6 lead to the following observation.

Proposition 4.11. The universal algebras $(\mathcal{I}_\sqcup, \sqcup)$ and $(\mathcal{I}_\sqcap, \sqcap)$ are semilattices.

In case $\mathbb{L}_1$ is a bounded chain, it holds that $\mathcal{I}_\sqcup = \mathcal{I}_\sqcap = \mathcal{F}$. Hence, if $\mathbb{L}_1$ is a chain, then the algebraic structures $(\mathcal{F}, \sqcup)$ and $(\mathcal{F}, \sqcap)$ are semilattices.

Finally, recall that a bounded lattice is a set equipped with two binary operations which satisfy the commutativity, associativity, absorption and identity laws. Theorems 4.3 and 4.4 and Propositions 4.10 and 4.11 lead to the central result of this paper.

Theorem 4.5. The universal algebra $\mathbb{L} = (\mathcal{N}_a \cap \mathcal{I} \cap \mathcal{C}, \sqcup, \sqcap, \mathbf{0}_a, \mathbf{1}_a)$ (with $a \in L_2$) is a bounded lattice if and only if $\mathbb{L}_1$ is a distributive lattice.

The preceding result justifies the name convolution lattice for the algebraic structure $\mathbb{L} = (\mathcal{N}_a \cap \mathcal{I} \cap \mathcal{C}, \sqcup, \sqcap, \mathbf{0}_a, \mathbf{1}_a)$ (with $a \in L_2$).

Remark 4.4. Note that the preceding theorem expresses that the convolution operations constitute a bounded lattice on the maximal set $\mathcal{N}_a \cap \mathcal{I} \cap \mathcal{C}$ (with $a \in L_2$) if and only if $\mathbb{L}_1$ is a distributive lattice. However, even if $\mathbb{L}_1$ is not distributive, we can still find a smaller set $\mathcal{G} \subset \mathcal{N}_a \cap \mathcal{I} \cap \mathcal{C}$, closed under the convolution operations, such that these operations constitute a bounded lattice on $\mathcal{G}$. For instance, one easily verifies that the sets $\mathcal{S}_a \subset \mathcal{N}_a \cap \mathcal{I} \cap \mathcal{C}$ (with $a \in L_2$) given by

$$\mathcal{S}_a = \{f \in \mathcal{F} \mid (\exists x^* \in L_1)(f(x^*) = a \,\text{and}\, (\forall x \in L_1)(x \neq x^* \Rightarrow f(x) = 0))\},$$

are closed under the convolution operations (whether or not $\mathbb{L}_1$ is distributive). Moreover, since $\mathbf{0}_a, \mathbf{1}_a \in \mathcal{S}_a$, we find that the algebraic structure $\mathbb{L} = (\mathcal{S}_a, \sqcup, \sqcap, \mathbf{0}_a, \mathbf{1}_a)$ (with $a \in L_2$) constitutes a bounded lattice independently of the distributivity of $\mathbb{L}_1$.

4.5 Distributivity laws

The distributivity of $\mathbb{L}_1$ plays a decisive role in the constitution of the bounded lattice $\mathbb{F} = (\mathcal{N}_a \cap \mathcal{I} \cap \mathcal{C}, \sqcup, \sqcap, \mathbf{0}_a, \mathbf{1}_a)$ (with $a \in L_2$). A natural question that arises is whether or not the convolution operators satisfy the distributivity laws. In general, the following inequalities hold.

Proposition 4.12. Let $\mathbb{L}_1$ be a distributive lattice and $f, g, h \in \mathcal{F}$. The following statements hold:

(i) $f \sqcup (g \sqcap h) \leq (f \sqcup g) \sqcap (f \sqcup h)$;

(ii) $f \sqcap (g \sqcup h) \leq (f \sqcap g) \sqcup (f \sqcap h)$.

Proof. We first prove statement (i). For any $x \in L_1$, due to the distributivity of $\mathbb{L}_1$, it holds that

$$\begin{aligned}(f \sqcup (g \sqcap h))(x) &= \bigvee_{u \vee (v \wedge w) = x} f(u) \wedge g(v) \wedge h(w) \\ &= \bigvee_{(u \vee v) \wedge (u \vee w) = x} (f(u) \wedge g(v)) \wedge (f(u) \wedge h(w)) \\ &\leq ((f \sqcup g) \sqcap (f \sqcup h))\,(x)\,.\end{aligned}$$

Dually, we now prove statement (ii). For any $x \in L_1$, due to the distributivity of $\mathbb{L}_1$, it holds that

$$\begin{aligned}(f \sqcap (g \sqcup h))(x) &= \bigvee_{u \wedge (v \vee w) = x} f(u) \wedge g(v) \wedge h(w) \\ &= \bigvee_{(u \wedge v) \vee (u \wedge w) = x} (f(u) \wedge g(v)) \wedge (f(u) \wedge h(w)) \\ &\leq ((f \sqcap g) \sqcup (f \sqcap h))\,(x)\,.\end{aligned}$$

□

In the following example, we show that the inequalities in Proposition 4.12 no longer hold in general when $\mathbb{L}_1$ is not distributive.

Example 4.5. Let $\mathbb{L}_1$ be the non-distributive lattice $\mathbb{N}_5$ and $\mathbb{L}_2 = \mathbb{C}_2$.

(i) Consider the functions $f_1, g_1, h_1 \in \mathcal{F}$ depicted in Figs. 4.6(a)–(c). The corresponding functions $f_1 \sqcup (g_1 \sqcap h_1)$ and $(f_1 \sqcup g_1) \sqcap (f_1 \sqcup h_1)$ are depicted in Figs. 4.6(d)–(e). One easily verifies that neither $f_1 \sqcup (g_1 \sqcap h_1) \leq (f_1 \sqcup g_1) \sqcap (f_1 \sqcup h_1)$ nor $f_1 \sqcup (g_1 \sqcap h_1) \geq (f_1 \sqcup g_1) \sqcap (f_1 \sqcup h_1)$.

(ii) Consider the functions $f_2, g_2, h_2 \in \mathcal{F}$ depicted in Figs. 4.6(f)–(h). The corresponding functions $f_2 \sqcap (g_2 \sqcup h_2)$ and $(f_2 \sqcap g_2) \sqcup (f_2 \sqcap h_2)$ are depicted in Figs. 4.6(i)–(j). One easily verifies that neither $f_2 \sqcap (g_2 \sqcup h_2) \leq (f_2 \sqcap g_2) \sqcup (f_2 \sqcap h_3)$ nor $f_2 \sqcap (g_2 \sqcup h_2) \geq (f_2 \sqcap g_2) \sqcup (f_2 \sqcap h_2)$.

In the following example, we show that the inequality in Proposition 4.12 can be strict.

Example 4.6. Let $\mathbb{L}_1 = \mathbb{M}_2$ and $\mathbb{L}_2 = [\![0, 1]\!]$.

(i) Consider the function $f_1 \notin \mathcal{I}$ depicted in Fig. 4.7(a) and the functions $g_1, h_1 \in \mathcal{F}$ depicted in Figs. 4.7(b)–(c). The corresponding functions $f_1 \sqcup (g_1 \sqcap h_1)$ and $(f_1 \sqcup g_1) \sqcap (f_1 \sqcup h_1)$ are depicted in Figs. 4.7(d)–(e). One easily verifies that $f_1 \sqcup (g_1 \sqcap h_1) < (f_1 \sqcup g_1) \sqcap (f_1 \sqcup h_1)$.

(ii) Consider the function $f_2 \notin \mathcal{I}$ depicted in Fig. 4.7(f) and the functions $g_2, h_2 \in \mathcal{F}$ depicted in Figs. 4.7(g)–(h). The corresponding functions $f_2 \sqcap (g_2 \sqcup h_2)$ and $(f_2 \sqcap g_2) \sqcup (f_2 \sqcap h_2)$ are depicted in Figs. 4.7(i)–(j). One easily verifies that $f_2 \sqcap (g_2 \sqcup h_2) < (f_2 \sqcap g_2) \sqcup (f_2 \sqcap h_2)$.

In the following theorem, we show that the inequalities in Proposition 4.12 turn into equalities when restricting to the set of functions that are idempotent and convex.

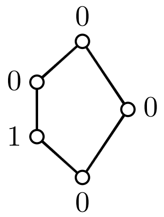
Theorem 4.6. Let $\mathbb{L}_1$ be a distributive lattice. If $f \in \mathcal{I} \cap \mathcal{C}$, then the following statements hold:

(i) $f \sqcup (g \sqcap h) = (f \sqcup g) \sqcap (f \sqcup h)$, for any $g, h \in \mathcal{F}$;

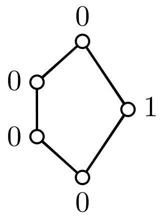
(ii) $f \sqcap (g \sqcup h) = (f \sqcap g) \sqcup (f \sqcap h)$, for any $g, h \in \mathcal{F}$.

Proof. Suppose $f \in \mathcal{I} \cap \mathcal{C}$. We first prove statement (i). Due to Proposition 4.12, it holds that $f \sqcup (g \sqcap h) \leq (f \sqcup g) \sqcap (f \sqcup h)$, so it only remains to prove that $(f \sqcup g) \sqcap (f \sqcup h) \leq f \sqcup (g \sqcap h)$, i.e., we need to verify that, for any $x \in L_1$, it holds that

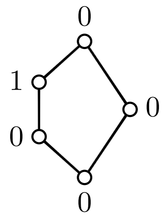
$$\begin{aligned}
((f \sqcup g) \sqcap (f \sqcup h))(x) &= \bigvee_{(u_1 \vee v) \wedge (u_2 \vee w) = x} f(u_1) \wedge g(v) \wedge f(u_2) \wedge h(w) \\
&= \bigvee_{(u_1 \vee v) \wedge (u_2 \vee w) = x} f(u_1) \wedge f(u_2) \wedge g(v) \wedge h(w) \\
&\leq \bigvee_{u \vee (v \wedge w) = x} f(u) \wedge g(v) \wedge h(w) = (f \sqcup (g \sqcap h))(x) \,.
\end{aligned}$$



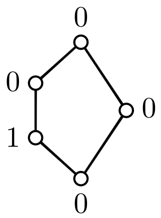
(a) Function f_1



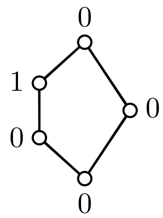
(b) Function g_1



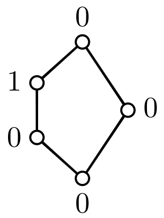
(c) Function h_1



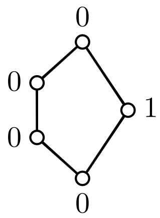
(d) Function $f_1 \sqcup (g_1 \sqcap h_1)$



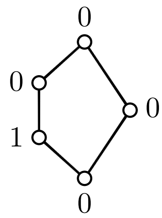
(e) Function $(f_1 \sqcup g_1) \sqcap (f_1 \sqcup h_1)$



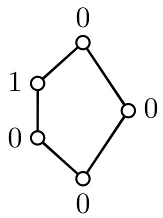
(f) Function f_2



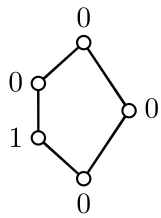
(g) Function g_2



(h) Function h_2

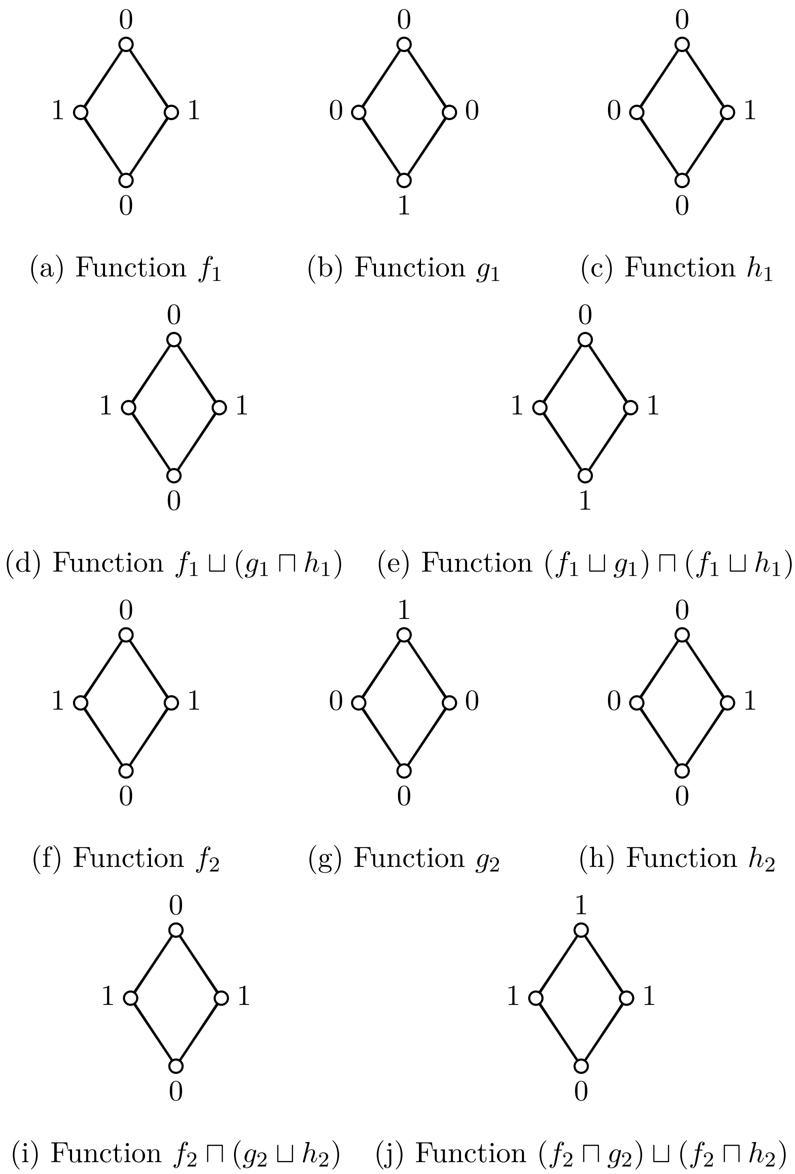


(i) Function $f_2 \sqcap (g_2 \sqcup h_2)$



(j) Function $(f_2 \sqcap g_2) \sqcup (f_2 \sqcap h_2)$

Figure 4.6: Graphical representation of the functions in Example 4.5: (a) the function f_1, (b) the function g_1, (c) the function h_1, (d) the corresponding function $f_1 \sqcup (g_1 \sqcap h_1)$, (e) the corresponding function $(f_1 \sqcup g_1) \sqcap (f_1 \sqcup h_1)$, (f) the function f_2, (g) the function g_2, (h) the function h_2, (i) the corresponding function $f_2 \sqcap (g_2 \sqcup h_2)$, and (h) the corresponding function $(f_2 \sqcap g_2) \sqcup (f_2 \sqcap h_2)$.



(a) Function f_1 (b) Function g_1 (c) Function h_1

(d) Function $f_1 \sqcup (g_1 \sqcap h_1)$ (e) Function $(f_1 \sqcup g_1) \sqcap (f_1 \sqcup h_1)$

(f) Function f_2 (g) Function g_2 (h) Function h_2

(i) Function $f_2 \sqcap (g_2 \sqcup h_2)$ (j) Function $(f_2 \sqcap g_2) \sqcup (f_2 \sqcap h_2)$

Figure 4.7: Graphical representation of the functions in Example 4.6: (a) the function f_1, (b) the function g_1, (c) the function h_1, (d) the corresponding function $f_1 \sqcup (g_1 \sqcap h_1)$, (e) the corresponding function $(f \sqcup g) \sqcap (f \sqcup h)$, (f) the function f_2, (g) the function g_2, (h) the function h_2, (i) the corresponding function $f_2 \sqcap (g_2 \sqcup h_2)$, and (j) the corresponding function $(f_2 \sqcap g_2) \sqcup (f_2 \sqcap h_2)$.

Since $f \in \mathcal{I}$, it holds that $f(u_1) \wedge f(u_2) \leq f(u_1 \vee u_2)$ and $f(u_1) \wedge f(u_2) \leq f(u_1 \wedge u_2)$. Hence, $f(u_1) \wedge f(u_2) \leq f(u_1 \wedge u_2) \wedge f(u_1 \vee u_2)$. This leads to

$$\begin{aligned} ((f \sqcup g) \sqcap (f \sqcup h))(x) &= \bigvee_{(u_1 \vee v) \wedge (u_2 \vee w) = x} f(u_1) \wedge f(u_2) \wedge g(v) \wedge h(w) \\ &\leq \bigvee_{(u_1 \vee v) \wedge (u_2 \vee w) = x} f(u_1 \wedge u_2) \wedge f(u_1 \vee u_2) \wedge g(v) \wedge h(w) . \end{aligned}$$

Taking into account that $u_1 \wedge u_2 \leq (u_1 \vee v) \wedge (u_2 \vee w) = x$, as well as $u_1 \wedge u_2 \leq u_1 \vee u_2$, it follows that

$$u_1 \wedge u_2 \leq x \wedge (u_1 \vee u_2) \leq u_1 \vee u_2 .$$

Since $f \in \mathcal{C}$, it holds that

$$\begin{aligned} ((f \sqcup g) \sqcap (f \sqcup h))(x) &\leq \bigvee_{(u_1 \vee v) \wedge (u_2 \vee w) = x} f(u_1 \wedge u_2) \wedge f(u_1 \vee u_2) \wedge g(v) \wedge h(w) \\ &\leq \bigvee_{(u_1 \vee v) \wedge (u_2 \vee w) = x} f(x \wedge (u_1 \vee u_2)) \wedge g(v) \wedge h(w) . \end{aligned}$$

Taking into account that $(u_1 \vee v) \wedge (u_2 \vee w) = x$, it holds that

$$\begin{aligned} & v \wedge w \leq (v \vee u_1) \wedge (w \vee u_2) = x \\ & x = (v \vee u_1) \wedge (w \vee u_2) \leq v \vee u_1 \\ & x = (v \vee u_1) \wedge (w \vee u_2) \leq w \vee u_2 . \end{aligned} \tag{4.2}$$

On the one hand, since $x \wedge (u_1 \vee u_2) \leq x$ and $v \wedge w \leq x$ (Eq. (4.2)), we find that

$$(x \wedge (u_1 \vee u_2)) \vee (v \wedge w) \leq x . \tag{4.3}$$

On the other hand, since L_1 is a distributive lattice, it follows that

$$\begin{aligned} (x \wedge (u_1 \vee u_2)) \vee (v \wedge w) &= (x \vee (v \wedge w)) \wedge ((u_1 \vee u_2) \vee (v \wedge w)) \\ &= (x \vee (v \wedge w)) \wedge ((u_1 \vee u_2) \vee v) \wedge ((u_1 \vee u_2) \vee w) \\ &= (x \vee (v \wedge w)) \wedge \underbrace{((u_1 \vee v) \vee u_2)}_{(*)} \wedge \underbrace{(u_1 \vee (u_2 \vee w))}_{(**)} \\ &\geq x \wedge x \wedge x = x , \end{aligned} \tag{4.4}$$

where $(*)$ and $(**)$ are greater than or equal to x due to Eq. (4.2).

Due to Eqs. (4.3) and (4.4), it holds that $(x\wedge(u_1\vee u_2))\vee(v\wedge w) = x$. Denoting $u = x\wedge(u_1\vee u_2)$, it follows that

$$((f \sqcup g) \sqcap (f \sqcup h))(x) \leq \bigvee_{(u_1\vee v)\wedge(u_2\vee w)=x} f(x \wedge (u_1 \vee u_2)) \wedge g(v) \wedge h(w)$$
$$\leq \bigvee_{u\vee(v\wedge w)=x} f(u) \wedge g(v) \wedge h(w) = (f \sqcup (g \sqcap h))(x)\,.$$

Dually, we now prove statement (ii). Due to Proposition 4.12, it holds that $f \sqcap (g \sqcup h) \leq (f \sqcap g) \sqcup (f \sqcap h)$, so it only remains to prove that $(f \sqcap g) \sqcup (f \sqcap h) \leq f \sqcap (g \sqcup h)$, i.e., we need to verify that, for any $x \in L_1$, it holds that

$$((f \sqcap g) \sqcup (f \sqcap h))(x) = \bigvee_{(u_1\wedge v)\vee(u_2\wedge w)=x} f(u_1) \wedge g(v) \wedge f(u_2) \wedge h(w)$$
$$= \bigvee_{(u_1\wedge v)\vee(u_2\wedge w)=x} f(u_1) \wedge f(u_2) \wedge g(v) \wedge h(w)\,.$$

Since $f \in \mathcal{I}$, it holds that $f(u_1) \wedge f(u_2) \leq f(u_1 \vee u_2)$ and $f(u_1) \wedge f(u_2) \leq f(u_1 \wedge u_2)$. Hence, $f(u_1) \wedge f(u_2) \leq f(u_1 \wedge u_2) \wedge f(u_1 \vee u_2)$. This leads to

$$((f \sqcap g) \sqcup (f \sqcap h))(x) = \bigvee_{(u_1\wedge v)\vee(u_2\wedge w)=x} f(u_1) \wedge f(u_2) \wedge g(v) \wedge h(w)$$
$$\leq \bigvee_{(u_1\wedge v)\vee(u_2\wedge w)=x} f(u_1 \wedge u_2) \wedge f(u_1 \vee u_2) \wedge g(v) \wedge h(w)\,.$$

Taking into account that $x = (u_1 \wedge v) \vee (u_2 \wedge w) \leq u_1 \vee u_2$, as well as $u_1 \wedge u_2 \leq u_1 \vee u_2$, it follows that

$$u_1 \wedge u_2 \leq x \vee (u_1 \wedge u_2) \leq u_1 \vee u_2\,.$$

Since $f \in \mathcal{C}$, it holds that

$$((f \sqcap g) \sqcup (f \sqcap h))(x) \leq \bigvee_{(u_1\wedge v)\vee(u_2\wedge w)=x} f(u_1 \wedge u_2) \wedge f(u_1 \vee u_2) \wedge g(v) \wedge h(w)$$
$$\leq \bigvee_{(u_1\wedge v)\vee(u_2\wedge w)=x} f(x \vee (u_1 \wedge u_2)) \wedge g(v) \wedge h(w)\,.$$

Taking into account that $(u_1 \wedge v) \vee (u_2 \wedge w) = x$, it holds that

$$\begin{aligned} &x = (v \wedge u_1) \vee (w \wedge u_2) \leq v \vee w\,, \\ &v \wedge u_1 \leq (v \wedge u_1) \vee (w \wedge u_2) = x\,, \\ &w \wedge u_2 \leq (v \wedge u_1) \vee (w \wedge u_2) = x\,. \end{aligned} \tag{4.5}$$

On the one hand, since $x \leq x \vee (u_1 \wedge u_2)$ and $x \leq v \vee w$ (Eq. (4.5)), we find that

$$x \leq (x \vee (u_1 \wedge u_2)) \wedge (v \vee w)\,. \tag{4.6}$$

On the other hand, since $\mathbb{L}_1$ is distributive, it follows that

$$\begin{aligned}(x \vee (u_1 \wedge u_2)) \wedge (v \vee w) &= (x \wedge (v \vee w)) \vee ((u_1 \wedge u_2) \wedge (v \vee w)) \\ &= (x \wedge (v \vee w)) \vee ((u_1 \wedge u_2) \wedge v) \vee ((u_1 \wedge u_2) \wedge w) \\ &= \underbrace{(x \wedge (v \vee w))} \vee \underbrace{((u_1 \wedge v) \wedge u_2)}_{(*)} \vee \underbrace{(u_1 \wedge (u_2 \wedge w))}_{(**)} \\ &\leq x \wedge x \wedge x = x\,,\end{aligned} \tag{4.7}$$

where $(*)$, $(**)$ are smaller than or equal to x due to Eq. (4.5).

Due to Eqs. (4.6) and (4.7), it holds that $(x \vee (u_1 \wedge u_2)) \wedge (v \vee w) = x$. Denoting $u = x \vee (u_1 \wedge u_2)$, it follows that

$$\begin{aligned}((f \sqcap g) \sqcup (f \sqcap h))(x) &\leq \bigvee_{(u_1 \wedge v) \vee (u_2 \wedge w) = x} f(x \vee (u_1 \wedge u_2)) \wedge g(v) \wedge h(w) \\ &\leq \bigvee_{u \wedge (v \vee w) = x} f(u) \wedge g(v) \wedge h(w) = (f \sqcap (g \sqcup h))(x)\,.\end{aligned}$$

□

Corollary 4.5. The universal algebra $\mathbb{L} = (\mathcal{N}_a \cap \mathcal{I} \cap \mathcal{C}, \sqcup, \sqcap, \mathbf{0}_a, \mathbf{1}_a)$ (with $a \in L_2$) is a bounded distributive lattice if and only if $\mathbb{L}_1$ is a distributive lattice.

4.6 Birkhoff systems

One easily verifies that the fulfilment of the absorption laws implies that fulfilment of the Birkhoff equation, i.e., $f \sqcup (f \sqcap g) = f$ and $f \sqcap (f \sqcup g) = f$ implies that $f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g)$. Consequently, a Birkhoff system is a more general universal algebra than a lattice. In this section we study whether or not there exists a bigger subset of lattice functions on which the convolution operations constitute a Birkhoff system.

Proposition 4.13. Let $\mathbb{L}_1$ be a distributive lattice. The equality $f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g)$ holds for any $g \in \mathcal{F}$ if and only if $f \in \mathcal{I}$.

Proof. $\Rightarrow$ Suppose that, for any $g \in \mathcal{F}$, it holds that $f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g)$, while $f \notin \mathcal{I}$.

We distinguish two different cases.

(a) Case 1: $f \notin \mathcal{I}_{\sqcup}$. Let $g = \mathbf{1}_{s_f}$. It holds that $f \sqcap g = f$ and $f \sqcup g = g$. Hence, it holds that

$$f \sqcup (f \sqcap g) = f \sqcup f$$

and

$$f \sqcap (f \sqcup g) = f \sqcup g = f\,.$$

Since $f \sqcup f \neq f$, it follows that $f \sqcup (f \sqcap g) \neq f \sqcap (f \sqcup g)$ $f \sqcup f = f$, a contradiction.

(b) Case 2: $f \notin \mathcal{I}_{\sqcap}$. Let $g = \mathbf{0}_{s_f}$. It holds that $f \sqcup g = f$ and $f \sqcap g = g$. Hence, it holds that

$$f \sqcup (f \sqcap g) = f \sqcup g = f\,,$$

and

$$f \sqcap (f \sqcup g) = f \sqcap f\,.$$

Since $f \neq f \sqcap f$, it follows that $f \sqcup (f \sqcap g) \neq f \sqcap (f \sqcup g)$, a contradiction.

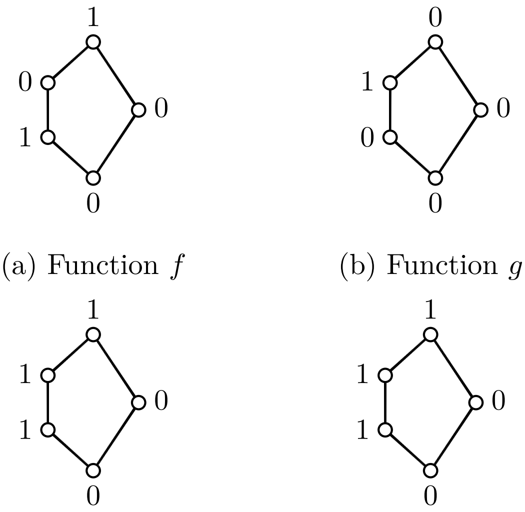
$\Leftarrow$ Let $f \in \mathcal{I}$. Due to the distributivity of $\mathbb{L}_1$ and the fact that $f \in \mathcal{I}_{\sqcup}$, for any $x \in L_1$, it holds that

$$\begin{aligned}
(f \sqcup (f \sqcap g))(x) &= \bigvee_{u_1 \vee (u_2 \wedge v) = x} f(u_1) \wedge f(u_2) \wedge g(v) \\
&= \bigvee_{(u_1 \vee u_2) \wedge (u_1 \vee v) = x} f(u_1) \wedge f(u_2) \wedge g(v) \\
&= \bigvee_{(u_1 \vee u_2) \wedge (u_1 \vee v) = x} f(u_1) \wedge f(u_2) \wedge f(u_1) \wedge g(v) \\
&\leq \bigvee_{(u_1 \vee u_2) \wedge (u_1 \vee v) = x} f(u_1 \vee u_2) \wedge f(u_1) \wedge g(v) \\
&\leq (f \sqcap (f \sqcup g))(x)\,.
\end{aligned}$$

Due to the distributivity of $\mathbb{L}_1$ and the fact that $f \in \mathcal{I}_{\sqcap}$, for any $x \in L_1$, it holds that

$$\begin{aligned}
(f \sqcap (f \sqcup g))(x) &= \bigvee_{u_1 \wedge (u_2 \vee v) = x} f(u_1) \wedge f(u_2) \wedge g(v) \\
&= \bigvee_{(u_1 \wedge u_2) \vee (u_1 \wedge v) = x} f(u_1) \wedge f(u_2) \wedge g(v) \\
&= \bigvee_{(u_1 \wedge u_2) \vee (u_1 \wedge v) = x} f(u_1) \wedge f(u_2) \wedge f(u_1) \wedge g(v) \\
&\leq \bigvee_{(u_1 \wedge u_2) \vee (u_1 \wedge v) = x} f(u_1 \wedge u_2) \wedge f(u_1) \wedge g(v) \\
&\leq (f \sqcup (f \sqcap g))(x)\,.
\end{aligned}$$

Taking into account both inequalities, it holds that $f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g)$. □



(a) Function f (b) Function g

(c) Function $f \sqcup (f \sqcap g)$ (d) Function $f \sqcap (f \sqcup g)$

Figure 4.8: Graphical representation of the lattice $\mathbb{L}_1$ and the functions in Example 4.7: (a) the function f, (b) the function g, (c) the corresponding function $f \sqcup (f \sqcap g)$, and (d) the corresponding function $f \sqcap (f \sqcup g)$

Interestingly, even if $\mathbb{L}_1$ is a non-distributive lattice, we can find functions that satisfy Birkhoff equation as we show in the following example.

Example 4.7. Let $\mathbb{L}_1 = \mathbb{N}_5$ and $\mathbb{L}_2 = [\![0,1]\!]$. Consider the functions $f, g \in \mathcal{F}$ depicted in Figs. 4.8(a)–(b). The corresponding functions $f \sqcup (f \sqcap g)$ and $f \sqcap (f \sqcup g)$ are depicted in Figs. 4.8(c)–(d). One easily verifies that $f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g) \neq f$.

Remark 4.5. Note that at this moment it is not clear whether the restriction to a distributive lattice can be dropped in Proposition 4.13. Indeed, even for the non-distributive lattices $\mathbb{N}_5$ and $\mathbb{M}_3$, the identity $f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g)$ turns out to hold for any $g \in \mathcal{F}$ and $f \in \mathcal{I}$. We leave this as an open problem to the interested reader.

Note that the Birkhoff equation is satisfied on the set $\mathcal{I}$ when $\mathbb{L}_1$ is distributive, while we have shown in Example 4.3(i) that the set of lattice functions $\mathcal{I}_{\sqcap}$ is not closed under the join-convolution. Consequently, even if we restrict to a distributive lattice $\mathbb{L}_1$, the set $\mathcal{I}$ is not closed under the convolution operations. This means that the convolution operations do not generate a Birkhoff system on $\mathcal{I}$. However, as we have mentioned before, if $\mathbb{L}_1$ is a bounded chain, then $\mathcal{F} = \mathcal{I}$ and we can state the following interesting result.

Corollary 4.6. Let $\mathbb{L}_1$ be a chain. Then the universal algebra $\mathcal{F} = (\mathcal{F}, \sqcup, \sqcap)$ is a Birkhoff system.

4.7 Conclusions of Chapter 4

In this chapter, we have studied for which subsets of functions the convolution operations introduced in Definition 3.1 satisfy the algebraic properties of a bounded lattice. However, we have shown that, in general, these sets are not closed under the convolution operations. We have proven that the convolution operations constitute a bounded lattice on the maximal subset $\mathcal{N}_a \cap \mathcal{I} \cap \mathcal{C}$ (for some $a \in L_2$) if and only if $\mathbb{L}_1$ is distributive. Hence, the distributivity of $\mathbb{L}_1$ is of primordial importance. Finally, we have studied the distributivity laws of the bounded convolution lattice and the possibility of the convolution of constituting a Birkhoff system.

In the following chapter we will study how this convolution operations can be used to define the connective disjunction "or" and the conjunction "and" for the proposed logical system.

Chapter 5

Logical systems

5.1 Introduction to the proposed logical system

The main goal of this dissertation is to propose the first steps into the direction of defining a logical system which encompasses several logical systems in the literature. This chapter is devoted to analyze which logical systems are encompassed in our proposal considering the disjunction and conjunction connectives defined using the convolution operations studied in Chapter 4.

We remind the reader that in the proposed logical system the degree of truth of a sentence is a function between two bounded lattices $\mathbb{L}_1$ and $\mathbb{L}_2$, i.e., the degree of truth of a sentence is determined by a lattice function $f \in \mathcal{F}(\mathbb{L}_1, \mathbb{L}_2)$. Considering a logical system $\mathcal{L}$, for our proposal the bounded lattice $\mathbb{L}_1$ is chosen as the lattice of truth values of the logical system $\mathcal{L}$, i.e., a representation of the possible truth values of the logical system $\mathcal{L}$. The lattice $\mathbb{L}_2$ is chosen as a lattice that represents degrees of possibility. Hence, by considering a function $f \in \mathcal{F}(\mathbb{L}_1, \mathbb{L}_2)$, the image $f(x)$ (for any $x \in L_1$) determines the degree of possibility of the considered sentence to have the truth value $x \in L_1$. The motivation for this representation lies in the representation of uncertainty in a flexible way to generate a system that encompasses several logical systems in the literature.

In Chapters 3 and 4 we have studied the suitability of the convolution operations in order to define the disjunction and conjunction connectives for the proposed logical system.

Specifically, in Chapter 4 we have studied the convolutions

$$(f \sqcup g)(x) = \bigvee_{u \vee v = x} f(u) \wedge g(v)$$

$$(f \sqcap g)(x) = \bigvee_{u \wedge v = x} f(u) \wedge g(v)\,.$$

A relevant aspect of most of the logical systems in the literature is that the disjunction and conjunction connectives constitute a bounded lattice. In Chapter 4, we have shown that for the convolution operations, the minimal restriction on the set of functions to constitute a bounded lattice is to consider a closed subset of $\mathcal{G} = \mathcal{N}_a \cap \mathcal{I} \cap \mathcal{C}$ (for some $a \in L_2$). However, $\mathcal{G}$ is not closed under the convolution operations unless the lattice $\mathbb{L}_1$ is distributive.

It is important to mention that for any $a \in L_2$ we have a different set of lattice functions $\mathcal{N}_a$, i.e., we have a different bounded lattice for any $a \in L_2$. An extraordinary set is $\mathcal{N}_0 = \{\underline{\mathbf{0}}\}$, which only contains the constant function $\underline{\mathbf{0}}$. From an algebraic point of view, the set $\mathcal{N}_0$ is of no interest. However, semantically it is of crucial importance. Since the lattice $\mathbb{L}_1$ represents all the possible truth values of the semantic logical system $\mathcal{L}$, the function $\underline{\mathbf{0}}$ determines that none of the truth values is possible, i.e., it determines that the semantic model is not adequate. On the contrary, a function $f \in \mathcal{N}_1$ determines that at least one truth value is possible. The set $\mathcal{N}_1$ is of leading importance in this chapter. We even consider a major restriction. We impose that there exists an element $x \in L_1$ whose image is 1, i.e., we consider the restriction to the set:

$$\mathcal{N}_1{}' = \{f \in \mathcal{F} \mid (\exists x \in L_1)(f(x) = 1)\}$$

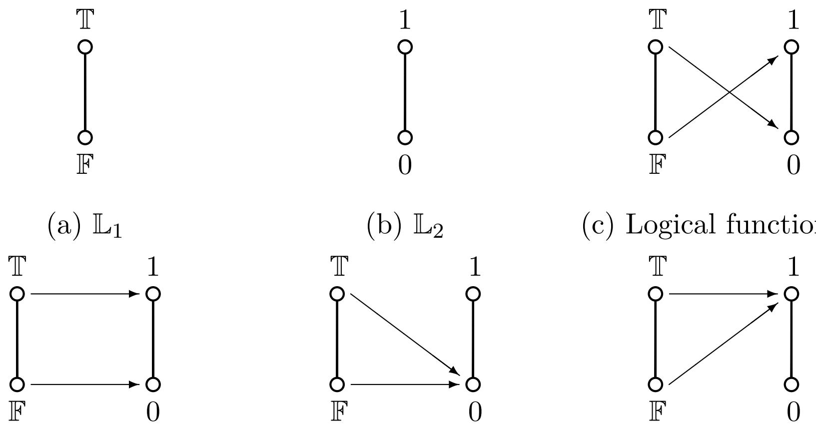
In the context of type-2 fuzzy logic, these functions are sometimes called strongly normal [39, 40]. Also the sets $\mathcal{S}_a \subset \mathcal{N}_a \cap \mathcal{I} \cap \mathcal{C}$ (with $a \in L_2$) introduced in Remark 4.4 and given by

$$\mathcal{S}_a = \{f \in \mathcal{F} \mid (\exists x^* \in L_1)(f(x^*) = a \,\text{and}\, (\forall x \in L_1)(x \neq x^* \Rightarrow f(x) = 0))\}\,,$$

are crucial to encompass several logical systems in the literature. We refer to the functions in the set $\mathcal{S}_1$ as singular functions.

5.2 Semantics of the proposed logical system

This section is devoted to clarify the semantics of the proposed logical system. Particularly, the semantics are illustrated focusing on the case of classical bivalent logic.



(a) $\mathbb{L}_1$ (b) $\mathbb{L}_2$ (c) Logical function f_1

(d) Logical function f_2 (e) Logical function f_3 (f) Logical function f_4

Figure 5.1: Graphical representation of $\mathbb{L}_1$ and $\mathbb{L}_2$ lattices in classical logic and the possible logical functions between them: (a) lattice $\mathbb{L}_1 = \mathbb{C}_2$, (b) lattice $\mathbb{L}_2 = \mathbb{C}_2$, (c) logical function f_1, (d) logical function f_2, (e) logical function f_3, and (f) logical function f_4.

In classical logic, there are only two possible truth values: false ($\mathbb{F}$) and true ($\mathbb{T}$). Hence, the lattice $\mathbb{L}_1$ in our system should be a set with only two elements. Moreover, with our assumption of the "truth scale" the only possibility of $\mathbb{L}_1$ is the chain of two elements ($\mathbb{F}$ and $\mathbb{T}$), i.e., $\mathbb{L}_1 = \{\mathbb{F}, \mathbb{T}\}$ as depicted in Fig. 5.1(a). Classical logic only models precise information, i.e., there is no uncertainty, so we only require $\mathbb{L}_2$ to be the chain of two elements (0 and 1), i.e., $\mathbb{L}_2 = \{0, 1\}$ as depicted in Fig. 5.1(b). Due to the specific nature of $\mathbb{L}_1$ and $\mathbb{L}_2$, there are only four possible functions in the set of functions $\mathcal{F}(\mathbb{L}_1, \mathbb{L}_2)$:

$$f_1(x) = \begin{cases} 1 & , \text{ if } x = \mathbb{F}\,, \\ 0 & , \text{ if } x = \mathbb{T}\,; \end{cases} \qquad f_2(x) = \begin{cases} 1 & , \text{ if } x = \mathbb{T}\,, \\ 0 & , \text{ if } x = \mathbb{F}\,. \end{cases}$$

$$f_3(x) = 0 \quad , \text{ for all } x \in \{\mathbb{F}, \mathbb{T}\}\,,$$
$$f_4(x) = 1 \quad , \text{ for all } x \in \{\mathbb{F}, \mathbb{T}\}\,,$$

as depicted in Figs. 5.1(c)–(f).

One easily verifies that all the functions are idempotent and convex. Moreover, it holds that $\mathcal{N}_0 = \{f_3\}$, $\mathcal{N}_1 = \{f_1, f_2, f_4\}$ and $\mathcal{S}_1 = \{f_1, f_2\}$.

Semantically, the function f_1 is a suitable representation of the classical truth value $\mathbb{F}$ since the logical function f_1 determines that the truth value $\mathbb{F}$ is possible ($f_1(\mathbb{F}) = 1$), while the truth value $\mathbb{T}$ is not possible ($f_1(\mathbb{T}) = 0$). Similarly, the function f_2 is a suitable representation of the classical truth value $\mathbb{T}$. One easily verifies that applying the convolution operations $\sqcup$ and $\sqcap$ to the functions f_1 and f_2 retrieves the usual disjunction and conjunction operations as in Table 2.3.2. Hence, by considering the lattices $\mathbb{L}_1 = \{\mathbb{F}, \mathbb{T}\}$ and $\mathbb{L}_2 = \{0, 1\}$ and the set of lattice functions $\mathcal{S}_1$, classical logic is encompassed in our proposed logical system.

Contrary to the case of f_1 and f_2, neither f_3 nor f_4 are valid representations of truth values in classical logic. On the one hand, as we have explained, $f_3 \in \mathcal{N}_0$ determines that the model is not adequate. On the other hand, f_4 determines that both truth values $\mathbb{F}$ and $\mathbb{T}$ are possible, i.e., it determines indecision ($\mathbb{U}$ in Łukasiewicz logic). One easily verifies that applying the convolution operations $\sqcup$ and $\sqcap$ to the functions f_1, f_2 and f_4 retrieves the usual disjunction and conjunction operations as in Table 2.4. Hence, the 3-valued logic of Łukasiewicz also arises naturally in our proposed logical system by considering the lattices $\mathbb{L}_1 = \{\mathbb{F}, \mathbb{T}\}$ and $\mathbb{L}_2 = \{0, 1\}$ and the set of lattice functions ${\mathcal{N}_1}'$.

From this example we can derive an important consequence of the proposed logical system. To encompass different logical systems in the literature, we will be forced to consider suitable restrictions on the set of lattice functions $\mathcal{F}(\mathbb{L}_1, \mathbb{L}_2)$. For example, with the lattices $\mathbb{L}_1 = \{\mathbb{F}, \mathbb{T}\}$ and $\mathbb{L}_2 = \{0, 1\}$ if we restrict to the set $\mathcal{S}_1$ we retrieve the classical logical system, while when we restrict to the set ${\mathcal{N}_1}'$, we retrieve the 3-valued logic of Łukasiewicz.

5.3 The case of singular functions

The set of functions $\mathcal{S}_a$ (for some $a \in L_2$) is closed under the convolution operations and the algebraic structure $\mathbb{L} = (\mathcal{S}_a, \sqcup, \sqcap, \mathbf{0}_a, \mathbf{1}_a)$ constitutes a bounded lattice independently of the distributivity of $\mathbb{L}_1$ (see Remark 4.4). In this section, we will show that several logical systems can be encompassed if we consider the set of singular functions $\mathcal{S}_1$.

Proposition 5.1. It holds that $\mathcal{S}_1$ and $\mathbb{L}_1$ are order isomorphic.

Proof. Let $f_x \in \mathcal{S}_1$ refer to the function defined as:

$$f_x(y) = \begin{cases} 1 & , \text{ if } y = x\,, \\ 0 & , \text{ otherwise }\,. \end{cases}$$

One easily verifies that the function

$$\psi : \mathcal{S}_1 \to \mathbb{L}_1$$
$$f_x \mapsto x$$

is an order isomorphism. □

As a consequence of the order isormorphism, the following result holds.

Corollary 5.1. Let $f_x, f_y \in \mathcal{S}_1$ and consider the order isomorphism ψ introduced in Proposition 5.1. The following statements hold.

(i) $\psi(f_x \sqcup f_y) = \psi(f_x) \vee \psi(f_y) = x \vee y$;

(ii) $\psi(f_x \sqcap f_y) = \psi(f_x) \wedge \psi(f_y) = x \wedge y$.

Proof. We first prove that $\psi(f_x \sqcup f_y) = \psi(f_x) \vee \psi(f_y) = x \vee y$. Consider the element $z \in L_1$ such that $z = x \vee y$. One easily verifies that $f_x \sqcup f_y = f_z$. Hence, it holds that

$$\psi(f_x \sqcup f_y) = \psi(f_z) = z\,.$$

Further, we find that

$$\psi(f_x) \vee \psi(f_y) = x \vee y = z\,,$$

and, hence, the equality holds.

Dually, we now prove that $\psi(f_x \sqcap f_y) = \psi(f_x) \wedge \psi(f_y) = x \wedge y$. Consider the element $z \in L_1$ such that $z = x \wedge y$. One easily verifies that $f_x \sqcap f_y = f_z$. Hence, it holds that

$$\psi(f_x \sqcap f_y) = \psi(f_z) = z\,.$$

Further, we find that

$$\psi(f_x) \wedge \psi(f_y) = x \wedge y = z\,,$$

and, hence, the equality holds. □

As a consequence of Corollary 5.1, the proposed logical system encompasses all the logical systems $\mathcal{L}$ that allow for a single truth value for every sentence and whose disjunction and conjunction operations are defined by sup and inf, respectively (or, max, and min if $\mathbb{L}_1$ is a chain and the supremum and infimum are attained). This means that with the proposed logical system, at least, the following logical systems in the literature can be encompassed:

- Classical logic;
- 3-valued logic of Łukasiewicz ;
- Belnap logic;
- Fuzzy logic.

It is important to mention that Belnap logic is a logical system where the truth values of the system are not a completely ordered set, i.e., the truth values do not constitute a chain (see Section 2.3.4 for more information). Hence, the choice in the set of lattice functions of $\mathbb{L}_1$ as a bounded lattice instead of as a bounded chain becomes crucial at this point.

We can again derive an important consequence of the proposed logical system. In Section 2.3.3 we have shown that the 3-valued logic of Łukasiewicz can be encompassed with the lattices $\mathbb{L}_1 = \{\mathbb{F}, \mathbb{T}\}$ and $\mathbb{L}_2 = \{0, 1\}$ by considering the set of lattice functions $\mathcal{N}_1{}'$. Similarly, from Corollary 5.1, the 3-valued logic of Łukasiewicz can be also encompassed with the lattices $\mathbb{L}_1 = \{\mathbb{F}, \mathbb{U}, \mathbb{T}\}$ and $\mathbb{L}_2 = \{0, 1\}$ by considering the set of lattice functions $\mathcal{S}_1$. Hence, the representation of the logical system may or may not be unique.

5.4 Retrieval of fuzzy logical systems

5.4.1 On the relationship between logic and set theory

There is a natural relationship between set theory and logic. The main reason is that the fundamental binary relation in set theory "whether or not an object is an element of a set", can be understood as a logical proposition "the object a is an element of the set A". Moreover, the logical connectives can be associated with basic operations in set theory. Specifically, the disjunction and conjunctions connectives are naturally associated with the union and intersection operations, while the negation connective is associated with the complementation of a set.

In classical set theory, the membership of an object to a set is determined in binary terms: the object a is a member of the set A or the object a is not a member of the set A. Hence, classical set theory can only be used for modelling precise information. Fuzzy set theory, as a generalization of set theory, arises as a solution for modelling non-precise or uncertain information. It is based on the assumption that the membership of an object to a set is determined in graded terms, i.e., instead of a binary membership, fuzzy sets allow for a gradual scale of membership degrees. In fuzzy set theory, the membership degree is described with a number in the unit interval $[0, 1]$, where 0 represents that the object is not an element of the set and 1 represents that the object is an element of the set. Mimicking the natural relationship between set theory and logic, fuzzy sets can be associated with a generalization of classical logic called fuzzy logic.

As explained in Section 2.3.5, aiming at solving different problems in the determination of fuzzy sets, several generalizations of these sets have appeared in the literature. One easily verifies that each generalization of fuzzy sets leads to a different model of fuzzy logic. The remaining of this chapter is devoted to analyze several generalizations of fuzzy sets that generate several logical systems that can be encompassed by our logical system. In order to avoid unnecessary reiterations, we only include the definition of the set with its union and intersection instead of the disjunction and conjunction connectives of the generated logical system.

5.4.2 Notion of a fuzzy set

In 1965, Zadeh introduced the notion of a fuzzy sets [81]. Its design was developed in order to generate a logical system that deals with uncertainty and imprecise information in engineering applications. Zadeh's ideas on fuzzy sets were soon applied to different areas such as artificial intelligence, natural language processing, decision making, expert systems, neural networks, control theory, etc.

Definition 5.1. A fuzzy set A on a non-empty universe of discourse X is a mapping $A : X \to [0,1]$, where the value $A(x)$ is referred to as the membership degree of the element x to the fuzzy set A.

We denote by $\mathrm{F}(X)$ the class of fuzzy sets on the universe X.

Proposition 5.2. Let X be a universe of discourse. The universal algebra $(\mathrm{F}(X), \cup, \cap)$, with the union and intersection operations defined for any $A, B \in \mathrm{F}(X)$ and for any $x \in X$ as:

$$(A \cup_{\mathrm{F}} B)(x) = \max(A(x), B(x)) \text{ , and}$$
$$(A \cap_{\mathrm{F}} B)(x) = \min(A(x), B(x)) \text{ .}$$

is a bounded lattice.

Theorem 5.1. Fuzzy logic, associated with the concept of fuzzy sets, can be encompassed in the proposed logical system by considering the lattices $\mathbb{L}_1 = [\![0,1]\!]$ and $\mathbb{L}_2 = \{0,1\}$ and the set of functions $\mathcal{S}_1$.

Proof. Let $A, B \in \mathrm{F}(X)$ and let $\mathbb{L}_1$ and $\mathbb{L}_2$ be the lattices $[\![0,1]\!]$ and $\{0,1\}$, respectively. For any $x \in X$, we consider the lattice functions $f_x^A, f_x^B \in \mathcal{S}_1$ defined as

$$f_x^A(y) = \begin{cases} 1 & , \text{ if } y = A(x)\,, \\ 0 & , \text{ otherwise } ; \end{cases} \qquad f_x^B(y) = \begin{cases} 1 & , \text{ if } y = B(x)\,, \\ 0 & , \text{ otherwise } . \end{cases}$$

We find that

$$(f_x^A \sqcup f_x^B)(y) = \begin{cases} 1 & , \text{ if } y = \max(A(x), B(x))\,, \\ 0 & , \text{ otherwise } ; \end{cases}$$

and

$$(f_x^A \sqcap f_x^B)(y) = \begin{cases} 1 & , \text{ if } y = \min(A(x), B(x))\,, \\ 0 & , \text{ otherwise } ; \end{cases}$$

which are the membership degrees to the fuzzy sets $A \cup_{\mathrm{F}} B$ and $A \cap_{\mathrm{F}} B$, respectively. □

5.4.3 Notion of an interval-valued fuzzy set

In 1975, Sambuc [64] presented the concept of Interval-Valued Fuzzy Set (IVFS) with the name of Φ-fuzzy set. One year later, Grattan-Guinness [36] established a definition of an IV membership function. In that same decade IVFSs appeared in the literature in various guises and it was not until the 1980s, with the work of Gorzalczany and Türksen [27, 31, 32, 33, 71, 72, 73] that the importance of these sets, as well as their name, was definitely established.

Let $L([0,1])$ refer to the set of all closed subintervals of $[0,1]$, i.e.

$$L([0,1]) = \{[\underline{x}, \overline{x}] \mid (\underline{x}, \overline{x}) \in [0,1]^2 \text{ and } \underline{x} \leq \overline{x}\}.$$

Definition 5.2. [64] An IVFS A on X is a mapping $A : X \to L([0,1])$. The membership degree of $x \in X$ to A is $A(x) = [\underline{A}(x), \overline{A}(x)] \in L([0,1])$, where the mappings $\underline{A} : X \to [0,1]$ and $\overline{A} : X \to [0,1]$ correspond to the lower and the upper bound of the membership interval $A(x)$, respectively.

We denote by $\mathrm{F_{IV}(X)}$ the class of IVFSs on the universe X.

Proposition 5.3. Let X be a universe of discourse. The universal algebra $(\mathrm{F_{IV}(X)}, \cup_{\mathrm{IV}}, \cap_{\mathrm{IV}})$, with the union and intersection operations defined, for any $A, B \in \mathrm{F_{IV}(X)}$ and for any $x \in X$, as:

$$A \cup_{\mathrm{IV}} B(x) = [\max(\underline{A}(x), \underline{B}(x)), \max(\overline{A}(x), \overline{B}(x))]\,,$$
$$A \cap_{\mathrm{IV}} B(x) = [\min(\underline{A}(x), \underline{B}(x)), \min(\overline{A}(x), \overline{B}(x))]\,,$$

is a bounded lattice.

Theorem 5.2. Interval-valued fuzzy logic, associated with the concept of interval-valued fuzzy sets, can be encompassed in the proposed logical system by considering the lattices $\mathbb{L}_1 = [\![0,1]\!]$ and $\mathbb{L}_2 = \{0,1\}$ and the set of functions $\mathcal{N}_1{}'$.

Proof. Let $A, B \in \mathrm{F_{IV}(X)}$ and let $\mathbb{L}_1$ and $\mathbb{L}_2$ be the lattices $[\![0,1]\!]$ and $\{0,1\}$, respectively. For any $x \in X$, we consider the lattice functions $f_x^A, f_x^B \in \mathcal{N}_1{}'$ defined as

$$f_i^A(y) = \begin{cases} 1 & \text{, if } y \in [\underline{A}(x), \overline{A}(x)]\,, \\ 0 & \text{, otherwise ;} \end{cases} \qquad f_i^B(x) = \begin{cases} 1 & \text{, if } y \in [\underline{B}(x), \overline{B}(x)]\,, \\ 0 & \text{, otherwise .} \end{cases}$$

We find that

$$(f_x^A \sqcup f_x^B)(y) = \begin{cases} 1 & \text{, if } y \in [\max(\underline{A}(x), \underline{B}(x)), \max(\overline{A}(x), \overline{B}(x))]\,, \\ 0 & \text{, otherwise ;} \end{cases}$$

and

$$(f_x^A \sqcap f_x^B)(y) = \begin{cases} 1 & \text{, if } y \in [\min(\underline{A}(x), \underline{B}(x)), \min(\overline{A}(x), \overline{B}(x))]\,, \\ 0 & \text{, otherwise ,} \end{cases}$$

which are the membership degrees to the fuzzy sets $A \cup_{\mathrm{IV}} B$ and $A \cap_{\mathrm{IV}} B$, respectively.

□

5.4.4 Notion of a set-valued fuzzy set

In 1976, Grattan-Guinness [36] defined Set-Valued Fuzzy Sets (SVFSs) as FS for which membership degrees are expressed as subsets of $[0,1]$. Formally, we have the following definition.

Definition 5.3. An SVFS A on X is a mapping $A : X \to \mathcal{P}(X) \setminus \{\emptyset\}$. The membership degree of $x \in X$ to A is a set $\emptyset \subset A_x \subseteq [0,1]$.

We denote by $\mathrm{F_{SV}(X)}$ the class of all SVFSs on X.

Proposition 5.4. Let X be a universe of discourse. The universal algebra $\mathrm{F_{SV}(X)}, \cup_{\mathrm{SV}}, \cap_{\mathrm{SV}})$, with the union and intersection operations defined, for any $A, B \in \mathrm{F_{SV}(X)}$ and for any $x \in X$, as:

$$A \cup_{\mathrm{SV}} B(x) = A_x \cup B_x \;,$$
$$A \cap_{\mathrm{SV}} B(x) = A_x \cap B_x \;,$$

is a bounded lattice.

In his paper, Grattan-Guinness [36] avoids the use of union and intersection, because as he explains in a short paragraph, the standard union and intersection in set theory is not convex. Specifically, he highlights the drawback that the union and intersection of two intervals may not be an interval with the standard union and intersection in set theory. However, the operations introduced in Proposition 5.4 can be found in the literature in similar studies. For example, in [80].

Another drawback of the operations $\cup_{\mathrm{SV}}$ and $\cap_{\mathrm{SV}}$ is that the union and intersection of FSs, as defined by Zadeh, are not retrieved. Grattan-Guinness did not consider this problem in his work, but a possible solution was provided by Torra [69].

In [69], Torra proposed to express membership degrees as subsets of $[0,1]$, giving birth to the so-called Hesitant Fuzzy Sets (HFSs), which he defined as "a function that when applied to X returns a subset of $[0,1]$" [69]. Clearly, this definition turns out to be exactly the same as SVFSs. However, in [69], and contrary to what Grattan-Guinness did, Torra proposes a definition of union and intersection for HFSs that extends those by Zadeh. We denote by $\mathrm{F_H(X)}$ the class of HFSs on the universe X.

Definition 5.4. Let X be a universe of discourse. The union and intersection operations can be defined, for any $A, B \in \mathrm{F_H(X)}$ and for any $x \in X$, as:

$$\begin{aligned} A \cup_{\mathrm{H}} B(x) =& \{t \in A_x \cup B_x \mid t \geq \max(\inf A_x, \inf B_x)\} \text{ , and} \\ A \cap_{\mathrm{H}} B(x) =& \{t \in A_x \cup B_x \mid t \leq \min(\sup A_x, \sup B_x)\} \text{ .} \end{aligned}$$

Theorem 5.3. Hesitant (or set-valued) fuzzy logic, associated with the concept of set-valued/hesitant fuzzy sets with the union and intersection operations introduced in Definition 5.4, can be encompassed in the proposed logical system by considering the lattices $\mathbb{L}_1 = [\![0,1]\!]$ and $\mathbb{L}_2 = \{0,1\}$ and the set of functions $\mathcal{N}_1{}'$.

Proof. Let $A, B \in \mathrm{F_H(X)}$ and let $\mathbb{L}_1$ and $\mathbb{L}_2$ be the lattices $= [\![0,1]\!]$ and $\mathbb{C}_2 = \{0,1\}$, respectively. For any $x \in X$ we consider the lattice functions $f_x^A, f_x^B \in \mathcal{N}_1{}'$ defined as

$$f_x^A(y) = \begin{cases} 1 & \text{, if } y \in A_x \text{ ,} \\ 0 & \text{, otherwise ;} \end{cases} \qquad f_x^B(x) = \begin{cases} 1 & \text{, if } y \in B_x \text{ ,} \\ 0 & \text{, otherwise .} \end{cases}$$

We find that

$$(f_x^A \sqcup f_x^B)(y) = \begin{cases} 1 & \text{, if } y \in \{t \in A_x \cup B_x \mid t \geq \max(\inf A_x, \inf B_x)\} \text{ ,} \\ 0 & \text{, otherwise ;} \end{cases}$$

and

$$(f_x^A \sqcap f_x^B)(y) = \begin{cases} 1 & , \text{ if } y \in \{t \in A_x \cup B_x \mid t \leq \min(\inf A_x, \inf B_x)\}\,, \\ 0 & , \text{ otherwise }\,, \end{cases}$$

which are the membership degrees to the fuzzy sets $A \cup_{\mathrm{H}} B$ and $A \cap_{\mathrm{H}} B$, respectively. $\square$

As far as we know, until now the problem of when the union and intersection of HFS constitute a bounded lattice was unsolved. However, taking into account the preceding theorem we already know that the union and intersection of HFSs constitute a bounded distributive lattice if we restrict to the set $\mathcal{N}_a \cap \mathcal{C}$ (for some $a \in [0,1]$).

5.4.5 Notion of a type-2 fuzzy set

Type-2 Fuzzy Sets (T2FSs) were introduced by Zadeh and, later on, deeply studied in [82] as solution to the problem of determining the membership degrees of the elements to the fuzzy sets. As a generalization of fuzzy sets they are sets for which membership degrees are expressed as a fuzzy set in the universe of discourse $[0,1]$. Formally, we have the following definition.

Definition 5.5. A T2FS A on X is a mapping $A : X \to \mathrm{FS}([0,1])$. The membership degree of $x \in X$ to A is a function $A_x : [0,1] \to [0,1]$.

Note that the membership degrees of T2FSs lie in the set of functions $\mathcal{F}([0,1],[0,1])$. We denote by $\mathrm{F_{T2}(X)}$ the class of T2FSs on the universe X.

Both operations of type-2 fuzzy sets presented in this dissertation are found in [24].

Proposition 5.5. Let X be a universe of discourse. The universal algebra $(\mathrm{F_{T2}(X)}, \cup_{\mathrm{T2}}, \cap_{\mathrm{T2}})$, with the union and intersection operations defined, for any $A, B \in \mathrm{F_{T2}(X)}$ and for any $x \in X$, as:

$$(A \cup_{\mathrm{T2}} B(x))(y) = \max\{A_x(y), B_x(y)\}\,, \qquad \text{for any } y \in [0,1] \text{ and}$$
$$(A \cap_{\mathrm{T2}} B(x))(y) = \min\{A_x(y), B_x(y)\}\,, \qquad \text{for any } y \in [0,1]\,.$$

is a bounded lattice.

Note that these operations can be found also outside the fuzzy community. For example, associated with the notion of probabilistic sets, these operations can be found in [45].

However, with the union and intersection introduced in Definition 5.5 , the classical definitions of union and intersection $\cup_{\mathrm{F}}$ and $\cap_{\mathrm{F}}$ introduced by Zadeh for FSs [21] are not retrieved. Alternative definitions of union and intersection have been provided for T2FSs by extending Zadeh's union and intersection [44, 56, 76].

Proposition 5.6. [56, 76] Let X be a universe of discourse. The universal algebra $(\mathrm{T2FS}(X), \sqcup_{\mathrm{T2}}, \sqcap_{\mathrm{T2}})$, with the union and intersection operations defined as:

$$(A \sqcup_{\mathrm{T2}} B)(x) = \bigvee_{y \vee z = x} A(y) \wedge B(z)$$
$$(A \sqcap_{\mathrm{T2}} B)(x) = \bigvee_{y \wedge z = x} A(y) \wedge B(z)$$

is a bounded lattice if the restriction to the set $\mathcal{N} \cap \mathcal{C}$ is considered.

Since we have based the convolutions operations of our logical system on the convolution operations of T2FSs, the following theorem is a direct consequence.

Theorem 5.4. Type-2 fuzzy logic, associated with the concept of type-2 fuzzy sets with the union and intersection operations introduced in Definition 5.6, can be encompassed in the proposed logical system by considering the lattices $\mathbb{L}_1 = \mathbb{L}_2 = [\![0, 1]\!]$ and the set of functions $\mathcal{F}$.

5.4.6 Notion of an $\mathbb{L}$-fuzzy set

Goguen [30] realized that, other than its lattice structure, there was no relevant reason to use the interval $[0, 1]$ in the definition of fuzzy sets. This observation led him to the introduction of the concept of an $\mathbb{L}$-fuzzy set.

Definition 5.6. Let $\mathbb{L}$ be a complete lattice. An $\mathbb{L}$-fuzzy set A on X is a mapping $A : X \to \mathbb{L}$.

Given a complete lattice $\mathbb{L}$, the class of $\mathbb{L}$-fuzzy sets on the universe X is denoted by $\mathbb{L}\text{-FS}(X)$. Note that, with this notation, if $\mathbb{L} = [\![0, 1]\!]$ (and considering the max and min operations), then $\mathrm{F}(X)$ is retrieved.

The set $\mathrm{F}_{\mathbb{L}}(\mathrm{X})$ can be endowed with a partial order relation, which is induced by the lattice structure of $\mathbb{L}$. Given $A, B \in \mathbb{L}\text{-FS}(X)$, $A \leq_{\mathbb{L}} B$ if the inequality $A(x) \leq_{\mathbb{L}} B(x)$ holds for every $x \in X$, where $\leq_{\mathbb{L}}$ denotes the order relation on the lattice $\mathbb{L}$.

Proposition 5.7. [30] Let X be a universe of discourse. The universal algebra $(\mathrm{F}_{\mathbb{L}}(\mathrm{X}), \cup_{\mathbb{L}}, \cap_{\mathbb{L}})$ with the union and intersection operations defined for any $A, B \in \mathrm{F}_{\mathbb{L}}(\mathrm{X})$ and for any $x \in X$, as:

$$A \cup_{\mathbb{L}} B(x) = A(x) \vee B(x) \text{ , and}$$

$$A \cup_{\mathbb{L}} B(x) = A(x) \wedge B(x) \text{ ,}$$

where $\vee$ and $\wedge$ are the join and meet operations of the lattice $\mathbb{L}$, is a complete lattice.

Theorem 5.5. $\mathbb{L}$-fuzzy logic, associated with the concept of $\mathbb{L}$-fuzzy sets with the union and intersection operations introduced in Proposition 5.7, can be encompassed in the proposed logical system by considering the lattices $\mathbb{L}_1 = \mathbb{L}$ and $\mathbb{L}_2 = \{0, 1\}$ and the set of functions $\mathcal{S}_1$.

Proof. Due to Corollary 5.1, the convolution operations on the set $\mathcal{S}_1$ are equivalent to the join and meet operations on the lattice $\mathbb{L}_1$. Hence, we retrieve the operation of $\mathbb{L}$-fuzzy logic. □

From Proposition 5.7 it is clear that FSs are a special case of $\mathbb{L}$-fuzzy sets for which $\mathbb{L} = [\![0, 1]\!]$ and the maximum and minimum take the role of the join and meet, respectively. Recall that most of the generalizations of fuzzy sets equipped with the union and intersection operations constitute a complete lattice. Hence, all these types of fuzzy sets are encompassed in $\mathbb{L}$-fuzzy sets and they can be retrieved in the proposed logical system. Particularly interesting are the cases of Atanassov intuitionistic fuzzy sets and interval-valued Atanassov intuitionistic fuzzy sets that we recall in the following remark.

Remark 5.1. In 1983, Atanassov presented [2] a new type of fuzzy sets called Atanassov intuitionistic fuzzy sets. This work was written in Bulgarian, while in 1986 he presented these ideas in English [3]. Given a universe of discourse X, an Atanassov intuitionistic fuzzy set associates with each element x of the universe a pair of numbers (μ_x, ν_x) that satisfy the restriction $\mu_x + \nu_x \leq 1$. The value μ_x is usually referred to as the membership degree of the element $x \in X$, while μ_x is usually referred to as the non-membership degree. At a first look, the semantics of the logical system associated with Atanassov intuitionistic fuzzy sets cannot be retrieved easily in our proposed logical system since we need some kind of "falsity-degree". However, one easily verifies that the union and intersection operations on Atanassov intuitionistic fuzzy sets constitute a complete lattice. Hence, Atanassov intuitionistic fuzzy sets are a particular instance of $\mathbb{L}$-fuzzy sets. This means that although the idea of "falsity degree"' is not retrieved in our proposed logical system, Atanassov-intuitionistic fuzzy logic can be encompassed as a particular instance of $\mathbb{L}$-fuzzy set.

Similar considerations can be made about interval-valued Atanassov intuitionistic fuzzy sets [4], which represents the uncertainty as a pair of intervals (one as membership degree and the other one as non-membership degree).

5.5 Conclusions of Chapter 5

In this chapter we have clarified the semantics of the proposed logical system by illustrating it in the case of classical bivalent-logic. We have shown that there is an order-isomorphism between the convolution operations on the set $\mathcal{S}$ and the operations of the lattice $\mathbb{L}_1$. Hence, the convolution operations allow us to retrieve all the logical systems where the truth values constitute a lattice and the disjunction and conjunction connectives are defined by the join and meet operations of the lattice $\mathbb{L}_1$. We have also proven that the sets of functions $\mathcal{N}_1{}'$ and $\mathcal{S}$ allow us to retrieve several of the logical systems in the literature, especially all those that turn up from fuzzy set theory. We highlight two important consequences of the developments of this chapter.

- We have shown that by considering the same lattices $\mathbb{L}_1$ and $\mathbb{L}_2$ but different restrictions on the set of functions we retrieve different logical systems.
- The representation of the logical systems may not be unique in the proposed framework.

Chapter 6

General conclusions and future research

6.1 Conclusions

In this dissertation we have taken the first steps into the direction of defining a new logical system. The main characteristic of the proposed logical system is that the truth values are lattice functions, i.e., functions in the set of functions $\mathcal{F}(\mathbb{L}_1, \mathbb{L}_2)$.

We have defined the disjunction and conjunction connectives using different convolution operations. Specifically, we have analyzed the behaviour of the convolution operations introduced in several contributions on type-2 fuzzy sets when we replace functions from the unit interval into itself with functions from a bounded lattice to a frame, i.e., we have considered the convolution operations defined on the set $\mathcal{F}(\mathbb{L}_1, \mathbb{L}_2)$ instead of on the set $\mathcal{F}([0,1],[0,1])$.

Due to the non-fulfillment of the idempotency laws, we have tried to modify the definition of the convolution operations and we have introduced the strict and the extended convolution operations. We have also shown that these modifications of the convolution operations come along with some other negative aspects. Since none of the definitions of the convolution operations satisfy the absorption laws and due to the negative aspects of the strict and extended convolution operations, we have gone back to the original definitions of convolution operations on type-2 fuzzy logic and we have studied whether or not there exists a subset of functions on which the convolution operations constitute a bounded lattice and/or a Birkhoff system. Specifically, we have analyzed which additional properties of the functions ensure that the algebraic laws of a bounded distributive lattice are satisfied. We have proven that the maximum subset of functions where the considered convolution operations fulfill all the algebraic laws of a bounded distributive lattice is $\mathcal{N}_a \cap \mathcal{I} \cap \mathcal{C}$ (with some $a \in L_2$). However, this subset is closed under the convolution operations if and only if the considered lattice of truth values $\mathbb{L}_1$ is distributive.

Finally, we have proven that the disjunction and conjunctions connectives defined using the convolutions operations allow us to retrieve the disjunction and conjunction connectives of many logical systems introduced in the literature. These logical systems are encompassed in our proposed one by considering different bounded lattices $\mathbb{L}_1$ and $\mathbb{L}_2$ and a suitable subset of functions $\mathcal{G}(\mathbb{L}_1, \mathbb{L}_2)$. The subset of functions $\mathcal{S}_1$ becomes crucial in this direction since it allows to retrieve most of the logical systems that define their disjunction and conjunction connectives, respectively, as the join and meet of the lattice constituted by the truth values. We have specifically focused on fuzzy logical systems. Moreover, we have shown that the representation of some logical systems in our proposed one is not unique.

6.2 Future research

Several open problems and points of further interest of the results of this dissertation are the following:

(a) From the strict and extended convolutions (Chapter 3):

   (i) A study of whether or not there exists a subset of lattice functions where the strict and extended convolution operations constitute a bounded distributive lattice or/and a Birkhoff system.

(b) From the algebraic structures of convolution operations (Chapter 4):

   (i) A deeper study of the convolution operations introduced in Definition 3.1 when $\mathbb{L}_1$ is not distributive. Note that this study will include:

      (a) the search for subsets $\mathcal{G} \subset \mathcal{N}_a \cap \mathcal{I} \cap \mathcal{C}$ that are closed under the convolution operations and such that the operations constitute a bounded lattice on $\mathcal{G}$;

      (b) the study of the Birkhoff equation.

   (ii) The search for specific classes of lattices (larger than chains) such that the set of idempotent functions is closed under the convolution operations.

   (iii) The study of the completeness of convolution lattices as well as the meet-continuity of the convolution operations in order to characterize when convolution lattices constitute a frame.

(c) From logical systems (Chapter 5):

(i) The analysis of the different representations of the logical systems defined in the literature studying the advantages and disadvantages of each representation in our logical system.

(ii) A study of Atanassov intuitionistic fuzzy logic in order to see if the semantics of these logical systems (truth-degree+falsity-degree) can be encompassed in our logical system. Note that since the union and intersection operations of Atanassov intuitionistic fuzzy sets are defined using the maximum and minimum operations, they can be encompassed in our logical system by considering the subset $\mathcal{S}_1$. Similarly, they are a particular case of $\mathbb{L}$-fuzzy logic. However, in this representation the semantics of the model are not taken into account.

Finally, after the disjunction "or" and the conjunction "and" connectives the next step into the direction of defining a logical system will be to study suitable endomorphisms to define the negation connective and the universal algebra that the connectives constitute in the logical system. However, this is not an easy task at all. Indeed, all our first approximations make us believe that the restriction of some kind of symmetry of the lattice $\mathbb{L}_1$ should be imposed.

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Appendix

Brief explanation

The necessary studies and research that lead to the elaboration of a doctoral thesis (dissertation) should give rise to scientific publications. Some of them have a direct relationship with the items covered in the main body of the dissertation. For instance, the main results of, say, a crucial chapter in a dissertation could perhaps generate a whole paper published in some suitable scientific journal. This is the case, for instance, of Chapter 4, which has been published as an article in the journal Fuzzy Sets and Systems [1].

However, during this period, and maybe having points in common but a not so direct relationship with the main results in the dissertation, several other results that have also led to the elaboration of some other scientific publications, have been object of our attention. Perhaps this can be considered as "lateral thinking"[1]. But, in our opinion, this lateral thinking also has undeniable interest. As a matter of fact, crucial facts in Science (as the famous anecdote attributed to Isaac Newton when discovering the Laws of Gravity after being hit by an apple that fell down on his head) can perhaps be considered as lateral thinking, too.

In other words, we may consider that the research work that leads to the elaboration of a dissertation leans on two axis, namely:

(i) a "vertical" one that works in depth, studying the main stream and lines of the problems and objectives considered throughout the body of the dissertation, and actually generating the title of it;

(ii) an "‘horizontal" one that could somewhat be considered as lateral thinking, that gives rise to developments of ideas that could have a partial or even punctual relationship with some concepts analyzed in the main part of the dissertation, but are then developed

[1] E. De Bono: Lateral thinking: a textbook of creativity. Penguin Books, Middlesex, England. 1970.

in a self-contained and independent manner, perhaps going out, so-to-say, of the main stream, or being not so directly related with the title. However, they still deserve interest, and should be considered, for sure, as products coming from the studies made in this process, even if we call them "by-products".

Bearing all this in mind, let us comment now which have been the papers that emanate from the studies made to conclude this dissertation. In this direction, we might opt for a chronological exposition of this papers and their main ideas, or, perhaps better, we may choose a thematic exposition. For several reasons, we have finally chosen the second possibility:

(i) the chronological order is doubtful, since the acceptation and/or publication of some papers could be strongly dependent on the journals and editorials policies: sometimes a paper is finally published long time after its elaboration, so that the chronological order of publication does not coincide with the order in which the ideas were thought and launched;

(ii) the thematic order shows perhaps better that aforementioned ideas of lateral thinking, by commenting how a point or punctual aspect covered in the main body of the dissertation gave rise to an independent development and, hence, to a published paper.

As a first motivation that generated several of our papers we may cite the *search for suitable orderings for remarkable families of fuzzy sets and/or their generalizations*. Taking into account that in the main body of the dissertation we study *lattices*, therefore a particular class of partial orders, it seems interesting to look for different lattice structures, and, once again in particular, total orders that can be defined on suitable generalizations of fuzzy sets. In this direction, in the paper [2] we dealt with the problem of defining total orders on interval-valued Atanassov intuitionisitc fuzzy sets. Then, in [3] we used those results to build intuitionistic OWA aggregation operators in this setting. Also, in the paper [4], we addressed a similar (but technically more complex) study in order to define suitable total orders on n-dimensional fuzzy sets.

Some other papers have several different motivations, instead of just one. Thus, in [5] we studied some kind of functional aggregation that can indeed be applied to type-2 fuzzy sets. Therefore, we may say that this paper has some points in common with the ones previously cited [2-4], since those papers were focused on generalizations of fuzzy sets (type-2 fuzzy sets, in particular). Also, in [5] we used an idea of restricting a domain in order to keep good properties. Namely, versions of intersection and union defining pointwise fusion operators. In a sense, this is parallel to the inspiration of Chapter 4 in the main body of the thesis.

There we were looking for smaller domains after analyzing, in Chapter 3 some drawbacks arising in some convolution operations. Furthermore,in this new paper [5], a clearer idea and motivation was the study of some *functional equations* that appear in a natural way when studying aggregation operators in a general setting. At this stage, we may still ask ourselves about which could be the relationship between functional equations and the main body of the dissertation... In this direction, first we may observe that, on a nonempty set X a bivariate map $F : X \times X \to X$ can actually be understood as a binary "algebraic" operation $*$ just by changing the notation, calling $x * y = F(x, y)$ for every $x, y \in X$. Needless to say, algebraic properties of the operation $*$ can immediately be translated into functional equations that are satisfied by the bivariate map F. To put only an example, if $*$ is associative, F satisfies the associativity functional equation, namely $F(F(x, y), z) = F(x, F(y, z)$ for every $x, y, z \in X$. Lattices can be defined as a particular case of algebraic structures with two basics operations (meet and join) , so that the main properties in the definition of a lattice can immediately be translated into functional equations. Bearing in mind this arguments, we may expect that some ideas related to functional equations arise in a natural way in the work and studies made in the preparation of this dissertation. And this indeed happened to be true. In fact, some ideas related to the consideration of the concept of *pointwise* aggregation operator analyzed in [5] were inspired by definitions given throughout the main body of the dissertation, as Definition 2.9 (pointwise join and pointwise meet of two given functions). Other different ideas related to functional equations gave rise to the publication [6], which was also inspired someway by the consideration of a convolution as a binary operation. Moreover, some aspects of the paper [6] are also partially related to some ideas that appear in the former papers [2,4], since they deal with the genesis of some total order (as well as of other different kinds of orderings) by means of the solutions of a suitable functional equation on two variables. Some relationship with the theory of generalizations of fuzzy sets also appears in the paper [3], so that it could also establish some links with ideas on fuzzy sets arising in the papers [2-4].

Finally, among the areas in which fuzzy set theory has played a relevant role, data similarity modelling is one of the most prominent. A further motivation leans on the comparison of different complex data [7,8], so giving rise to measurements and distances (or entropies) between different sets of data. In this direction, type-2 fuzzy sets [7] and radial data [8] are also considered, so that these new papers can also be related someway with some of the aforementioned ones [2-5].

To conclude, we may mention that some of these papers have already received a high number of references (i.e., they have been cited in other papers arising in the JCR system of impact factors) in a short period of time, since they have quite recently been issued and published. It is worth mentioning the case of paper [2], that has received 10 citations in one year.

Publications

In this section we include a list of the most important publications. Since they have not been covered in the main body of this dissertation, we also include a copy of those publications that can be considered as "lateral thinking".

[1] L. De Miguel, H. Bustince, B. De Baets. Convolution Lattices. *Fuzzy Sets and Systems,* in-press, doi: https://doi.org/10.1016/j.fss.2017.04.017.

[2] L. De Miguel, H. Bustince, J. Fernandez, E. Induráin, A. Kolesárová, R. Mesiar. Construction of admissible linear orders for interval-valued Atanassov intuitionistic fuzzy sets with an application to decision making. *Information Fusion,* 27, 189–197, 2016.

[3] L. De Miguel, H. Bustince, B. Pękala, U. Bentkowska, I. Da Silva, B. Bedregal, R. Mesiar, G. Ochoa. Interval-valued Atanassov intuitionistic OWA aggregations using admissible linear orders and their application to decision making. *IEEE Transactions on Fuzzy Systems,* 24, 6, 1586–1597, 2016.

[4] L. De Miguel, M. Sesma-Sara, M. Elkano, M. Asiain, H. Bustince. An algorithm for group decision making using n-dimensional fuzzy sets, admissible orders and OWA operators. *Information Fusion,* 37, 126–131, 2017.

[5] L. De Miguel, M. J. Campión, E. Induráin, J.C. Candeal, D. Paternain. Pointwise aggregation of maps: Its structural functional equation and some applications to social choice theory. *Fuzzy Sets and Systems*, in-press, doi: https://doi.org/10.1016/j.fss.2016.05.010.

[6] M. J. Campión, L. De Miguel, R. Catalán, E. Induráin, J. Abrísqueta Binary relations coming from solutions of functional equations: orderings and fuzzy subsets. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, Accepted March 2017.

[7] L. De Miguel, H. Santos, M. Sesma-Sara, B. Bedregal, A. Jurio, H. Bustince, H. Hagras. Type-2 Fuzzy Entropy-Sets. *IEEE Transactions on Fuzzy Systems,* in-press, doi:10.1109/TFUZZ.2016.2593497.

[8] C. Marco-Detchart, J. Cerron, L. De Miguel, C. Lopez-Molina, H. Bustince, M. Galar A framework for radial data comparison and its application to fingerprint analysis *Applied Soft Computing,* 46, 246–259, 2016.

Construction of admissible linear orders for interval-valued Atanassov intuitionistic fuzzy sets with an application to decision making

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Abstract

In this work we introduce a method for constructing linear orders between pairs of intervals by using aggregation functions. We adapt this method to the case of interval-valued Atanassov intuitionistic fuzzy sets and we apply these sets and the considered orders to a decision making problem.

Keywords: Interval-valued Atanassov intuitionistic fuzzy set; Interval linear order; Interval-valued Atanassov intuitionistic multi-expert decision making.

1. Introduction

In decision making problems it may happen that, after the exploitation phase, the best alternatives are equally ranked and it is not possible to decide which one is the best. It has been noticed [1] that these troubles often appear when the entries of the considered fuzzy preference matrix are close to 0.5, that is, when the experts have doubts about their preferences of some alternatives over the others. In this situation, the systematic use of extensions of fuzzy sets has been shown to be a really useful tool [2]. Among those fuzzy sets, interval-valued fuzzy sets(IVFSs) [3–5] or, equivalently, Atanassov intuitionistic fuzzy sets(AIFs) [6] play indeed a crucial role.

In some special cases, despite the fact of using IVFSs and AIFs, still remain problems that are similar to those encountered in the previous ones. For these new last situations we may use the interval-valued Atanassov intuitionistic fuzzy sets (IVAIFSs) [7]. Besides, the use of intervals to represent membership and non-membership has, from our point of view, a double advantage:

1. If we want to model environments where there exist non-comparable elements, it will be enough to use classical partial orders between intervals. This is not the case in this work.
2. If we must represent ignorance [8] associated to the datum given by an expert, we can understand the length of the intervals as a representation of such ignorance. If, in these cases, we need to be able to compare any two data, then we can use any of the linear orders we consider here.

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Once the decision of using IVAIFSs to deal with a decision making problem has been reached, we should choose, accordingly, a linear order between pairs of intervals. In this way, we will select as the best option the alternative which is associated to the largest pair of intervals, with respect to the considered linear order.

Moreover, in decision making problems we must also aggregate the information furnished by the experts by means of aggregation functions [9–11].

All these considerations have led us to aim the following objectives:

(1) To use aggregation functions for building linear orders for pairs of intervals whose end-points belong to the unit interval;
(2) To study methods for constructing linear orders on the set of IVAIFSs;
(3) To deal with the exploitation phase of decision making problems through IVAIFSs, by using the previously built linear orders.

The structure of this paper is the following. In Section 2 we introduce the notation and recall some well-known notions. In Sections 3-4, we construct two classes of linear orders between pairs of intervals. Section 5 contains an application of our theoretical results to group decision making. In particular, we provide two algorithms. Some concluding remarks as well as suggestions for further research close the paper.

2. Previous concepts and results

We start by recalling some well-known concepts that will be useful for subsequent developments throughout the paper.

2.1. On orders and partially ordered sets

Given a partially ordered set (poset) $(P, \preceq)$, we say that

a) 1_P is the top of the poset if for all $x \in P$ it holds $x \preceq 1_P$.

b) 0_P is the bottom of the poset if for all $x \in P$ it holds $0_P \preceq x$.

In case they exist, 1_P and 0_P are unique.

Let $K([0,1]) \subset \mathbb{R}^2$ be given by

$$K([0,1]) = \{(\underline{x}, \overline{x}) \in [0,1] \times [0,1] | \, \underline{x} \leq \overline{x}\}$$

and let $L([0,1])$ be the set of all closed subintervals of the unit interval, that is

$$L([0,1]) = \{\mathbf{x} | \mathbf{x} = [\underline{x}, \overline{x}] \text{ such that } 0 \leq \underline{x} \leq \overline{x} \leq 1\}.$$

There is a straightforward bijection $i : K([0,1]) \longrightarrow L([0,1])$ given by $i((\underline{x}, \overline{x})) = [\underline{x}, \overline{x}] = \mathbf{x}$. Through this bijection, the partial order on $\mathbb{R}^2$, $(a,b) \leq_2 (c,d)$ if and only if $a \leq c$ and $b \leq d$ induces an equivalent partial order on $L([0,1])$, namely,

$$\mathbf{x} \preceq_2 \mathbf{y} \text{ iff } \underline{x} \leq \underline{y} \text{ and } \overline{x} \leq \overline{y} \,. \tag{1}$$

In this way, $(L([0,1]), \preceq_2)$ is a poset whose bottom and top are, respectively, $\mathbf{0} = [0,0]$ and $\mathbf{1} = [1,1]$. In fact, the bijection above is a lattice isomorphism [1].

[1]This kind of sets, namely $K([0,1])$ and $L([0,1])$ have already been used, suitably equipped with some order and latticial structure [12, 13], to construct some universal codomain where it was possible to represent different kinds of orderings as, e.g., total preorders, interval-orders and semiorders by means of a single function that preserves the ordinal structure. The bijection $i : K([0,1]) \longrightarrow L([0,1])$ has also been considered in those approaches, and some other similar bijections and/or latticial isomorphims as well as order isotonies have also been introduced accordingly. By the way, another universal codomain to represent different kinds of orderings, which is essentially equivalent to $K([0,1])$, consists of triangular and symmetric fuzzy numbers. For further information see [14–17].

We refer as $(L([0,1]))^2$, to the universe of pairs of intervals, that is,

$$(L([0,1]))^2 = \{(\mathbf{x},\mathbf{y}) = ([\underline{x},\overline{x}],[\underline{y},\overline{y}]) \text{ with } \underline{x},\overline{x},\underline{y},\overline{y} \in [0,1]\}\,.$$

Similarly to what happens in the case of $\mathbb{R}^2$ and $L([0,1])$, the partial order on $\mathbb{R}^4$, given by $(a_1,b_1,c_1,d_1) \leq_4 (a_2,b_2,c_2,d_2)$ if and only if $a_1 \leq a_2$ and $b_1 \leq b_2$ and $c_1 \leq c_2$ and $d_1 \leq d_2$, also induces an equivalent partial order $\preceq_4$ on $(L([0,1]))^2$, given by

$$(\mathbf{x_1},\mathbf{y_1}) \preceq_4 (\mathbf{x_2},\mathbf{y_2}) \text{ if and only if } \underline{x}_1 \leq \underline{x}_2 \text{ and } \overline{x}_1 \leq \overline{x}_2 \text{ and } \underline{y}_1 \leq \underline{y}_2 \text{ and } \overline{y}_1 \leq \overline{y}_2\,. \tag{2}$$

In this way, $((L([0,1]))^2, \preceq_4)$ becomes a poset whose bottom and top are, respectively, $(\mathbf{0},\mathbf{0}) = ([0,0],[0,0])$ and $(\mathbf{1},\mathbf{1}) = ([1,1],[1,1])$.

Definition 2.1. *[18] An order $\preceq$ on $L([0,1])$ is said to be admissible if it is linear and refines the order $\preceq_2$, i.e., it is a linear order satisfying that for all $\boldsymbol{x},\boldsymbol{y} \in L([0,1])$ such that $\boldsymbol{x} \preceq_2 \boldsymbol{y}$ it holds $\boldsymbol{x} \preceq \boldsymbol{y}$.*

Example 2.1. *The lexicographic orders on $L([0,1])$, given by*

- $\boldsymbol{x} \preceq_{lex1} \boldsymbol{y}$ *if and only if $(\underline{x} < \underline{y})$ or $(\underline{x} = \underline{y}$ and $\overline{x} \leq \overline{y})$ (lexicographic-1 order), and*
- $\boldsymbol{x} \preceq_{lex2} \boldsymbol{y}$ *if and only if $(\overline{x} < \overline{y})$ or $(\overline{x} = \overline{y}$ and $\underline{x} \leq \underline{y})$ (lexicographic-2 order),*

are admissible.

2.2. Extensions of fuzzy sets

Definition 2.2. *[6] Let U be a nonempty set usually called a universe. An Atanassov's Intuitionistic Fuzzy Set (AIFS) F over U is given by*

$$F = \{\langle u, \mu_F(u), \nu_F(u)\rangle | u \in U\}$$

where $\mu_F : U \to [0,1]$ defines the membership degree of the element $u \in U$ to F and $\nu_F : U \to [0,1]$ defines its nonmembership degree to the same set F. Besides, the functions μ_F and ν_F satisfy that, for all $u \in U$, $\mu_F(u) + \nu_F(u) \leq 1$.

The pair $(\mu_F(u), \nu_F(u))$ is called an intuitionistic pair, $\mathcal{L}([0,1])$ being the set of all possible intuitionistic pairs, i.e.,

$$\mathcal{L}([0,1]) = \{\mathbf{a} \mid \mathbf{a} = (a_1,a_2),\, a_1,a_2 \in [0,1] \text{ and } a_1 + a_2 \leq 1\}.$$

In [6], Atanassov introduced a partial order in the universe of AIFSs.

Definition 2.3. *Let F_1, F_2 be two AIFSs. According to the order given by Atanassov in [6]*

$$F_1 \leq F_2 \text{ if and only if for all } u \in U, \mu_{F_1}(u) \leq \mu_{F_2}(u) \text{ and } \nu_{F_1}(u) \geq \nu_{F_2}(u)\,.$$

Definition 2.4. *[7] Let U be a universe. An Interval-Valued Atanassov Intuitionistic Fuzzy Set (IVAIFS) G over U is given by*

$$G = \{\langle u, \mathrm{m_G}(u), \mathrm{n_G}(u)\rangle | u \in U\}$$

where $\mathrm{m_G} : U \to L([0,1])$ defines the membership degree of the element $u \in U$ to F and $\mathrm{n_G} : U \to L([0,1])$ defines its nonmembership degree to the same universe U. Moreover, for all $u \in U$, the sum of the upper boundary values of $\mathrm{m_G}(u)$ and $\mathrm{n_G}(u)$ must be lower than or equal to 1.

The pair $(\mathrm{m_G}(u), \mathrm{n_G}(u))$ is called an interval-valued intuitionistic pair, being $\mathcal{L}_{IV}([0,1])$ the set of all possible interval-valued intuitionistic pairs, i.e.,

$$\mathcal{L}_{IV}([0,1]) = \{(\mathbf{x},\mathbf{y}),\ \text{with } \mathbf{x},\mathbf{y} \in L([0,1]) \text{ and } \overline{x} + \overline{y} \leq 1\}.$$

Remark 1. Note that $\mathcal{L}_{IV}([0,1])$ consists of special types of intervals, while $(\mathcal{L}([0,1]))^2$ is a set of all possible intuitionistic pairs.

Definition 2.5. *Let G_1, G_2 be two IVAIFSs. According to the order given by Atanassov in [7], $G_1 \preceq G_2$ if and only if, for all $u \in U$,*

$$\mathrm{m_{G1}}(u) \preceq_2 \mathrm{m_{G2}}(u) \quad \textit{and} \quad \mathrm{n_{G2}}(u) \preceq_2 \mathrm{n_{G1}}(u)\,,$$

where $\preceq_2$ is the partial order on $L([0,1])$ given in Equation (1).

2.3. Aggregation functions

Definition 2.6. *Given a poset $(P, \preceq_P)$ with bottom 0_P and top 1_P, an aggregation function M on P w.r.t the order $\preceq_P$ (also known as an $\preceq_P$-aggregation function) is a mapping $M : P^n \to P$ satisfying*

- $M(0_P, \ldots, 0_P) = 0_P, \quad M(1_P, \ldots, 1_P) = 1_P$, *and*
- $M(x_1, \ldots, x_n) \preceq_P M(y_1, \ldots, y_n)$ *for* $(x_1, \ldots, x_n) \preceq_P (y_1, \ldots, y_n)$

where $(x_1, \ldots, x_n) \preceq_P (y_1, \ldots, y_n)$ holds if and only if $x_i \preceq_P y_i$ for all $i \in \{1, \ldots, n\}$.

This definition extends the usual one for the unit interval $[0,1]$. For further information see [19].

Proposition 2.1. *[18] Let $B_1, B_2 : [0,1]^2 \to [0,1]$ be two continuous aggregation functions, such that for all $(p_1,p_2),(q_1,q_2) \in K([0,1])$, the equalities $B_1(p_1,p_2) = B_1(q_1,q_2)$ and $B_2(p_1,p_2) = B_2(q_1,q_2)$ only hold provided that $(p_1,p_2) = (q_1,q_2)$.*

The order $\preceq_{B_1,B_2}$ on $L([0,1])$, given by

$$\boldsymbol{x} \preceq_{B_1,B_2} \boldsymbol{y} \text{ if and only if } B_1(\underline{x},\overline{x}) < B_1(\underline{y},\overline{y}) \text{ or else } (B_1(\underline{x},\overline{x}) = B_1(\underline{y},\overline{y}) \text{ and } B_2(\underline{x},\overline{x}) \le B_2(\underline{y},\overline{y})),$$

is an admissible order on $L([0,1])$.

The following results can be found in [9, 11, 20, 21].

Definition 2.7. *A function $T : [0,1]^2 \to [0,1]$ is called a t-norm if it is symmetric, associative, increasing with respect to the order $\le$ and $T(x,1) = x$ for all $x \in [0,1]$.*

Definition 2.8. *A function $S : [0,1]^2 \to [0,1]$ is called a t-conorm if it is symmetric, associative, increasing with respect to the order $\le$ and $S(x,0) = x$ for all $x \in [0,1]$.*

A strictly decreasing and continuous function $n : [0,1] \to [0,1]$ such that $n(0) = 1$ and $n(1) = 0$ is called a strict negation. If, in addition, it is involutive (that is, $n(n(x)) = x$ for all $x \in [0,1]$), then n is said to be a strong negation. A t-norm T is dual to a t-conorm S (and vice-versa) with respect to a strong negation n if $T(x,y) = n(S(n(x),n(y)))$ for all $x, y \in [0,1]$.

3. Admissible orders on $(L([0,1]))^2$

Although a partial order is enough to define aggregation functions, some special classes of aggregations actually require to have at hand a linear order. Examples of such classes are Choquet integrals and Sugeno integrals. The order given by Atanassov for IVAIFSs is partial, which is a undeniable handicap in the adaptation of such classes of aggregation operators to the IVAI setup. In this section we define the admissible linear orders on $(L([0,1]))^2$, generalizing the concept of admissible orders on $L([0,1])$.

Definition 3.1. *An order $\preceq$ on $(L([0,1]))^2$ is said to be admissible if it is a linear and refines the order $\preceq_4$ in Eq. (2), i.e., it is linear order satisfying that for all $(\boldsymbol{x_1}, \boldsymbol{y_1}), (\boldsymbol{x_2}, \boldsymbol{y_2}) \in (L([0,1]))^2$, $(\boldsymbol{x_1}, \boldsymbol{y_1}) \preceq_4 (\boldsymbol{x_2}, \boldsymbol{y_2})$ implies $(\boldsymbol{x_1}, \boldsymbol{y_1}) \preceq (\boldsymbol{x_2}, \boldsymbol{y_2})$.*

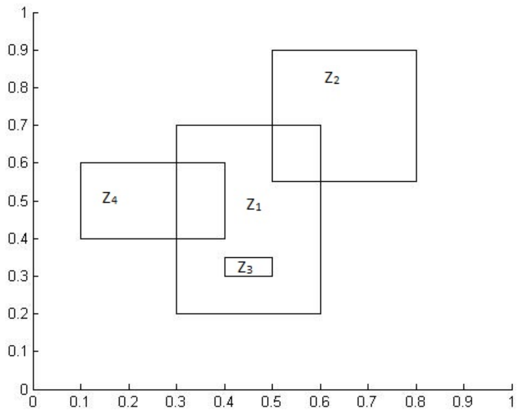

Figure 1: Pairs of intervals

The elements $z_i = (\mathbf{x_i}, \mathbf{y_i}) \in (L([0,1]))^2$ can be visualized in a straightforward manner. Since $\mathbf{x_i}, \mathbf{y_i} \in L([0,1])$, each pair of intervals can be drawn as a rectangle for which the first interval lies in the horizontal axis and the second interval lies in the vertical one. In such a representation, the following statements hold true:

- The wider the first interval, the wider the rectangle.
- The wider the second interval, the higher the rectangle.

As a consequence, the area of the rectangle will be directly proportional to the width of the intervals. Furthermore, for any $z_1, z_2 \in (L([0,1]))^2$, $z_1 \preceq_4 z_2$ if and only if each corner of the rectangle of z_2 is located above and on the right side of its corresponding corner in the rectangle z_1.

Example 3.1. *Let* $z_1 = ([0.3, 0.6], [0.2, 0.7])$, $z_2 = ([0.5, 0.8], [0.55, 0.9])$, $z_3 = ([0.4, 0.5], [0.3, 0.35])$, $z_4 = ([0.1, 0.4], [0.4, 0.6])$*. The intervals can be represented in the unit square* $[0,1]^2$ *as in Fig. 1. In that figure some visual interpretations can be drawn. For example, we have that the intervals of* z_1 *are wider than those of any other* z_i*, since its area is significantly greater. Alternatively, we have that* $z_i \preceq_4 z_2$ *for* $i \in \{1, 3, 4\}$*, since the corners of* z_2 *are located above and on the right side w.r.t the other rectangles. Similarly, we can deduce that* z_1, z_3 *and* z_4 *are incomparable in terms of* $\preceq_4$.

In [18], Bustince et al. introduced a construction method of admissible orders on $L([0,1])$ by using two aggregation functions. Such method can also be generalized to handle elements in $(L([0,1]))^2$.

Proposition 3.1. *Let* $A =< A_1, A_2, A_3, A_4 >$ *be four aggregation functions* [2], $A_i : [0,1]^4 \to [0,1]$ *such that for all* $(\boldsymbol{p}, \boldsymbol{q}), (\boldsymbol{r}, \boldsymbol{s}) \in (L([0,1]))^2$ *the equalities* $A_i(\underline{p}, \overline{p}, \underline{q}, \overline{q}) = A_i(\underline{r}, \overline{r}, \underline{s}, \overline{s})$ *for all* $i = \{1, \ldots 4\}$ *only hold if* $(\boldsymbol{p}, \boldsymbol{q}) = (\boldsymbol{r}, \boldsymbol{s})$.

An admissible order $\preceq_A$ *on* $(L([0,1]))^2$ *can be defined as follows* $(\boldsymbol{x_1}, \boldsymbol{y_1}) \preceq_A (\boldsymbol{x_2}, \boldsymbol{y_2})$ *if and only if one of the (mutually exclusive) following conditions is satisfied.*

i) $A_1(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) < A_1(\underline{x}_2, \overline{x}_2, \underline{y}_2, \overline{y}_2)$;

ii) $A_1(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = A_1(\underline{x}_2, \overline{x}_2, \underline{y}_2, \overline{y}_2)$ *and* $A_2(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) < A_2(\underline{x}_2, \overline{x}_2, \underline{y}_2, \overline{y}_2)$;

[2]Warning: notice that here the order of appearence of the $A_i's$ counts. See also Remark 2

iii) $A_1(\underline{x}_1,\overline{x}_1,\underline{y}_1,\overline{y}_1) = A_1(\underline{x}_2,\overline{x}_2,\underline{y}_2,\overline{y}_2)$ *and* $A_2(\underline{x}_1,\overline{x}_1,\underline{y}_1,\overline{y}_1) = A_2(\underline{x}_2,\overline{x}_2,\underline{y}_2,\overline{y}_2)$ *and* $A_3(\underline{x}_1,\overline{x}_1,\underline{y}_1,\overline{y}_1) < A_3(\underline{x}_2,\overline{x}_2,\underline{y}_2,\overline{y}_2)$*;*

iv) $A_1(\underline{x}_1,\overline{x}_1,\underline{y}_1,\overline{y}_1) = A_1(\underline{x}_2,\overline{x}_2,\underline{y}_2,\overline{y}_2)$ *and* $A_2(\underline{x}_1,\overline{x}_1,\underline{y}_1,\overline{y}_1) = A_2(\underline{x}_2,\overline{x}_2,\underline{y}_2,\overline{y}_2)$ *and* $A_3(\underline{x}_1,\overline{x}_1,\underline{y}_1,\overline{y}_1) = A_3(\underline{x}_2,\overline{x}_2,\underline{y}_2,\overline{y}_2)$ *and* $A_4(\underline{x}_1,\overline{x}_1,\underline{y}_1,\overline{y}_1) \leq A_4(\underline{x}_2,\overline{x}_2,\underline{y}_2,\overline{y}_2)$.

Proof. The order $\preceq_A$ refines $\leq_4$ since every A_i is an aggregation function. Moreover, the linearity is assured since the four equalities of A_i only hold simultaneously if $(\mathbf{x_1},\mathbf{y_1}) = (\mathbf{x_2},\mathbf{y_2})$. The transitivity follows from the transitivity of the standard order on $[0,1]$.

Remark 2. Notice that any permutation of the aggregation functions A_i also produces an admissible order different from the former one.

Remark 3. In [18] it was proven that an admissible order on $K([0,1])$ can not be induced by a single function. Clearly, this result also holds true since we are working in a larger space.

Henceforward, we use the order generated by four aggregation functions (in Prop 3.1). Thus, all the ideas to be introduced till the end of this section refer to such family of admissible orders named 4-admissible.

Example 3.2. *The lexicographic orders can be constructed from the four projections.*

1. *The standard lexicographic order: let* A_i *be the aggregation function that maps to the* i*-th component (i.e. the* i*-th projection). In that case,* $(\boldsymbol{x_1},\boldsymbol{y_1}) \preceq_A (\boldsymbol{x_2},\boldsymbol{y_2})$ *if and only if*
   - $(\underline{x}_1 < \underline{x}_2)$*, or*
   - $(\underline{x}_1 = \underline{x}_2$ *and* $\overline{x}_1 < \overline{x}_2)$*, or*
   - $(\underline{x}_1 = \underline{x}_2$*,* $\overline{x}_1 = \overline{x}_2$ *and* $\underline{y}_1 < \overline{y}_2)$*, or*
   - $(\underline{x}_1 = \underline{x}_2$*,* $\overline{x}_1 = \overline{x}_2$*,* $\underline{y}_1 = \underline{y}_2$ *and* $\overline{y}_1 \leq \overline{y}_2)$*.*
2. *The reversed lexicographic order: let* A_i *be the aggregation function that maps to the* $(5-i)$*-th component (i.e. the* $(5-i)$*-th projection). In that case,* $(\boldsymbol{x_1},\boldsymbol{y_1}) \preceq_A (\boldsymbol{x_2},\boldsymbol{y_2})$ *if and only if*
   - $(\overline{y}_1 < \overline{y}_2)$*, or*
   - $(\overline{y}_1 = \overline{y}_2$ *and* $\underline{y}_1 < \underline{y}_2)$*, or*
   - $(\overline{y}_1 = \overline{y}_2$*,* $\underline{y}_1 = \underline{y}_2$ *and* $\overline{x}_1 < \overline{x}_2)$*, or*
   - $(\overline{y}_1 = \overline{y}_2$*,* $\underline{y}_1 = \underline{y}_2$*,* $\overline{x}_1 = \overline{x}_2$ *and* $\underline{x}_1 \leq \underline{x}_2)$*.*
3. *Any other permutation of the projections gives rise to an admissible order where we compare the components in a predetermined order.*

Proposition 3.2. *Let* $A =< A_1, A_2, A_3, A_4 >$ *be four aggregation functions given by*

$$A_i(\underline{x}_1,\overline{x}_1,\underline{y}_1,\overline{y}_1) = a_i\underline{x}_1 + b_i\overline{x}_1 + c_i\underline{y}_1 + d_i\overline{y}_1\,,$$

with $a_i, b_i, c_i, d_i \in [0,1]$*,* $a_i + b_i + c_i + d_i = 1$ *and*

$$|D| = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \neq 0\,.$$

Then (and only then), the order generated by the aggregation functions A_i *is a 4-admissible order.*

Proof.
The functions A_i are weighted arithmetic means. Let $([\underline{x}_1, \overline{x}_1], [\underline{y}_1, \overline{y}_1]), ([\underline{x}_2, \overline{x}_2], [\underline{y}_2, \overline{y}_2]) \in (L([0,1]))^2$, such that

$$a_i\underline{x}_1 + b_i\overline{x}_1 + c_i\underline{y}_1 + d_i\overline{y}_1 = a_i\underline{x}_2 + b_i\overline{x}_2 + c_i\underline{y}_2 + d_i\overline{y}_2$$

for $i \in \{1,\ldots,4\}$. Because of the regularity of D, both linear systems have a unique and common solution, i.e., $(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = (\underline{x}_2, \overline{x}_2, \underline{y}_2, \overline{y}_2)$. The result now follows from Prop. 3.1.

Example 3.3. *Let A contain the following aggregation functions:*

- $A_1(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \frac{3}{8}\underline{x}_1 + \frac{3}{8}\overline{x}_1 + \frac{1}{8}\underline{y}_1 + \frac{1}{8}\overline{y}_1$;
- $A_2(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \frac{10}{20}\underline{x}_1 + \frac{5}{20}\overline{x}_1 + \frac{3}{20}\underline{y}_1 + \frac{2}{20}\overline{y}_1$;
- $A_3(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \frac{1}{20}\underline{x}_1 + \frac{10}{20}\overline{x}_1 + \frac{8}{20}\underline{y}_1 + \frac{1}{20}\overline{y}_1$;
- $A_4(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \frac{1}{4}\underline{x}_1 + \frac{1}{4}\overline{x}_1 + \frac{1}{4}\underline{y}_1 + \frac{1}{4}\overline{y}_1$.

Since $|D| = -0.0069$, the order generated by A, as in Prop. 3.1, is a 4-admissible order.

Remark 4. Notice that the value of the determinant is close to 0 but this is due to the fact that all the elements of the matrix are smaller than 1.

The construction of admissible orders through a 4-tuple of weighted arithmetic means has an interesting geometrical interpretation. If we consider A in the form of the corresponding four weighting vectors which generate $A_1, \ldots, A_4$, i.e.,

$$A \approx R = \{< a_1, b_1, c_1, d_1 >, < a_2, b_2, c_2, d_2 >, < a_3, b_3, c_3, d_3 >, < a_4, b_4, c_4, d_4 >\}$$

the condition in Prop. 3.2 means that R is a basis of the vector space $\mathbb{R}^4$. Hence, to any basis R of $\mathbb{R}^4$ which consists of weighting vectors there is a unique admissible order $\preceq_A$ constructed by means of the corresponding weighted means.

Finally, after changing the basis, the values of interval-valued intuitionistic pairs in the new basis, (which are now in $[0,1]^4$), are ordered through the standard lexicographic order.

Proposition 3.3. *Let a tuple $A = \langle A_1, \ldots, A_4\rangle$ of aggregation functions generate an admissible order $\preceq_A$. Let $B_i : [0,1]^2 \longrightarrow [0,1]$, $i \in \{1,\ldots,4\}$ be four aggregation functions such that*

- $A_i(\underline{x}, \overline{x}, \underline{y}, \overline{y}) = B_i(\underline{x}, \overline{x})$ *for* $i \in \{1,2\}$, *and*
- $A_j(\underline{x}, \overline{x}, \underline{y}, \overline{y}) = B_j(\underline{y}, \overline{y})$ *for* $j \in \{3,4\}$.

Then, $(\boldsymbol{x_1}, \boldsymbol{y_1}) \preceq_A (\boldsymbol{x_2}, \boldsymbol{y_2})$ *if and only if*

i) $(\boldsymbol{x_1} \prec_{B_1,B_2} \boldsymbol{x_2})$, *or*

ii) $(\boldsymbol{x_1} = \boldsymbol{x_2}$ *and* $\boldsymbol{y}_1 \preceq_{B_3,B_4} \boldsymbol{y_2})$,

where $\preceq_{B_i,B_j}$ is the order on $L([0,1])$ generated in Prop. 2.1.

Proof. It is straightforward.

Notice that, if we use $B_1 = B_3$ and $B_2 = B_4$, the result is a 4-admissible order where we combine the standard lexicographic order with the order $\preceq_{B_1,B_2}$. The resulting order acts as follows: first we compare the intervals using $\preceq_{B_1,B_2}$ and, only if they are equal, we compare the second interval with that same order ($\preceq_{B_1,B_2}$). For instance, the standard lexicographic order can be seen as the composition of the lexicographic-1 order between intervals combined with itself.

Alternatively, notice that, if $A_i(\underline{x},\overline{x},\underline{y},\overline{y}) = B_i(\underline{y},\overline{y})$ for $i \in \{1,2\}$, and $A_j(\underline{x},\overline{x},\underline{y},\overline{y}) = B_j(\underline{x},\overline{x})$ for $j \in \{3,4\}$, then the resulting order is also 4-admissible.

A well-known class of binary aggregation functions is that of Atanassov's operators $\mathbb{K}_\alpha$ given by $\mathbb{K}_\alpha(a,b) = a + \alpha(b-a)$ with $\alpha \in [0,1]$.

In our particular case, the inputs being intervals, an Atanassov's operator acting on the endpoints of the intervals yields a point inside the corresponding intervals.

Example 3.4. *Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,1]$, with $\alpha_1 \neq \alpha_2$ and $\alpha_3 \neq \alpha_4$. Let $A =< A_1, \dots, A_4 >$ be four aggregation functions given by*

- *$A_i(\underline{x}_1,\overline{x}_1,\underline{y}_1,\overline{y}_1) = \mathbb{K}_{\alpha_i}(\underline{x}_1,\overline{x}_1)$, for $i \in \{1,2\}$, and*
- *$A_j(\underline{x}_1,\overline{x}_1,\underline{y}_1,\overline{y}_1) = \mathbb{K}_{\alpha_j}(\underline{y}_1,\overline{y}_1)$, for $j \in \{3,4\}$.*

The tuple A generates a 4-admissible order that renders in $(\boldsymbol{x_1},\boldsymbol{y_1}) \preceq_A (\boldsymbol{x_2},\boldsymbol{y_2})$ if and only if

- *$(\boldsymbol{x_1} \prec_{\mathbb{K}_{\alpha_1},\mathbb{K}_{\alpha_2}} \boldsymbol{x_2})$, or*
- *$(\boldsymbol{x_1} =_{\mathbb{K}_{\alpha_1},\mathbb{K}_{\alpha_2}} \boldsymbol{x_2}$ and $\boldsymbol{y_1} \preceq_{\mathbb{K}_{\alpha_3},\mathbb{K}_{\alpha_4}} \boldsymbol{y_2})$.*

From the construction in Example 3.4, we can retrieve some well-known orders. For example, if $\{\alpha_1,\alpha_2\} = \{0,1\}$ and $\{\alpha_3,\alpha_4\} = \{0,1\}$, we obtain lexicographic orders. Moreover, all these 4-admissible orders are particular examples of the construction in Prop. 3.2, with $c = d = 0$ for A_1, A_2 and $a = b = 0$ for A_3 and A_4.

In [18] it was proven that given an $\alpha \in [0,1)$ then all admissible orders $\preceq_{\alpha,\beta}$ on $L([0,1])$ with $\beta > \alpha$ coincide. Then, different aggregation functions could generate the same admissible order. This also affects to admissible orders generated as in Prop. 3.2. For instance,

$$|D_1| = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{vmatrix} \neq 0\,, \qquad |D_2| = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{vmatrix} \neq 0$$

generate the same order.

4. IVAIF-admissible order on $\mathcal{L}_{IV}([0,1])$

The admissible orders defined in Section 3 refine the partial order $\preceq_4$. However, any of them could also refine the partial order given by Atanassov for IVAIFS [7]. In this section, we define a new family of linear orders with a crucial additional feature, namely, they refine Atanassov's partial order.

We remind the reader that in Atanassov's partial order, given two elements $(\mathbf{x_1},\mathbf{y_1}),(\mathbf{x_2},\mathbf{y_2}) \in \mathcal{L}_{IV}([0,1])$,

$$(\mathbf{x_1},\mathbf{y_1}) \preceq (\mathbf{x_2},\mathbf{y_2}) \text{ if and only if } \underline{x}_1 \leq \underline{x}_2,\ \ \overline{x}_1 \leq \overline{x}_2,\ \ \underline{y}_1 \geq \underline{y}_2, \text{ and } \ \overline{y}_1 \geq \overline{y}_2\,. \tag{3}$$

Definition 4.1. *An order $\preceq$ on $\mathcal{L}_{IV}([0,1])$ is said to be an IVAIF-admissible order if it is a linear order and refines the partial order given by Atanassov for IVAIFS (Eq.(3)).*

Notice that, if we have an IVAIF-admissible order on $\mathcal{L}_{IV}([0,1])$, as in Def. 4.1, then the bottom of $(\mathcal{L}_{IV}([0,1]),\preceq)$ is $(\mathbf{0},\mathbf{1})$ and the top is $(\mathbf{1},\mathbf{0})$.

As in Section 3, we can generate a visualization of the elements in $\mathcal{L}_{IV}([0,1]) \subset (L([0,1]))^2$ that captures the behaviour of the admissible orders in Def. 4.1. Following the visualization rules in

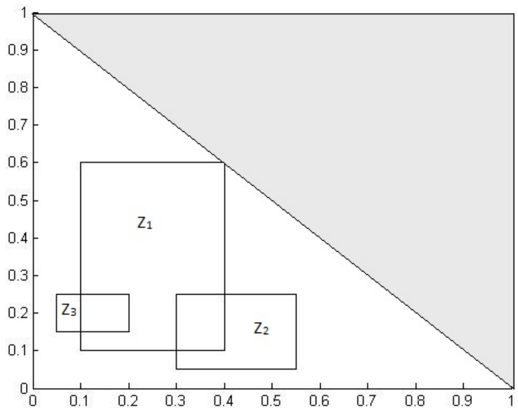

Figure 2: Visual representation of interval-valued intuitionistic pairs. The white zone represents the subset of $\mathcal{L}_{IV}([0,1])$ in which such pairs are allowed.

Fig. 1 we have that, for any two elements z_1, z_2 in $\mathcal{L}_{IV}([0,1])$, $z_1 \preceq z_2$ if and only if the corners of z_2 are individually located below and to the right of those of z_1. For example, in Fig. 2, we have given $z_1 = ([0.1, 0.4], [0.1, 0.6]), z_2 = ([0.3, 0.55], [0.05, 0.25]), z_3 = ([0.05, 0.2], [0.15, 0.25]) \in \mathcal{L}_{IV}([0,1])$. Visually, it is evident that $z_1 \preceq z_2$ and $z_3 \preceq z_2$, but also that z_1 and z_3 are not comparable with the partial order in Def. 4.1. Notice that, in this visualization, no element is allowed to be in the grey zone of the rectangle in Fig. 2 due to the restrictions in the definitions of the membership and nonmembership degrees in an interval-valued intuitionistic pair.

In the sequel, two different constructions of IVAIF-admissible orders are introduced.

Proposition 4.1. *Let $B =< B_1, B_2, B_3, B_4 >$ be four aggregation functions $B_i : [0,1]^4 \longrightarrow [0,1]$ which generate the orders $\preceq_{B_1,B_2}$ and $\preceq_{B_3,B_4}$ on $L([0,1])$ as in Prop. 2.1. Then the order relation $\preceq_B^{IV}$ given by*

$$(\boldsymbol{x}_1, \boldsymbol{y}_1) \preceq_B^{IV} (\boldsymbol{x}_2, \boldsymbol{y}_2) \text{ if and only if } \boldsymbol{x}_1 \prec_{B_1,B_2} \boldsymbol{x}_2 \text{ or } (\boldsymbol{x}_1 = \boldsymbol{x}_2 \text{ and } \boldsymbol{y}_2 \preceq_{B_3,B_4} \boldsymbol{y}_1)$$

is an IVAIF-admissible order.

Proof. The linearity of $\preceq_B^{IV}$ is straight as $\mathcal{L}_{IV}([0,1]) \subset (L([0,1]))^2$. In addition, it refines the partial order given by Atanassov due to the fact that the order relation, $\preceq_{B_3,B_4}$, has been reversed.

In particular, if $B_1 = B_3$ and $B_2 = B_4$, then $\preceq_{B_1,B_2} = \preceq_{B_3,B_4}$ and, consequently, the same order is used to compare both intervals although in the second one the order is reversed.

Proposition 4.2. *Let $A =< A_1, A_2, A_3, A_4 >$ be four aggregation functions, $A_i : [0,1]^4 \to [0,1]$ such that for all $(p_1, p_2, p_3, p_4), (q_1, q_2, q_3, q_4) \in [0,1]^4$ the equalities $A_i(p_1, p_2, p_3, p_4) = A_i(q_1, q_2, q_3, q_4)$ for all $i \in \{1, \ldots, 4\}$ only hold if $(p_1, p_2, p_3, p_4) = (q_1, q_2, q_3, q_4)$.*

An IVAIF*-admissible order $\preceq_A^{IV}$ on $\mathcal{L}_{IV}([0,1])$, is defined as follows: $(\boldsymbol{x}_1, \boldsymbol{y}_1) \preceq_A^{IV} (\boldsymbol{x}_2, \boldsymbol{y}_2)$ if and only if one of the following (mutually exclusive) conditions is satisfied.*

i) $A_1(\underline{x}_1, \overline{x}_1, 1 - \underline{y}_1, 1 - \overline{y}_1) < A_1(\underline{x}_2, \overline{x}_2, 1 - \underline{y}_2, 1 - \overline{y}_2)$,

ii) $A_1(\underline{x}_1, \overline{x}_1, 1 - \underline{y}_1, 1 - \overline{y}_1) = A_1(\underline{x}_2, \overline{x}_2, 1 - \underline{y}_2, 1 - \overline{y}_2)$ *and*
$A_2(\underline{x}_1, \overline{x}_1, 1 - \underline{y}_1, 1 - \overline{y}_1) < A_2(\underline{x}_2, \overline{x}_2, 1 - \underline{y}_2, 1 - \overline{y}_2)$,

iii) $A_1(\underline{x}_1, \overline{x}_1, 1 - \underline{y}_1, 1 - \overline{y}_1) = A_1(\underline{x}_2, \overline{x}_2, 1 - \underline{y}_2, 1 - \overline{y}_2)$,
$A_2(\underline{x}_1, \overline{x}_1, 1 - \underline{y}_1, 1 - \overline{y}_1) = A_2(\underline{x}_2, \overline{x}_2, 1 - \underline{y}_2, 1 - \overline{y}_2)$, *and*
$A_3(\underline{x}_1, \overline{x}_1, 1 - \underline{y}_1, 1 - \overline{y}_1) < A_3(\underline{x}_2, \overline{x}_2, 1 - \underline{y}_2, 1 - \overline{y}_2)$,

iv) $A_1(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) = A_1(\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$,
$A_2(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) = A_2(\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$,
$A_3(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) = A_3(\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$, *and*
$A_4(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) \leq A_4(\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$.

Proof. The linearity is warranted because the equalities only hold if $(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) = (\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$.

To check the second condition (that of refining the partial order) in the statement of Def. 4.1, notice that if

$$\underline{x}_1 \leq \underline{x}_2, \ \ \overline{x}_1 \leq \overline{x}_2, \ \ \underline{y}_1 \geq \underline{y}_2, \text{ and } \ \overline{y}_1 \geq \overline{y}_2 .$$

then

$$\underline{x}_1 \leq \underline{x}_2, \ \ \overline{x}_1 \leq \overline{x}_2, \ \ 1-\underline{y}_1 \leq 1-\underline{y}_2, \text{ and } \ 1-\overline{y}_1 \leq 1-\overline{y}_2,$$

so consequently $A_i(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) \leq A_i(\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$ for all $i \in \{1, \ldots, 4\}$.

From now on we name the order generated by four aggregation functions (as in Prop. 4.2) 4-IVAIF-admissible order.

Remark 5. Given $\mathbf{y} \in L([0,1])$, it follows that $(1-\underline{y}, 1-\overline{y}) \in L^*([0,1])$, where

$$L^*([0,1]) = \{\mathbf{s} | \mathbf{s} = (s_1, s_2) \text{ such that } 0 \leq s_2 \leq s_1 \leq 1\}.$$

Then in Prop. 4.2 it is enough that to see, given $(\mathbf{p_1}, \mathbf{q_1}) = ([\underline{p}_1, \overline{p}_1], (\underline{q}_1, \overline{q}_1))$ and $(\mathbf{p_2}, \mathbf{q_2}) = ([\underline{p}_2, \overline{p}_2], (\underline{q}_2, \overline{q}_2))$ in $L([0,1]) \times L^*([0,1])$, the equalities $A_i(\underline{p}_1, \overline{p}_1, \underline{q}_1, \overline{q}_1) = A_i(\underline{p}_2, \overline{p}_2, \underline{q}_2, \overline{q}_2)$ hold if and only if $(\mathbf{p_1}, \mathbf{q_1}) = (\mathbf{p_2}, \mathbf{q_2})$.

However, in order to simplify notation we have imposed a slightly stronger restriction. Anyway, all the given examples in Section 3 satisfy it.

Let a tuple $A = \langle A_1, \ldots, A_4 \rangle$ of aggregation functions generate an admissible order. Let $B_i : [0,1]^2 \longrightarrow [0,1]$ be four aggregations such that

- $A_i(\underline{x}, \overline{x}, \underline{y}, \overline{y}) = B_i(\underline{x}, \overline{x})$ for $i \in \{1,2\}$, and
- $A_j(\underline{x}, \overline{x}, \underline{y}, \overline{y}) = B_j(\underline{y}, \overline{y})$ for $j \in \{3,4\}$,

Then the orders $\preceq_A^{IV}$ and $\preceq_B^{IV}$ may be different. To guarantee that they are actually different it is enough that $B_3(\underline{y}_1, \overline{y}_1) < B_3(\underline{y}_2, \overline{y}_2)$ and simultaneously $B_3(1-\underline{y}_1, 1-\overline{y}_1) > B_3(1-\underline{y}_2, 1-\overline{y}_2)$ hold true for some $\mathbf{y}_1, \mathbf{y}_2 \in L([0,1])$.

For instance, let $B_3(\underline{y}, \overline{y}) = \underline{y}\overline{y}$. Here, we have that for $\mathbf{y}_1 = [0.2, 0.2]$ and $\mathbf{y}_2 = [0.1, 0.9]$

$$B_3(0.2, 0.2) = 0.04 < 0.09 = B_3(0.1, 0.9)$$
$$B_3(0.8, 0.8) = 0.64 > 0.09 = B_3(0.9, 0.1).$$

Proposition 4.3. *Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,1]$, with $\alpha_1 \neq \alpha_2$ and $\alpha_3 \neq \alpha_4$. If*

- $A_i(\underline{x}, \overline{x}, \underline{y}, \overline{y}) = K_{\alpha_i}(\underline{x}, \overline{x})$ *for* $i \in \{1,2\}$, *and*
- $A_j(\underline{x}, \overline{x}, \underline{y}, \overline{y}) = K_{\alpha_j}(\underline{y}, \overline{y})$ *for* $j \in \{3,4\}$,

then the tuple $A = \langle A_1, \ldots, A_4 \rangle$ generates a 4-IVAIF admissible order that is equal to $\preceq_B^{IV}$ being $B =< \mathbb{K}_{\alpha_1}, \mathbb{K}_{\alpha_2}, \mathbb{K}_{\alpha_3}, \mathbb{K}_{\alpha_4} >$.

Proof. The fact that the aggregation functions satisfy the conditions to generate a 4-IVAIF order is a simple calculation. To prove the equality between the two orders notice that in this case the conditions *i*) and *ii*) of the order $\preceq_A^{IV}$ are exactly equal to $\mathbf{x}_1 \preceq_{\mathbb{K}_{\alpha_1}, \mathbb{K}_{\alpha_2}} \mathbf{x}_2$. Then, it is enough to prove that for all γ,

$$\mathbb{K}_\gamma(1-a_1, 1-b_1) < \mathbb{K}_\gamma(1-a_2, 1-b_2)$$

is equivalent to $\mathbb{K}_\gamma(a_1, b_1) > \mathbb{K}_\gamma(a_2, b_2)$.

But

$$\begin{aligned}
\mathbb{K}_\gamma(1-a_1, 1-b_1) &< \mathbb{K}_\gamma(1-a_2, 1-b_2)\\
&\Leftrightarrow 1-a_1+\gamma(1-b_1-(1-a_1)) < 1-a_2+\gamma(1-b_2-(1-a_2))\\
&\Leftrightarrow 1-a_1+\gamma(a_1-b_1) < 1-a_2+\gamma(a_2-b_2)\\
&\Leftrightarrow a_2-\gamma(a_2-b_2) < a_1-\gamma(a_1-b_1)\\
&\Leftrightarrow a_2+\gamma(b_2-a_2) < a_1+\gamma(b_1-a_1)\\
&\Leftrightarrow \mathbb{K}_\gamma(a_2,b_2) < \mathbb{K}_\gamma(a_1,b_1),
\end{aligned}$$

so the proof is complete.

Example 4.1. *Let $\preceq_A^{IV}$ be the order generated by $A =< \mathbb{K}_{0.25}, \mathbb{K}_{0.75}, \mathbb{K}_{0.25}, \mathbb{K}_{0.75} >$. Consider the elements $z_1 = ([0.15, 0.35], [0.2, 0.5])$ and $z_2 = ([0.15, 0.35], [0.1, 0.9]) \in \mathcal{L}_{IV}([0, 1])$. Since their membership degrees are identical we only need to compare their nonmembership degrees.*

In fact,

$$\mathbb{K}_{0.25}(0.2, 0.5) = 0.2 + 0.25 \cdot (0.5 - 0.2) = 0.275 < 0.3 = 0.1 + 0.25 \cdot (0.9 - 0.1) = \mathbb{K}_{0.25}(0.1, 0.9)$$

and $([0.15, 0.35], [0.1, 0.9]) \preceq_A^{IV} ([0.15, 0.35], [0.2, 0.5])$.

5. Application to Decision Making

Decision making problems may be summarized as follows. We have a set of p alternatives:

$$Z = \{z_1, \cdots, z_p\}$$

and a set of $n > 2$ experts:

$$E = \{e_1, \cdots, e_n\}.$$

Each of the latter provides her/his preferences on the former set of alternatives by means of a preference relation in the following way:

$$r_{el} = \begin{pmatrix} - & r_{(el)12} & \cdots & r_{(el)1p} \\ r_{(el)21} & - & \cdots & r_{(el)2p} \\ \cdots & \cdots & - & \cdots \\ r_{(el)p1} & \cdots & \cdots & - \end{pmatrix}. \qquad (4)$$

Here $r_{(el)ij}$, with $i \neq j$, expresses to what extent the expert l (with $l \in \{1, \cdots, n\}$) prefers the alternative z_i over the alternative z_j.

We must reach a decision of selecting either an alternative or a set of alternatives, which is (are) optimal as regards the experts assessments.

In [20], it is stated that the resolution of a group decision making problem consists of two steps:

(1) Uniform representation of information. In this phase, the heterogeneous information for the problem (the information can be represented by means of preference orderings or utility functions or fuzzy preference relations) is translated into homogeneous information by means of different transformation functions (see [22]).

(2) Application of a selection procedure. This procedure consists of two phases:

  (2.1) Aggregation phase. A collective preference structure is built from the set of individual homogeneous preference structures.

  (2.2) Exploitation phase. A given method is applied to the collective preference structure to obtain a selection of alternatives.

We use the theoretical developments in previous sections in the exploitation phase of the group decision making problem considered by Nayagam [23]. In particular, we consider the adaptation of this problem done by Zhang et al in [24]. In this adptation, authors consider that *there exists a panel with four possible alternatives for investment:*

(1) z_1 *is a car company,*

(2) z_2 *is a food company,*

(3) z_3 *is a computer company,*

(4) z_4 *is an arms company.*

It is necessary to choose the best company for investment.

Let the data in [24] be our collective preference matrix. In the exploitation phase we use the voting method which consists in aggregating the values in each row of the collective matrix R_c in such a way that, at the end, we have as many values (pairs of intervals) as rows. Since these latter values are not comparable through the partial order, we will select the alternative associated to the largest pair, according to a considered linear order.

$$R_c = \begin{pmatrix} - & ([0.4,0.5],[0.3,0.4]) & ([0.4,0.6],[0.2,0.4]) & ([0.1,0.3],[0.5,0.6]) \\ ([0.6,0.7],[0.2,0.3]) & - & ([0.6,0.7],[0.2,0.3]) & ([0.4,0.8],[0.1,0.2]) \\ ([0.3,0.6],[0.3,0.4]) & ([0.5,0.6],[0.3,0.4]) & - & ([0.4,0.5],[0.1,0.3]) \\ ([0.7,0.8],[0.1,0.2]) & ([0.6,0.7],[0.1,0.3]) & ([0.3,0.4],[0.1,0.2]) & - \end{pmatrix}.$$

To aggregate the values of each row of R_c we use the concept of interval-valued intuitionistic t-norms.

Definition 5.1. *A mapping* $\mathbb{T} : (\mathcal{L}_{IV}([0,1]))^2 \to \mathcal{L}_{IV}([0,1])$ *is an interval-valued intuitionistic t-norm if it is symmetric, associative, increasing with respect to the partial order* $\preceq$ *given by Atanassov (also called monotone) and* $\mathbb{T}((\mathbf{x},\mathbf{y}),(\mathbf{1},\mathbf{0})) = (\mathbf{x},\mathbf{y})$.

It is easy to see that, if we take the classical product t-norm, $T_P(x,y) = x \cdot y$, and its dual t-conorm with respect to the standard negation, $S_P(x,y) = x + y - x \cdot y$, the following expression is an interval-valued inuitionistic t-norm: $\mathbb{T}((\mathbf{x},\mathbf{y}),(\mathbf{z},\mathbf{t})) = ([\underline{x}\cdot\underline{z}, \overline{x}\cdot\overline{z}], [\underline{y}+\underline{t}-\underline{y}\cdot\underline{t}, \overline{y}+\overline{t}-\overline{y}\cdot\overline{t}])$.

Applying $\mathbb{T}$ to each row of R_c we get a new matrix, say Rg, given by:

$$Rg = \begin{pmatrix} z_1 = ([0.016, 0.090], [0.720, 0.856]) \\ z_2 = ([0.144, 0.392], [0.424, 0.608]) \\ z_3 = ([0.060, 0.180], [0.559, 0.748]) \\ z_4 = ([0.126, 0.224], [0.271, 0.552]) \end{pmatrix}.$$

In this setting, as regards the partial order $\preceq$, it follows

$$z_1 \preceq z_3 \preceq z_2 \text{ and } z_1 \preceq z_3 \preceq z_4,$$

but z_2 and z_4 are not comparable.

For this reason we consider the 4-IVAIF-admissible order $\preceq_A^{IV}$ defined through the following aggregation functions.

- $A_1(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \frac{2}{20}\underline{x}_1 + \frac{2}{20}\overline{x}_1 + \frac{8}{20}\underline{y}_1 + \frac{8}{20}\overline{y}_1$
- $A_2(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \frac{10}{20}\underline{x}_1 + \frac{5}{20}\overline{x}_1 + \frac{3}{20}\underline{y}_1 + \frac{2}{20}\overline{y}_1$
- $A_3(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \frac{1}{20}\underline{x}_1 + \frac{10}{20}\overline{x}_1 + \frac{8}{20}\underline{y}_1 + \frac{1}{20}\overline{y}_1$

- $A_4(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \frac{1}{4}\underline{x}_1 + \frac{1}{4}\overline{x}_1 + \frac{1}{4}\underline{y}_1 + \frac{1}{4}\overline{y}_1$.

With this order, we have $z_1 \preceq_A^{IV} z_3 \preceq_A^{IV} z_2 \preceq_A^{IV} z_4$ and the selected firm is arms company.

However, as it happened in [18] different 4-IVAIFS-admissible orders can lead to different rankings and hence the selection of the best alternative for a given decision making problem can be forced. For instance, in our case if we take $\preceq_A^{IV}$ with $A =< \mathbb{K}_{0.25}, \mathbb{K}_{0.75}, \mathbb{K}_{0.25}, \mathbb{K}_{0.75} >$ it comes out that the best alternative is the second one, since $z_1 \preceq_A^{IV} z_3 \preceq_A^{IV} z_4 \preceq_A^{IV} z_2$. Nevertheless, for the order $([\underline{x}_1, \overline{x}_1], [\underline{y}_1, \overline{y}_1]) \preceq_A^{IV} ([\underline{x}_2, \overline{x}_2], [\underline{y}_2, \overline{y}_2])$ if and only if

- $(\overline{x}_1 < \overline{x}_2)$, or
- $(\overline{x}_1 = \overline{x}_2$ and $\underline{x}_1 < \underline{x}_2)$, or
- $(\overline{x}_1 = \overline{x}_2$, $\underline{x}_1 = \underline{x}_2$ and $\overline{y}_1 > \overline{y}_2)$, or else
- $(\overline{x}_1 = \overline{x}_2$, $\underline{x}_1 = \underline{x}_2$, $\overline{y}_1 = \overline{y}_2$ and $\underline{y}_1 \geq \underline{y}_2)$

we have that $z_1 \preceq_A^{IV} z_3 \preceq_A^{IV} z_4 \preceq_A^{IV} z_2$ and the best alternative is the second one.

To cope with this situation the following algorithm takes different 4-IVAIFS-admissible orders into account simultaneously.

(1) To select several linear orders built with the methods developed in the previous sections.
(2) For each order, to apply in the exploitation phase the voting method with the same aggregations. For instance, $\mathbb{T} = (T_P, S_P)$.
(3) To select the alternative which appears as the best placed in the majority of all the so-obtained rankings.

In our considered problem, the chosen alternative through this algorithm is the second one. That is, we must invest our money in a food company. Clearly, the nature of the problem will impose the number of linear orders to be considered and/or the conditions that will force us to use alternative methods.

6. Conclusions

In this work we have studied how to construct linear orders between pairs of intervals on $L([0, 1])$ that can be used to construct linear orders in Atanassov interval-valued intuitionistic fuzzy sets. We have applied this operator to group decision making problems giving two algorithms, the first one for a particular linear order and the second one which mixes different linear orders.

As a possible development for future research, somewhat related to the main ideas introduced throughout the present manuscript, we point out the introduction of different orderings on families of intervals of the real line could be also analyzed from the point of view of extensions of the canonical ordering of the real line to a superset (namely, $L([0, 1])$) following a suitable set of criteria established a priori. The real line can be immediately embedded into $L([0, 1])$ by just considering each real number x as the degenerate interval $[x, x]$).

A similar typical problem corresponds to the extension of linear orders from a finite set to its power set. Indeed, although it is always possible to extend a linear order from a given finite set U to its power set, a typical question that gave rise to some classical papers from the 1970's on (see e.g. [25–28]), is whether or not it is possible to perform an extension that follows a list of criteria imposed a priori. Sometimes, the extension is not possible because the criteria used are, so-to-say, contradictory. But, in addition, there are other situations in which the extension is not possible because of a combinatorial explosion which, due to the bigger cardinality of the power set of U, does not leave room to rank all the terms of the power set in an extended linear order, accomplishing all the criteria. Perhaps the most famous result in this direction is the so-called Kannai - Peleg impossibility theorem (see [26]).

However, when the extension does not affect to the whole power set, but to some suitable superset (smaller than the power set), perhaps it may still happen that an extension accomplishing aprioristic criteria is possible, after all. As far as we know, an analysis of this kind where we start with the canonical order of the real line (instead of a linear order on a finite set), and try to extend it to the set of closed intervals of real numbers, following some list of criteria that have been set beforehand, is an open problem.

We leave for future works the interpretation of the length of the intervals in a given decision making problem and its relation with ignorance functions and possibility theory.

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Interval-valued Atanassov intuitionistic OWA aggregations using admissible linear orders and their application to decision making

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***Abstract*—Based on the definition of admissible order for interval-valued Atanassov intuitionistic fuzzy sets, we study OWA operators in these sets distinguishing between the weights associated to the membership and those associated to the non-membership degree which may differ from the latter. We also study Choquet integrals for aggregating information which is represented using interval-valued Atanassov intuitionistic fuzzy sets. We conclude with two algorithms to choose the best alternative in a decision making problem when we use this kind of sets to represent information.**

***Index Terms*—Interval-valued Atanassov intuitionistic fuzzy set; interval-valued Atanassov intuitionistic OWA operator; Unbalanced interval-valued Atanassov intuitionistic OWA operator; interval-valued Atanassov intuitionistic Choquet Integral.**

I. Introduction

OWA operators [1] and Choquet integrals [2] are often used for fusing information. The goal of this paper is to extend these operators for using Interval-Valued Atanassov Intuitionistic Fuzzy Sets (IVAIFSs)[3] to represent the information. To achieve this goal, we focus on the definition and construction of a particular class of linear orders on IVAIFSs called admissible and defined in [4].

We know that, in order to aggregate information represented by means of IVAIFSs, we should aggregate the intervals which represent the membership, on one hand, and the intervals which represent the non-membership, on the other hand. This fact has suggested us to propose the use of Unbalanced OWA operators, in the sense that we use admissible orders and different weight vectors for aggregating the membership and the non-membership values.

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Besides, it is known that for some decision making problems, experts may have problems to provide exact numerical values to represent their preferences and non-preferences between the different alternatives. In these cases, some authors advise for the use of intervals ([5], [6], [7], [8]) to represent such preferences and non-preferences. In this situation, a suitable option is to represent the information by means of IVAIFSs. To do so, it is necessary an appropriate theoretical development of aggregation functions such as OWAs and Choquet integrals [9]. We discuss the usefulness of our theoretical developments in the last part of this work, where we present two algorithms to select the best alternative in a multi-expert decision making problem where the preferences are given as IVAIFSs.

This work is organized as follows. In Preliminaries we discuss several concepts which are going to be used along the paper. In Section III, we recall the definition and two methods of construction of interval-valued Atanassov intuitionistic fuzzy admissible orders using aggregation functions. Next, we develop OWA operators for interval-valued Atanassov intuitionistic vectors in Section IV while in Section V we propose Unbalanced OWA operators for IVAIFSs. In Section VI, we introduce the discrete Choquet integral for these sets and we present two algorithms which make use of admissible orders, OWA operators and Choquet integrals when we are dealing with IVAIFSs in a multi-expert decision making problem in Section VII. We finish with some concluding remarks and references.

II. Preliminaries

In this section we introduce some preliminary notions in order to fix notation. We denote $L([0,1])$ to the set of all closed subintervals of the unit interval, that is,

$$L([0,1]) = \{\mathbf{x} = [\underline{x_1}, \overline{x_1}] \mid 0 \leq \underline{x_1} \leq \overline{x_1} \leq 1\}\,.$$

Since Zadeh [10] introduced the concept of fuzzy sets different generalizations have been defined, see [11]. In particular, Interval-Valued Atanassov Intuitionistic Fuzzy Sets (IVAIFSs) generalize the concept of intuitionistic fuzzy sets given in 1986 [12] by Atanassov. In this work we introduce the notions of OWA operator and discrete Choquet integral for IVAIFSs. Some other studies on these sets are [13], [14], [15].

Definition 2.1: ([3]) Let $U \neq \emptyset$ be a universe. An Interval-Valued Atanassov Intuitionistic Fuzzy Set (IVAIFS) G over U is the set

$$G = \{(u, \mathbf{x}_u, \mathbf{y}_u) \mid u \in U\}$$

where $\mathbf{x}_u = [\underline{x(u)}, \overline{x(u)}]$, $\mathbf{y}_u = [\underline{y(u)}, \overline{y(u)}] \in L([0,1])$ state, respectively, the membership degree and the nonmembership degree of u to G and they satisfy that for all $u \in U$, $\overline{x(u)} + \overline{y(u)} \leq 1$.

Pairs $(\mathbf{x}_u, \mathbf{y}_u)$ are called Interval-Valued Atanassov Intuitionistic Fuzzy pairs (IVAIF-pairs), while the set of all possible IVAIF-pairs is denoted

$$\mathcal{L}_{IV}([0,1]) = \{z = (\mathbf{x}, \mathbf{y}) \mid \mathbf{x}, \mathbf{y} \in L([0,1]) \text{ and } \overline{x} + \overline{y} \leq 1\}\ .$$

Given an IVAIFS G, $z_G(u) = (\mathbf{x}_{G,u}, \mathbf{y}_{G,u})$ denotes the IVAIF-pair associated with the referential element $u \in U$. However, for the sake of simplicity, $z_G(u)$ is abbreviated to $z = (\mathbf{x}, \mathbf{y})$ when possible.

Definition 2.2: Let G_1, G_2 be two IVAIFSs. Atanassov proposed an order $\leq$ such that $G_1 \leq G_2$ if and only if, for all $u \in U$,

$$\mathbf{x}_{G_1,u} \preceq_2 \mathbf{x}_{G_2,u} \text{ and } \mathbf{y}_{G_1,u} \succeq_2 \mathbf{y}_{G_2,u}, \qquad (1)$$

where $\preceq_2$ is the partial order on $L([0,1])$, given by

$$[\underline{p_1}, \overline{p_1}] \preceq_2 [\underline{p_2}, \overline{p_2}] \text{ if and only if } \underline{p_1} \leq \underline{p_2} \text{ and } \overline{p_1} \leq \overline{p_2}\,. \quad (2)$$

Aggregation functions [16], [17] are a frequently used tool in fuzzy logic with their extensions and their applications.

Definition 2.3: Let $(P, \preceq)$ be a Partially Ordered Set (*i.e.*, a poset) with bottom 0_P and top 1_P. An n-aggregation function with respect to the order $\preceq$ is a mapping $M : P^n \to P$ such that

- $M(0_P, \ldots, 0_P) = 0_P$ and $M(1_P, \ldots, 1_P) = 1_P$,
- if $(x_1, \ldots, x_n) \preceq (y_1, \ldots, y_n)$,
  then $M(x_1, \ldots, x_n) \preceq M(y_1, \ldots, y_n)$,

where $(x_1, \ldots, x_n) \preceq (y_1, \ldots, y_n)$ if and only if $x_i \preceq y_i$ for all $i \in \{1, \ldots, n\}$.

Note that Definition 2.3 extends the definition of aggregation operator on the unit interval $[0,1]$ (see [18]). In particular, in this work we study some specific aggregation functions that are OWA operators and Choquet integrals.

Definition 2.4: [1] Let $w = (w_1, \ldots, w_n) \in [0,1]^n$ with $w_1 + \ldots + w_n = 1$. The Ordered Weighted Averaging (OWA) operator associated with w is a mapping $OWA_w : [0,1]^n \longrightarrow [0,1]$ given by

$$OWA_w(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i x_{(i)}$$

where $x_{(i)}$, $i \in \{1, \ldots, n\}$, denotes the i-th greatest component of $(x_1, \ldots, x_n)$.

Definition 2.5: Let $U \neq \emptyset$ be a finite universe. A fuzzy measure m is a mapping $m : 2^U \longrightarrow [0,1]$ such that $m(\emptyset) = 0$, $m(U) = 1$ and $m(F_1) \leq m(F_2)$ whenever $F_1 \subset F_2$ for $F_1, F_2 \in 2^U$.

Definition 2.6: Let m be a fuzzy measure on a non-empty finite universe $U = \{u_1, \ldots, u_n\} \neq \emptyset$. The discrete Choquet integral of a fuzzy set $\mu : U \longrightarrow [0,1]$ with respect to m is defined as

$$C_m(\mu) = \sum_{i=1}^{n} \mu(u_{\sigma(i)}) m(\{u_{\sigma(i)}, \ldots, u_{\sigma(n)}\}) - \mu(u_{\sigma(i)}) m(\{u_{\sigma(i+1)}, \ldots, u_{\sigma(n)}\})\,,$$

where σ is a permutation such that $\mu(u_{\sigma(1)}) \leq \mu(u_{\sigma(2)}) \leq \ldots \leq \mu(u_{\sigma(n)})$ and, $m(\{u_{\sigma(n+1)}, u_{\sigma(n)}\}) = 0$ by convention.

As defined in [19], a fuzzy measure is called symmetric if $m(F)$ depends only on the cardinality of the set $F \subseteq U$, i.e., for any $F_1, F_2 \subseteq U$ if $|F_1| = |F_2|$ then $m(F_1) = m(F_2)$. OWA operators are a special class of Choquet integrals where the fuzzy measure associated is symmetric.

A relevant characteristic of OWA operators and Choquet integrals is the fact that they require a linear order on the set of inputs. Hence, although the definition of aggregation function is grounded on partial orders, linear orders play a very relevant role in the generation of some of these operators. In the development of the paper, some results of [9], [20] are applied.

An order $\leq$ on $L([0,1])$ is called *admissible* if it is linear and satisfies that, for all $\mathbf{x}, \mathbf{y} \in L([0,1])$, such that $\mathbf{x} \preceq_2 \mathbf{y}$ then $\mathbf{x} \leq \mathbf{y}$ [20].

Proposition 2.1: [20] Let $B_1, B_2 : [0,1]^2 \mapsto [0,1]$ be two continuous aggregation functions, such that, for all $\mathbf{x} = [\underline{x}, \overline{x}], \mathbf{y} = [\underline{y}, \overline{y}] \in L([0,1])$, the equalities $B_1(\underline{x}, \overline{x}) = B_1(\underline{y}, \overline{y})$ and $B_2(\underline{x}, \overline{x}) = B_2(\underline{y}, \overline{y})$ hold if and only if $\mathbf{x} = \mathbf{y}$.

If the order $\leq_{B_{1,2}}$ on $L([0,1])$ is defined by $\mathbf{x} \leq_{B_{1,2}} \mathbf{y}$ if and only if

$$B_1(\underline{x}, \overline{x}) < B_1(\underline{y}, \overline{y}) \quad \text{or} \quad (B_1(\underline{x}, \overline{x}) = B_1(\underline{y}, \overline{y}) \text{ and } B_2(\underline{x}, \overline{x}) \leq B_2(\underline{y}, \overline{y}))$$

then $\leq_{B_{1,2}}$ is an admissible order on $L([0,1])$.

In [9], this class of linear orders on $L([0,1])$ is used to extend the definition of OWA operators for interval-valued fuzzy sets.

Definition 2.7: Let $\leq$ be an admissible order on $L([0,1])$, and let $w = (w_1, \ldots, w_n) \in [0,1]^n$, with $w_1 + \ldots + w_n = 1$. The Interval-Valued OWA operator associated with $\leq$ and w is a mapping $IVOWA_{[\leq,w]} : (L([0,1]))^n \longrightarrow L([0,1])$ given by

$$IVOWA_{[\leq,w]}([a_1, b_1], \ldots, [a_n, b_n]) = \sum_{i=1}^{n} w_i \cdot [a_{(i)}, b_{(i)}]$$

where $[a_{(i)}, b_{(i)}]$, $i = 1, \ldots, n$ denotes the i-th greatest of the inputs with respect to the order $\leq$, and the operations are $w \cdot [a, b] = [wa, wb]$ and $[x_1, y_1] + [x_2, y_2] = [x_1 + x_2, y_1 + y_2]$.

III. IVAIF-ADMISSIBLE ORDERS

Since the aim of the paper is to define OWA operators and Choquet Integrals for IVAIFS, the generation of linear orders plays a crucial role. The linear orders we based on the rest of the work were presented in [4] where a study on linear orders similar to the previous ones on $L([0,1])$ [9], but focused on IVAIFS is done.

An IVAIF-admissible order $\leq$ on $\mathcal{L}_{IV}([0,1])$ is a linear order which refines Atanassov's partial order in Eq. (1). We recall a method to construct IVAIF-admissible orders [4] .

Proposition 3.1: Let $B = \langle B_1, B_2, B_3, B_4\rangle$ be a set of four aggregation functions, $B_i : [0,1]^2 \longrightarrow [0,1]$ such that they generate admissible orders $\leq_{B_{1,2}}, \leq_{B_{3,4}}$ on $L([0,1])$ as in the Proposition 2.1. An IVAIF-admissible order $\leq_B$ on $(L([0,1]))^2$ is such that $(\mathbf{x}_1, \mathbf{y}_1) \leq_B (\mathbf{x}_2, \mathbf{y}_2)$ if an only if

$$\mathbf{x}_1 <_{B_{1,2}} \mathbf{x}_2 \quad \text{or} \quad (\mathbf{x}_1 =_{B_{1,2}} \mathbf{x}_2 \text{ and } \mathbf{y}_1 \geq_{B_{3,4}} \mathbf{y}_2)\,.$$

In particular, if $B_1 = B_3$ and $B_2 = B_4$, the orders $\leq_{B_{1,2}}$ and $\leq_{B_{3,4}}$ are equal: the same order is used to compare both intervals although in the second one the order is reversed.

Proposition 3.2: Let $A = \langle A_1, A_2, A_3, A_4\rangle$ be a set of four aggregation functions, $A_i : [0,1]^4 \to [0,1]$ such that, for all $(p_1, q_1, r_1, s_1), (p_2, q_2, r_2, s_2) \in [0,1]^4$ the equalities $A_i(p_1, q_1, r_1, s_1) = A_i(p_2, q_2, r_2, s_2)$ for all $i \in \{1, \ldots, 4\}$ only hold simultaneously if $(p_1, q_1, r_1, s_1) = (p_2, q_2, r_2, s_2)$.

Then, the relation $(\mathbf{x}_1, \mathbf{y}_1) \leq_A (\mathbf{x}_2, \mathbf{y}_2)$ if and only if one of the (mutually exclusive) conditions is satisfied

i) $A_1(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) < A_1(\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$.

ii) $A_1(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) = A_1(\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$ and
$A_2(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) < A_2(\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$.

iii) $A_1(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) = A_1(\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$,
$A_2(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) = A_2(\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$ and
$A_3(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) < A_3(\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$.

iv) $A_1(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) = A_1(\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$,
$A_2(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) = A_2(\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$,
$A_3(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) = A_3(\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$ and
$A_4(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) \leq A_4(\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$

is an IVAIF-admissible order on $\mathcal{L}_{IV}([0,1])$.

Example 3.3: Particular examples of $\leq_A$ orders are the following.

1) Let Q_i refer to an aggregation function given by
$$Q_i(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = a_i \underline{x} + b_i \overline{x} + c_i \underline{y} + d_i \overline{y}\,,$$
such that $a_i, b_i, c_i, d_i \geq 0$, $a_i, b_i, c_i, d_i \geq 0$, $a_i + b_i + c_i + d_i = 1$. Let $A = \langle Q_1, Q_2, Q_3, Q_4\rangle$ be the set of four of such aggregation functions satisfying
$$|D| = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \neq 0\,.$$
Since the aggregation functions satisfy the conditions of Proposition 3.2, they generate an IVAIF-admissible order. We denote this particular class of orders by $\leq_Q$.

2) Let $i \in \{1,2\}$, $j \in \{3,4\}$ and $k_i, k_j \in [0,1]$. Then the functions:
   - $K_i(\underline{x}, \overline{x}, \underline{y}, \overline{y}) = \underline{x} + k_i(\overline{x} - \underline{x})$, $i \in \{1,2\}$ and
   - $K_j(\underline{x}, \overline{x}, \underline{y}, \overline{y}) = \underline{y} + k_j(\overline{y} - \underline{y})$, $j \in \{3,4\}$

   are aggregation functions since $\underline{x} \leq \overline{x}$ and $\underline{y} \leq \overline{y}$.

   The orders generated by sets of the type $A = \langle K_1, K_2, K_3, K_4\rangle$ are particular instances of $\leq_Q$ orders where
$$D = \begin{pmatrix} 1-k_1 & k_1 & 0 & 0 \\ 1-k_2 & k_2 & 0 & 0 \\ 0 & 0 & 1-k_3 & k_3 \\ 0 & 0 & 1-k_4 & k_4 \end{pmatrix}$$
   whose determinant is $|D| = (k_2 - k_1)(k_4 - k_3)$. If $k_1 \neq k_2$ and $k_3 \neq k_4$ then the aggregation functions satisfy the conditions of the Proposition 3.2 and they generate an IVIAIF-admissible order.

   We denote this particular class of orders by $\leq_K$.

3) Let Π_i refer to the i-th projection of a 4-place vector. A general lexicographic order, denoted by $\leq_\Pi$ is constructed from the four projections. For instance:
   - the lexicographic 1 order is generated by the set $A = \langle \Pi_1, \Pi_2, \Pi_3, \Pi_4\rangle$ that we denote with $\leq_{\Pi_1}$;
   - the composed lexicographic 2 order is generated by the set $A = \langle \Pi_2, \Pi_1, \Pi_4, \Pi_3\rangle$ that we denote with $\leq_{\Pi_2}$.

   Notice that also the projections are particular instances of K_i or K_j aggregation functions for $k_i, k_j \in \{0,1\}$. Thereby, the determinant of $|D|$ is 1 or -1 since each row and each column of the matrix D is generated by all zeros except one element whose value is 1. For instance, the matrix D associated with the composed lexicographic 2 order is
$$D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

IV. Interval-valued Atanassov's intuitionistic OWA operators

The original definition of OWA operator was given by Yager [1]. In this section we generalize this definition on interval-valued Atanassov intuitionistic fuzzy sets using IVAIF-admissible orders and we study under which conditions they satisfy monotonicity.

Definition 4.1: Let $\leq$ be an IVAIF-admissible order on $\mathcal{L}_{IV}([0,1])$ and let $w = (w_1, \ldots, w_n) \in [0,1]^n$, with $w_1 + \ldots + w_n = 1$. The Interval-Valued Atanassov's Intuitionistic OWA (IVAIOWA) operator associated with w and $\leq$ is a mapping $(\mathcal{L}_{IV}([0,1]))^n \mapsto \mathcal{L}_{IV}([0,1])$ defined by

$$\begin{aligned} IVAIOWA_{[w,\leq]}(z_1, \ldots, z_n) &= \sum_{i=1}^n w_i \cdot z_{(i)} \\ &= \left(\sum_{i=1}^n w_i \cdot \mathbf{x}_{(i)}, \sum_{i=1}^n w_i \cdot \mathbf{y}_{(i)}\right) \\ &= \left(\sum_{i=1}^n w_i \cdot [\underline{x}_{(i)}, \overline{x}_{(i)}], \sum_{i=1}^n w_i \cdot [\underline{y}_{(i)}, \overline{y}_{(i)}]\right) \end{aligned}$$

where $z_{(i)}$ denotes the i-th greatest IVAIF-pair of the inputs $(z_1, \ldots, z_n)$ with respect to the order $\leq$ on $\mathcal{L}_{IV}([0,1])$ and the interval product and sum are the same as used in Definition 2.7.

Notice that $w_1\underline{x}_{(1)} + \ldots + w_n\underline{x}_{(n)} \leq w_1 + \ldots + w_n = 1$ (indeed, the same holds for $\overline{x}$, $\underline{y}$, and $\overline{y}$). The monotonicity of real-valued weighted arithmetic means ensures that each of the components of the result yield by an IVAIOWA is an interval. In addition, the result is always an IVAIF-pair, since

$$\sum_{i=1}^{n} w_i\overline{x}_{(i)} + \sum_{i=1}^{n} w_i\overline{y}_{(i)} = \sum_{i=1}^{n} w_i(\overline{x}_{(i)} + \overline{y}_{(i)}) \leq \sum_{i=1}^{n} w_i = 1\,.$$

Example 4.1:

Let $w = (0.15, 0.35, 0.5)$ and let $\leq_Q$ be the order generated by the set $A = \langle Q_1, Q_2, Q_3, Q_4\rangle$ as in the Example 3.3 with

- $Q_1(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \frac{3}{8}\underline{x}_1 + \frac{3}{8}\overline{x}_1 + \frac{1}{8}\underline{y}_1 + \frac{1}{8}\overline{y}_1,$
- $Q_2(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \frac{10}{20}\underline{x}_1 + \frac{5}{20}\overline{x}_1 + \frac{3}{20}\underline{y}_1 + \frac{2}{20}\overline{y}_1,$
- $Q_3(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \frac{1}{20}\underline{x}_1 + \frac{10}{20}\overline{x}_1 + \frac{8}{20}\underline{y}_1 + \frac{1}{20}\overline{y}_1,$
- $Q_4(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \frac{1}{4}\underline{x}_1 + \frac{1}{4}\overline{x}_1 + \frac{1}{4}\underline{y}_1 + \frac{1}{4}\overline{y}_1.$

Taking into account that
$([0.8, 1], [0, 0]) \geq_Q ([0, 0.5], [0, 0.5]) \geq_Q ([0, 0.3], [0.2, 0.7])$
it holds

$$\begin{aligned} &IVAIOWA_{[w,\leq_Q]}(([0, 0.3], [0.2, 0.7]), ([0.8, 1], [0, 0]),\\ &\quad ([0, 0.5], [0, 0.5])) = ([0.12, 0.475], [0.1, 0.525]). \end{aligned}$$

Remark 1: It is important to mention:

1) Due to the characteristics of the orders above, in general, it is not true that

$$\begin{aligned} &IVAIOWA_{[w,\leq]}(z_1, \ldots, z_n)\\ &\quad = (IVOWA_{[w,\leq_{B_{1,2}}]}(\mathbf{x_1}, \ldots, \mathbf{x_n}),\\ &\qquad IVOWA_{[w,\leq_{B_{1,2}}]}(\mathbf{y_1}, \ldots, \mathbf{y_n}))\,. \quad (3) \end{aligned}$$

In fact, this is not necessarily true even if the order is $\leq_B$, with $B = \langle B_1, B_2, B_1, B_2\rangle$. For example, consider the weight vector $w = (0.2, 0.5, 0.3)$ and the $\leq_{\Pi_1}$ order on $\mathcal{L}_{IV}([0, 1])$. Taking into account that $([0.2, 0.8], [0, 0]) \geq_{\Pi_1} ([0.1, 0.2], [0.4, 0.8]) \geq_{\Pi_1} ([0, 0.3], [0.5, 0.7])$ (see item 3 of Example 3.3) it holds

$$\begin{aligned} &IVAIOWA_{[w,\leq_{\Pi_1}]}(([0.2, 0.8], [0, 0]),\\ &\quad ([0, 0.3], [0.5, 0.7]), ([0.1, 0.2], [0.4, 0.8]))\\ &\qquad = ([0.09, 0.35], [0.35, 0.61]) \end{aligned}$$

Similarly, since $[0.2, 0.8] \geq_{\Pi_{1,2}} [0.1, 0.2] \geq_{\Pi_{1,2}} [0, 0.3]$ and $[0.5, 0.7] \geq_{\Pi_{1,2}} [0.4, 0.8] \geq_{\Pi_{1,2}} [0, 0]$ with $\Pi_{1,2} = \langle \Pi_1, \Pi_2\rangle$ then
$IVOWA([0.2, 0.8], [0, 0.3], [0.1, 0.2]) = [0.09, 0.35]$ and
$IVOWA([0, 0], [0.5, 0.7], [0.4, 0.8]) = [0.3, 0.54]$.
However, $([0.09, 0.35], [0.35, 0.61]) \neq ([0.09, 0.35], [0.3, 0.54])$.

2) Due to the monotonicity of OWAs, if we apply fixed weights to $\underline{x}$, $\overline{x}$, $\underline{y}$, $\overline{y}$, we produce the corresponding intervals. For example, taking the weight vector $w = (0.2, 0.5, 0.3)$, such that we have the following results:

- $OWA_w(0.2, 0, 0.1) = 0.09$
- $OWA_w(0.8, 0.3, 0.2) = 0.37$
- $OWA_w(0, 0.5, 0.4) = 0.3$
- $OWA_w(0, 0.7, 0.8) = 0.51,$

we observe that the IVAI-pair $([0.09, 0.37], [0.3, 0.51]) \neq ([0.09, 0.35], [0.4, 0.6])$
In fact, we obtain the first one, by ordering the extreme values of the intervals and not the intervals (see Aumann [21]).
This, or some others approaches generalizing the theoretical results on Atanassov Intuitionistic Fuzzy Sets such as [22] would be another manners of generalizing OWA operators over IVAIFSs but in this work we focus on studying the first approach of OWA operator given in Def 4.1. However, this example lets some open questions: *What are the differences of these definitions? Are there any relation between the orders? Are there any orders which generate the same OWA operator for different definitions?*

Proposition 4.2: Let be the IVAIOWA operator associated with $\leq$ and w. Given an order $\leq_{B_{1,2}}$ on $L([0, 1])$ there exist w' and w'' two permutations of the vector w induced by $\leq_{B_{1,2}}$ such that

$$\begin{aligned} &IVAIOWA_{[w,\leq]}(z_1, \ldots, z_n)\\ &\quad = (IVOWA_{[w',\leq_{B_{1,2}}]}(\mathbf{x_1}, \ldots, \mathbf{x_n}),\\ &\qquad IVOWA_{[w'',\leq_{B_{1,2}}]}(\mathbf{y_1}, \ldots, \mathbf{y_n})). \quad (4) \end{aligned}$$

Proof. Straight. Besides, if $\mathbf{x}_{(i)}$ is the i-th greatest element according to $\leq_{B_{1,2}}$ then $w_i' = w_j$ where j is the position of the IVAIF-pair whose first interval is $\mathbf{x}_{(i)}$ through the $\leq$ order of IVAIF-pairs, namely, $z_{(j)} = (\mathbf{x}_{(i)}, \mathbf{y}^*)$.

Analogously, if $\mathbf{y}_{(i)}$ is the i-th greatest element according to $\leq_{B_{1,2}}$ then $w_i'' = w_j$ where j is the position of the IVAIF-pair whose second interval is $\mathbf{y}_{(i)}$ through the $\leq$ order of IVAIF-pairs, namely, $z_{(j)} = (\mathbf{x}^*, \mathbf{y}_{(i)})$.

Corollary 4.3: Let be $\leq_B$ with $B = \langle B_1, B_2, B_1, B_2\rangle$ and $\leq_{B_{1,2}}$ the orders used in the Prop. 4.2. Then $w' = w$ satisfies Eq. (4).

Proof. Remember that $\leq_B$ uses first $\leq_{B_{1,2}}$ on $L([0, 1])$ to compare the membership interval so that the elements in the set $\{\mathbf{x}_i \mid i \in \{1, \ldots, n\}\}$ are ordered in a decreasing order.

OWA operators, with the standard order between real numbers, are a special class of aggregation functions. However, as it occurs with IVOWA operators ([9]), IVAIOWA operators do not necessarily satisfy monotonicity.

Example 4.4: Let $B = \langle B_1, B_2, B_1, B_2\rangle$ with $B_1(x_1, x_2) = x_1x_2$ and $B_2(x_1, x_2) = \frac{x_1+x_2}{2}$.

Let $z_1 = ([0, 1], [0, 0])$, $z_2 = ([0.2, 0.35], [0.4, 0.6])$ and $w = (0.9, 0.1)$.

Taking into account that $z_2 >_B z_1$ it holds

$$IVAIOWA_{[w,\leq_B]}(z_1, z_2) = ([0.18, 0.415], [0.36, 0.54])\,.$$

However, if we take $\tilde{z}_1$ given by $\tilde{z}_1 = ([0.1, 0.15], [0.7, 0.8]) >_B z_1 = ([0, 1], [0, 0])$ it holds

$z_2 \geq_B \tilde{z}_1$ and consequently,

$$IVAIOWA_{[w,\leq_B]}(\tilde{z}_1, z_2) = ([0.19, 0.33], [0.43, 0.62]) .$$

Since $([0.19, 0.33], [0.43, 0.62]) <_B ([0.18, 0.415], [0.36, 0.54])$ the monotonicity of the operator does not hold.

For sake of simplicity, given M a 4-aggregation function we write $M(z)$ where z is the IVAIF-pair given by $z = ([\underline{x}, \overline{x}], [\underline{y}, \overline{y}])$ to denote the image of $M(\underline{x}, \overline{x}, 1-\underline{y}, 1-\overline{y})$.

Proposition 4.5: Let be $\leq_Q$ the admissible order generated as in the Example 3.3 then

$$Q_i(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, z_n)) = w_1 Q_i(z_{(1)}) + \ldots + w_n Q_i(z_{(n)}) , \quad (5)$$

where $z_{(i)}$ denotes the i-th greatest IVAIF-pair of the inputs $(z_1, \ldots, z_n)$ through the order $\leq_Q$ on $\mathcal{L}_{IV}([0,1])$.

Proof. Straight.

Remark 2: In the Proposition 4.5, for $i = 1$ we have

$$Q_1(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, z_n)) = OWA_w(Q_1(z_1), \ldots, Q_1(z_n)) .$$

However, this does not necessarily hold for any other index.

Proposition 4.6: Let $\leq_Q$ be the order generated as in the Example 3.3 and $w \in (0,1]^n$. An IVAIOWA operator on $\mathcal{L}_{IV}([0,1])$ associated with $\leq_Q$ and w is an aggregation function.

Proof. In order to simplify notation, we assume that the IVAIF-pairs $(z_1, \ldots, z_n)$ are ordered in a decreasing way with respect to the order $\leq_Q$, i.e., $z_1 \geq_Q z_2 \geq_Q \ldots \geq_Q z_n$. Notice that since IVAIOWA operators are symmetric, we do not lose generality by this assumption.

The boundary conditions are straight, but the monotonicity of the function needs to be proven. Let us assume that the IVAIOWA is not monotone. Then, there exist $\hat{z}_i$ satisfying that $z_i \leq_Q \hat{z}_i$ and such that

$$IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, z_i, \ldots, z_n) >_Q IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, \hat{z}_i, \ldots, z_n) .$$

If $z_i = \hat{z}_i$ then $IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, z_i, \ldots, z_n) =_Q IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, \hat{z}_i, \ldots, z_n)$ so we require $z_i <_Q \hat{z}_i$. There exist four different cases:

i) $Q_1(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, z_i, \ldots, z_n)) > Q_1(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, \hat{z}_i, \ldots, z_n))$

ii) $Q_1(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, z_i, \ldots, z_n)) = Q_1(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, \hat{z}_i, \ldots, z_n))$ and $Q_2(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, z_i, \ldots, z_n)) > Q_2(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, \hat{z}_i, \ldots, z_n))$.

iii) $Q_1(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, z_i, \ldots, z_n)) = Q_1(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, \hat{z}_i, \ldots, z_n))$ and $Q_2(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, z_i, \ldots, z_n)) = Q_2(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, \hat{z}_i, \ldots, z_n))$ and $Q_3(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, z_i, \ldots, z_n)) > Q_3(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, \hat{z}_i, \ldots, z_n))$.

iv) $Q_1(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, z_i, \ldots, z_n)) = Q_1(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, \hat{z}_i, \ldots, z_n))$ and $Q_2(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, z_i, \ldots, z_n)) = Q_2(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, \hat{z}_i, \ldots, z_n))$ and $Q_3(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, z_i, \ldots, z_n)) = Q_3(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, \hat{z}_i, \ldots, z_n))$ and $Q_4(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, z_i, \ldots, z_n)) > Q_4(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, \hat{z}_i, \ldots, z_n))$.

We tackled them individually:

i) If $Q_1(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, z_i, \ldots, z_n)) > Q_1(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, \hat{z}_i, \ldots, z_n))$, then by the Remark 2,

$$\begin{aligned} Q_1(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, z_i, \ldots, z_n)) &= \\ OWA(Q_1(z_1), \ldots, Q_1(z_i), \ldots, Q_1(z_n)) &> \\ OWA(Q_1(z_1), \ldots, Q_1(\hat{z}_i), \ldots, Q_1(z_n)) & \\ = Q_1(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, \hat{z}_i, \ldots, z_n)). & \end{aligned}$$

By the increasing monotonicity of OWA operators this implies $Q_1(z_i) > Q_1(\hat{z}_i)$ which contradicts $z_i \leq_Q \hat{z}_i$.

ii) If $Q_1(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, z_i, \ldots, z_n)) = Q_1(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, \hat{z}_i, \ldots, z_n))$, then by Eq. (5), $Q_1(z_i) = Q_1(\hat{z}_i)$. Two cases can be further discriminated:

- If the order of the IVAIF-pairs has not changed, by the Proposition 4.6, then

$$\begin{aligned} Q_2(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, z_i, \ldots, z_n)) &= \\ w_1 Q_2(z_1) + \ldots + w_i Q_2(z_i) + \ldots w_n Q_2(z_n) &> \\ w_1 Q_2(z_1) + \ldots + w_i Q_2(\hat{z}_i) + \ldots w_n Q_2(z_n) &= \\ Q_2(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, \hat{z}_i, \ldots, z_n)) , & \end{aligned}$$

  which implies $Q_2(z_i) > Q_2(\hat{z}_i)$, in contradiction with $z_i \leq_Q \hat{z}_i$.

- If the order of the IVAIF-pairs has changed r positions, then it holds true

$$Q_1(\hat{z}_i) = Q_1(z_{i-r}) = \ldots = Q_1(z_{i-1}) = Q_1(z_i)$$

  and

$$Q_2(\hat{z}_i) \geq Q_2(z_{i-r}) \geq \ldots \geq Q_2(z_{i-1}) \geq Q_2(z_i). \quad (6)$$

  However,

$$\begin{aligned} & Q_2(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, z_i, \ldots, z_n)) \\ & = w_1 Q_2(z_1) + \ldots + w_i Q_2(z_i) + \ldots + w_n Q_2(z_n) \\ & > w_1 Q_2(z_1) + \ldots + w_{i-r} Q_2(\hat{z}_i) + w_{i-(r-1)} Q_2(z_{i-r}) \\ & + \ldots + w_i Q_2(z_{i-1}) + w_{i+1} Q_2(z_{i+1}) + \ldots + w_n Q_2(z_n) \\ & = Q_2(IVAIOWA_{[w,\leq_Q]}(z_1, \ldots, \hat{z}_i, \ldots, z_n)). \end{aligned}$$

  This implies that

$$\begin{aligned} & w_{i-r} Q_2(z_{i-r}) + w_{i-(r-1)} Q_2(z_{i-(r-1)}) + \ldots + w_i Q_2(z_i) \\ & > w_{i-r} Q_2(\hat{z}_i) + w_{i-(r-1)} Q_2(z_{i-r}) + \ldots + w_i Q_2(z_{i-1}), \end{aligned}$$

  which is in contradiction with (6).

  Items iii) and iv) are similar to item ii).

V. Unbalanced Interval-valued Intuitionistic OWA operators

In Def. 4.1 IVAIOWA operators used a fixed weight vector for both membership and nonmembership degrees. This is certainly practical, but at some situations it might be desirable to treat them differently. In this section we study the extension of IVAIOWA operators to cope with different weight vectors for the membership and nonmembership degrees.

Definition 5.1: Let $\leq$ be an IVAIF-admissible order on $\mathcal{L}_{IV}([0,1])$ and let $w, v \in [0,1]^n$ with $w_1 + \ldots + w_n = 1$ and $v_1 + \ldots + v_n = 1$. An Unbalanced IVAIOWA operator associated with w, v and $\leq$ is a mapping $(\mathcal{L}_{IV}([0,1]))^n \longrightarrow (L([0,1]))^2$ given by

$$UIVAIOWA_{[w,v,\leq]}(z_1,\ldots,z_n) = \left(\sum_{i=1}^{n} w_i \cdot [\underline{x}_{(i)}, \overline{x}_{(i)}], \sum_{i=1}^{n} v_i \cdot [\underline{y}_{(i)}, \overline{y}_{(i)}]\right), \quad (7)$$

where $([\underline{x}_{(i)}, \overline{x}_{(i)}], [\underline{y}_{(i)}, \overline{y}_{(i)}])$ denotes the i-th greatest of the inputs $(z_1, \ldots, z_n)$ with respect to the order $\leq$ on $\mathcal{L}_{IV}([0,1])$ and the interval product and sum are the same as used in Definition 2.7.

Remark 3: IVAIOWA operators in Section 3 are particular instances of Unbalanced IVAIOWA operators with $w = v$.

Notice that if there exist an index i such that $w_i < v_i$, then there is an index j such that $w_j > v_j$.

Next, we study the conditions under which Unbalanced IVAIOWA operators are aggregation functions, that is:

- They satisfy the boundary conditions.
- They are monotonic.
- The co-domain is $\mathcal{L}_{IV}([0,1])$, i.e. , the image of n IVAIF-pairs is always an IVAIF-pair. This is satisfied if

$$\sum_{i=1}^{n} w_i \overline{x}_{(i)} + \sum_{i=1}^{n} v_i \overline{y}_{(i)} \leq 1.$$

Theorem 5.1: Unbalanced IVAIOWA operators always satisfy the boundary conditions.

Proof. The boundary conditions imply that

$$\begin{aligned} &UIVAIOWA_{[w,v,\leq]}(([1,1],[0,0]),\ldots,([1,1],[0,0])) \\ &= \left(\sum_{i=1}^{n} w_i \cdot [1,1], \sum_{i=1}^{n} v_i \cdot [0,0]\right) \\ &= ([1,1],[0,0]) \end{aligned}$$

which is satisfied due to $\sum_{i=1}^{n} w_i = 1$, and

$$\begin{aligned} &UIVAIOWA_{[w,v,\leq]}(([0,0],[1,1]),\ldots,([0,0],[1,1])) \\ &= \left(\sum_{i=1}^{n} w_i \cdot [0,0], \sum_{i=1}^{n} v_i \cdot [1,1]\right) \\ &= ([0,0],[1,1]) , \end{aligned}$$

which is satisfied due to $\sum_{i=1}^{n} v_i = 1$.

By the Proposition 4.6, the IVAIOWA operators associated with orders $\leq_Q$ (as in the Example 3.3) and weight vectors $w = v$ satisfy monotonicity. This property is not guaranteed for weight vectors $w \neq v$.

Example 5.2: Let $w = (0.1, 0.9)$, and $v = (0.9, 0.1)$ and let the order $\leq_Q$ be given by the four aggregation functions that follow:

- $Q_1(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \frac{1}{4}\underline{x}_1 + \frac{1}{4}\overline{x}_1 + \frac{1}{4}\underline{y}_1 + \frac{1}{4}\overline{y}_1$,
- $Q_2(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \overline{x}_1$,
- $Q_3(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \underline{y}_1$,
- $Q_4(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \overline{y}_1$.

Then, although $([0,0],[0,0]) <_Q ([0.5,0.5],[0.5,0.5])$, it holds

$$\begin{aligned} &UIVAIOWA_{[w,v,\leq_Q]}(([0,0],[0,0]),([0,0],[1,1])) \\ &= ([0,0],[0.1,0.1]) > ([0.05,0.05],[0.55,0.55]) \\ &UIVAIOWA_{[w,v,\leq_Q]}(([0.5,0.5],[0.5,0.5]),([0,0],[1,1])) . \end{aligned}$$

Consequently, $UIVAIOWA_{[w,v,\leq_Q]}$ is not monotonic.

Proposition 5.3: Let be $\leq_K$ an order generated as in the Example 3.3. Then the following holds

$$\begin{aligned} &K_i(UIVAIOWA_{[w,v,\leq_K]}(z_1,\ldots,z_n)) \\ &= w_1 K_i(z_{(1)}) + \ldots + w_n K_i(z_{(n)}), \quad i \in \{1,2\} \text{ and} \end{aligned}$$

$$\begin{aligned} &K_j(UIVAIOWA_{[w,v,\leq_K]}(z_1,\ldots,z_n)) \\ &= w_1 K_j(z_{(1)}) + \ldots + w_n K_j(z_{(n)}), \quad j \in \{3,4\}. \end{aligned}$$

Proof. Straight.

Proposition 5.4: Let be $\leq_K$ the order generated as in the Example 3.3, and let $w, v \in (0,1]^n$ with $w_1 + \ldots + w_n = 1$ and $v_1 + \ldots + v_n = 1$. Then the $UIVAIOWA_{[w,v,\leq_K]}$ operator satisfies monotonicity.

Proof. Considering the Proposition 5.3, the proof is almost analogous to that of the Proposition 4.6.

Next, we study when the image of Unbalanced IVAIOWA operators is guaranteed to be IVAIF-pairs, i.e., $\left(\sum_{i=1}^{n} w_i \overline{x}_{(i)} + \sum_{i=1}^{n} v_i \overline{y}_{(i)} \leq 1\right)$. Note that this is trivially satisfied with $w = v$, and in the remainder of this section we only consider $w \neq v$.

Besides, when $\overline{y}_i = 1 - \overline{x}_i$ the equation is reduced to

$$\sum_{i=1}^{n} w_i \overline{x}_{(i)} + \sum_{i=1}^{n} v_i (1 - \overline{x}_{(i)}) \ \leq 1 ,$$

which is equivalent to

$$\left(\sum_{i=1}^{n} w_i \overline{x}_{(i)} \leq \sum_{i=1}^{n} v_i \overline{x}_{(i)}\right). \quad (8)$$

Proposition 5.5: Let be $\leq_K$ an order generated as in the Example 3.3 with $k_1 \in (0,1)$, and let $w, v \in (0,1]^n$ with $w_1 + \ldots + w_n = 1$ and $v_1 + \ldots + v_n = 1$. Then, the operator $UIVAIOWA_{[w,v,\leq_K]}$ is not always an IVAIF-pair.

Proof. We define the set of indexes

$$\begin{aligned} I &= \{j \mid w_{j_1} = v_{j_1}\} , \\ J_1 &= \{j \mid w_{j_1} < v_{j_1}\} \text{ and} \\ J_2 &= \{j \mid w_{j_2} > v_{j_2}\} . \end{aligned}$$

As $w \neq v$, we have $J_1 \neq \emptyset$ and $J_2 \neq \emptyset$. Let $j_0 = \min J_1 \cup J_2$.

As w, v sum 1 and $\{1, \ldots, j_0 - 1\} \subseteq I$, then

$$\sum_{k=1}^{j_0-1} w_k = \sum_{k=1}^{j_0-1} v_k \quad \text{and} \quad \sum_{k=j_0}^{n} w_k = \sum_{k=j_0}^{n} v_k. \tag{9}$$

Let us show that there are always n IVAIF-pairs whose image does not satisfy Eq. (8). We separate the proof in two different cases.

- If $j_0 \in J_2$, then $w_{j_0} > v_{j_0}$. We choose the IVAIF-pairs

  i) $z_i = ([1;1],[0;0])$ for $i \in \{1, \ldots, j_0\}$;
  ii) $z_i = ([0;0],[1;1])$ for $i \in \{j_0+1, \ldots, n\}$.

  These IVAIF-pairs are top and bottom in $\mathcal{L}_{IV}([0,1])$ and they are ordered in a decreasing order. Since $\overline{x}_{(i)} = 0$ for $i \in \{j_0+1, \ldots, n\}$ Eq. (8) results it

  $$\sum_{i=1}^{j_0} w_i \leq \sum_{i=i}^{j_0} v_i\,.$$

  However, this only holds if

  $$\sum_{i=1}^{j_0-1} w_i + w_{j_0} \leq \sum_{i=i}^{j_0-1} v_i + v_{j_0}.$$

  By Eq. (9), this is equivalent to $w_{j_0} \leq v_{j_0}$, which is in contradiction with $j_0 \in J_2$.
- If $j_0 \in J_1$, since $w_{j_0} < v_{j_0}$ then

  $$\sum_{i=j_0+1}^{n} w_k > \sum_{i=j_0+1}^{n} v_k. \tag{10}$$

  We choose the IVAIF-pairs

  i) $z_i = ([1,1],[0,0])$ for $i \in \{1, \ldots, j_0-1\}$;
  ii) $z_{j_0} = ([0.4,0.4],[0.6,0.6])$;
  iii) $z_i = ([0,a],[0,1-a])$ for $i \in \{j_0+1, \ldots, n\}$, with $a \in (0.4,1]$ satisfying $a < \frac{0.4}{k_1}$ (this can hold as $k_1 \neq 1$).

  Notice that the IVAIF-pairs are ordered in a decreasing order. From 1 to $j_0 - 1$ the IVAIF-pairs are the top. Besides $z_{j_0} > z_{j_0+1}$ since $K_1([0.4,0.4],[0.6,0.6]) = 0.4 + k_1(0.4-0.4) = 0.4$ and $K_1([0,a],[0,1-a]) = 0 + k_1(a-0) = k_1 a$ but $k_1 a < k_1 \frac{0.4}{k_1} = 0.4$. As the last ones are equal they are also ordered.
  These IVAIF-pairs do not satisfy Eq. (8).

  $$\sum_{k=1}^{j_0-1} w_k + 0.4 w_{j_0} + \sum_{k=j_0+1}^{n} a w_k \leq \sum_{k=1}^{j_0-1} v_k + 0.4 v_{j_0} + \sum_{k=j_0+1}^{n} a v_k.$$

  First, by Eq.(9) the expression is reduced to

  $$0.4 w_{j_0} + \sum_{k=j_0+1}^{n} a w_k \leq 0.4 v_{j_0} + \sum_{k=j_0+1}^{n} a v_k.$$

  The expression can be rewritten

  $$0.4 \sum_{k=j_0}^{n} w_k + (a-0.4) \sum_{k=j_0+1}^{n} w_k \leq 0.4 \sum_{k=j_0}^{n} v_k + (a-0.4) \sum_{k=j_0+1}^{n} v_k$$

  if and only if

  $$(a - 0.4) \sum_{k=j_0+1}^{n} w_k \leq (a - 0.4) \sum_{k=j_0+1}^{n} v_k\,,$$

  where the equivalence is due to Eq. (9).
  Since $(a-0.4) > 0$, the expression is reduced to

  $$\sum_{k=j_0+1}^{n} w_k \leq \sum_{k=j_0+1}^{n} v_k\,,$$

  which is in contradiction with Eq. (10).
  Consequently, the image of the n IVAIF-pairs is not an IVAIF-pair and Unbalanced IVAIOWA operators could not be an aggregation function (since the domain and co-domain are different sets).

Remark 4: In [20], it was proven that given $k_1 \in (0,1]$, all the admissible orders on $L([0,1])$ with $k_2 < k_1$ are equivalent. In this case, for the particular case of Unbalanced IVAIOWA operators, only $k_1 = 1$ can lead to well-defined operators. As a consequence, all the possible admissible orders are given by $(\mathbf{x_1}, \mathbf{y_1}) \leq_B (\mathbf{x_2}, \mathbf{y_2})$ if and only if

- $(\overline{x}_1 < \overline{x}_2$), or
- $(\overline{x}_1 = \overline{x}_2$ and $\underline{x}_1 < \underline{x}_2)$, or
- $(\overline{x}_1 = \overline{x}_2$, $\underline{x}_1 = \underline{x}_2$, and
  $K_3(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) > K_3(\underline{x}_2, \overline{x}_2, \underline{y}_2, \overline{y}_2)$, or
- $(\overline{x}_1 = \overline{x}_2$, $\underline{x}_1 = \underline{x}_2$,
  $K_3(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = K_3(\underline{x}_2, \overline{x}_2, \underline{y}_2, \overline{y}_2)$ and
  $K_4(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) \geq K_4(\underline{x}_2, \overline{x}_2, \underline{y}_2, \overline{y}_2))$

for some $k_3, k_4 \in [0,1]$ with $k_3 \neq k_4$.

We refer to these orders as $\leq_O$.

Next, we study the conditions under which the orders $\leq_O$ define an aggregation function.

Lemma 5.6: Let be $w, v \in (\mathbb{R}^+)^n$. Then the following statements are equivalent.

i) $\sum_{j=1}^{i} w_j \leq \sum_{j=1}^{i} v_j$ for all $i \in \{1, \ldots, n\}$.
ii) $\sum_{j=1}^{n} w_j t_j \leq \sum_{j=1}^{n} v_j t_j$ for all $t_j \in [0,1]$ such that $t_1 \geq t_2 \geq \ldots \geq t_n \geq 0$.

Proof. We first prove that i) implies ii). As

$$w_1 \leq v_1 \text{ then } a_1 w_1 \leq a_1 v_1 \text{ for all } a_1 \geq 0.$$

$w_1 + w_2 \leq v_1 + v_2$ then

$$a_2(w_1 + w_2) \leq a_2(v_1 + v_2) \text{ for all } a_2 \geq 0.$$

$$\ldots$$

$w_1 + \ldots + w_n \leq v_1 + \ldots + v_n$ then
$a_n(w_1 + \ldots + w_n) \leq a_n(v_1 + \ldots + v_n)$ for all $a_n \geq 0$.

If we sum

$$(a_1 + \ldots + a_n)w_1 + (a_2 + \ldots + a_n)w_2 + \ldots + a_n w_n \leq (a_1 + \ldots + a_n)v_1 + (a_2 + \ldots + a_n)v_2 + \ldots + a_n v_n \quad \text{for all } a_1, \ldots, a_n \geq 0.$$

Taking $t_1 = (a_1 + \ldots + a_n)$, $t_2 = (a_2 + \ldots + a_n), \ldots, t_n = a_n$ it satisfies ii).

To prove that ii) implies i), given $i \in \{1, \ldots, n\}$ take $t_1 = t_2 = \ldots = t_i = 1$ and $t_{i+1} = t_{i+2} = \ldots = t_n = 0$.

Finally, we have the following characterization of Unbalanced IVAIOWA operators.

Theorem 5.7: Let $w, v \in [0,1]^n$ with $w_1 + \ldots + w_n = 1$ and $v_1 + \ldots + v_n = 1$. Then the following statements are equivalent:

i) Unbalanced IVAIOWA operators associated with w, v and the orders $\leq_O$ is an aggregation function;

ii) $\sum_{i=1}^{n} w_i t_i \leq \sum_{i=1}^{n} v_i t_i$ for all $i = 1, \ldots, n$ for all $t_i \in [0,1]$ such that $t_1 \geq t_2 \geq \ldots \geq t_n \geq 0$.

Proof. Let us show that i) implies ii). Suppose Unbalanced IVAIOWA operator is well defined. Then it satisfies Eq.(8) for the right endpoints of intervals $\sum_{i=1}^{n} w_i \overline{x}_{(i)} \leq \sum_{i=1}^{n} v_i \overline{x}_{(i)}$. Taking $t_i = \overline{x}_{(i)}$, it satisfies ii) since $\overline{x}_{(i)} \geq \overline{x}_{(j)}$ (because the order used is $\leq_O$).

Finally let us show that ii) implies i). First of all ii) can be rewritten as

$$\sum_{i=1}^{n} (w_i - v_i) t_i \leq 0, \text{ for all } t_i \in [0,1] \quad \text{such that } t_1 \geq t_2 \geq \ldots \geq t_n \geq 0. \tag{11}$$

Let $z_i = (\mathbf{x}_i, \mathbf{y}_i)$, $i = 1, \ldots, n$, be n IVAIF-pairs. The expression of Unbalanced IVAIOWA operator associated with w, v and one order $\leq_O$ is an aggregation function.

$$UIVAIOWA_{[w,v,\leq_O]}(z_1, \ldots, z_n) = \left(\sum_{i=1}^{n} w_i \cdot [\underline{x}_{(i)}, \overline{x}_{(i)}], \sum_{i=1}^{n} v_i \cdot [\underline{y}_{(i)}, \overline{y}_{(i)}] \right) = (\mathbf{x}, \mathbf{y})$$

where $\overline{x}_{(1)} \geq \overline{x}_{(2)} \geq \ldots \geq \overline{x}_{(n)}$ due to the order $\leq_O$.

Considering that $\overline{x}_{(i)} + \overline{y}_{(i)} \leq 1$ and $v_1 + v_2 + \ldots + v_n = 1$ then

$$\begin{aligned} &\overline{x} + \overline{y} = \\ &w_1 \overline{x}_{(1)} + w_2 \overline{x}_{(2)} + \ldots + w_n \overline{x}_{(n)} + v_1 \overline{y}_{(1)} + v_2 \overline{y}_{(2)} + \ldots + v_n \overline{y}_{(n)} \\ &\leq w_1 \overline{x}_{(1)} + w_2 \overline{x}_{(2)} + \ldots + w_n \overline{x}_{(n)} \\ &+ v_1(1 - \overline{x}_{(1)}) + v_2(1 - \overline{x}_{(2)}) + \ldots + v_n(1 - \overline{y}_{(n)}) \\ &= 1 + (w_1 - v_1)\overline{x}_1 + (w_2 - v_2)\overline{x}_2 + \ldots + (w_n - v_n)\overline{x}_n \leq 1, \end{aligned}$$

where the last inequality is due to Eq. 11).

Corollary 5.8: Let $w, v \in (0,1]^n$ with $w_1 + \ldots + w_n = 1$ and $v_1 + \ldots + v_n = 1$. Then the following statements are equivalent:

i) Unbalanced IVAIOWA operators associated with w, v and the orders $\leq_O$;

ii) $\sum_{j=1}^{i} w_j \leq \sum_{j=1}^{i} v_j$ for all $i \in \{1, \ldots, n\}$

Proof. Straight by the Lemma 5.6 and the Theorem 5.7.

Example 5.9: Take $w = (0.3, 0.2, 0.5)$, $v = (0.5, 0.1, 0.4)$ and the composed lexicographic 2 order $\leq_O$ which corresponds to $k_1 = k_3 = 1$ and $k_2 = k_4 = 0$. Then for $z_1 = ([0, 0.3], [0.5, 0.7])$, $z_2 = ([0.2, 0.8], [0, 0.2])$ and $z_3 = ([0.1, 0.2], [0.75, 0.8])$, since $z_2 \geq_O z_1 \geq_O z_3$, it holds

$$UIVAIOWA_{[w,v,\leq_O]}(z_1, z_2, z_3) = ([0.11, 0.4], [0.35, 0.49]),$$

which is an IVAIF-pair.

VI. Discrete Interval-valued Atanassov intuitionistic Choquet Integral

In this section, using IVAIF-admissible orders we define discrete Choquet integrals for IVAIFS.

Definition 6.1: Let m be a fuzzy measure of a non-empty finite universe $U = \{u_1, \ldots, u_n\} \neq \emptyset$. The discrete Choquet integral of G (an IVAIFS), with respect to an admissible order $\leq$ on $\mathcal{L}_{IV}([0,1])$ is given by

$$C_{[m,\leq]}(G) = \sum_{i=1}^{n} \Big(G(u_{\sigma(i)}) m(\{u_{\sigma(i)}, \ldots, u_{\sigma(n)}\}) - G(u_{\sigma(i)}) m(\{u_{\sigma(i+1)}, \ldots, u_{\sigma(n)}\}) \Big)$$

where $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ is a permutation such that $G(u_{\sigma(1)}) \leq G(u_{\sigma(2)}) \leq \ldots \leq G(u_{\sigma(n)})$ and $m(\{u_{\sigma(n+1)}, u_{\sigma(n)}\}) = 0$, by convention.

If $G(u_i) = (\mathbf{x}_i, \mathbf{y}_i)$ for all $i = 1, \ldots, n$ and $m_i = m(\{u_{\sigma(i)}, \ldots, u_{\sigma(n)}\}) - m(\{u_{\sigma(i+1)}, \ldots, u_{\sigma(n)}\})$ for $i = 1, \ldots, n$, the expression can be rewritten as

$$C_{[m,\leq]}(G) = \left(\left[\sum_{i=1}^{n} \underline{x}_{\sigma(i)} m_i, \sum_{i=1}^{n} \overline{x}_{\sigma(i)} m_i \right], \left[\sum_{i=1}^{n} \underline{y}_{\sigma(i)} m_i, \sum_{i=1}^{n} \overline{y}_{\sigma(i)} m_i \right] \right).$$

Remark 5: If we use a symmetric fuzzy measure [23] in the integral Choquet for IVAIFS, we also recover IVAIOWA operators defined in Section 3.

Remember that if the order used is $\leq_Q$ as in the Example 3.3 then $IVAIOWA$ operators are monotonic. However, this is not true in Choquet Integrals as proves the next example.

Example 6.1: Let $U = \{u_1, u_2\}$ and G be an IVAIFS such that $G(u_1) = ([0, 0.3], [0.1, 0.2])$ and $G(u_2) = ([0.1, 0.5], [0, 0.4])$.

Let m be the fuzzy measure given for each $V \subset U$ by

$$m(V) = \left(\frac{\sum_{u_i \in V} \underline{x}_i + \overline{x}_i + \underline{y}_i + \overline{y}_i}{\sum_{u_j \in U} \underline{x}_j + \overline{x}_j + \underline{y}_j + \overline{y}_j} \right)^2$$

where $G(u_i) = ([\underline{x}_i, \overline{x}_i], [\underline{y}_i, \overline{y}_i)]$.

Consider the order $\leq_{\Pi_2}$, namely, the composed lexicographic 2 order:

$(\mathbf{x_1}, \mathbf{y_1}) \leq_{\Pi_2} (\mathbf{x_2}, \mathbf{y_2})$ if and only if

- $(\overline{x}_1 < \overline{x}_2$), or

- ($\overline{x}_1 = \overline{x}_2$ and $\underline{x}_1 < \underline{x}_2$), or
- ($\overline{x}_1 = \overline{x}_2$, $\underline{x}_1 = \underline{x}_2$, and $\overline{y}_1 > \overline{y}_1$), or
- ($\overline{x}_1 = \overline{x}_2$, $\underline{x}_1 = \underline{x}_2$, $\overline{y}_1 = \overline{y}_1$ and $\overline{y}_1 \geq \overline{y}_1$)

The Choquet integral of G is

$$C_{[m,\leq_{\Pi_2}]}(G) = ([0,0.3]\cdot(1-0.39)+[0.1,0.5]\cdot(0.39-0), [0.1,0.2]\cdot(1-0.39)+[0,0.4]\cdot(0.39-0)) = ([0.039,0.378],[0.061,0.278]).$$

If we take the IVAIF-pair $G'(u_2) = ([0,0.6],[0,0]) \geq_{\Pi_2} ([0.1,0.5],[0,0.4]) = G(u_2)$, then

$$C_{[m,\leq_{\Pi_2}]}(G') = ([0,0.3]\cdot(1-0.25)+[0,0.6]\cdot(0.25-0), [0.1,0.2]\cdot(1-0.25)+[0,0]\cdot(0.25-0)) = ([0,0.375],[0.075,0.15]).$$

But
$([0,0.375],[0.075,0.15]) <_{\Pi_2} ([0.039,0.378],[0.061,0.278])$
and the Choquet integral is not monotonic.

Open Problem: *When are discrete Choquet integrals monotone? Notice that they depend on the measure m, that can depend on the values of the inputs as in the Example 6.1.*

VII. Application to multi-expert decision making

Multi-expert decision making consists of choosing an alternative out of a given set $U = \{u_1,\ldots,u_p\}$, $(p \geq 2)$, according to the pairwise preferences given by some experts $E = \{e_1,\ldots,e_n\}$, $(n > 2)$. The concordances and discordances of such preferences must be taken into account in the process of choosing the best-possible alternative. Frequently, experts have difficulties in defining and quantifying their preferences between pairs of alternatives. In order to solve these difficulties, decision-making algorithms allow increasingly elaborated expressions of preference [24], [25] . In this work we consider the case where the expression of the preference of the experts is given by IVAIF-pairs.

A. Algorithms for interval-valued intuitionistic preference relations

B. Algorithms for interval-valued intuitionistic preference relations

An interval-valued Atanassov intuitionistic fuzzy preference relation R_{IVAIF} on U is a mapping $U\times U \longrightarrow \mathcal{L}_{IV}([0,1])$ such that $R_{\text{IVAIF}}(u_i,u_j)$ represents the desirability of the alternative u_i over alternative u_j. For each of such IVAIF-pairs the first interval denotes the degree of preference of u_i over u_j, while the second one represents the non-preference of u_i over u_j.

A multi-expert decision making algorithm takes as input the opinion of multiple experts. Each of such experts $e \in E$ expresses his preferences as an interval-valued Atanassov intuitionistic fuzzy relation, which is denoted R^e_{IVAIF}.

$$R^e_{\text{IVAIF}} = \begin{pmatrix} - & z_{12} & \cdots & z_{1p} \\ z_{21} & - & \cdots & z_{2p} \\ \vdots & \ddots & \ddots & \vdots \\ z_{p1} & \cdots & z_{p(p-1)} & - \end{pmatrix}.$$

Note that the elements in the main diagonal are unset, since they represent preference of each alternative over itself. For the sake of simplicity, we take $z_{ij} = (\mathbf{x}_{ij_e}, \mathbf{y}_{ij_e})$.

The method for choosing one alternative $u' \in U$ is depicted in Algorithm 1. The *a priori* information consists of the set of alternatives U, the preference relations generated by the experts and the weight vector and admissible orders used by the IVAIOWA operator and IVAIF Choquet integral, respectively. The result is expressed as the preferred alternative u'. The algorithm has two main phases, namely *information fusion* and *exploitation*. In the information fusion phase, the so-called collective preference relation is created. This relation fusions the preferences of each of the experts for each pair of alternatives. In the exploitation phase the algorithm models the global desirability of each of the alternatives. In order to do so, it computes a fuzzy measure for each of the alternatives (*i.e.* for each row in the collective preference relation). This fuzzy measure is further used in an IVAIF Choquet integral to produce a global desirability value z_i, which recalls the preference of the alternative u_i over all of the other alternatives. Finally, the alternative u_i whose z_i is maximum is taken as preferred alternative (if more than one alternative produces such maximum, any of them can be taken as preferred alternative).

Data: A set of alternatives U, a set of relations R^e_{IVAIF}, a weight vector $w \in (0,1]^n$, an IVAIF-admissible order $\leq_Q$
Result: A preferred alternative $u' \in U$

```
// 1- Information fusion: Creating a
   collective preference relation
for each position (i,j) ∈ U × U do
    R^c_IVAIF(i,j) ←
      IVAIOWA_[w,≤Q](R^e1_IVAIF(i,j), ..., R^en_IVAIF(i,j));
end
// 2- Exploitation
for each row i of R^c_IVAIF do
    // 2.1- Build the fuzzy measures m_i
    A_i = P({1,...,p} \ i);
    for each A' ∈ A_i do
        m_i(A') ←
          ( (Σ_{j∈A'} x̲_ij + x̄_ij + (1 − y̲_ij) + (1 − ȳ_ij)) / (Σ_{l=1, l≠i}^p x̲_il + x̄_il + (1 − y̲_il) + (1 − ȳ_il)) )^2 ;
    end
end
// 2.2- Aggregate the matrix row-wise
for each row i of R^c_IVAIF do
    z_i ← C_[m_i,≤Q](R^c_IVAIF(i,j)) with j ∈ {1,...,n} \ {i};
end
// 2.3- Select the most preferred
   alternative
u' ← u_k such that z_k is maximum;
```

Algorithm 1: First algorithm for multi-expert decision making using interval-valued Atanassov intuitionistic fuzzy preference relations.

The method in Algorithm 1 is fairly simple and powerful, but can also suffer from unexpected behaviours. This is due to the non-monotonicity of Choquet integrals through the IVAIF-admissible orders $\leq_Q$. Hence, an increase of the values in the i-th row of R_{IVAIF_c} might potentially lead to a reduction of the value z_i. Put to interpretable terms, this means that an increase of the preferences of a given alternative over the others can lead to a reduction of its global desirability. Although in some situations this fact might have no impact on the final choice, it is certainly undesirable. This issue is solved by the Algorithm 2. It modifies the exploitation phase of Algorithm 1 by aggregating the fuzzy R^c_{IVAIF} through an Unbalanced IVAIOWA operator. However, the orders must be restricted to $\leq_O$.

Data: A set of alternatives U, a set of relations R^e_{IVAIF}, a weight vector $w_1 \in (0,1]^n$, two vectors w_2, v_2 satisfying the conditions in the Corollary 5.8, an IVAIF-admissible order $\leq_O$
Result: A preferred alternative $u' \in U$

```
// 1- Information fusion: Creating a
   collective preference relation
```

for *each position* $(i,j) \in U \times U$ **do**
  $R^c_{\text{IVAIF}}(i,j) \leftarrow IVAIOWA_{[w,\leq_O]}(R^{e1}_{\text{IVAIF}}(i,j), \ldots, R^{e_n}_{\text{IVAIF}}(i,j))$;
end

```
// 2- Exploitation
// 2.1- Aggregate the matrix row-wise
```

for *each row* i *of* R^c_{IVAIF} **do**
  $z_i \leftarrow UIVAIOWA_{[w_2,v_2,\leq_O]}(R^c_{\text{IVAIF}}(i,j))$ with $j \in \{1,\ldots,n\} \setminus \{i\}$;
end

```
// 2.2- Select the most preferred
   alternative
```

$u' \leftarrow u_k$ such that z_k is maximum;

Algorithm 2: Second algorithm for multi-expert decision making using interval-valued Atanassov intuitionistic fuzzy preference relations.

Note that although in Algorithm 2 the non-monotonicity of Choquet integral is solved, it impose the order to be in the class of $\leq_O$ which is more restrictive than $\leq_Q$ class.

C. Example of multi-expert decision making

Let $\{z_1, \ldots, z_4\}$ represent four alternatives on which three experts provide their personal preferences. The preference relations obtained for each of the experts are depicted in Table I.

We intend to take a decision on the best possible option using the weight vector $w = (0.3, 0.4, 0.3)$, which gives more importance to the intermediate IVAIF-pair, i.e., we give more importance to the neutral expert (neither the optimistic, nor the pessimistic).

The collective matrix if the Lexicographic-1 order ($\leq_{\Pi_1}$) is chosen in Algorithm 1, is given in Table II.

After exploitation phase, the global desirability values are

$$z_1 = ([0.4299, 0.5592], [0.231\ , 0.355\ ])\,,$$
$$z_2 = ([0.3826, 0.5162], [0.257\ , 0.3448])\,,$$
$$z_3 = ([0.3024, 0.4201], [0.3272, 0.4227])\,, \text{ and}$$
$$z_4 = ([0.3551, 0.6636], [0.181\ , 0.2692])\,.$$

In this way, through the Lexicographic-1 order, the preferred option is u_1, followed by u_2, u_4 and u_3. The question remains open on whether other orders would yield the same decision on the existing preferences. The algorithm has been repeated using

- $\leq_{\Pi_2}$ as in the Example 3.3.
- $\leq_K$ generated by K_1, K_2, K_3, K_4 with $k_1 = k_3 = \frac{3}{4}$ and $k_2 = k_4 = \frac{1}{4}$.
- $\leq_Q$ with Q_1, Q_2, Q_3, Q_4 given by

$$Q_1(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \frac{1}{4}\underline{x}_1 + \frac{1}{4}\overline{x}_1 + \frac{1}{4}\underline{y}_1 + \frac{1}{4}\overline{y}_1\,,$$
$$Q_2(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \frac{10}{20}\underline{x}_1 + \frac{3}{20}\overline{x}_1 + \frac{3}{20}\underline{y}_1 + \frac{4}{20}\overline{y}_1\,,$$
$$Q_3(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \frac{2}{10}\underline{x}_1 + \frac{2}{10}\overline{x}_1 + \frac{3}{10}\underline{y}_1 + \frac{3}{10}\overline{y}_1 \text{ and}$$
$$Q_4(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \frac{1}{10}\underline{x}_1 + \frac{4}{10}\overline{x}_1 + \frac{1}{10}\underline{y}_1 + \frac{4}{10}\overline{y}_1\,.$$

The global desirability of the alternatives with each of the orders is as follows:

- Order $\leq_{\Pi_1}$: $z_1 \geq z_2 \geq z_4 \geq z_3$;
- Order $\leq_{\Pi_2}$: $z_4 \geq z_1 \geq z_2 \geq z_3$;
- Order $\leq_K$: $z_1 \geq z_4 \geq z_3 \geq z_2$;
- Order $\leq_Q$: $z_4 \geq z_1 \geq z_2 \geq z_3$.

So, depending on the chosen IVAIF-admissible order Alternatives 1 and 4 can be depicted. If non-monotonicity of IVAIF Choquet is affecting the result, we decide to run Algorithm 2. Taking the order $\leq_{\Pi_2}$ and the weight vectors $w_2 = (0.3, 0.2, 0.5)$ and $v_2 = (0.5, 0.1, 0.4)$ (which satisfy the conditions of Corollary 5.8). The final values of Unbalanced IVAIOWA operator are

$$z_1 = ([0.4265, 0.5640], [0.1930, 0.3280]),$$
$$z_2 = ([0.3460, 0.4917], [0.2334, 0.3252]),$$
$$z_3 = ([0.3470, 0.4397], [0.2445, 0.3115]),$$
$$z_4 = ([0.4455, 0.6395], [0.1820, 0.2868]).$$

Consequently, u_4 is preferred over all the other alternatives by Algorithm 2.

VIII. Conclusions

In this work we have analyzed the extension of OWA operators and discrete Choquet integral to cope with IVAIFSs. This has led to the proposal of novel definitions of IVAIOWA operators, Unbalanced IVAIOWA operators and IVAIF Choquet integrals. In the definition of these operators we have considered the possibility of choosing different weight vectors for the membership and non-membership. We have also studied the role of the IVAIF-admissible orders of IVAIF-pairs, more specifically the impact of such orders in the monotonicity of the IVAIOWA operators. For illustrative purposes, we have presented examples of application in the context of multi-expert decision making, considering two different algorithms in which the novel operators can take a relevant role.

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$$R^{e_1}_{IVAIF} = \begin{pmatrix} - & [0.38, 0.38], [0.57, 0.62] & [0.64, 0.68], [0.24, 0.3] & [0.49, 0.71], [0.14, 0.22] \\ [0.3, 1], [0, 0] & - & [0.11, 0.23], [0.64, 0.68] & [0.2, 0.3], [0, 0.05] \\ [0, 0.1], [0.6, 0.68] & [0.16, 0.27], [0.3, 0.55] & - & [0.6, 0.62], [0, 0.1] \\ [0.28, 0.76], [0.22, 0.22] & [1, 1], [0, 0] & [0.07, 0.29], [0.6, 0.71] & - \end{pmatrix}$$

$$R^{e_2}_{IVAIF} = \begin{pmatrix} - & [0.3, 0.46], [0, 0.5] & [0.1, 0.43], [0.5, 0.57] & [0.86, 0.92], [0, 0.05] \\ [0.9, 0.9], [0, 0.1] & - & [0.14, 0.36], [0.2, 0.6] & [0.6, 0.67], [0.1, 0.21] \\ [0, 0.1], [0.3, 0.4] & [0.16, 0.27], [0.7, 0.73] & - & [0.4, 0.5], [0.5, 0.5] \\ [0.61, 0.76], [0, 0.11] & [0.5, 0.6], [0.2, 0.4] & [0.22, 0.76], [0.1, 0.18] & - \end{pmatrix}$$

$$R^{e_3}_{IVAIF} = \begin{pmatrix} - & [0.4, 0.6], [0, 0] & [0.71, 0.83], [0, 0.1] & [0.15, 0.3], [0.48, 0.6] \\ [0.9, 0.9], [0, 0] & - & [0.1, 0.15], [0.7, 0.84] & [0.3, 0.4], [0.5, 0.52] \\ [0.8, 0.84], [0.06, 0.1] & [0.26, 0.6], [0.31, 0.4] & - & [0.9, 0.95], [0, 0.02] \\ [0.12, 0.46], [0, 0.3] & [0.2, 0.23], [0.7, 0.74] & [0.74, 0.93], [0, 0] & - \end{pmatrix}$$

TABLE I
PREFERENCES RELATIONS GIVEN BY THE THREE EXPERTS

$$R^{c}_{IVAIF} = \begin{pmatrix} - & [0.362, 0.47], [0.228, 0.398] & [0.499, 0.65], [0.246, 0.321] & [0.499, 0.65], [0.2, 0.283] \\ [0.72, 0.93], [0, 0.04] & - & [0.116, 0.245], [0.526, 0.704] & [0.36, 0.451], [0.23, 0.286] \\ [0.24, 0.322], [0.318, 0.394] & [0.19, 0.369], [0.423, 0.559] & - & [0.63, 0.683], [0.15, 0.196] \\ [0.331, 0.67], [0.088, 0.211] & [0.56, 0.609], [0.29, 0.382] & [0.331, 0.67], [0.22, 0.285] & - \end{pmatrix}.$$

TABLE II
COLLECTIVE PREFERENCE RELATION USING LEXICOGRAPHIC-1 ORDER FOR FUSING THE INFORMATION.

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An algorithm for group decision making using n-dimensional fuzzy sets, admissible orders and OWA operators

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Abstract

In this paper we propose an algorithm to solve group decision making problems using n-dimensional fuzzy sets, namely, sets in which the membership degree of each element to the set is given by an increasing tuple of n elements. The use of these sets has naturally led us to define admissible orders for n-dimensional fuzzy sets, to present a construction method for those orders and to study OWA operators for aggregating the tuples used to represent the membership degrees of the elements. In these conditions, we present an algorithm and apply it to a case study, in which we show that the exploitation phase which appears in many decision making methods can be omitted by just considering linear orders between tuples.

Keywords: fuzzy multisets, n-dimensional fuzzy sets, OWA operator, decision-making

1. Introduction

A multiple criteria group decision making problem consists in choosing a solution A_i out of a set of p ($p \geq 2$) alternatives according to the evaluations, given by n decision makers e_k ($k \in \{1, \ldots, n\}$), to each alternative with respect to q criteria. Thus, we have that:

1. The evaluations for the alternative A_i ($1 \leq i \leq p$) with respect to the criterion C_1 are given by the tuple $(d_{i1}^{e_1}, d_{i1}^{e_2}, \ldots, d_{i1}^{e_n})$, where $d_{i1}^{e_k} \in [0, 1]$ represents the evaluation of the decision maker e_k for the alternative A_i ($1 \leq i \leq p$) with respect to the criterion C_1.
2. The evaluations for the alternative A_i ($1 \leq i \leq p$) with respect to the criterion C_2 are given by the tuple $(d_{i2}^{e_1}, d_{i2}^{e_2}, \ldots, d_{i2}^{e_n})$, where $d_{i2}^{e_k} \in [0, 1]$ represents the evaluation of the decision maker e_k for the alternative A_i ($1 \leq i \leq p$) with respect to the criterion C_2.
3. We proceed analogously for every criteria.

In this manner, we can represent the problem using multisets (see [1, 2]), i.e., sets of this form:

$$D = \{D_{ij} = (d_{ij}^{e_1}, d_{ij}^{e_2}, \ldots, d_{ij}^{e_n}) \mid i \in \{1, \ldots, p\},\ j \in \{1, \ldots, q\}\},$$

where each element is a tuple D_{ij} of n elements consisting of the evaluations of the criteria.

The use of different generalizations of fuzzy sets is frequent to model the uncertainty inherent in many decision making and consensus problems [3–7]. Moreover, in most of these problems, the order in which the decision makers provide their evaluation does not have an impact in the election of the solution. Bearing that in mind, for our problem we can consider a particular case

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of multisets, the so-called n-dimensional fuzzy sets [8], where the membership of each element is given by a tuple of n numbers in $[0,1]$ increasingly ordered.

When solving decision making problems, a numerical value is usually associated to each alternative and the solution is taken as the alternative with the greatest value [9–15]. However, in the cases where the resolution uses interval-valued fuzzy sets or Atanassov's intuitionistic fuzzy sets [16–18], each alternative is associated to an interval or to a pair of numbers, respectively. In these cases, we are compelled to use linear orders for intervals or pairs of numbers (see [19, 20]), so that the solution is given by the greatest interval or pair of numbers. Since, in the selected context, n-tuples are used, we need to define a linear order to compare n-tuples.

With all previous considerations, our objectives for this work are:

1. To present the concept of admissible order for n-dimensional fuzzy sets.
2. To give a construction method for admissible orders using aggregation functions [21–23].
3. To extend to n-dimensional fuzzy sets the concept of OWA operators (which are always associated to a linear order).
4. To design a decision making algorithm using n-dimensional fuzzy sets and n-tuple OWA operators.
5. To justify our theoretical developments with an illustrative example applying the proposed algorithm.

Some of the most widely used methods for solving multiple criteria decision making problems consist of two phases [9, 10, 12, 13]: the aggregation phase and the exploitation phase. In these methods, each decision maker represents his/her evaluations by means of preference relations (matrices) whose inputs are the d_{ij} values. So we have as many preference relations as decision makers.

In the aggregation phase, an aggregation function is chosen in order to aggregate the n preference relations (matrices) to produce a single matrix: the *collective* matrix. This collective matrix has as many rows as alternatives and as many columns as considered criteria. In the exploitation phase, an aggregation function is also selected for aggregating the elements of the collective matrix row by row to get one single number for each row. In the final step of the exploitation phase, we get as many numbers as alternatives and we take as solution the alternative associated to the greatest of these numbers.

One advantage of the method that we propose in this work is that we may omit the exploitation phase. This is due to the fact that the aggregation of the collective matrix produces a tuple for each alternative, so it is enough to order these tuples in a decreasing way according to a linear order so that we can choose as solution the first ranked tuple, i.e., the greatest tuple with respect to the linear order.

The possibility of omitting the exploitation phase is very relevant due to the fact that we do not need to reduce the elements of each row of the collective matrix to a single value and, hence, we do not modify the original data provided by the decision makers. This means that our results are obtained more straightforwardly from the evaluations of the decision makers than in those methods where the two phases are considered. These considerations are further developed in the last section, devoted to the application of our algorithm.

The structure of the work is as follows: in Section 2 we recall some preliminary notions about admissible orders and the extensions of fuzzy sets. In Section 3 we study the theoretical concepts that are required for the development of our model. Firstly, we generalize the concept of admissible order for n-dimensional fuzzy sets and present a construction method. Secondly, we introduce the concept of MOWA operator, studying its monotonicity with respect to a certain admissible order. An algorithm for decision making problems that makes use of all previous concepts is presented in Section 4, while in Section 5, we apply this algorithm in an illustrative example in the context of a multiple criteria group decision making problem. We finish in Section 6 with some conclusions and directions for future research.

2. Preliminaries

We first introduce some theoretical notions in order to fix the notation for the subsequent sections. Let O^n be the set of increasing n-tuples on $[0,1]$, namely, the set

$$O^n = \{\mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n \mid x_1 \leq x_2 \leq \ldots \leq x_n\}.$$

We recall that there is a natural partial order $\preceq$ on $O^n \subseteq \mathbb{R}^n$ given by $(x_1, \ldots, x_n) \preceq (y_1, \ldots, y_n)$ if and only if $x_i \leq y_i$, $1 \leq i \leq n$. In this way, $(O^n, \preceq)$ is a complete lattice and $(0, \ldots, 0)$ and $(1, \ldots, 1)$ are the bottom and top elements of the partial order, respectively.

Fuzzy multisets are a generalization of fuzzy sets which were defined in [2] by Yager. Like many other generalizations, the aim of these sets lies on the formalization of a representation to deal with imprecision, inexactness, ambiguity, or uncertainty intrinsic to many problems. In particular, in the case of fuzzy multisets, a fixed number n of membership values is assigned to each element. Taking into account that in a group decision making problem we have as many evaluations as decision makers, fuzzy multisets are suitable models for these problems. In the case of fuzzy multisets, the different membership values are considered as a set, not as an n-tuple, since they are not necessarily ordered. If the values of the membership degree of each element are ordered in an increasing way, fuzzy multisets are called n-dimensional fuzzy sets.

Definition 1. *[8] Let U be a nonempty set usually called a universe. A n-dimensional fuzzy set A over U is given by*

$$A : U \mapsto O^n$$

where $A(u)$ denotes the membership degree of the element $u \in U$ to A.

Note that usual fuzzy sets are a specific example of a n-dimensional fuzzy set with $n = 1$. Analogously, interval-valued fuzzy sets [24] can be seen as an example of 2-dimensional fuzzy sets.

Given an element $u \in U$, we denote the n-dimensional membership tuple of the element u to the n-dimensional fuzzy set A by $A(u) \in O^n$. Moreover, it is worth mentioning that we recover fuzzy multisets when $[0,1]^n$ is considered instead of O^n.

In this work, due to the selected context, anonymity is a key point in the implemented algorithm. We consider a multiple criteria group decision making problem where each decision maker gives a evaluation about each alternative with respect to each criterion in terms of a fuzzy membership degree. We select a context where all the decision makers' evaluations are valued equally, independently of their identity. In this way, the n decision maker's values are sorted producing a single n-dimensional fuzzy set.

As we have mentioned before, our construction method is underpinned in aggregation functions. These functions, which play a crucial role in both applied and theoretical fields, were originally defined in the unit interval $[0,1]$. However, they can be readily extended to any poset [25].

Definition 2. *An aggregation function M is a mapping $M : [0,1]^n \to [0,1]$ satisfying*

- $M(0, \ldots, 0) = 0, \quad M(1, \ldots, 1) = 1$, *and*
- *for all n-tuples $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in [0,1]^n$ such that $x_i \leq y_i$, for all $1 \leq i \leq n$ then $M(x_1, \ldots, x_n) \leq M(y_1, \ldots, y_n)$.*

The aim of this study is to generalize the concept of OWA operators to deal with n-dimensional fuzzy sets. Let us first recall their definition in $[0,1]$.

Definition 3. *[26] Let w be a weighting vector, i.e., $w = (w_1, \ldots, w_m) \in [0,1]^m$ such that $w_1 + \ldots + w_m = 1$. The Ordered Weighted Averaging (OWA) operator associated to w is a mapping $OWA_w : [0,1]^m \longrightarrow [0,1]$ given by*

$$OWA_w(x_1, \ldots, x_m) = \sum_{i=1}^{m} w_i x_{(i)}\,,$$

where $x_{(i)}$, denotes the i-th greatest component of the vector $(x_1, \ldots, x_m)$.

Note that although aggregation functions can be defined on a strict partially ordered set, OWA operators require all the elements to be comparable, i.e., OWA operators require a linear order to be properly defined. Nevertheless, recent studies in the literature have proposed definitions for these operators in more general lattices [27].

3. Admissible orders and OWA operators on fuzzy multisets

The notion of admissible order was first introduced in [16] for interval-valued fuzzy sets and later on in [28] for interval-valued Atanassov's intuitionistic fuzzy sets. In this section, we first generalize the notion of admissible order to the setting of n-dimensional fuzzy sets showing some particular examples. We also provide a construction method for these orders which makes use of appropriate aggregation functions on O^n.

We start defining admissible orders on O^n.

Definition 4. *A linear order $\leq_L$ on O^n is called admissible if for all $\mathbf{x}, \mathbf{y} \in O^n$ satisfying $x_i \leq y_i$ for all $1 \leq i \leq n$ then $\mathbf{x} \leq_L \mathbf{y}$.*

Example 1.

- *As a first example of admissible order on O^n for every $n \geq 1$ we consider the first lexicographical order (with respect to the first variable), $\mathbf{x} \leq_L \mathbf{y}$ if the i-th component of $\mathbf{x}$ is strictly less than the i-th component of $\mathbf{y}$ ($i \in \{1, \ldots, n\}$), whereas $x_j = y_j$ for every $j < i$.*
- *For $n = 2$ the Xu and Yager order ([29]) is defined by*

$$(x_1, x_2) \leq (y_1, y_2) \text{ if and only if } \quad \frac{x_1 + x_2}{2} < \frac{y_1 + y_2}{2} \text{ or }$$
$$\left(\frac{x_1 + x_2}{2} = \frac{y_1 + y_2}{2} \text{ and } x_2 - x_1 < y_2 - y_1 \right)$$

We are interested in those admissible orders which can be obtained by means of appropriate aggregation functions. In particular, we consider the following result.

Definition 5. *Let $\boldsymbol{M} = (M_1, \ldots, M_n)$ be a sequence of n aggregation functions $M_i : [0,1]^n \to [0,1]$. Given $\mathbf{x}, \mathbf{y} \in O^n$,*

- *$\mathbf{x} <_{\boldsymbol{M}} \mathbf{y}$ if and only if there exists k with $1 \leq k \leq n$ such that $M_j(\mathbf{x}) = M_j(\mathbf{y})$ for all $1 \leq j \leq k-1$ and $M_k(\mathbf{x}) < M_k(\mathbf{y})$.*
- *$\mathbf{x} \leq_{\boldsymbol{M}} \mathbf{y}$ if and only if $\mathbf{x} <_{\boldsymbol{M}} \mathbf{y}$ or $\mathbf{x} = \mathbf{y}$.*

Proposition 1. *Let $\boldsymbol{M} = (M_1, \ldots, M_n)$ be a sequence of n aggregation functions $M_i : [0,1]^n \to [0,1]$. The order relation $\mathbf{x} \leq_{\boldsymbol{M}} \mathbf{y}$ is an admissible order on O^n if and only if the functions M_i satisfy*

$$(M_i(\mathbf{x}) = M_i(\mathbf{y}), \text{ for all } 1 \leq i \leq n) \Leftrightarrow \mathbf{x} = \mathbf{y}. \tag{1}$$

Proof. It is a straightforward calculation. □

Example 2. *The lexicographic orders can be constructed as before from the n projections given by $\pi_i(x_1, \ldots, x_n) = x_i$.*

For example, the first lexicographical order is generated taking $M_i = \pi_i$. But observe that, if we consider any permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ and we take $M_i = \pi_{\sigma(i)}$, then we get different examples of lexicographic orders which are different from each other.

In order to get examples of admissible orders on O^n we consider aggregation functions which are defined in terms of linear expressions, and, more specifically, in terms of weighted arithmetic means.

Proposition 2. *Let $\boldsymbol{M} = (M_1, \ldots, M_n)$ be a sequence of n aggregation functions given by*

$$M_i(x_1, \ldots, x_n) = \alpha_{i1}x_1 + \alpha_{i2}x_2 + \ldots + \alpha_{in}x_n\,, \quad 1 \leq i \leq n\,, \tag{2}$$

such that $\alpha_{i1}+\alpha_{i2}+\ldots+\alpha_{in} = 1$ with $\alpha_{ij} \in [0,1]$ for all $1 \leq j \leq n$. The order $\leq_{\boldsymbol{M}}$ is an admissible order on O^n if and only if the $n \times n$ matrix A given by

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{2n} & \ldots & \alpha_{nn} \end{pmatrix}$$

is regular.

Proof. Notice that Eq. (1) can be rewritten as $A(\mathbf{x}-\mathbf{y}) = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{y}$, which is equivalent to A being regular. $\square$

Example 3. *Let $\leq_{\boldsymbol{M}}$ be the order generated by the following functions M_i:*

- $M_1(x_1, \ldots x_5) = \frac{1}{10}x_1 + \frac{1}{5}x_2 + \frac{1}{5}x_3 + \frac{1}{4}x_4 + \frac{1}{4}x_5$,
- $M_2(x_1, \ldots x_5) = \frac{3}{10}x_1 + \frac{1}{5}x_2 + \frac{1}{2}x_5$,
- $M_3(x_1, \ldots x_5) = \frac{3}{5}x_1 + \frac{1}{10}x_2 + \frac{1}{10}x_3 + \frac{1}{10}x_4 + \frac{1}{10}x_5$,
- $M_4(x_1, \ldots x_5) = \frac{1}{5}x_2 + \frac{3}{10}x_3 + \frac{3}{10}x_4 + \frac{1}{5}x_5$,
- $M_5(x_1, \ldots x_5) = \frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_4 + \frac{1}{4}x_5$.

It is a simple calculation to see that the matrix A generated by the coefficients of the aggregation function is a regular matrix. Hence, the order relation $\leq_{\boldsymbol{M}}$ is an admissible order and we can compare, for instance, $\mathbf{x} = (0.2, 0.4, 0.9, 1, 1)$ and $\mathbf{y} = (0, 0.6, 0.8, 1, 1)$. In fact, $\mathbf{y} <_{\boldsymbol{M}} \mathbf{x}$ since $M_1(\mathbf{y}) = 0.78 = M_1(\mathbf{x})$ and $M_2(\mathbf{y}) = 0.62 < 0.64 = M_2(\mathbf{x})$.

Once we have introduced the concept of admissible orders on O^n, we can define OWA operators on this set. Firstly, we generalize the concept of aggregation function on O^n.

Definition 6. *Let $\leq_L$ be an admissible order on O^n. An aggregation function M on O^n, is a mapping $M : (O^n)^m \to O^n$ satisfying*

- $M(\mathbf{0}, \ldots, \mathbf{0}) = \mathbf{0}$, $\quad M(\mathbf{1}, \ldots, \mathbf{1}) = \mathbf{1}$, *and*
- *for all* $(\mathbf{x_1}, \ldots, \mathbf{x_m}), (\mathbf{y_1}, \ldots, \mathbf{y_m}) \in (O^n)^m$ *such that* $\mathbf{x_1} \leq_L \mathbf{y_1}$, $\ldots$, $\mathbf{x_m} \leq_L \mathbf{y_m}$ *then* $M(\mathbf{x_1}, \ldots, \mathbf{x_m}) \leq_L M(\mathbf{y_1}, \ldots, \mathbf{y_m})$.

Definition 7. *Let w be a weighting vector and let $\leq_L$ be an admissible order. The OWA operator associated to w and $\leq_L$ is a mapping $(O^n)^m \mapsto O^n$ defined by*

$$MOWA_{[w, \leq_L]}(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \sum_{i=1}^{m} w_i \mathbf{x}_{(i)}$$

where $\mathbf{x}_{(i)}$ denotes the i-th greatest n-dimensional fuzzy value of the inputs $(\mathbf{x}_1 \ldots, \mathbf{x}_m)$ with respect to the order $\leq_L$ on O^n and $w_i\mathbf{x} = (w_i x_1, \ldots, w_i x_n)$.

Example 4.

- *If we take $w = (1, 0, \ldots, 0)$, then we recover the maximum operator.*

- *If we take $w = (0, 0, \dots, 0, 1)$, then we recover the minimum operator.*

Notice that the MOWA operator is well defined, namely, the image of m elements in O^n is a new element in O^n due to the increasingness of the weighted arithmetic mean.

In the usual fuzzy setting, OWA operators play a crucial role since their monotonicity enables to classify OWA operators as a particular class of aggregation functions. In the following, we study the monotonicity of MOWA operators with respect to an order generated as in Prop. 2.

Theorem 1. *Let $w = (w_1, \dots, w_m)$ be a weighting vector such that $w_i > 0$ for all $1 \le i \le m$ and let $\le_{\mathbf{M}}$ be an admissible order on O^n generated as in Prop. 2. Then the MOWA operator is an increasing function.*

Proof. Let us show that if $\mathbf{x}_i \le_{\mathbf{M}} \mathbf{x}'_i$ then

$$MOWA_{[w,\le_{\mathbf{M}}]}(\mathbf{x}_1, \dots, \mathbf{x}_i, \dots, \mathbf{x}_m) \le MOWA_{[w,\le_{\mathbf{M}}]}(\mathbf{x}_1, \dots, \mathbf{x}'_i, \dots, \mathbf{x}_m).$$

It holds trivially if $\mathbf{x}_i = \mathbf{x}'_i$ so we only need to prove the case $\mathbf{x}_i <_{\mathbf{M}} \mathbf{x}'_i$.

If $\mathbf{x}_i <_{\mathbf{M}} \mathbf{x}'_i$ then there is an index $1 \le j \le n$ such that

$$M_k(\mathbf{x}_i) = M_k(\mathbf{x}'_i) \text{ for all } k \le j-1 \text{ and } M_j(\mathbf{x}_i) < M_j(\mathbf{x}'_i). \tag{3}$$

Notice that if $j = 1$ the condition is reduced to $M_1(\mathbf{x}_i) < M_1(\mathbf{x}'_i)$.

Moreover, due to the fact that the functions which generate the order $\le_{\mathbf{M}}$ are weighted arithmetic means, it holds that

$$M_k(MOWA_{[w,\le_{\mathbf{M}}]}(\mathbf{x}_1, \dots, \mathbf{x}'_i, \dots, \mathbf{x}_m)) = \sum_{h=1}^{m} w_h M_k(\mathbf{x}_{(h)}) \,.$$

Without loss of generality, we suppose the n-dimensional fuzzy values are ordered in a decreasing way, i.e., $\mathbf{x}_1 \ge_{\mathbf{M}} \mathbf{x}_2 \ge_{\mathbf{M}} \dots \ge_{\mathbf{M}} \mathbf{x}_m$.

We distinguish two different cases:

- If the n-dimensional fuzzy value $\mathbf{x}'_i$ has not altered the order of the n-dimensional fuzzy values, namely, $\mathbf{x}_{i-1} \ge_{\mathbf{M}} \mathbf{x}'_i \ge_{\mathbf{M}} \mathbf{x}_{i+1}$, then it holds that

$$\begin{aligned} &M_k(MOWA_{[w,\le_{\mathbf{M}}]}(\mathbf{x}_1, \dots, \mathbf{x}_i, \dots \mathbf{x}_m)) - M_k(MOWA_{[w,\le_{\mathbf{M}}]}(\mathbf{x}_1, \dots, \mathbf{x}'_i, \dots \mathbf{x}_m)) \\ &= \sum_{h=1}^{m} w_h M_k(\mathbf{x}_h) - \left(w_i M_k(\mathbf{x}'_i) + \sum_{h \ne i} w_h M_k(\mathbf{x}_h) \right) \\ &= w_i \left(M_k(\mathbf{x}_i) - M_k(\mathbf{x}'_i) \right) \,, \end{aligned}$$

  which, due to the Eq. (3), equals to 0 for all $1 \le k \le j-1$ and is less than 0 for the index j. Hence,

$$MOWA_{[w,\le_{\mathbf{M}}]}(\mathbf{x}_1, \dots, \mathbf{x}_i, \dots \mathbf{x}_m) < MOWA_{[w,\le_{\mathbf{M}}]}(\mathbf{x}_1, \dots, \mathbf{x}'_i, \dots \mathbf{x}_m) \,.$$

- If the n-dimensional fuzzy value $\mathbf{x}'_i$ has altered the order of the n-dimensional fuzzy values in l positions, namely, $\mathbf{x}_{i-l-1} \ge_{\mathbf{M}} \mathbf{x}'_i \ge_{\mathbf{M}} \mathbf{x}_{i-l} \ge_{\mathbf{M}} \dots \ge_{\mathbf{M}} \mathbf{x}_{i-1} \ge_{\mathbf{M}} \mathbf{x}_i \ge_{\mathbf{M}} \mathbf{x}_{i+1}$ for some $l \ge 1$, we find that

$$\begin{aligned} &M_k(MOWA_{[w,\le_{\mathbf{M}}]}(\mathbf{x}_1, \dots, \mathbf{x}_i, \dots \mathbf{x}_m)) - M_k(MOWA_{[w,\le_{\mathbf{M}}]}(\mathbf{x}_1, \dots, \mathbf{x}'_i, \dots \mathbf{x}_m)) \\ &= \sum_{h=1}^{m} w_h M_k(\mathbf{x}_h) - \left(w_{i-l} M_k(\mathbf{x}'_i) + \sum_{h=i-l+1}^{i} w_h M_k(\mathbf{x}_{h-1}) + \sum_{\substack{h \le i-l-1 \\ \text{or} \\ h \ge i+1}} w_h M_k(\mathbf{x}_h) \right) \\ &= w_{i-l} \left(M_k(\mathbf{x}_{i-l}) - M_k(\mathbf{x}'_i) \right) + \sum_{h=i-l+1}^{i} w_h \left(M_k(\mathbf{x}_h) - M_k(\mathbf{x}_{h-1}) \right) . \end{aligned} \tag{4}$$

Further, since the n-dimensional fuzzy values are ordered in a decreasing way, it follows that $M_1(\mathbf{x}'_i) \geq M_1(\mathbf{x}_{i-l}) \geq M_1(\mathbf{x}_{i-l+1}) \ldots \geq M_1(\mathbf{x}_{i-1}) \geq M_1(\mathbf{x}_i)$. Moreover, due to Eq. (3), it holds that $M_1(\mathbf{x}_i) = M_1(\mathbf{x}'_i)$ and, hence, $M_1(\mathbf{x}'_i) = M_1(\mathbf{x}_{i-l}) = M_1(\mathbf{x}_{i-l+1}) = \ldots = M_1(\mathbf{x}_{i-1}) = M_1(\mathbf{x}_i)$.

Iteratively,

$$M_k(\mathbf{x}'_i) = M_k(\mathbf{x}_{i-l}) = M_k(\mathbf{x}_{i-l+1}) = \ldots = M_k(\mathbf{x}_{i-1}) = M_k(\mathbf{x}_i) \text{ for all } 1 \leq k \leq j-1. \quad (5)$$

Besides,

$$M_j(\mathbf{x}'_i) \geq M_j(\mathbf{x}_{i-l+1}) \geq \ldots \geq M_j(\mathbf{x}_{i-1}) \geq M_j(\mathbf{x}_i) \quad (6)$$

where at least one of the inequalities is strict since, by Eq. (3), it holds that $M_j(\mathbf{x}'_i) > M_j(\mathbf{x}_i)$.

Using Eqs. (5) and (6) in Eq. (4), we find that

$$M_k(MOWA_{[w,\leq_{\mathbf{M}}]}(\mathbf{x}_1,\ldots,\mathbf{x}_i,\ldots\mathbf{x}_m)) = M_k(MOWA_{[w,\leq_{\mathbf{M}}]}(\mathbf{x}_1,\ldots,\mathbf{x}'_i,\ldots\mathbf{x}_m)) \text{ for all } 1 \leq k \leq j-1, \text{ and}$$

$$M_j(MOWA_{[w,\leq_{\mathbf{M}}]}(\mathbf{x}_1,\ldots,\mathbf{x}_i,\ldots\mathbf{x}_m)) < M_j(MOWA_{[w,\leq_{\mathbf{M}}]}(\mathbf{x}_1,\ldots,\mathbf{x}'_i,\ldots\mathbf{x}_m)).$$

Hence, $MOWA_{[w,\leq_{\mathbf{M}}]}(\mathbf{x}_1,\ldots,\mathbf{x}_i,\ldots\mathbf{x}_m) < MOWA_{[w,\leq_{\mathbf{M}}]}(\mathbf{x}_1,\ldots,\mathbf{x}'_i,\ldots\mathbf{x}_m)$.

$\square$

Notice that since $MOWA_{[w,\leq_{\mathbf{M}}]}(\mathbf{0},\ldots,\mathbf{0}) = \sum_{i=1}^m w_i\mathbf{0} = \mathbf{0}$ and $MOWA_{[w,\leq_{\mathbf{M}}]}(\mathbf{1},\ldots,\mathbf{1}) = \sum_{i=1}^m w_i\mathbf{1} = \mathbf{1}$, MOWA operators are aggregation functions with respect to the order $\leq_{\mathbf{M}}$ generated as in Prop. 2.

4. An algorithm for group decision making using MOWA operators

Multiple criteria group decision making consists in choosing an alternative out of a given set $A = \{A_1,\ldots,A_p\}$ $(p \geq 2)$ according to the evaluations given by a group of decision makers $E = \{e_1,\ldots,e_n\}$ $(n \geq 2)$ with respect to some criteria $C = \{C_1,\ldots,C_q\}$ $(q \geq 2)$. Thus, we can generate a matrix $D = (D_{ij})_{p\times q}$ of memberships of fuzzy multisets, where D_{ij} denotes the n-tuple of evaluations of the decision makers about alternative A_i under the criterion C_j.

Once the order and the weighting vector are set, the following algorithm, which is schematically represented in Figure 1, can be applied. Notice that this procedure maintains all the evaluations provided by the decision makers, as in [30].

Step 1. To generate the matrix D whose elements D_{ij} are n-dimensional fuzzy values; this step consists in generating an ordered tuple with the n evaluations of the decision makers.

Step 2. To generate an order $\leq_{\mathbf{M}}$, selecting a sequence of $(M_1,\ldots,M_n)$ of aggregation functions that satisfy the conditions in Prop. 2.

Step 3. To select the weighting vector w of q components; one for each criterion.

Step 4. To apply the $MOWA$ operator to each row of the matrix D using the order $\leq_{\mathbf{M}}$ in **Step 2** and the weighting vector w in **Step 3**.

Step 5. To select as the best alternative the greatest n-dimensional fuzzy value with respect to the order in **Step 2**.

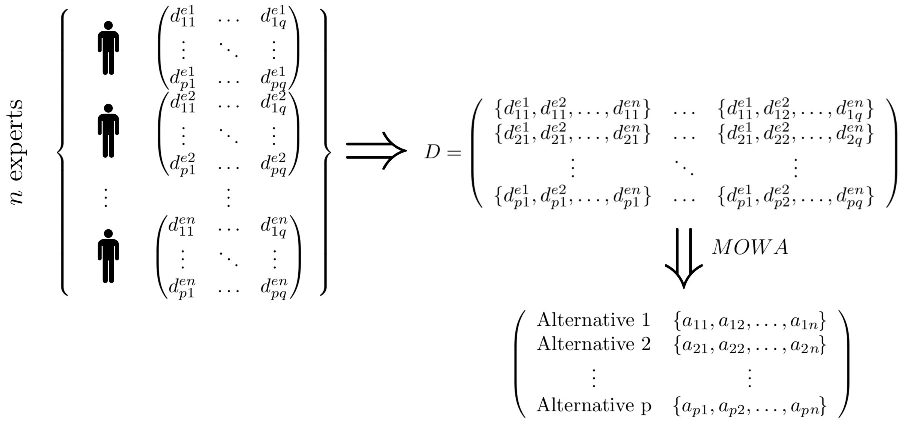

Figure 1: Schematic representation of Algorithm 1

Remark. The transformation of fuzzy multisets into n-dimensional fuzzy values ensures anonymity. In this manner, it does not matter which decision maker has provided each value of the fuzzy multiset and all of them are treated equally.

The output of Algorithm 1 can differ greatly depending upon Steps 2 and 3. The parameters with influence in such steps (that is, the aggregation functions used for the linear order and the weighing vector) become very relevant for the result of the algorithm; hence, their setting ought to be adapted depending on the specific problem.

It is worth mentioning that, in real scenarios, the assignment of non homogeneous weights to decision makers is rather common, and is simply done in order to weight their level of expertise or simply their relevance in the decision making process. In Algorithm 1, this cannot be directly done, as long as the weights are applied to the data according to their sorting, not to the relevance of the expert that provided them. For example, in a scenario in which experts tend to be optimistic, it seems appropriate to select weighing vectors empowering the lowest ranked elements, i.e., the first elements of the tuple. That is, using weighting vectors with decreasing values, so that the highest ranked (hence, more optimistic) evaluations receive less influence in the final decision.

5. Illustrative example

Ye et al. introduced in [31] a multiple criteria group decision making problem adapted from [13]. In this section, we show that Algorithm 1 is also a suitable option to solve that problem.

The practical example consists in determining the best company for investment. Four possible companies are considered: a car company A_1, a food company A_2, a computer company A_3 and an arm company A_4. Three decision makers are asked about their opinions with respect to three criteria: the risk analysis C_1, the growth analysis C_2 and the environmental impact analysis C_3. We take the same weighting vector as in [31], namely, $w = (0.35, 0.25, 0.4)$.

The main difference between our approach and Ye's [31] lies on the use of a different generalization of fuzzy sets. While in [31] dual hesitant fuzzy sets are considered, in our framework we make use of 3-dimensional fuzzy sets. The former have both membership and nonmembership degrees and the latter only membership degrees, so, for the practical example, we only consider the values of the membership degrees from [31]. Another difference is that dual hesitant fuzzy sets do not permit repeated membership values. Therefore, if some decision makers' evaluations coincide, the value is taken into account only once. Nevertheless, in our method the value can be repeated as many times as decision makers coincide.

We generate the decision makers' evaluations accordingly to the data in [31]:

$$\begin{array}{ccc} e_1 & e_2 & e_3 \\ \begin{pmatrix} 0.5 & 0.4 & 0.2 \\ 0.4 & 0.7 & 0.6 \\ 0.4 & 0.5 & 0.6 \\ 0.7 & 0.6 & 0.3 \end{pmatrix} & \begin{pmatrix} 0.4 & 0.6 & 0.1 \\ 0.6 & 0.6 & 0.7 \\ 0.6 & 0.5 & 0.5 \\ 0.6 & 0.6 & 0.4 \end{pmatrix} & \begin{pmatrix} 0.3 & 0.4 & 0.3 \\ 0.7 & 0.7 & 0.4 \\ 0.3 & 0.6 & 0.6 \\ 0.8 & 0.7 & 0.3 \end{pmatrix} \end{array}$$

We apply our method, described in Algorithm 1, to solve the problem.

Step 1. We generate the matrix D where the elements are 3-dimensional fuzzy values, namely, increasing 3-tuples:

$$D = \begin{pmatrix} \{0.3, 0.4, 0.5\} & \{0.4, 0.4, 0.6\} & \{0.1, 0.2, 0.3\} \\ \{0.4, 0.6, 0.7\} & \{0.6, 0.7, 0.7\} & \{0.4, 0.6, 0.7\} \\ \{0.3, 0.4, 0.6\} & \{0.5, 0.5, 0.6\} & \{0.5, 0.6, 0.6\} \\ \{0.6, 0.7, 0.8\} & \{0.6, 0.6, 0.7\} & \{0.3, 0.3, 0.4\} \end{pmatrix}.$$

Step 2. The order $\leq_{\mathbf{M}}$ considered is generated by

- $M_1(x_1, x_2, x_3) = \frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3$;
- $M_2(x_1, x_2, x_3) = \frac{1}{2}x_1 + \frac{1}{2}x_2$;
- $M_3(x_1, x_2, x_3) = \frac{1}{4}x_1 + \frac{1}{2}x_2 + \frac{1}{4}x_3$;

which satisfy the conditions of Prop. 2.

Step 3. The selected weighting vector is $w = (0.35, 0.25, 0.4)$ (as in [31]).

Step 4. We apply the $MOWA$ operator associated to $\leq_{\mathbf{M}}$ and w whose result is

$$\begin{pmatrix} \text{Company 1} & \longrightarrow & \{0.255, 0.32, 0.455\} \\ \text{Company 2} & \longrightarrow & \{0.47, 0.635, 0.7\} \\ \text{Company 3} & \longrightarrow & \{0.42, 0.495, 0.6\} \\ \text{Company 4} & \longrightarrow & \{0.48, 0.515, 0.615\} \end{pmatrix}$$

Step 5. We order the alternatives with respect to the selected order $\leq_{\mathbf{M}}$:

$$\text{Company 2} \geq_{\mathbf{M}} \text{Company 4} \geq_{\mathbf{M}} \text{Company 3} \geq_{\mathbf{M}} \text{Company 1}\,.$$

We select Company 2, which means that the best company to invest in is the car company.

Notice that, since the developed illustrative example is introduced in [31], we have a fixed weighting vector. We have no extra information about the problem and, hence, we set the aggregation functions (which generate the linear order of Step 2) arbitrarily. In some others studies, such as in [19, 28], a comparison between the different solutions using some different orders is made. However, since our sole intention is to show the validity of our proposal, we have only shown the results corresponding to one order.

The best alternative according to Algorithm 1 coincides with best alternative in [31]. However, the treatment of the data is different. Let us highlight the main advantages of the proposed algorithm.

On the one hand, using n-dimensional fuzzy sets the anonymity between the decision makers is assured. Values are the only relevant information, without taking into account the identities of the decision makers. Moreover, if some decision makers coincide in their evaluations, we are able to consider the repeated values avoiding the loss of information that some other systems suffer from.

Besides, n-dimensional fuzzy sets do not need the duality membership/non membership degree and consequently, our algorithm derives the same result using less information.

On the other hand, most of the works that consider generalizations of fuzzy sets make use of partial orders. In this direction, novel studies are trying to generate linear orders in most of the generalizations of fuzzy sets, but they require a study of the monotonicity with respect to the considered linear order. A first study about OWA operators in n-dimensional fuzzy sets as well as the study of their monotonicity with respect to certain admissible orders is found as a theoretical base for the proposed algorithm.

Finally, we solve the problem with the standard aggregation and exploitation phases in order to show that the solutions coincide.

Aggregation phase: we take the OWA operator with weighting vector $w = (0.35, 0.25, 0.4)$ as in [31].

$$C = \begin{pmatrix} 0.395 & 0.47 & 0.195 \\ 0.55 & 0.66 & 0.55 \\ 0.43 & 0.535 & 0.56 \\ 0.695 & 0.635 & 0.333 \end{pmatrix}$$

Exploitation phase: we take the OWA operator with weighting vector $w = (0.35, 0.25, 0.4)$ as in [31].

$$\begin{pmatrix} \text{Company 1} & \longrightarrow & 0.341 \\ \text{Company 2} & \longrightarrow & 0.589 \\ \text{Company 3} & \longrightarrow & 0.502 \\ \text{Company 4} & \longrightarrow & 0.533 \end{pmatrix}$$

$$\text{Consequently,} \quad \text{Company 2} \geq \text{Company 4} \geq \text{Company 3} \geq \text{Company 1}\,.$$

It is clear that with our method, we actually do not need to carry the exploitation phase out so we need to modify the original data less than in the method which consists of both phases.

6. Conclusions

In order to resemble the behavior of membership degrees of fuzzy sets, novel studies generating linear orders for the generalization of fuzzy sets have been presented. However, the linearity of the orders compel us to revise the concept of aggregation functions studying their monotonicity.

In this direction, this work introduces the concept of admissible order for n-dimensional fuzzy sets as well as a construction method for these orders. It is worth mentioning that if the considered membership values are not ordered, the generalizations are also suitable for fuzzy multisets without noteworthy effort.

We also introduce some operators for n-dimensional fuzzy sets, denoted by MOWA, which resemble OWA operators on fuzzy sets. Moreover, we prove that they are increasing functions with respect to a particular class of admissible orders generated by weighted arithmetic means.

Finally, we present an algorithm for multiple criteria group decision making problem using n-dimensional fuzzy sets so that the election of the solution is made by taking the alternative associated to the greatest tuple with respect to the considered admissible order. In order to construct the solution tuple we use the previously introduced OWA operators. Another advantage of our proposal is that it allows to omit the exploitation phase in decision making problems, so the procedure to solve these problems becomes simpler.

For future work, linear orders modify the concept of increasingness in aggregation functions and, hence, a theoretical effort must be done to define and generalize this notion in the different generalizations of fuzzy sets.

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Pointwise aggregation of maps: its structural functional equation and some uses in social choice theory

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Abstract

We study an structural functional equation that is directly related to the pointwise aggregation of a finite number of maps from a nonempty given set into another. First we establish links between pointwise aggregation and invariance properties. Then, paying attention to the particular case of aggregation operators of a finite number of real-valued functions, we characterize several special kinds of aggregation operators, as strictly monotone modifications of projections. As a case study, we introduce a first approach of type-2 fuzzy sets via fusion operators, with an application to image processing.

We develop some applications and possible uses related to the analysis of properties of social evaluation functionals in social choice, showing that those functionals can actually be described by using methods that derive from this setting.

Keywords: Aggregation operators, functional equations, pointwise aggregation, real-valued functions, type-2 fuzzy sets, applications to social choice.
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1. Introduction and motivation

Given a finite collection of maps $\{f_1, f_2, \ldots, f_n\}$ from a set X into another set Y, a new map $f_{n+1} : X \to Y$ obtained someway from the given ones $f_1, f_2, \ldots, f_n$ is usually said to be an aggregation of those maps. A typical example is the arithmetic mean $f_{n+1} = \frac{f_1+\ldots+f_n}{n}$ of n real-valued functions (here $Y = \mathbb{R}$, the real line).

Having these ideas in mind, we wonder which is the information we need in order to obtain the aggregating map f_{n+1}. Given an element $x \in X$, to compute $f_{n+1}(x)$ sometimes we should know the maps $f_1, \ldots, f_n$ as a whole, maybe in all the points of its domain, or at least in several points different from x. But it may also happen that in order to get $f_{n+1}(x)$ we only need to have at hand the values of $f_1, \ldots, f_n$ at that point x. That is: $f_{n+1}(x)$ directly comes from $f_1(x), \ldots, f_n(x)$, and we may then assume that there exists a map $G : Y^n \to Y$ such that $f_{n+1}(x) = G(f_1(x), \ldots, f_n(x))$ holds true for every $x \in X$. Then we say that f_{n+1} depends *pointwise* on the collection $\{f_1, f_2, \ldots, f_n\}$.

Let us consider the following illustrative example:

Example 1. Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ denote two real valued functions on one single real variable. Let $f_3, f_4 : \mathbb{R} \to \mathbb{R}$ respectively be defined, for every $x \in \mathbb{R}$, through the functional equations $f_3(x) = f_1(x) + f_2(x)$ and $f_4(x) = f_1(2x) + f_2(3x)$. Despite both functional equations looking similar at first glance, from a structural point of view they are quite different. The reason is that, working with the former one, in order to know the value of the map f_3 at a point $x \in \mathbb{R}$ we only need to know the values that f_1 and f_2 take *at the same point* x. However, in the second equation, to determine $f_4(x)$ it is not enough to know the values of $f_1(x)$ and $f_2(x)$. As a matter of fact, we need to know the values of f_1 and f_2 at *point(s) different from* x.

This nuance is essential in our approach throughout the present paper. The first case corresponds to the so-called *pointwise aggregation* of maps. The second case, despite still corresponding to aggregation of maps, cannot be called “pointwise”, at least a priori.

In the present paper, given two abstract (nonempty) sets X, Y as well as a natural number n and n maps $\{f_1, f_2, \dots, f_n : X \to Y\}$, we study how to aggregate those maps to obtain a new one, say, $f_{n+1} : X \to Y$ in a way such that the value of f_{n+1} at a point $x \in X$ depends only on the values $\{f_1(x), f_2(x), \dots, f_n(x)\}$ at the same point x. This is what we call *a pointwise aggregation* of $\{f_1, \dots, f_n\}$.

The structure of the paper goes as follows:

In Section 2 we formalize the notion of pointwise aggregation of maps. In Section 3 we study the main structural functional equation linked to the pointwise aggregations of maps. The particular case of real-valued functions is studied in Section 4. In Section 5 we analyze a relevant case study, namely type-2 fuzzy sets via fusion operators. In Section 6 we discuss several applications and uses in social choice theory. We conclude with a final Section 7 that includes suggestions for further research.

2. Previous concepts and notation

Definition 1. Let X, Y denote two (nonempty) sets. Let n be a natural number. Let Y^X denote the set of maps from X into Y, that is, $Y^X = \{f : X \to Y\}$. Let $(f_1, \dots, f_n) \in (Y^X)^n$ stand for an n-tuple of maps from X into Y. A map $f_{n+1} \in Y^X$ is said to be:

(i) An *aggregation* of $(f_1, \dots, f_n)$ if there exists a map $T : (Y^X)^n \to Y^X$ such that $f_{n+1} = T(f_1, \dots, f_n)$. In this case the map T is said to be an *n-dimensional aggregation operator*.

(ii) A *pointwise aggregation* of $(f_1, \dots, f_n)$ if there exists a map $W : Y^n \to Y$ such that $f_{n+1}(x) = W(f_1(x), \dots, f_n(x))$ holds for every $x \in X$. In this case, the map W is said to be a *pointwise n-dimensional aggregator*, whereas the functional equation $f_{n+1}(x) = W(f_1(x), \dots, f_n(x))$ is said to be the *structural functional equation of pointwise aggregation of maps.*

Remark 1. (i) When $n = 1$, we use the word *modifier* instead of aggregator. It is implicitly understood that when we aggregate maps, at least two maps are involved in the process.

(ii) In general it is *not* true that an aggregation f_{n+1} of n maps $f_1, \dots, f_n$ from a set X into another set Y is actually a pointwise aggregation (see the Example 2 below).

(iii) In general, the order of the elements $f_1, \dots, f_n$ in the n-tuple is relevant for the description of the problem.

Example 2. Let $X = Y = \mathbb{N}$ be the set of natural numbers. Given a bijection $f_1 : \mathbb{N} \to \mathbb{N}$, define $f_2(n) = f_1(2n)$, for every $n \in \mathbb{N}$. Thus f_2 is a modifier of f_1, but the values that f_1 takes on each odd natural number play no role in the generation of f_2. Therefore f_2 is not a pointwise modifier of f_1.

Definition 2. Let X, Y be two nonempty sets. Let $T : (Y^X)^n \to Y^X$ denote an n-dimensional aggregation operator of maps from X into Y. Then T is said to be: *representable* if there is a map $W : Y^n \to Y$ such that $T(f_1, \dots, f_n)(x) = W(f_1(x), \dots, f_n(x))$ holds for every $x \in X$ and every n-tuple $(f_1, \dots, f_n) \in (Y^X)^n$.

Example 3. As in Example 2, in general, an n-dimensional aggregation operator from a set X into another set Y may fail to be representable. To see this, let $X = Y = \mathbb{R}$ be the real line, and consider any bijection $f : \mathbb{R} \to \mathbb{R}$. It is clear that given a real number $x \in \mathbb{R}$, the value $f(x)$ has no relationship (in general) with the value $f(-x)$. In other words: despite knowing x and $f(x)$, we cannot guess the value $f(-x)$. Therefore, the modifier T defined by $T(f)(x) = f(-x)$ for every $x \in \mathbb{R}$ and every function $f : \mathbb{R} \to \mathbb{R}$ is not representable.

Remark 2. (i) The concept of a representable aggregation operator has been considered in [13] for the particular case X finite and $Y = \mathbb{R}$, and in [14], for an arbitrary set X and $Y = \mathbb{R}$.

(ii) Given a representable n-dimensional aggregation operator $T : (Y^X)^n \to Y^X$, we immediately observe that the functional equation that naturally arises as $T(f_1, \ldots, f_n)(x) = W(f_1(x), \ldots, f_n(x))$ has a solution W that *only depends on the given operator* T. In particular, for any fixed n-tuple of functions $(f_1, \ldots, f_n) \in (Y^X)^n)$, by calling $f_{n+1} = T(f_1, \ldots, f_n)$, it follows that the map W satisfies the structural functional equation of pointwise aggregation of maps.

3. Representability of operators and the structural functional equation associated to pointwise aggregation of maps

Given two nonempty sets X and Y, our aim is to characterize the representable n-dimensional aggregation operators $T : (Y^X)^n \to Y^X$. In other words, we are interested in finding the solutions (if any) of the structural functional equation $T(f_1, \ldots, f_n)(x) = W(f_1(x), \ldots, f_n(x))$ with $x \in X$ and $(f_1, \ldots, f_n) \in (Y^X)^n$, paying attention to those solutions that only depend on T.

Remark 3. Observe that, conceptually, the problem of solving a structural functional equation that corresponds to pointwise aggregation of maps, namely $f_{n+1}(x) = W(f_1(x), \ldots, f_n(x))$ for every $x \in X$, in which *all the maps* $f_1, \ldots, f_n, f_{n+1}$ *are already known a priori*, is totally different from, starting from an n-dimensional aggregation operator T and setting $T(f_1, \ldots, f_n) = f_{n+1}$ for every n-tuple of maps $(f_1, \ldots, f_n) \in (Y^X)^n$, searching for a map $W : Y^n \to Y$ such that $f_{n+1}(x) = W(f_1(x), \ldots, f_n(x))$ holds for every $x \in X$ and every $(f_1, \ldots, f_n) \in (Y^X)^n$. That is: in this second situation *the maps* $f_1, \ldots, f_n$ *are generic, and obviously not given a priori, so that* W *only depends on* T.

Nevertheless, it is also important to notice that if W is a solution of a structural functional equation of pointwise aggregation, we may try to study if W gives rise to an n-dimensional operator T such that $f_{n+1}(x) = W(f_1(x), \ldots, f_n(x))$ for every $x \in X$, *independently of* $f_1, \ldots, f_n$, where f_{n+1} denotes $T(f_1, \ldots, f_n)$. When this happens, T is representable.

We are now ready to introduce some characterizations of the representability of n-dimensional aggregation operators from a set X into another set Y. To do so, first we introduce a definition.

Definition 3. Let X, Y be two nonempty sets. Let $T : (Y^X)^n \to Y^X$ be an n-dimensional aggregation operator. Then T is said to be:

(i) *fully independent* if it holds that $(f_1(x), \ldots, f_n(x)) = (g_1(t), \ldots g_n(t)) \Rightarrow T(f_1, \ldots, f_n)(x) = T(g_1, \ldots, g_n)(t)$, for every $x, t \in X$ and $(f_1, \ldots, f_n)$, $(g_1, \ldots, g_n) \in (Y^X)^n$.

(ii) *independent as regards maps* if $(f_1(x), \ldots, f_n(x)) = (g_1(x), \ldots g_n(x)) \Rightarrow T(f_1, \ldots, f_n)(x) = T(g_1, \ldots, g_n)(x)$ for every $x \in X$ and $(f_1, \ldots, f_n)$, $(g_1, \ldots, g_n) \in (Y^X)^n$.

(iii) *pointwise independent* whenever $(f_1(x), \ldots, f_n(x)) = (f_1(t), \ldots f_n(t)) \Rightarrow T(f_1, \ldots, f_n)(x) = T(f_1, \ldots, f_n)(t)$ for every $x, t \in X$ and $(f_1, \ldots, f_n) \in (Y^X)^n$.

The next result generalizes Theorem 3.4 of [14].

Theorem 1. Let X, Y be two nonempty sets. Let $T : (Y^X)^n \to Y^X$ be an n-dimensional aggregation operator. The following statements are equivalent:

(i) T is representable,

(ii) T is fully independent,

(iii) T is independent as regards maps, and pointwise independent.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are straightforward. To prove (iii) $\Rightarrow$ (i), let $(y_1, \ldots, y_n) \in Y^n$. Let $c_{y_i} : X \to Y$ be the constant map defined by $c_{y_i}(x) = y_i$ for every $x \in X$ $(i = 1, \ldots, n)$. Fix an element $x_0 \in X$. Define now $W : Y^n \to Y$ as $W(y_1, \ldots, y_n) = T(c_{y_1}, \ldots, c_{y_n})(x_0)$. Observe that the choice of x_0 is irrelevant here, since T is pointwise independent. In order to see that W represents T, fix $x \in X$ and $(f_1, \ldots, f_n) \in (Y^X)^n$. Let $c_i : X \to Y$ be the constant map given by $c_i(t) = f_i(x)$ for every $t \in X$ $(i = 1, \ldots, n)$. Since T is independent as regards maps, it follows that $T(f_1, \ldots, f_n)(x) = T(c_1, \ldots, c_n)(x)$. But, by definition of W, we also have that $T(c_1, \ldots, c_n)(x) = W(f_1(x), \ldots, f_n(x))$. Therefore $T(f_1, \ldots, f_n)(x) = W(f_1(x), \ldots, f_n(x))$ and we are done. $\square$

For the particular case of modifiers of maps from a set X into itself, we obtain the following corollary.

Corollary 1. Let X be a nonempty set and let $T : X^X \to X^X$ be a modifier. If T is representable, then the map $W = T(1_X) \in X^X$, where 1_X denotes the identity map, is actually a representation of T.

Proof. Notice that T is fully independent by Theorem 1. Thus, given $t \in X$ and $f \in X^X$, we have that $f \circ 1_X(t) = f(t) = 1_X(f(t))$. Hence $T(1_X)(f(t)) = T(f \circ 1_X)(t) = T(f)(t)$. Therefore $W(f(t)) = T(1_X)(f(t)) = T(f \circ 1_X)(t) = T(f)(t)$ holds for every $t \in X$, so that T is representable by means of W. $\square$

Example 4. Let X, Y be two nonempty sets. The following n-dimensional aggregation operators from X into Y are obviously representable:

(i) each *projection* $\pi_i : (Y^X)^n \to Y^X$, where $\pi_i(f_1, \ldots, f_n) = f_i$ for every $(f_1, \ldots, f_n) \in (Y^X)^n$ $(i = 1, \ldots, n)$,

(ii) each *constant operator* mapping any n-tuple $(f_1, \ldots, f_n) \in (Y^X)^n$ to a (fixed a priori) map $g : X \to Y$.

Moreover, any n-ary operation in Y immediately gives rise to a representable n-dimensional aggregation operator from X into Y. Indeed, given a map $H : Y^n \to Y$, it is clear that the n-dimensional aggregation operator $T_H : (Y^X)^n \to Y^X$ given by $T_H(f_1, \ldots, f_n)(x) = H(f_1(x), \ldots, f_n(x))$ for every $x \in X$ and $(f_1, \ldots, f_n) \in (Y^X)^n$ is representable through H.

Remark 4. Despite the operators mentioned in Example 4 (i) above being a trivial case of representable aggregation operators, they play a crucial role in many contexts coming from miscellaneous applications where it is important to detect those operators that are actually projections. Thus, in many contexts that study systems of voting or ranking of objects, arising in social choice theory, it is usual to consider a set Y of m elements called alternatives. Assuming that each voter ranks these alternatives by means of a linear order (also known as a *profile*) that reflects her/his individual preferences, we have at hand a set of n maps from $X = \{1, \ldots, m\}$ to Y. (Here n is the number of voters). If f_i is the map that corresponds to the agent i $(i = 1, \ldots, n)$, we shall understand that she/he prefers the alternative $f_i(1)$ as her/his best one, $f_i(2)$ as her/his second best and so on. In this context, a *social rule* is a map from $(Y^X)^n$ to (Y^X), that tries to aggregate the individual preferences into a social one, so defining a new ranking (a linear order or a *social profile*) on the set of alternatives, based on the individual profiles.

In order to obtain some suitable social ranking, that is, one that preserves (in some sense) the individual preferences, several classical models arising in social choice try to impose common sense restrictions to the social rules. However, in most of these models, the unique possible social rules are the so-called *dictatorial* ones, namely, the projections. (See e.g. [3, 4, 20, 23, 31, 33, 34] for a further account). We discuss some applications in this direction later on, in Section 6.

In the spirit of Definition 3 and Theorem 1 we finish this section by considering that an n-tuple of functions from a set X into another set Y has been *fixed*. We furnish a result concerning the structural functional equation of pointwise aggregation.

Theorem 2. Let X, Y be two nonempty sets. Let $(f_1, \ldots, f_n) \in (Y^X)^n$ denote a fixed n-tuple of maps from X into Y. Let $f_{n+1} : X \to Y$ be a map. The following statements are equivalent:

(i) There exists a solution $W : Y^n \to Y$ of the structural functional equation of pointwise aggregation so that $f_{n+1}(x) = W(f_1(x), \ldots, f_n(x))$ holds for every $x \in X$.

(ii) The implication $(f_1(x), \ldots, f_n(x)) = (f_1(t), \ldots f_n(t)) \Rightarrow f_{n+1}(x) = f_{n+1}(t)$ holds true for all $x, t \in X$.

Proof. The implication (i) $\Rightarrow$ (ii) is obvious. To prove that (ii) $\Rightarrow$ (i), we choose an element $y_0 \in Y$, and define W as follows: given $(y_1, \ldots y_n) \in Y^n$ we declare that $W(y_1, \ldots y_n) = f_{n+1}(x)$ if there exists $x \in X$ such that $(y_1, \ldots y_n) = (f_1(x), \ldots f_n(x))$; otherwise, $W(y_1, \ldots y_n) = y_0$. $\square$

4. Pointwise aggregation of real-valued functions

As we have seen in Example 4, given two nonempty sets X and Y the set of representable n-dimensional aggregation operators from X into Y could be too big. Consequently, we do not try to give a list of all the representable n-dimensional aggregation operators from a set X into another set Y. Not even we intend to describe general solutions of the structural functional equation of pointwise aggregation. *Particular results*, paying an special attention to the aggregation of real-valued functions, or at least, maps from a set X into another set Y such that either X or Y (or both) is the real line $\mathbb{R}$.

To start with, we analyze some invariance properties that are closely related to the representability of n-dimensional aggregation operators from a set X into the real line $\mathbb{R}$. To do so, first we introduce some previous definitions.

Definition 4. Let A denote a nonempty set. Each element of A^A, the set of all maps from A into itself, is said to be a *transformation* of the set A.

Definition 5. Let X, Y denote two nonempty sets. Let $T : (Y^X)^n \to Y^X$ be an n-dimensional aggregation operator. Let $\phi_1, \ldots, \phi_n, \gamma : Y \to Y$ stand for $n + 1$ (fixed) transformations of Y. Let $g : X \to Y$ be a (fixed) map from X into Y. Then the operator T is said to be:

(i) *g-constant* if for any n-tuple of maps $(f_1, \ldots, f_n) \in (Y^X)^n$ it holds that $T(f_1, \ldots, f_n) = g$.

(ii) *strongly constant* (or *trivial*) if it is g-constant such that the corresponding map $g : X \to Y$ is also a constant map, that is, there exists $y_0 \in Y$ with $g(x) = y_0$ for every $x \in X$. (In other words: $T(f_1, \ldots, f_n)(x) = y_0$ for every $(f_1, \ldots, f_n) \in (Y^X)^n$ and every $x \in X$).

(iii) *$(\phi_1, \ldots, \phi_n, \gamma)$-invariant* if for every n-tuple $(f_1, \ldots, f_n) \in (Y^X)^n$ it holds that $T(\phi_1 \circ f_1, \ldots, \phi_n \circ f_n) = \gamma \circ T(f_1, \ldots, f_n)$.

In the same way, if F, G are two (fixed) nonempty subsets of the set Y^Y of transformations of Y, the operator T is said to be:

(iv) *(F, G)-invariant* if for every $\{\phi_1, \ldots, \phi_n\} \subseteq F$ and $(f_1, \ldots, f_n) \in (Y^X)^n$ there is a map $\gamma \in G$, which depends on $\{\phi_1, \ldots, \phi_n\}$ and $(f_1, \ldots, f_n)$, such that $T(\phi_1 \circ f_1, \ldots, \phi_n \circ f_n) = \gamma \circ T(f_1, \ldots, f_n)$.

Remark 5. It should be noted that the concept of invariance as defined in [14] is slightly different from ours (see also Definition 1(2) in [13] for the particular case X finite).

Definition 6. Let Y denote a nonempty set. A subset $F \subseteq Y^Y$ of transformations of Y is said to be *stable* if it satisfies the following properties:

(a) Each map $f \in F$ is a bijection whose inverse f^{-1} also belongs to the set F.

(b) The identity map 1_Y (given by $1_Y(t) = t$ for every $t \in Y$) belongs to F.

(c) The subset F is closed under composition of maps, that is $g \circ f \in F$, for every $f, g \in F$.

Example 5. Let $\mathbb{R}$ be the real line. The collection $\mathcal{I}$ of *strictly increasing* functions from $\mathbb{R}$ into $\mathbb{R}$ is stable. In addition, the collection $\mathcal{A}$ of *positively affine* functions from $\mathbb{R}$ into $\mathbb{R}$, given by $\mathcal{A} = \{f \in \mathbb{R}^{\mathbb{R}}$: there exists $a > 0$ and $b \in \mathbb{R}$ such that $f(x) = ax + b$ for every $x \in \mathbb{R}\}$ is stable, too.

Definition 7. Let X, Y denote two nonempty sets. Let $F \subseteq Y^Y$ be a stable subset of transformations of Y. Given two n-tuples $(f_1, \ldots, f_n)$ and $(g_1, \ldots, g_n) \in (Y^X)^n$ of maps from X to Y, we say that $(f_1, \ldots, f_n)$ and $(g_1, \ldots, g_n)$ are F*-equivalent* if there exist n maps $\phi_1, \ldots, \phi_n \in F$ such that $(g_1, \ldots, g_n) = (\phi_1 \circ f_1, \ldots, \phi_n \circ f_n)$.

Remark 6. Notice that by Definition 6 the relationship of F-equivalence just defined is indeed an equivalence relation on $(Y^X)^n$. Moreover, comparing with Definition 5, we observe that an n-dimensional aggregation operator $T \in (Y^X)^n$ is (F, G)-invariant if and only if for every $(f_1, \ldots, f_n)$ and $(g_1, \ldots, g_n) \in (Y^X)^n$ that are F-equivalent, it holds that $T(f_1, \ldots, f_n)$ and $T(g_1, \ldots, g_n) \in Y^X$ are G-equivalent.

We are ready now to introduce a result that links pointwise aggregation to invariance properties.[1]

Theorem 3. Let X be a set with at least two points and let $T : (\mathbb{R}^X)^n \to \mathbb{R}^X$ be an n-dimensional aggregation operator. Let $\mathcal{A}$ (respectively, $\mathcal{I}$) denote the set of all positively affine (respectively, strictly increasing) transformations from $\mathbb{R}$ into $\mathbb{R}$. The following statements are equivalent:

(i) The aggregation operator T is $(\mathcal{A}, \mathcal{I})$-invariant and representable,

(ii) Either T is a strongly constant operator, or there exists $j \in \{1, \ldots, n\}$ and a strictly monotone (i.e.: either strictly increasing or strictly decreasing) function $g : \mathbb{R} \to \mathbb{R}$ such that $T(f_1, \ldots, f_n) = g \circ f_j$, for every $(f_1, \ldots, f_n) \in (\mathbb{R}^X)^n$.

Proof. The implication $(ii) \Rightarrow (i)$ is straightforward. Therefore, we only prove the fact $(i) \Rightarrow (ii)$.

To do so, first we observe that by the representability of T there exists a function $W : \mathbb{R}^n \to \mathbb{R}$ such that $T(f_1, \ldots, f_n)(x) = W(f_1(x), \ldots, f_n(x))$ holds for every $(f_1, \ldots, f_n) \in (\mathbb{R}^X)^n$ and every $x \in X$.

Let us see that the function W satisfies that:

$W(u_1, \ldots, u_n) \leq W(v_1, \ldots, v_n) \Leftrightarrow W(a_1u_1 + b_1, \ldots, a_nu_n + b_n) \leq W(a_1v_1 + b_1, \ldots, a_nv_n + b_n)$ for any $(u_1, \ldots, u_n), (v_1, \ldots, v_n),\ (b_1, \ldots, b_n) \in \mathbb{R}^n$ and $(a_1, \ldots, a_n) \in (0, +\infty)^n$.

Indeed, the converse implication is obvious by taking $a_1 = \ldots = a_n = 1$ and $b_1 = \ldots = b_n = 0$. In order to prove the direct implication, let $(u_1, \ldots, u_n), (v_1, \ldots, v_n),\ (b_1, \ldots, b_n) \in \mathbb{R}^n$ and $(a_1, \ldots, a_n) \in (0, +\infty)^n$ be fixed. Take now $(f_1, \ldots, f_n) \in (\mathbb{R}^X)^n$ and $x, y \in X$ such that, for each $i \in \{1, \ldots, n\}$, $f_i(x) = u_i$ and $f_i(y) = v_i$. Notice that, since we are assuming that the cardinality of X is strictly bigger than 1, this is always possible. So, $W(u_1, \ldots, u_n) = T(f_1, \ldots, f_n)(x) \leq T(f_1, \ldots, f_n)(y) = W(v_1, \ldots, v_n)$. Consider, for each $i \in \{1, \ldots, n\}$, the transformation $\phi_i(t) = a_it + b_i \in \mathcal{A}$. Then, by $(\mathcal{A}, \mathcal{I})$-invariance, there is a strictly increasing transformation, say $\gamma \in \mathcal{I}$, such that $T(\phi_1 \circ f_1, \ldots, \phi_n \circ f_n) = \gamma \circ T(f_1, \ldots, f_n)$. So, $T(\phi_1 \circ f_1, \ldots, \phi_n \circ f_n)(x) = \gamma \circ$

[1] In addition, in [14] the concept of a *comparison meaningful*, with respect to independent ordinal scales, aggregation operator is studied. For the particular case of real-valued functions of n variables, this concept turns out to be a fundamental axiom in measurement theory (see [26, 27] for further details).

$T(f_1,\ldots,f_n)(x) = \gamma(T(f_1,\ldots,f_n)(x)) \leq \gamma(T(f_1,\ldots,f_n)(y)) = \gamma \circ T(f_1,\ldots,f_n)(y) = T(\phi_1 \circ f_1,\ldots,\phi_n \circ f_n)(y)$.

Now, for each $z \in X$, it holds that $T(\phi_1 \circ f_1,\ldots,\phi_n \circ f_n)(z) = W(\phi_1 \circ f_1(z),\ldots,\phi_n \circ f_n(z)) = W(a_1 f_1(z)+b_1,\ldots,a_n f_n(z)+b_n)$. Thus, $W(a_1 f_1(x)+b_1,\ldots,a_n f_n(x)+b_n) = T(\phi_1 \circ f_1,\ldots,\phi_n \circ f_n)(x) \leq T(\phi_1 \circ f_1,\ldots,\phi_n \circ f_n)(y) = W(a_1 f_1(y)+b_1,\ldots,a_n f_n(y)+b_n)$, and the inequality is proved.

Let us see now that this last fact implies that either $W : \mathbb{R}^n \to \mathbb{R}$ is a constant function, or, alternatively, there exist both $j \in \{1,\ldots n\}$ and a strictly monotonic function $g : \mathbb{R} \to \mathbb{R}$ such that $W(t_1,\ldots,t_n) = g(t_j)$ holds true for every n-tuple $(t_1,\ldots,t_n) \in \mathbb{R}^n$. To that end, consider the total preorder, say $\precsim_W$, on $\mathbb{R}^n$ defined by W as follows: $(s_1,\ldots,s_n) \precsim_W (t_1,\ldots,t_n) \Leftrightarrow W(s_1,\ldots,s_n) \leq W(s_1,\ldots,s_n)$.

By the properties of W, using Theorem 2 in [24] as well as Theorem 3 and its subsequent results in [12], it follows that either $\precsim_W$ is trivial (and consequently W is a constant function), or $\precsim_W$ is non-trivial and, in this case, either there is $j \in \{1,\ldots,n\}$ such that $(s_1,\ldots,s_n) \precsim_W (t_1,\ldots,t_n) \Leftrightarrow s_j \leq t_j$ holds for every $(s_1,\ldots,s_n)$, $(t_1,\ldots t_n) \in \mathbb{R}^n$, or, in a dual way, there exists $j \in \{1,\ldots,n\}$ such that $(s_1,\ldots,s_n) \precsim_W (t_1,\ldots,t_n) \Leftrightarrow t_j \leq s_j$ holds for any $(s_1,\ldots,s_n)$, $(t_1,\ldots t_n) \in \mathbb{R}^n$.

Therefore, in the former of these two last situations in which $\precsim_W$ is non-trivial, there exists a strictly increasing function $g : \mathbb{R} \to \mathbb{R}$ such that $W = g \circ \pi_j$, where $\pi_j : \mathbb{R}^n \to \mathbb{R}$ stands here for the projection on the j-th component, that is, $\pi_j(t_1,\ldots,t_n) = t_j$ for every n-tuple $(t_1,\ldots,t_n) \in \mathbb{R}^n$.

And in the second of these two situations, where $\precsim_W$ is non-trivial, there exists a strictly decreasing function $h : \mathbb{R} \to \mathbb{R}$ such that $W = h \circ \pi_j$.

The proof concludes immediately, by just taking into account that the operator T is representable through W. □

Remark 7. If the cardinality of X is 1, $\mathbb{R}^X$ can be identified with $\mathbb{R}$. Thus, an aggregation operator $T : (\mathbb{R}^X)^n \to \mathbb{R}^X$ clearly reduces to a function from $\mathbb{R}^n$ into $\mathbb{R}$. Therefore, in this case, any aggregation operator is trivially representable.

Moreover, $(\mathcal{A},\mathcal{I})$-invariance is also trivially met when X is a singleton. So, in this case, the statement reads as follows:

<<*For a function* $T : \mathbb{R}^n \to \mathbb{R}$ *the following assertions are equivalent:*

(1) $T(u_1,\ldots,u_n) \leq T(v_1,\ldots,v_n) \Leftrightarrow T(a_1u_1+b_1,\ldots,a_nu_n+b_n) \leq T(a_1v_1+b_1,\ldots,a_nv_n+b_n)$ *holds true, for any* n*-tuples* $(u_1,\ldots,u_n)$, $(v_1,\ldots,v_n)$, $(b_1,\ldots,b_n) \in \mathbb{R}^n$ *and* $(a_1,\ldots,a_n) \in (0,+\infty)^n$,

(2) Either T *is a constant function, or there exists* $j \in \{1,\ldots,n\}$ *and a strictly monotone function* $g : \mathbb{R} \to \mathbb{R}$ *such that* $T(u_1,\ldots,u_n) = g(u_j)$, *for every* $(u_1,\ldots,u_n) \in \mathbb{R}^n$.>>

An immediate consequence of Theorem 3 appears when the operator T satisfies a property of *idempotence* (see Definition 8 below). As a matter of fact, the subsequent Corollary 2 characterizes projections among n-dimensional operators from a set X, whose cardinality is strictly bigger than 1, to the real line $Y = \mathbb{R}$.

Definition 8. Let X, Y be two nonempty sets. An n-dimensional operator $T : (Y^X)^n \to Y^X$ is said to be *idempotent* if for every map $f \in Y^X$ it holds that $T(f,\ldots,f) = f$.

Corollary 2. Let $T : (\mathbb{R}^X)^n \to \mathbb{R}^X$ be an n-dimensional aggregation operator, where X has, at least, two points. The following statements are equivalent:

(i) The operator T is $(\mathcal{A},\mathcal{I})$-invariant, idempotent and representable,

(ii) The operator T is a projection: there exists $j \in \{1,\ldots,n\}$ such that $T(f_1,\ldots,f_n) = f_j$, for every $(f_1,\ldots,f_n) \in (\mathbb{R}^X)^n$.

Remark 8. The assumption of X having at least two points cannot be dropped from the statement of Corollary 2. Indeed, if X is a singleton, then, as already mentioned in Remark 7 (v), an aggregation operator T is a function from $\mathbb{R}^n$ into $\mathbb{R}$. Note that the function $T : \mathbb{R}^n \to \mathbb{R}$ defined as $T(u_1, \ldots, u_n) = max(u_i)$, for every $(u_1, \ldots, u_n) \in \mathbb{R}^n$, $i \in \{1, \ldots, n\}$, is $(\mathcal{A}, \mathcal{I})$-invariant, idempotent and representable. Nevertheless, it is not a projection. A version of Corollary 2, in the case that X is a singleton, can be obtained by adding to the idempotence property the fulfillment of the condition $W(u_1, \ldots, u_n) \leq W(v_1, \ldots, v_n) \Leftrightarrow W(a_1u_1+b_1, \ldots, a_nu_n+b_n) \leq W(a_1v_1+b_1, \ldots, a_nv_n+b_n)$ for any $(u_1, \ldots, u_n), (v_1, \ldots, v_n)$, $(b_1, \ldots, b_n) \in \mathbb{R}^n$ and $(a_1, \ldots, a_n) \in (0, +\infty)^n$.

5. A case study: type-2 fuzzy sets via fusion operators

5.1. Fusion of type-2 fuzzy sets

Since Zadeh introduced the concept of a fuzzy set in 1965 [36], the interest for the different possible extensions of such sets has been increasingly growing, both from a theoretical and from an applied point of view [9]. Among the extensions of fuzzy sets some of the most relevant ones are the interval-valued fuzzy sets [32], the Atanassov's intuitionistic fuzzy sets [5] and the type-2 fuzzy sets [37], which encompass the two previous ones. In this work we focus on the latter, having in mind the results introduced in Section 2 and Section 3 and with an eye kept in the application in fields such as image processing or decision making (see Section 6).

As an extension of the concept of a fusion operator relative to fuzzy sets (functions that take m values (membership degrees) in $[0,1]$ and give back a new value in $[0,1]$), and taking into account that the membership values of the elements in a type-2 fuzzy set are given in terms of new fuzzy sets over the universe $[0,1]$ (that is, by means of functions defined over $[0,1]$) we define fusion operators for type-2 fuzzy sets as mappings that take m functions from $[0,1]$ to $[0,1]$ into a new function in the same domain. That is, functions of the type $F : \left([0,1]^{[0,1]}\right)^m \to [0,1]^{[0,1]}$. Our goal is to study these functions in a way as general as possible, since in principle no restriction is imposed to the membership values of a type-2 fuzzy set. So we do not require a priori any property such as continuity, monotonicity, symmetry, etc. (See also [11] for more details).

In this subsection we recall the concepts of a fuzzy set and a type-2 fuzzy set. Throughout this paper we denote by X a nonempty set that represents the universe of discourse.

Definition 9. A fuzzy set A on the universe X is defined as the subset of the Cartesian product $X \times [0,1]$ given by $A = \{(x, \mu_A(x)) | x \in X\}$, where $\mu_A : X \longrightarrow [0,1]$ is said to be the *membership function* (also known as the *indicator*) of A with respect to the universe X. given $x \in X$, the value $\mu_A(x) \in [0,1]$ is called the *membership degree* of the element x as regards X.

In [37], Zadeh introduced the concept of a type-2 fuzzy set as a generalization of a fuzzy set (also called type-1 fuzzy set). In type-2 fuzzy sets, the corresponding notion of the membership degree of an element with respect to the universe considered is given by a fuzzy set whose universe is again $[0,1]$. That is, the membership degree of an element to a type-2 fuzzy set becomes now a function that belongs to $[0,1]^{[0,1]}$, the set of all possible functions from $[0,1]$ to $[0,1]$. The mathematical formalization of the notion of type-2 fuzzy set was made in [17, 30].

Definition 10. A type-2 fuzzy set A on the universe X is defined as $A = \{(x, \mu_A(x)) | x \in X\}$, where $\mu_A : X \to [0,1]^{[0,1]}$.

As a particular case of Definition 1 and Definition 2 the following concepts are introduced in this context.

Definition 11. Let $(f_1, \ldots, f_m) \in ([0,1]^{[0,1]})^m$ stand for a m-tuple of maps from $[0,1]$ into $[0,1]$.

(i) A map $f_{m+1} \in [0,1]^{[0,1]}$ is said to be a *fusion* of $(f_1, \ldots, f_m)$ if there exists a map $F : ([0,1]^{[0,1]})^m \to [0,1]^{[0,1]}$ such that $f_{m+1} = F(f_1, \ldots, f_m)$. In this case the map F is said to be an *m-dimensional fusion operator.*

(ii) A map $f_{m+1} \in [0,1]^{[0,1]}$ is said to be a *pointwise fusion* of $(f_1, \dots, f_m)$ if there exists a function $W : [0,1]^m \to [0,1]$ satisfying that $f_{m+1}(x) = W(f_1(x), \dots, f_m(x))$ for every $x \in [0,1]$. In this case, the map W is said to be a *pointwise m-dimensional fusion operator*, whereas the functional equation $f_{m+1}(x) = W(f_1(x), \dots, f_m(x))$ is said to be the *structural functional equation of pointwise fusion operators.*

(iii) Let $F : ([0,1]^{[0,1]})^m \to [0,1]^{[0,1]}$ denote an m-dimensional fusion operator of maps from $[0,1]$ into $[0,1]$. Then F is said to be *representable* if there is a map $W : [0,1]^n \to [0,1]$ such that $F(f_1, \dots, f_m)(x) = W(f_1(x), \dots, f_m(x))$ holds true for every $x \in [0,1]$ and $(f_1, \dots, f_m) \in ([0,1]^{[0,1]})^m$.

Remark 9. Obviously Definition 3 and the subsequent results (Theorem 1, etc.) stated in Section 2 can also be used in this setting, now working with type-2 fuzzy sets and/or fusion operators.

5.2. Union and intersection of type-2 fuzzy sets as fusion operators

In this subsection we focus on two key concepts to deal with type-2 fuzzy sets, namely the union and the intersection. Recall that the union and intersection of two type-2 fuzzy sets is a new type-2 fuzzy set. Therefore, we can interpret the union and intersection of type-2 fuzzy sets as a special case of fusion of type-2 fuzzy sets.

It is important to say that it does not exist a unique definition for union and intersection of type-2 fuzzy sets. However, the operations considered in this work cover several cases since they act in the same way. For each element in the universe X, we use a function that fuses the membership functions of that element to each set. So these operations can be seen as fusion operators $F : \left([0,1]^{[0,1]}\right)^2 \to [0,1]^{[0,1]}$.

Considering type-2 fuzzy sets as a special case of L-fuzzy sets launched by Goguen [21], the union and intersection of type-2 fuzzy sets is stated leaning on the union and intersection of fuzzy sets, as follows [22].

Definition 12. Let $f_1, f_2 \in [0,1]^{[0,1]}$ be two functions. The operations (respectively called *union and intersection*) $\cup, \cap : \left([0,1]^{[0,1]}\right)^2 \to [0,1]^{[0,1]}$ are respectively defined as $(f_1 \cup f_2)(x) = \max(f_1(x), f_2(x))$ and$(f_1 \cap f_2)(x) = \min(f_1(x), f_2(x))$ for every x in the unit interval $[0,1]$.

The following result is straightforward.

Proposition 1. The mappings $\cup, \cap : \left([0,1]^{[0,1]}\right)^2 \to [0,1]^{[0,1]}$ are representable fusion operators for all $f_1, f_2 \in [0,1]^{[0,1]}$.

The problem with the previous definition of union and intersection of type-2 fuzzy sets is that these concepts do not retrieve the usual union and intersection of fuzzy sets [18]. In order to avoid this trouble, another definition of union and intersection of type-2 fuzzy sets was introduced in this literature, based on Zadeh's extension principle [18, 28, 30, 35].

Definition 13. Let $f_1, f_2 \in [0,1]^{[0,1]}$ be two maps. The operations (again, respectively called *union and intersection*, but in the sense of Zadeh's extension principle) $\sqcup, \sqcap : \left([0,1]^{[0,1]}\right)^2 \to [0,1]^{[0,1]}$ are respectively defined as $(f_1 \sqcup f_2)(x) = sup\{(f_1(y) \wedge f_2(z)) : y \vee z = x\}$ and $(f_1 \sqcap f_2)(x) = sup\{(f_1(y) \wedge f_2(z)) : y \wedge z = x\}$ for every $x \in [0,1]$.

(Here, given $a, b \in [0,1]$, the standard latticial notation goes as follows: $a \vee b = max\{a,b\}$, whereas $a \wedge b = min\{a,b\}$).

Remark 10. Observe that the fusion operators $\sqcup$ and $\sqcap$ are completely different from $\cup$ and $\cap$. In general, for any $f_1, f_2 \in [0,1]^{[0,1]}$ it is not possible to know the values $(f_1 \sqcup f_2)(x)$ and $(f_1 \sqcap f_2)(x)$ by knowing only the values of $f_1(x)$ and $f_2(x)$.

Proposition 2. In general, the mappings $\sqcup, \sqcap : \left([0,1]^{[0,1]}\right)^2 \to [0,1]^{[0,1]}$ fail to be representable fusion operators.

Nevertheless, we may still notice that the union and intersection of fuzzy sets (in the sense of Zadeh's extension principle) indeed become pointwise fusion operators on a suitable (smaller) restricted domain.

Theorem 4. The following statements hold true:

(i) If $f_1, f_2 \in [0,1]^{[0,1]}$ are increasing mappings then the operation $\sqcup$ is a pointwise fusion operator.

(ii) If $f_1, f_2 \in [0,1]^{[0,1]}$ are decreasing mappings then the operation $\sqcap$ is a pointwise fusion operator.

Proof. (See also [35])

(i) For the case $\sqcup$, let us see that $\{(y,z)|y \vee z = x\} = \{(x,z)|z \leq x\} \cup \{(y,x)|y \leq x\}$
In the first situation, namely for $\{(x,z)|z \leq x\}$, since the function f_2 is increasing, we get $f_2(z) \leq f_2(x)$ for all $z \leq x$. So, $f_1(x) \wedge f_2(z) \leq f_1(x) \wedge f_2(x)$ for all $z \leq x$. In particular, $\sup_{z \leq x}\{f_1(x) \wedge f_2(z)\} \leq f_1(x) \wedge f_2(x)$. Moreover, since the point (x,x) lies in the set considered, we have that $\sup_{z \leq x}\{f_1(x) \wedge f_2(z)\} = f_1(x) \wedge f_2(x)$.
In a similar way, in the second situation, for $\{(y,x)|y \leq x\}$, it holds that $\sup_{y \leq x}\{f_1(y) \wedge f_2(x)\} = f_1(x) \wedge f_2(x)$.
Therefore $(f_1 \sqcup f_2)(x) = \sup\{(f_1(y) \wedge f_2(z) : y \vee z = x\} =$
$\vee(\sup_{z \leq x}\{f_1(x) \wedge f_2(z)\}, \sup_{y \leq x}\{f_1(y) \wedge f_2(x)\}) = f_1(x) \wedge f_2(x)$.
Hence the union is a pointwise fusion operator.

(ii) This case, for the intersection $\sqcap$, is handled in an entirely analogous way to the case $\sqcup$ just discussed. □

5.3. Applications to Image Processing

A grayscale image of $M \times N$ pixels (each pixel having an intensity value from 0 to $L-1$) can be interpreted as a function f defined on the unit square $[0,1] \times [0,1]$ and taking values in the unit interval $[0,1]$ (normalizing the number of rows, columns and intensities). Each point $(x,y) \in [0,1] \times [0,1]$ is usually called a *pixel*. The value $f(x,y) \in [0,1]$ would be here a graduation of gray between 0 (or totally black) and 1 (or totally white).

Similarly, an image in the RGB color scheme could be interpreted as the superposition of three images, each of them using one of the basic colors, namely red, green and blue.

Proposition 3. A function $F : [0,1] \times [0,1] \to [0,1]$ can also be equivalently interpreted as a type-2 fuzzy set on the universe $X = [0,1]$, the unit interval.

Proof. The function F is identified with the map $F^* : [0,1] \to [0,1]^{[0,1]}$ such that, given any element $x \in [0,1]$, $F^*(x)$ is actually a function $F^*(x) : [0,1] \to [0,1]$ satisfying that $[F^*(x)](y) = F(x,y)$ holds true for every $y \in [0,1]$. □

Remark 11. Bearing this equivalence in mind, each procedure that merges someway two different grayscale images, into a new one, could accordingly be interpreted as a suitable fusion of two type-2 fuzzy sets f_1 and f_2 into a new one f_3. Using the notation $X = [0,1]$ and $Y = [0,1] \times [0,1]$, the functions f_1, f_2, f_3 are indeed type-2 fuzzy sets and belong to Y^X. And a procedure T such that $f_3 = T(f_1, f_2)$ would correspond to a fusion operator $T : (Y^X)^2 \to Y^X$.

Moreover, whenever T is *representable* (see Theorem 1 in Section 3 as well as Proposition 1 and Theorem 4 in Section 5.3) we also have that there exists a map $W : [0,1]^{[0,1]} \times [0,1]^{[0,1]} \to [0,1]^{[0,1]}$ such that $f_3(x) = T(f_1,f_2)(x) = W(f_1(x), f_2(x))$ holds true for every $x \in [0,1]$ and every pair (f_1, f_2) of type-2 fuzzy sets on the universe $[0,1]$.

If f_1, f_2, f_3 are indeed type-2 fuzzy sets on $[0,1]$ and T is a procedure such that $f_3 = T(f_1, f_2)$, due to the equivalence stated in Proposition 3, we may consider f_1, f_2, f_3 as images in grayscale, that is, as functions $F_1, F_2, F_3 : [0,1] \times [0,1] \to [0,1]$. Notice that for every $x, y \in [0,1]$ and

$i \in \{1,2,3\}$ we have that $F_i(x,y) = [f_i(x)](y)$ holds true. In addition, if T is a fusion operator $T : (Y^X)^2 \to Y^X$ with $X = [0,1]$ and $Y = [0,1] \times [0,1]$, such that $T(f_1, f_2) = f_3$, it gives rise to another aggregation operator, say $T' : (X^Y)^2 \to X^Y$, such that $T'(F_1, F_2) = F_3$. As a matter of fact, given $x, y \in [0,1]$ it follows that $T'(F_1,F_2)(x,y) = F_3(x,y) = [f_3(x)](y) = [T(f_1,f_2)(x)](y)$.

Remark 12. The obvious interpretation at this stage is the following: if the procedure T' that merges any two images F_1 and F_2 to get a new one $F_3 = T'(F_1, F_2)$ is representable, then in order to obtain the new image F_3 we do not need to have at hand, a priori, the entire images F_1 and F_2. Instead, we can work pointwise, so obtaining for any pixel $(x,y) \in [0,1] \times [0,1]$ the corresponding amount of gray, $F_3(x,y)$, in terms of $F_1(x,y)$ and $F_2(x,y)$ only.

This is the mathematical explanation that supports a wide sort of typical procedures often used in Image Processing to obtain enhanced and fused images.

6. Some applications and uses in social choice theory

6.1. Columns vs. rows in the Arrowian setting

Let $X = \{1, \ldots, m\}$ be the set of the first m natural numbers. A map $f : X \to X$ can be described through a column vector of m elements such that the i-th entry of this column vector is $f(i) \in X$, with i varying from 1 to m. If we have at hand n such maps, say $\{f_1, \ldots, f_n\}$ from X into X, then we can jointly represent them by a $m \times n$ matrix whose entry in the i-th row and j-th column is $f_j(i)$. Here $i \in \{1, \ldots, m\}$ whereas $j \in \{1, \ldots, n\}$. A n-dimensional operator $T : (X^X)^n \to X^X$ can be understood as a *rule* that operates with the *columns* $\{f_1, \ldots, f_n\}$ of each of those matrices $m \times n$ to accordingly get a new column vector $T(f_1, \ldots, f_n) \in X^X$.

However, when T is *representable*, due to the satisfaction of the structural functional equation of pointwise aggregation, there exist a map $W : X^n \to X$ such that $T(f_1, \ldots, f_n)(i) = W(f_1(i), \ldots, f_n(i))$ holds for every $i \in \{1, \ldots m\}$. This means that we can get the column vector $T(f_1, \ldots, f_n)$ working directly on each *row* of the $m \times n$ matrix that defines T.

All this has important uses in *voting theory*.

In fact, as already pointed out in Remark 4, the famous *Arrow's impossibility theorem in social choice theory* (see e.g. [3, 4, 20, 23, 25, 31, 33, 34]) deals with conditions on the rankings of preferences that n agents define on m objects, in a way that each individual ranks the m objects by means of a linear (total) order, thus defining a column vector that represents her/his preferences.

Therefore, in this procedure there are n such columns, so that column j where j varies from 1 to n reflects the preferences of the individual j. In these contexts, the preferences of the agents are usually defined through *permutations* (that is, bijective maps) stated on the given set of m objects or *alternatives*. Consequently, we can visualize all the preference rankings established by the n agents using a $m \times n$ matrix, in such a way that each column corresponds to an individual preference, as remarked before.

A social choice aggregation rule (in the Arrowian sense) is then an n-dimensional aggregation operator that acts over the columns of such a matrix in order to define a social ranking, also given by means of a suitable *permutation* on the set of m objects. In addition, some conditions of, say, common sense are assumed to be satisfied by the aggregation operator [2].

Kenneth J. Arrow established in [3, 4] a set of conditions, all of them being quite natural, that, perhaps surprisingly, imply that the only possible social preference that accomplishes all such conditions of aggregation is one of the columns of the given $m \times n$ matrix, so that the social preference is indeed a *projection*: it must coincide with the preference of at least one of the m agents, who could be considered as a *dictator* in the social choice theory context and terminology. (See also [34]).

[2] To put an obvious example, when all the columns of the matrix are identical, or, in other words, provided that all the agents have defined exactly the same ranking, the social preference should compulsorily agree with such columns, thus respecting the unanimity.

Thus, Arrow's theorem is based on *vertical* conditions (i.e., conditions on the *individual profiles* that correspond to *columns* of the matrix) imposed on the aggregation rules.

However, as already studied e.g. in [31], sometimes it is possible to achieve similar results, now based on *horizontal* aggregation conditions (i.e., conditions on the *rows* of the matrix), expressed in terms of the absolute position of an alternative in a preference ranking. As a matter of fact, it is possible to obtain a *positional version of Arrow's theorem* for the case when the possibility of indifference is removed (so that preferences are always strict), see [31] for further details.

This last possibility obviously links with the results introduced in the present manuscript. Indeed, if a social rule, say T, is defined through a *representable* n-dimensional aggregation operator (not necessarily being a projection), it is then obvious that it can be re-obtained working by rows instead of by columns, due to the satisfaction of the structural functional equation of pointwise agregation.

6.2. Social evaluation operators

Now we consider a social choice model that takes as primitive the concept of a social evaluation functional. Basically, a social evaluation functional is an aggregation operator that maps a n-tuple of (individual) real-valued functions defined on an unstructured set X into a (social) real-valued function defined on X. (See [7, 29] for an excellent account of the role played by utility methods in social choice theory. See also [1, 2, 10] to find some other functional equations related to utility theory and/or social choice. For further details on social evaluation functionals, see [13]).

Let X be a nonempty set (usually called the set of alternatives) and let $n \geq 1$ be a natural number (number of agents). A typical function, from X into $\mathbb{R}$, corresponding to the agent j, is denoted by f_j. As in the previous sections, the set of all real-valued functions from X into $\mathbb{R}$ is denoted by $\mathbb{R}^X$.

A typical n-tuple of real-valued functions is denoted by $F = (f_1, \ldots, f_n)$ and $(\mathbb{R}^X)^n$ stands for the set of all those n-tuples.

For a given $x \in X$ and $F = (f_1, \ldots, f_n) \in (\mathbb{R}^X)^n$, $F(x)$ denotes the n-tuple $(f_1(x), \ldots f_n(x)) \in \mathbb{R}^n$.

Let $\mathcal{A}$ stand for the set of positively affine maps from $\mathbb{R}^n$ into $\mathbb{R}^n$. Given n-tuples $a = (a_1, \ldots, a_n) \in (0, +\infty)^n$ and $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$ a typical element (that we denote by A^a_b) of the set $\mathcal{A}$ satisfies that $A^a_b(x_1, \ldots, x_n) = (a_1x_1 + b_1, \ldots, a_nx_n + b_n)$, for every $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

Given $A^a_b \in \mathcal{A}$ and $F = (f_1, \ldots, f_n) \in (\mathbb{R}^X)^n$, let $A^a_bF \in (\mathbb{R}^X)^n$ denote the n-tuple of maps such that $A^a_bF(x) = (a_1\ f_1(x) + b_1, \ldots, a_n\ f_n(x) + b_n)$, for every $x \in X$.

Definition 14. A *social evaluation functional* is an n-dimensional aggregation operator $T : (\mathbb{R}^X)^n \to \mathbb{R}^X$ that associates a real-valued function $T(f_1, \ldots, f_n) \in \mathbb{R}^X$ to any n-tuple $F = (f_1, \ldots, f_n)$ in the domain $(\mathbb{R}^X)^n$. It should be noticed that, since the domain of a social evaluation functional is $(\mathbb{R}^X)^n$, the so-called property of *universality of domain* is implicitly assumed. A social evaluation functional T is said to be *trivial* if there exists a constant real-valued map $f_0 \in \mathbb{R}^X$ such that $T(f_1, \ldots, f_n) = f_0$, for all $(f_1, \ldots, f_n) \in (\mathbb{R}^X)^n$ and $f_0(t) = r_0 \in \mathbb{R}$, for every $t \in X$ (here r_0 is the constant value that f_0 takes on X). In other words, the aggregation operator T is strongly constant in the sense of Definition 5 (ii).

We now present a basic definition that essentially translates the usual welfarist axioms involved in the definition of a social welfare functional to the setting of social evaluation functionals.

Definition 15. A social evaluation functional $T : (\mathbb{R}^X)^n \to \mathbb{R}^X$ is said to satisfy:

(1) the property of *welfarism* if there is a function $W : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $T(f_1, \ldots, f_n)(x) = W(f_1(x), \ldots, f_n(x))$, for every $(f_1, \ldots, f_n) \in (\mathbb{R}^X)^n$ and $x \in X$ (i.e., T is representable in the sense of Definition 2),

(2) the property of *information invariance with respect to independent interval scales* if $T(f_1, \ldots, f_n)(x) \leq T(f_1, \ldots, f_n)(y) \Longleftrightarrow T(A^b_aF)(x) \leq T(A^b_aF)(y)$, for every $F = (f_1, \ldots, f_n) \in (\mathbb{R}^X)^n$, $x, y \in X$ and $A^b_a \in \mathcal{A}$,

(3) the *weak Pareto* property, if for every $F = (f_1, \dots, f_n) \in (\mathbb{R}^X)^n$ and $x, y \in X$ such that $f_j(x) < f_j(y)$ holds for every $j \in \{1, \dots, n\}$, it holds that $T(f_1, \dots, f_n)(x) < T(f_1, \dots, f_n)(y)$,

(4) the *Pareto* property if it already satisfies the weak Pareto property and, in addition, for every $F = (f_1, \dots, f_n) \in (\mathbb{R}^X)^n$ and $x, y \in X$ such that $f_j(x) \leq f_j(y)$ holds for every $j \in \{1, \dots, n\}$, it holds that $T(f_1, \dots, f_n)(x) \leq T(f_1, \dots, f_n)(y)$,

(5) the *strong Pareto* property if for every $F = (f_1, \dots, f_n) \in (\mathbb{R}^X)^n$ and $x, y \in X$ such that $f_j(x) \leq f_j(y)$ holds for every $j \in \{1, \dots, n\}$ and $f_k(x) < f_k(y)$ holds for some $k \in \{1, \dots, n\}$, then it holds that $T(f_1, \dots, f_n)(x) < T(f_1, \dots, f_n)(y)$,

(6) the *strong dictatorship* property if there is an individual $i \in \{1, \dots, n\}$ (the strong dictator) such that $T(f_1, \dots, f_n)(x) \leq T(f_1, \dots, f_n)(y)$ if and only if $f_i(x) \leq f_i(y)$, for every $F = (f_1, \dots, f_n) \in (\mathbb{R}^X)^n$ and $x, y \in X$,

(7) the *inverse strong dictatorship* property if there is $i \in \{1, \dots, n\}$ (the inverse strong dictator) such that $T(f_1, \dots, f_n)(x) \leq T(f_1, \dots, f_n)(y)$ if and only if $f_i(y) \leq f_i(x)$, for every $F = (f_1, \dots, f_n) \in (\mathbb{R}^X)^n$ and $x, y \in X$,

(8) the *unanimity* property if $T(f, \dots, f) = f$, for every $f \in \mathbb{R}^X$,

(9) the *anonymity* property if $T(f_{\sigma(1)}, \dots, f_{\sigma(n)}) = T(f_1, \dots, f_n)$, for every permutation σ of the set $\{1, \dots, n\}$.

As a direct consequence of the ideas used in the proof of Theorem 3, the following results are reached. We only sketch the proof of Theorem 5.

Theorem 5. Let X be a set with at least two points. For a social evaluation functional $T : (\mathbb{R}^X)^n \to \mathbb{R}^X$, the following assertions are equivalent:

(i) T satisfies welfarism and information invariance with respect to independent interval scales,

(ii) T is trivial, or it satisfies strong dictatorship, or it satisfies inverse strong dictatorship.

Proof. The proof of the converse implication $(ii) \Rightarrow (i)$ is easy. Indeed, if T is trivial, then it is sufficient to take $W(x_1, \dots, x_n) = r_0$, for some fixed $r_0 \in \mathbb{R}$, in order to see that the assumptions in (i) hold. If T is strongly dictatorial, then take $W(x_1, \dots, x_n) = x_i$, for every $(x_1, \dots, x_n) \in \mathbb{R}^n$, i being the strong dictator, and observe that all the conditions in (i) hold too. If T is strongly inversely-dictatorial, then take $W(x_1, \dots, x_n) = -x_i$, for every $(x_1, \dots, x_n) \in \mathbb{R}^n$, i being the inverse strong dictator, to conclude. So, we concentrate on the proof of the direct implication $(i) \Rightarrow (ii)$. To that end, consider a real-valued function $W : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $T(f_1, \dots, f_n)(x) = W((f_1(x), \dots, f_n(x))$, for every $(f_1, \dots, f_n) \in (\mathbb{R}^X)^n$ and $x \in X$. Notice that W exists because T satisfies welfarism. Then, since T also satisfies information invariance with respect to independent interval scales, it is straightforward to see that W satisfies the key property in the proof of Theorem 3, namely, $W(u_1, \dots, u_n) \leq W(v_1, \dots, v_n) \Leftrightarrow W(a_1u_1 + b_1, \dots, a_nu_n + b_n) \leq W(a_1v_1 + b_1, \dots, a_nv_n + b_n)$ for any $(u_1, \dots, u_n), (v_1, \dots, v_n), (b_1, \dots, b_n) \in \mathbb{R}^n$ and $(a_1, \dots, a_n) \in (0, +\infty)^n$.

So, as in the proof of Theorem 3, we have that either W is constant, or there are both $i \in \{1, \dots, n\}$ and a strictly monotone function $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that $W(x_1, \dots, x_n) = g(x_i)$, for every $(x_1, \dots, x_n) \in \mathbb{R}^n$. If W is constant, then T is trivial. Otherwise, if g is strictly increasing, then, for every $(f_1, \dots, f_n)(\mathbb{R}^X)^n$ and $x, y \in X$, it follows that $T(f_1, \dots, f_n)(x) = W(f_1(x), \dots, f_n(x)) = g(f_i(x)) \leq T(f_1, \dots, f_n)(y) = W(f_1(y), \dots, f_n(y)) = g(f_i(y))$ if and only if $f_i(x) \leq f_i(y)$. Therefore T satisfies the property of strong dictatorship, i acting as a strong dictator. Alternatively, if g is strictly decreasing, then, in a similar manner, we see that T satisfies the property of inverse strong-dictatorship, i acting now as an inverse strong dictator. This finishes the proof. □

Corollary 3. Let X be a set with at least two points. For a social evaluation functional $T : (\mathbb{R}^X)^n \to \mathbb{R}^X$, the following assertions are equivalent:

(i) T satisfies welfarism, information invariance with respect to independent interval scales and the Pareto property (respectively, the weak Pareto property).

(ii) Either T is trivial, or it satisfies strong dictatorship (respectively, T satisfies strong dictatorship).

Corollary 4. Let X be a set with at least two points. For a social evaluation functional $T : (\mathbb{R}^X)^n \to \mathbb{R}^X$, the following assertions are equivalent:

(i) T satisfies welfarism, information invariance with respect to independent interval scales and unanimity,

(ii) There is $i \in \{1, \dots, n\}$ such that $T(f_1, \dots, f_n) = f_i$, for all $(f_1, \dots, f_n) \in (\mathbb{R}^X)^n$.

Remark 13.

(i) Notice that, unlike the usual statements encountered in the social choice literature where it is required that the cardinality ("size") of X is, at least, 3, our statements (Theorem 5 and Corollaries 3 to 4) only demand that this cardinality is, at least, 2. In addition, n is allowed to be 1.

(ii) In the case that the X is a singleton, then a social evaluation functional $T : (\mathbb{R}^X)^n \to \mathbb{R}^X$, turns out to be a function from $\mathbb{R}^n$ into $\mathbb{R}$. In this case, both properties welfarism as well as information invariance with respect to independent interval scales are trivially met (see Remark 7 (v) and Remark 8) and, therefore, they play no role. So, in order to provide similar statements to those of Theorem 5 and Corollaries 3 to 4, if X is a singleton, we have to replace these two conditions by the fulfillment of condition (1) that appears in Remark 7 (v).

(iii) It is a direct consequence of Theorem 5 above that if X has at least two elements, then a social evaluation functional that satisfies the properties of welfarism, information invariance with respect to independent interval scales and anonymity, is, a fortiori, trivial.

Using a counter-argument in the statement of Theorem 5, the following impossibility result can be achieved.

Theorem 6. Let $n > 1$. For a set X with at least two points no social evaluation functional exists that satisfies welfarism, information invariance with respect to independent interval scales and the strong Pareto property.

Remark 14. If X is a singleton and $n > 1$, then a similar statement to that of Theorem 6 above can be provided by taking into account the observation done in Remark 13 (ii).

7. Final comments

In this final section, we outline some ideas that could lead to a further development of the analysis made throughout this manuscript.

A suggestion could be the consideration of *continuity* properties of n-dimensional aggregation operators from a nonempty set X into another set Y, provided that X and Y are endowed with suitable topologies (see e.g. [13]). By the way, the set Y^X can also be endowed with a topology as, for instance, the classical compact-open topology (see e.g. pp. 257 and ff. in [19] for further details). Finally, on $(Y^X)^n$ we may consider the product topology (of the compact-open topology on each factor).

If we pay attention to *representable* operators, and depending on the topologies considered on X and Y, the continuity of a map $W : Y^n \to Y$ that represents an aggregation operator $T : (Y^X)^n \to Y^X$ is intimately related to the continuity of T.

To put an example in this direction, let $Y = \mathbb{R}$ be the real line endowed with the usual (Euclidean) topology. Let $X = \{1, \ldots, m\}$. Endow X with the discrete topology. It is then clear that Y^X can be identified with $\mathbb{R}^m$. Moreover, we can assume that, after this identification, the set Y^X is also endowed with the Euclidean topology, because this topology and the compact-open topology on Y^X agree in this case when X is finite and $Y = \mathbb{R}$. Furthermore $(Y^X)^n$, endowed with the product topology, can also be identified with $\mathbb{R}^{mn}$ with the usual Euclidean topology. Notice also that $\mathbb{R}^{mn}$ can be viewed as the collection of all $m \times n$ matrices whose entries are real numbers. (Compare this with Remark 4 and the body of Section 6 above).

In this setting, the following straightforward result arises:

Theorem 7. Let $X = \{1, \ldots, m\}$, endowed with the discrete topology. Let $Y = \mathbb{R}$ the real line, endowed with the usual topology. Let $T : (Y^X)^n \to Y^X$ be a representable n-dimensional aggregation operator. Let $W : \mathbb{R}^n \to \mathbb{R}$ be a function that represents T, so that $T(f_1, \ldots, f_n)(x) = W(f_1(x), \ldots, f_n(x))$ holds for every $(f_1, \ldots, f_n) \in (Y^X)^N$ and every $x \in X$. Identify $(Y^X)^n$ to $\mathbb{R}^{mn}$, and Y^X to $\mathbb{R}^m$ and endow each of them with its respective usual Euclidean topology. Then, the operator T is continuous if and only if the function W is.

As in Section 6, this new topological setting would immediately have applications to *social choice theory*. For instance, in the *topological Chichilnisky model* arising in social choice theory, a nonempty set Z endowed with a topology τ is considered. Provided that n is a natural number, a *social Chichilnisky n-rule* is just a continuous map $F : Z^n \to Z$ satisfying the following two conditions:

(i) $F(z, \ldots, z) = z$ for every $z \in Z$,

(ii) $F(z_{\sigma_1}, \ldots, z_{\sigma_n}) = F(z_1, \ldots, z_n)$ for any permutation σ acting on the set $\{1, \ldots, n\}$.

This is actually an *abstract* model[3], so that it is (a priori) irrelevant here if each element $z \in Z$ is interpreted as an individual preference defined through a total order on a set of alternatives, or if it is a different kind of ordering on the set of alternatives, or even if it is just the best alternative that an individual has selected as her/his best option. All this depends on the particular context considered. Perhaps, the most remarkable feature to be pointed out at this stage is that Chichilnisky models furnish a theoretical and totally abstract setting, alternative to the Arrowian model. Moreover, depending on the particular context to be explored, either preference orderings or just best individual alternatives are understood as points of a topological space. It is important to notice here that the set Z may now be infinite. It can be proved that the existence or nonexistence of a social Chichilnisky n-rule strongly depends on the topology which Z is endowed with. For instance, when (Z, τ) is a finite cellular complex, social Chichilnisky n-rules do exist for every natural number n if and only if the topological space (Z, τ) is contractible. (See [10, 16] for further details).

In the context of the present manuscript, if X and Y are nonempty sets, each of them endowed with a topology, we may wonder if there exist some operator from $(Y^X)^n$ into Y^X that is representable through a continuous map $W : Y^n \to Y$ such that "it preserves the diagonal of Y", that is, $W(y, \ldots, y) = y$ for every $y \in Y$ and, in addition, "it is not affected by permutations", that is, $W(y_{\sigma_1}, \ldots, y_{\sigma_n}) = W(y_1, \ldots, y_n)$ for every permutation σ of the set $\{1, \ldots, n\}$. As commented before, the topology on Y may provoke the non-existence of such a map W (see e.g. [15]).

[3]As a matter of fact, the theoretical background of the topological Chichilnisky model had already appeared much earlier in the analysis of the so-called "topological spaces that admit a topological n-mean", see e.g. [6].

As another interesting stream for further research we suggest matching the ideas of Section 6 (applications to social choice) with some sort of fuzzy approach, in the spirit of [29] and perhaps using some results introduced in Section 5 related to type-2 fuzzy sets.

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BINARY RELATIONS COMING FROM SOLUTIONS OF FUNCTIONAL EQUATIONS: ORDERINGS AND FUZZY SUBSETS

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1. Introduction and motivation

Let X stand for a nonempty set. Consider a binary relation $\mathcal{R}$ defined on X. Thus $\mathcal{R}$ is a subset of the Cartesian product $X \times X$. Given $x, y \in X$, the notation $x\mathcal{R}y$ stands for $(x, y) \in \mathcal{R}$.

In various appealing applications (e.g. when dealing with individual preferences in Utility Theory in Economics, see Ref. 8), it is typical to use a binary relation $\mathcal{R}$ to define, at least implicitly, a qualitative scale on the set X. Thus, if $x\mathcal{R}y$ we may understand that the element y is "at least as good as" the element x. Moreover, it is also typical to convert –when possible– qualitative scales into numerical or quantitative ones, so that the binary relation $\mathcal{R}$ appears as characterized and controlled by means of one or two real-valued functions defined on X, from which we may retrieve $\mathcal{R}$. A typical example is a binary relation $\mathcal{R}$ and a real-valued function $u : X \to \mathbb{R}$ such that $x\mathcal{R}y \Leftrightarrow u(x) \leq u(y)$ holds true for all $x, y \in X$. However, we may also want to compare two elements $x, y \in X$ by means of a real-valued bivariate map $F : X \times X \to \mathbb{R}$ such that, for instance, the fact $F(x, y) > 0$ expresses again the idea of the element y being at least as good as the element x. But, in this alternative approach, it gives us additional information, namely, it measures how better is the element x when confronted to the element y. When, in addition, F satisfies certain functional equations (e.g.: $F(x, y) + F(y, z) = F(x, z)$), the binary relation $\mathcal{R}_F$ induced by F as $x\mathcal{R}_F y \Leftrightarrow F(x, y) > 0$ could be a particular kind of ordering defined on X. Some

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of these relationships between different kinds of orderings, its numerical representation –if any– through real-valued functions, and some bivariate map that satisfies a suitable functional equation and retrieves the ordering by means of its associated binary relation $\mathcal{R}_F$, were already launched in a paper by Olóriz et al. issued in 1998. [1]

Unlike the order of ideas followed in that seminal paper, in the present manuscript we directly start by considering some real-valued bivariate map $F : X \times X \to \mathbb{R}$.[1, 2] Associated to F in a natural way, we may consider the binary relation $\mathcal{R}_F$ given by $x\mathcal{R}_F y \Leftrightarrow F(x, y) > 0 \quad (x, y \in X)$. Then we analyze how the satisfaction of some classical functional equations by the given bivariate function F will obviously have a decisive influence on the properties of its associated binary relation $\mathcal{R}_F$.

The contents and structure of the manuscript go as follows:

After the Introduction and motivation (Section 1), in the subsequent Section 2 of Preliminaries we include some previous background on binary relations on a set –in particular orderings of different kinds– as well as on classical functional equations, too. Then in Section 3 we pass to consider properties of the binary relation associated to a real-valued bivariate map. In Section 4 we analyze the converse problem, namely, given a binary relation $\mathcal{R}$ on a set, we wonder whether or not $\mathcal{R}$ may coincide with the associated binary relation $\mathcal{R}_F$ that comes from a suitable bivariate map F satisfying some particular functional equation. In Section 5 we discuss possible interpretations and extensions of this approach to the fuzzy setting. A final Section 6 of concluding remarks, open problems and suggestions for further research closes the paper.

2. Preliminaries

2.1. Binary relations

Definition 1. A binary relation $\mathcal{R}$ on a nonempty set X is a subset of the Cartesian product $X^2 = X \times X$. Given two elements $x, y \in X$, we will use the standard notation $x\mathcal{R}y$ to express that the pair (x, y) belongs to $\mathcal{R}$.

Naturally associated to a binary relation $\mathcal{R}$ on a set X, we will also deal with the binary relations $\mathcal{R}^c$ and $\mathcal{R}^{-1}$ on X, respectively given by $\mathcal{R}^c = X^2 \setminus \mathcal{R}$, and by $x\mathcal{R}^{-1}y \iff y\mathcal{R}x, \quad (x, y \in X)$.

A binary relation $\mathcal{R}$ defined on a set X is said to be

i) *reflexive* if $\Delta \subseteq \mathcal{R}$, with $\Delta = \{(x, x) : x \in X\}$ (here Δ is said to be the *diagonal* of X^2),

ii) *irreflexive* if $\mathcal{R} \cap \Delta = \emptyset$,

iii) *symmetric* if $\mathcal{R}$ and $\mathcal{R}^{-1}$ coincide,

iv) *antisymmetric* if $\mathcal{R} \cap \mathcal{R}^{-1} \subseteq \Delta$,

v) *asymmetric* if $\mathcal{R} \cap \mathcal{R}^{-1} = \emptyset$,

vi) *total* (or *complete*) if $\mathcal{R} \cup \mathcal{R}^{-1} = X^2$,

vii) *transitive* if $x\mathcal{R}y \wedge y\mathcal{R}z \Rightarrow x\mathcal{R}z$ for every $x, y, z \in X$,

viii) *negatively transitive* if $\mathcal{R}^c$ is transitive.

Given two binary relations $\mathcal{R}$, $\mathcal{S}$ on X, its *composition* $\mathcal{R} \circ \mathcal{S}$ is a new binary relation on X, defined as follows: For any pair $(x, y) \in X^2$, we declare that $x\ (\mathcal{R} \circ \mathcal{S})\ y$ holds true –equivalently, we say that the pair (x, y) belongs to $\mathcal{R} \circ \mathcal{S} \subseteq X \times X$– whenever there exists $z \in X$ such that (x, z) belongs to $\mathcal{R} \subseteq X \times X$, whereas (z, y) belongs to $\mathcal{S} \subseteq X \times X$. The composition of binary relations is associative. Given a natural number n, we will use the standard notation $\mathcal{R}^n$ to denote the composition $\mathcal{R} \circ \ldots$ (n-times) $\ldots \circ \mathcal{R}$.

The binary relation $\mathcal{R}$ is said to be *acyclic* if $\mathcal{R}^n \cap \Delta = \emptyset$ holds true for every natural number n.

The *transitive closure* $\bar{\mathcal{R}}$ of a binary relation $\mathcal{R}$ is defined as $\bar{\mathcal{R}} = \bigcup_{n=1}^{\infty} \mathcal{R}^n$. It is clear that $\bar{\mathcal{R}}$ is transitive.

Remark 1. Notice that the composition of binary relations is not commutative in general, that is, $\mathcal{R} \circ \mathcal{S}$ may or may not be different from $\mathcal{S} \circ \mathcal{R}$. Observe also that a binary relation $\mathcal{R}$ is transitive if and only if $\mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R}$. Also, it is asymmetric if and only if $(\mathcal{R} \circ \mathcal{R}) \cap \Delta = \mathcal{R}^2 \cap \Delta = \emptyset$. Furthermore $\mathcal{R}$ is acyclic if and only if its transitive closure $\bar{\mathcal{R}}$ is irreflexive.

2.2. Orderings on a set

In the particular case of different classes of *orderings* on a set, the standard notation for these kinds of binary relations is different. We include it here for sake of completeness.

Definition 2. A *preorder* $\precsim$ on a nonempty set X is a binary relation on X which is reflexive and transitive.

An antisymmetric preorder is said to be a *partial order*. A *total preorder* $\precsim$ on a set X is a preorder such that if $x, y \in X$ then $x \precsim y$ or $y \precsim x$ holds. An antisymmetric total preorder is said to be a *total order*. A total order is also called a *linear order*, and a totally ordered set $(X, \precsim)$ is also said to be a *chain*.

If $\precsim$ is a preorder on X, then as usual we denote the associated *asymmetric* relation by $\prec$ and the associated *equivalence* relation by $\sim$ and these are defined, respectively, by $x \prec y \iff (x \precsim y) \wedge \neg(y \precsim x)$ and by $x \sim y \iff (x \precsim y) \wedge (y \precsim x)$. Also, the associated *dual* preorder $\precsim_d$ is defined by $x \precsim_d y \iff y \precsim x$. The asymmetric part of a linear order (respectively, of a total preorder) is said to be a *strict linear order* (respectively, a *strict total preorder*). Notice also that when $\precsim$ is a linear order we can also define $\prec$ as follows: $x \prec y \iff (x \precsim y) \wedge (x \neq y)$.

A *total preorder* $\precsim$ on a set X is said to be *representable* if there exists a real-valued map $u : X \to \mathbb{R}$ such that, for any $x, y \in X$, we have $x \precsim y \Leftrightarrow u(x) \leq u(y)$. The map u is said to be a *utility function* or an *order-isomorphism*.

Definition 3. An *interval order* $\prec$ on a nonempty set X is an asymmetric binary relation on X such that $(x \prec y) \wedge (z \prec t) \Rightarrow (x \prec t) \vee (z \prec y)$ $(x, y, z, t \in X)$. Associated to it, we will define the binary relation $\precsim$ given by $a \precsim b \Leftrightarrow \neg(b \prec a)$. This relation $\precsim$ is called the *weak preference* relative to $\prec$. By the way, $\prec$ is also called a *strict preference* defined on X. In addition, the binary relation $\sim$ defined by $a \sim b \Leftrightarrow (a \precsim b) \wedge (b \precsim a)$ is said to be the *indifference* relative to $\prec$.

Given a nonempty set X endowed with an interval order $\prec$, then $\prec$ is said to be a *semiorder* if, for every $a, b, c, d \in X$ it holds true that $a \prec b \prec c$ implies that $a \prec d$ or $d \prec c$ holds true. This condition is usually known as the *semitransitivity property.*

An interval order $\prec$ on a set X is said to be *representable* if there exist two real-valued maps $u, v : X \to \mathbb{R}$ such that given $x, y \in X$ it holds true that $x \prec y \Leftrightarrow v(x) < u(y)$.[1, 2]

Furthermore, a semiorder $\prec$ on a set X is said to be *representable in the sense of Scott and Suppes* if there exists a real-valued map $f : X \to \mathbb{R}$ and a strictly positive constant $k > 0$ such that given $x, y \in X$ it holds true that $x \prec y \Leftrightarrow f(x) + k < f(y)$.[3, 4] Notice that $k > 0$ acts here as a *threshold* of discrimination.

Remark 2. The intuition behind the definition of an interval order, is that the elements x, y, z, t that appear in the condition, also known as the *Ferrers property*, given by $(x \prec y) \wedge (z \prec t) \Rightarrow (x \prec t) \vee (z \prec y)$ behave as if they were the end-points of two intervals of the real line $\mathbb{R}$. Indeed, given two intervals $[x, y]$ and$[z, t]$ of the real line, with respect to the usual order $\leq$ in $\mathbb{R}$ it is evident that the condition $(x < y) \wedge (z < t) \Rightarrow (x < t) \vee (z < y)$ holds true. This justifies the nomenclature *interval order* for this kind of orderings on an abstract nonempty set X.

In addition, the intuition behind the concept of a semiorder could be the following: suppose that a, b, c, d are points in a line and, given $x, y \in X$, we interpret $x \prec y$ as "x is at least one inch far from y". Notice that "one inch" acts here as, so-to-say, a threshold of discrimination.

It is well-known that given an interval order $\prec$ on a set X, the associated relations $\precsim$ and $\sim$ may fail to be transitive.[5–7] However, the strict preference $\prec$ is always transitive. Moreover, a property usually known as *pseudo transitivity* namely $s \prec t \precsim x \prec u \Rightarrow s \prec u$ for every $s, t, x, u \in X$ also holds true.[2, 6–9]

2.3. Some classical functional equations in two variables

Let X be a nonempty set. Let $F : X \times X \to \mathbb{R}$ be a real-valued bivariate map defined on X.

Definition 4. The bivariate map F is said to satisfy:

i) the *hemisymmetry functional equation* if $F(x, y) + F(y, x) = 0$ holds for every $x, y \in X$, [10]

ii) the *Sincov functional equation* if $F(x, y) + F(y, z) = F(x, z)$ holds for every $x, y, z \in X$, [11–15]

iii) the *Sincov functional equation (second version)* if $F(x, y) + F(y, z) + F(z, x) = 0$ holds for every $x, y, z \in X$,

iv) the *separability functional equation* (see e.g. pp. 122 and ff. in Ref. 16) if $F(x, y) + F(y, z) = F(x, z) + F(y, y)$ holds for every $x, y, z \in X$, [16]

v) the *restricted separability functional equation* if $F(x, y) + F(y, z) = F(x, z) + F(t, t)$ holds for every $x, y, z, t \in X$, [17]

vi) the *Ferrers functional equation* if $F(x, y) + F(z, t) = F(x, t) + F(z, y)$ holds for every $x, y, z, t \in X$,

vii) the *semitransitivity functional equation* if $F(x, y) + F(y, z) = F(x, t) + F(t, z)$ holds for every $x, y, z, t \in X$,

viii) the 3-*circuit functional equation* if $F(x, y) + F(y, z) + F(z, x) = F(x, z) + F(z, y) + F(y, x)$ holds for every $x, y, z \in X$.[18]

Among the aforementioned functional equations several hierarchies and equivalences appear, as next Proposition 1 shows.

Proposition 1. Let X be a nonempty set and $F : X \times F \to \mathbb{R}$ a bivariate map. The following statements hold true:

i) Both versions of Sincov functional equation are equivalent.

ii) The separability equation is equivalent to the Ferrers equation.

iii) The restricted separability equation is equivalent to the semitransitivity equation.

iv) If F satisfies the semitransitivity equation, then it satisfies the Ferrers equation. The converse is not true, in general.

v) If F satisfies the Sincov equation, then it satisfies the semitransitivity functional equation. The converse is not true, in general.

Proof. To prove part i), observe that if $F(x,y)+F(y,z) = F(x,z)$ holds true for every $x,y,z \in X$, then it immediately follows that $F(x,x) = 0 = F(x,y) + F(y,x) \quad (x,y \in X)$. Therefore $F(x,y) + F(y,z) + F(z,x) = F(x,y) + F(y,z) - F(x,z) = 0$. Conversely, if $F(x,y) + F(y,z) + F(z,x) = 0$ holds true for every $x,y,z \in X$, we see again that $F(x,x) = 0 = F(x,y) + F(y,x) \quad (x,y \in X)$. Thus $F(x,y) + F(y,z) = -F(z,x) = F(x,z)$.

To prove part ii), notice that If $F(x,y)+F(y,z) = F(x,z)+F(y,y)$ holds true for every $x,y,z \in X$, then we have that $F(x,y) + F(z,t) = F(x,y) + F(z,y) - F(y,y) + F(y,t) = F(z,y) + F(x,t)$. Conversely, if $F(x,y) + F(z,t) = F(x,t) + F(z,y)$ holds true for every $x,y,z,t \in X$, it follows by just taking $y = z$ in the above equality, that F satisfies the separability equation.

In order to prove part iii), assume now that $F(x,y) + F(y,z) = F(x,z) + F(t,t)$ holds true for every $x,y,z,t \in X$. Taking $y = t$ we have $F(x,t) + F(t,z) = F(x,z) + F(t,t)$, too. Hence $F(x,y) + F(y,z) = F(x,t) + F(t,z)$ holds true for every $x,y,z,t \in X$. Conversely, if $F(x,y) + F(y,z) = F(x,t) + F(t,z)$ holds $(x,y,z,t \in X)$, then $F(a,a) + F(a,a) = F(a,b) + F(b,a) = F(b,a) + F(a,b) = F(b,b) + F(b,b)$, so that $F(a,a) = F(b,b)$ holds for every $a,b \in X$. Hence $F(x,y)+F(y,z) = F(x,z)+F(z,z) = F(x,z)+F(t,t)$ holds true for every $x,y,z,t \in X$. Therefore F satisfies the restricted separability equation.

To prove iv) just observe that If satisfies the semitransitivity equation, then by parts iii) and ii) it satisfies the Ferrers equation. To see that the converse is not true, let $X = (0,+\infty)$ and define $F(x,y) = e^y - x^2 - e^x \quad (x,y \in X)$. It is straightforward to see that F satisfies the separability equation, but fails to satisfy the semitransitivity equation.

Finally, to prove part v), observe that by definition and the proof of part i), any solution of the Sincov satisfies the semitransitivity functional equation. To prove that the converse is not true, let again $X = (0,+\infty)$ and define $F(x,y) = y - x - 1 \quad (x,y \in X)$. A direct checking shows that F satisfies the semitransitivity equation, but not the Sincov functional equation. □

Concerning the solutions of some of the main functional equations introduced in Definition 4, the following results appear.

Proposition 2. i) A bivariate map $F : X \times X \to \mathbb{R}$ satisfies the Sincov functional equation if there exists a real-valued function $G : X \to \mathbb{R}$ such that $F(x,y) = G(y) - G(x)$ holds true for every $x,y \in X$.

ii) A bivariate map $F : X \times X \to \mathbb{R}$ satisfies the Ferrers functional equation (or, equivalently, the separability equation) if there exist two real-valued functions $G,H : X \to \mathbb{R}$ such that $F(x,y) = G(y) - H(x)$ holds true for every $x,y \in X$.

iii) A bivariate map $F : X \times X \to \mathbb{R}$ satisfies the semitransitivity functional equation (or, equivalently, the restricted separability equation) if there exists a real-valued function $G : X \to \mathbb{R}$ and a constant $K \in \mathbb{R}$ such that $F(x,y) = G(y) - G(x) + K$ holds true for every $x,y \in X$.

Proof. The expression of the solutions of a Sincov functional equation is a well-known old result.[11, 12]

Regarding the separability equation, the corresponding result is also well-known in this theory (see e.g. pp. 122 and ff. in Ref. 16). Indeed, if we fix an element $a \in X$ and we write $G(x) = F(x,x) - F(x,a)$ and $H(x) = -F(x,a)$ we immediately see that $F(x,y) = G(y) - H(x)$ and F satisfies the separability functional equation.

Finally, regarding the semitransitivity equation, we proceed in the same way, namely we fix an element $a \in X$ and we write $G(x) = F(x,x) - F(x,a)$ and $H(x) = -F(x,a)$. However, we should notice that in this new situation, where F satisfies the restricted separability equation, it is straightforward to see that $F(x,x) = F(t,t)$ holds true for every $x,t \in X$ as in the proof of Proposition 1, so that there exists a constant $K \in \mathbb{R}$ such that $F(x,x) = K \quad (x \in X)$. Thus for every $x, y \in X$ we have that $H(x) = G(x) - K \quad (x \in X)$. Hence $F(x, y) = G(y) - H(x) = G(y) - G(x) + K$. □

As regards the 3-circuit functional equation, the following result appears.

Proposition 3. Let X be a nonempty set. Let $F : X \times X \to \mathbb{R}$ be a real-valued bivariate map defined on X. Then F satisfies the 3-circuit functional equation if and only if the bivariate map $G : X \times X \to \mathbb{R}$ defined by $G(x, y) = F(x, y) - F(y, x) \quad (x, y \in X)$ satisfies the Sincov functional equation.

Proof. Assume that F satisfies the 3-circuit functional equation. Define $G(x, y) = F(x, y) - F(y, x) \quad (x, y \in X)$. Then for any $x, y, z \in X$ we have that $G(x, y) + G(y, z) = [F(x, y) - F(y, x)] + [F(y, z) - F(z, y)]$. By hypothesis, $F(x, y) + F(y, z) + F(z, x) = F(x, z) + F(z, y) + F(y, x)$. Hence $[F(x, y) - F(y, x)] + [F(y, z) - F(z, y)] = F(x, z) - F(z, x) = G(x, z)$. Thus G satisfies the Sincov functional equation.

Conversely, if G satisfies the Sincov functional equation then $G(x, y) + G(y, z) = G(x, z)$ holds true for every $x, y, z \in$. Equivalently, we have that $[F(x, y) - F(y, x)] + [F(y, z) - F(z, y)] = F(x, z) - F(z, x)$. Hence $F(x, y) + F(y, z) + F(z, x) = F(x, z) + F(z, y) + F(y, x) \quad (x, y, z \in X)$. Therefore F satisfies the 3-circuit functional equation. □

3. Binary relations associated to bivariate maps on a set

Given a binary relation $\mathcal{R}$ on a nonempty set X, we may immediately interpret $\mathcal{R}$ through a bivariate map $F : X \times X \to \mathbb{R}$. As a matter of fact, the characteristic function of the binary relation $\mathcal{R} \subseteq X \times X$ works. In this case, $F(x, y) = 1 \Leftrightarrow (x, y) \in \mathcal{R}$ and $F(x, y) = 0 \Leftrightarrow (x, y) \notin \mathcal{R}$. However, this F may fail to satisfy suitable additional properties, as, for instance, to be the solution of some classical functional equation.

In this section we analyze the converse situation, so-to-say. Namely, we begin with a bivariate map $F : X \times X \to \mathbb{R}$, and we define its associated binary relation $\mathcal{R}_F$ by declaring that $(x, y) \in \mathcal{R}_F$ holds true if and only if $F(x, y) > 0$. Needless to say, provided that F satisfies certain additional properties, its associated binary relation $\mathcal{R}_F$ will *a fortiori* feature some related special characteristics.

To put an obvious example, we may notice that if F vanishes on the diagonal Δ, then $\mathcal{R}_F$ is irreflexive.

Therefore, we may analyze here those particular properties of $\mathcal{R}_F$ that come from suitable restrictions on F. In particular, we will study how the fact of F satisfying some classical functional equations –as the ones described above– gives rise to appealing additional properties on $\mathcal{R}_F$.

Proposition 4. Let X denote a nonempty set and $F : X \times X \to \mathbb{R}$ a bivariate map. Let $\mathcal{R}_F$ the binary relation defined on X by means of F, as follows: $x\mathcal{R}_F y \Leftrightarrow F(x, y) > 0 \quad (x, y \in X)$. The following statements hold true:

i) If $F(x, x) > 0$ holds for every $x \in X$ then $\mathcal{R}_F$ is reflexive.

ii) If $F(x, x) \leq 0$ holds for every $x \in X$ then $\mathcal{R}_F$ is irreflexive.

iii) If $F(x, y) + F(y, x) = 0$ holds for every $x, y \in X$ then $\mathcal{R}_F$ is asymmetric.

iv) If $F(x, y) \cdot F(y, z) = F(x, z)$ holds for every $x, y, z \in X$ then $\mathcal{R}_F$ is transitive.

v) If F satisfies the Sincov functional equation, then $\mathcal{R}_F$ is asymmetric and negatively transitive. It is actually a strict total preorder (i.e.: $\mathcal{R}_F$ is the asymmetric part of a total preorder).

vi) If $F(x, x) \leq 0$ holds for every $x \in X$ and F satisfies the Ferrers functional equation, then $\mathcal{R}_F$ is an interval order.

vii) If $F(x,x) \leq 0$ holds for every $x \in X$ and F satisfies the semitransitivity functional equation, then $\mathcal{R}_F$ is a semiorder.

Proof. The former four statements are clear.

To prove v) notice that if F satisfies the Sincov functional equation, namely $F(x,y)+F(y,z) = F(x,z) \quad (x,y,z \in X)$, by taking $x = y = z$ we get $F(x,x) = 0$ for every $x \in X$. Hence, $F(x,y) + F(y,x) = F(x,x) = 0$ holds for every $x, y \in X$, so that F is asymmetric. Moreover, if given $x, y, z \in X$ neither $x\mathcal{R}_F y$ nor $y\mathcal{R}_F z$ hold true, then $F(x,y) \leq 0$ and also $F(y,z) \leq 0$ hold true. Therefore, since $F(x,z) = F(x,y) + F(y,z) \leq 0$. Thus $(x,z) \notin \mathcal{R}_F$, so that F is negatively transitive.

To prove vi) notice that F is irreflexive. Hence $F(x,y) + F(y,x) = F(x,x) + F(y,y) \leq 0$ holds true for every $x, y \in X$. Hence if $F(x,y) > 0$ we have a fortiori that $F(y,x) \leq 0$, so that F is asymmetric, too. Finally, the Ferrers condition for $\mathcal{R}_F$ immediately follows from the fact of F satisfying the Ferrers equation. Therefore $\mathcal{R}_F$ is an interval order.

To prove vii), first we observe that $\mathcal{R}_F$ is already an interval order, by part vi). Moreover, it satisfies the semitransitivity property because F satisfies the semitransitivity equation. Hence $\mathcal{R}_F$ is indeed a semiorder.

□

Remark 3. The equation $F(x,y) \cdot F(y,z) = F(x,z) \quad (x,y,z \in X)$ that appears in part iv) is just one among many other functional equations that imply transitivity. For instance, it is also clear that when $F(x,y) + F(y,z) = F(x,z)$ or alternatively when $F(x,z) = \min\{F(x,y), F(y,z)\}$ holds for every $x, y, z \in X$, then $\mathcal{R}_F$ is transitive, too.

4. Binary relations that come from solutions of functional equations

In this section we analyze to what extent the converses of the different statements involved in Proposition 4 are true. That is, if $\mathcal{R}$ is a binary relation on a set X, we wonder if there exists a real-valued bivariate map $F : X \times X \to \mathbb{R}$ such that, on the one hand, F satisfies some of the restrictions that correspond to the statements in Proposition 4, and, on the other hand, $\mathcal{R}$ coincides with $\mathcal{R}_F$.

4.1. Some general results: reflexivity and asymmetry

Let X be a nonempty set.

Proposition 5. i) Suppose that $\mathcal{R}$ is a reflexive binary relation on X. Let $\mu_{\mathcal{R}}$ stand for the characteristic function of $\mathcal{R}$ on $X \times X$, that is $\mu_{\mathcal{R}}(x,y) = 1 \Leftrightarrow x\mathcal{R}y$ and $\mu_{\mathcal{R}}(x,y) = 0 \Leftrightarrow x\mathcal{R}^c y$. Define $F : X \times X \to \mathbb{R}$ as $F(x,y) = \mu_R(x,y) \quad (x,y \in X)$. Then $\mathcal{R}$ and $\mathcal{R}_F$ coincide. Furthermore, $F(x,x) = 1$ holds for every $x \in X$.

ii) Suppose that $\mathcal{R}$ is an irreflexive binary relation on X, let $\mu_{\mathcal{R}}$ denote again the characteristic function of $\mathcal{R}$. Define $F : X \times X \to \mathbb{R}$ as $F(x,y) = \mu_R(x,y) \quad (x,y \in X)$. Then $\mathcal{R}$ and $\mathcal{R}_F$ coincide. Moreover, $F(x,x) = 0$ holds for every $x \in X$.

iii) Suppose that $\mathcal{R}$ is an asymmetric binary relation on X. Define $F : X \times X \to \mathbb{R}$ by declaring that $F(x,x) = 0$ for every $x \in X$; $F(x,y) = 1$ if $(x,y) \in \mathcal{R}$; $F(x,y) = -1$ if $(y,x) \in \mathcal{R}$, and $F(x,y) = 0$ if $x \neq y$ and neither (x,y) nor (y,x) belong to $\mathcal{R} \subset X^2$. Then $\mathcal{R}$ coincides with $\mathcal{R}_F$. Furthermore $F(x,y) + F(y,x) = 0 \quad (x,y \in X)$.

Proof. To prove part i) notice that $x\mathcal{R}y \Leftrightarrow \mu_{\mathcal{R}}(x,y) = 1 \Leftrightarrow F(x,y) = 1 \Leftrightarrow F(x,y) > 0 \Leftrightarrow x\mathcal{R}_F y$ holds true for every $x, y \in X$. So $\mathcal{R}$ and $\mathcal{R}_F$ coincide. In addition, for every $x \in X$ we have that $x\mathcal{R}x$ holds true because $\mathcal{R}$ is reflexive. Therefore $F(x,x) = 1$ for every $x \in X$.

To prove part ii) we proceed in the same way as in the proof of part i). However we have now that, given $x \in X$, $x\mathcal{R}x$ never holds since $\mathcal{R}$ is irreflexive. Therefore $F(x,x) = 0$ for every $x \in X$.

Finally, to prove part iii), we observe that, by definition of F, $x\mathcal{R}y \Leftrightarrow F(x,y)=1 \Leftrightarrow F(x,y)>0 \Leftrightarrow x\mathcal{R}_F y$ holds true for every $x,y \in X$. So $\mathcal{R}$ and $\mathcal{R}_F$ coincide. Moreover, F is asymmetric so that given $x,y \in X$ we have that $(x,x) \notin \mathcal{R}$ and $(x,y) \in \mathcal{R} \Rightarrow (y,x) \notin \mathcal{R}$. Thus, again by definition of F, $F(x,x)=0$ as well as $F(x,y)+F(y,x)=0$ hold true for every $x,y \in X$.

□

Remark 4. The last construction shows that some functional equations do indeed characterize some kinds of binary relations. For instance, asymmetry can be characterized through the existence of the solution of the functional equation of hemisymmetry.

Nevertheless, even for quite common properties as transitivity, negative transitivity or acyclicity among others, it remains as an open problem to find a functional equation that characterizes the considered property on binary relations. Instead we have some necessary and/or sufficient conditions (as e.g. the sufficient condition for transitivity that appears in part iv) of Proposition 4). Surprisingly, there is a gap among these characterizations, in the sense that the next properties of binary relations that have actually been characterized through functional equations are much more restrictive. Most of them are related to different kinds of orderings (total preorders, interval orders and semiorders). We study them in the next subsection.

By the way, concerning acyclicity we might try to interpret it in terms of irreflexivity of the associated transitive closure. But unfortunately, this idea does not lead to a happy end, because we cannot retrieve a given binary relation $\mathcal{R}$ from its transitive closure $\bar{\mathcal{R}}$. Indeed, it is easy to see that several different binary relations could have the same transitive closure.

4.2. Numerical representability of orderings versus solutions of classical functional equations

Let X be a nonempty set. Let $\precsim$ denote a total preorder defined on X. A direct checking shows that the asymmetric part $\prec$ of $\precsim$ is asymmetric and negatively transitive. This actually characterizes total preorders. That is, if $\prec$ is an asymmetric and negatively transitive binary relation defined on X, then the binary relation $\precsim$ given by $x \precsim y \Leftrightarrow \neg(y \prec x)$ is actually a total preorder on X. Asymmetric plus negatively transitive is equivalent to strict total preorder.

Proposition 6. Let X be a nonempty set. Let $\precsim$ be a total preorder on X. Then the following statements are equivalent:

i) The total preorder $\precsim$ is representable by means of a utility function $u : X \to \mathbb{R}$ such that $x \precsim y \Leftrightarrow u(x) \leq u(y) \quad (x,y \in X)$.

ii) There exists a real-valued bivariate map $F : X \times X \to \mathbb{R}$ that satisties the Sincov functional equation and, in addition, $x \prec y \Leftrightarrow F(x,y) > 0$ holds true for every $x,y \in X$.

Proof. Despite it is a well-known result we include here its proof for the sake of completeness.[1, 2, 15]

To see that i) implies ii), just notice that if u represents $\precsim$ then $x \prec y \Leftrightarrow u(x) < u(y) \Leftrightarrow u(y)-u(x) > 0$ holds true for every $x,y \in X$. It is now clear that the bivariate map $F : X^2 \to \mathbb{R}$ given by $F(x,y) = u(y)-u(x)$ satisfies the Sincov functional equation. Finally $x \prec y \Leftrightarrow F(x,y) > 0 \quad (x,y \in X)$.

For the converse, let us consider a bivariate map $F : X \times X \to \mathbb{R}$ such that F satisfies the Sincov functional equation and $F(x,y) > 0 \Leftrightarrow x \prec y \quad (x,y \in X)$. Now fix an element $a \in X$ and define $u : X \to \mathbb{R}$ by declaring that $u(x) = F(a,x) \quad (x \in X)$. Thus $x \precsim y \Leftrightarrow \neg(y \prec x) \Leftrightarrow F(y,x) \leq 0 \Leftrightarrow F(y,a)+F(a,x) \leq 0 \Leftrightarrow -F(a,y)+F(a,x) \leq 0 \Leftrightarrow F(a,x) \leq F(a,y) \Leftrightarrow u(x) \leq u(y)$ holds true for every $x,y \in X$. Therefore u is an order-isomorphism that represents the given total preorder $\precsim$.

□

Remark 5. Proposition 6 does *not* carry as a consequence that any asymmetric and negatively transitive binary relation $\prec$ appears as the associated binary relation $\mathcal{R}_F$ of a bivariate map F that satisfies the Sincov functional equation. The reason is that the corresponding total preorder

$\precsim$ could *fail to be representable* through a real-valued order isomorphism (utility function). Non-representable total preorders do exist. A classical example is the *lexicographic plane*, namely $X = \mathbb{R}^2$ endowed with the linear order $\preceq_L$ given by $(a,b) \preceq_L (c,d) \Leftrightarrow (a < c) \vee (a = c,\ b \leq d)$ for every $(a,b),(c,d) \in \mathbb{R}^2$, the Euclidean plane.[8, 19]

However, what is true is that any solution $F : X \times X \to \mathbb{R}$ of the Sincov functional equation immediately gives rise to a representable total preorder $\precsim$ on X by just declaring $x \precsim y \Leftrightarrow F(y,x) \leq 0 \quad (x, y \in X)$. As a matter of fact, here the associated asymmetric binary relation $\prec$ (strict total preorder) coincides with $\mathcal{R}_F$.

Let us see now what happens with other different kinds of orderings on a nonempty set X.

Proposition 7. Let X be a nonempty set. Let $\prec$ be an interval order on X. Then the following statements are equivalent:

i) The interval order $\prec$ is representable by means of a pair of real-valued functions $u, v : X \to \mathbb{R}$ such that $x \prec y \Leftrightarrow v(x) < u(y) \quad (x, y \in X)$.

ii) There exists a real-valued bivariate map $F : X \times X \to \mathbb{R}$ that satisfies the separability functional equation and, in addition, $x \prec y \Leftrightarrow F(x,y) > 0$ holds true for every $x, y \in X$.

Proof. This fact was proved in Ref. 1 and Ref. 2. We also include here the proof for the sake of completeness.

To see that i) implies ii), just notice that if the pair (u, v) represents the interval order $\prec$ then $x \prec y \Leftrightarrow v(x) < u(y) \Leftrightarrow u(y) - v(x) > 0$ holds true for every $x, y \in X$. It is now clear that the bivariate map $F : X^2 \to \mathbb{R}$ given by $F(x,y) = u(y) - v(x)$ satisfies the separability functional equation $F(x,y) + F(y,z) = F(x,z) + F(y,y) \quad (x, y, z \in X)$. Finally, we have that $x \prec y \Leftrightarrow F(x,y) > 0 \quad (x, y \in X)$.

To prove the converse implication, let us consider a bivariate map $F : X \times X \to \mathbb{R}$ such that F satisfies the separability functional equation and $F(x,y) > 0 \Leftrightarrow x \prec y \quad (x, y \in X)$. Now fix an element $a \in X$ and define $u, v : X \to \mathbb{R}$ by declaring that $u(x) = F(a,x)\ ; v(x) = F(a,a) - F(x,a)\ (x \in X)$. Thus $x \prec y \Leftrightarrow F(x,y) > 0 \Leftrightarrow F(x,a) + F(a,y) - F(a,a) > 0 \Leftrightarrow u(y) - v(x) > 0$ holds true for every $x, y \in X$. Therefore the pair (u, v) represents the interval order $\prec$.

□

Remark 6. As was the case for Proposition 6, this new Proposition 7 is *not* saying that any interval order $\prec$ on a nonempty set X appears as the associated binary relation $\mathcal{R}_F$ of a bivariate map F that satisfies the separability functional equation. Again, the reason is that there exist interval orders that fail to be representable through a pair of real-valued functions. An example is the interval order $\prec$ defined on the real line $\mathbb{R}$ by declaring that $x \prec y \Leftrightarrow x+1 \leq y \quad (x, y \in \mathbb{R})$.[1, 2]

However, given a bivariate function $F : X \times X \to \mathbb{R}$ that satisfies the separability functional equation (or its equivalent Ferrers equation) and, in addition $F(x,x) \leq 0$ holds for every $x \in X$, immediately gives rise to a representable interval order $\prec$, by just defining $x \prec y \Leftrightarrow F(x,y) > 0 \quad (x, y \in X)$. Again, here $\prec$ agrees with $\mathcal{R}_F$.

Regarding semiorders, we also get a result that is similar to the ones got for total preorders and interval orders in Propositions 6 and 7.

Proposition 8. Let X be a nonempty set. Let $\prec$ be a semiorder on X. Then the following statements are equivalent:

i) The semiorder $\prec$ is representable in the sense of Scott and Suppes [3] by means of a real-valued function $u : X \to \mathbb{R}$ such that $x \prec y \Leftrightarrow u(x) + 1 < u(y) \quad (x, y \in X)$.

ii) There exists a real-valued bivariate map $F : X \times X \to \mathbb{R}$ that satisfies the semitransitivity functional equation as well as $F(x,x) = -1$ for every $x \in X$, and, in addition, $x \prec y \Leftrightarrow F(x,y) > 0$ holds true for every $x, y \in X$.

Proof. Let us prove that i) implies ii): Fix a Scott-Suppes representation of $\prec$ with threshold 1 by means of the function $u : X \to \mathbb{R}$. Consider the bivariate map $F : X^2 \to \mathbb{R}$ given by $F(x, y) = u(y) - u(x) - 1 \quad (x, y) \in X$. In particular, $F(x, x) = -1$ for all $x \in X$. Moreover it is clear that F satisfies the semitransitivity equation. Finally, given $x, y \in X$ we have that $x \prec y \Leftrightarrow u(x) + 1 < u(y) \Leftrightarrow u(y) - u(x) - 1 > 0 \Leftrightarrow F(x, y) > 0$. Therefore $\prec$ and $\mathcal{R}_F$ coincide.

To prove the converse, assume that there exists a bivariate function $F : X \times X \to \mathbb{R}$ accomplishing the semitransitivity functional equation, and such that $F(x, x) = -1, \quad x \prec y \Leftrightarrow F(x, y) > 0$ hold true for every $x, y \in X$. Fix an element $a \in X$ and define the function $u : X \to \mathbb{R}$ given by $u(x) = F(a, x)$. Thus, because of F satisfying the semitransitivity equation, it follows that $x \prec y \Leftrightarrow F(x, y) > 0 \Leftrightarrow F(x, a) + F(a, y) - F(a, a) > 0 \Leftrightarrow F(x, x) + F(a, a) - F(a, x) + F(a, y) - F(a, a) > 0 \Leftrightarrow F(x, x) - F(a, x) + F(a, y) > 0 \Leftrightarrow -1 - F(a, x) + F(a, y) > 0 \Leftrightarrow u(y) - u(x) - 1 > 0 \Leftrightarrow u(x) + 1 < u(y) \quad (x, y \in X)$. Therefore u generates a Scott-Suppes representation of $\prec$ with threshold 1, that is $x \prec y \Leftrightarrow u(x) + 1 < u(y) \quad (x, y \in X)$.[20] □

Remark 7. Once more, as in Propositions 6 and 7, now we are *not* saying that any semiorder $\prec$ on a nonempty set X appears as the associated binary relation $\mathcal{R}_F$ of a suitable bivariate map F satisfying the separability functional equation. Indeed, there exist semiorders that do not admit a Scott-Suppes representation. An example is the one already introduced in Remark 6, namely the binary relation $\prec$ defined on the real line $\mathbb{R}$ by $x \prec y \Leftrightarrow x + 1 \leq y \quad (x, y \in \mathbb{R})$. That binary relation is not only an interval order, but actually a semiorder. Another typical example, of a different nature, is $X = \{0\} \bigcup \{\frac{1}{n+1} : n \in \mathbb{N}\}$ endowed with the Euclidean strict linear order $<$ of the real line. Considered as a semiorder, $<$ does not admit a Scott-Suppes representation.[1, 2, 4, 21, 22] However, we may immediately build semiorders on a nonempty set X so that they are representable in the sense of Scott and Suppes, or equivalently as stated in Proposition 8, are controlled by means of a suitable solution F of the semitransitivity funcitonal equation. Indeed, starting from any real valued function $u : X \to \mathbb{R}$, and taking the map $F : X^2 \to \mathbb{R}$ given by $F(x, y) = u(y) - u(x) - 1$, we immediately observe that $\mathcal{R}_F$ is a semiorder satisfying those additional requirements.

In view of Propositions 6, 7 and 8, it is important to say which total preorders, interval orders and semiorders are representable. In this direction, the corresponding characterizations have already been achieved in the literature. To do so, we first introduce some necessary definitions.

Definition 5. Associated to an interval order $\prec$ defined on a nonempty set X, we shall consider two new binary relations.

These binary relations are said to be the *traces* of $\prec$. They are respectively denoted by $\prec^*$ (*left trace*) and $\prec^{**}$ (*right trace*), and defined as follows: $x \prec^* y \Leftrightarrow x \prec z \precsim y$ for some $z \in X$, and similarly $x \prec^{**} y \Leftrightarrow x \precsim z \prec y$ for some $z \in X \quad (x, y \in X)$.

Remark 8. We denote $x \precsim^* y \Leftrightarrow \neg(y \prec^* x), \quad x \sim^* y \Leftrightarrow x \precsim^* y \precsim^* x$, and $x \precsim^{**} y \Leftrightarrow \neg(y \prec^{**} x)$ and $x \sim^{**} y \Leftrightarrow x \precsim^{**} y \precsim^{**} x \quad (x, y \in X)$. Both the binary relations $\precsim^*$ and $\precsim^{**}$ are total preorders on X. Moreover, the indifference relation $\sim$ associated to the interval order $\prec$ is transitive if and only if $\precsim^*$, $\precsim^{**}$ and $\precsim$ coincide. In this case $\precsim$ is actually a total preorder on X.[2]

Definition 6. Let X be a nonempty set. A total preorder $\precsim$ defined on X is said to be *perfectly separable* if there exists a countable subset $D \subseteq X$ such that for every $x, y \in X$ with $x \prec y$ there exists $d \in D$ such that $x \precsim d \precsim y$.

An interval order $\prec$ defined on X is said to be *interval order separable* if there exists a countable subset $D \subseteq X$ such that for every $x, y \in X$ with $x \prec y$ there exists $d \in D$ such that $x \precsim^* d \prec y$.

A semiorder $\prec$ defined on X is said to be *regular with respect to sequences* if there are no $x, y \in X$ and sequences $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}$ of elements in the set X such that $x \prec \ldots \prec x_{n+1} \prec x_n \prec \ldots \prec x_3 \prec x_2 \prec x_1$ happens or, dually, $y_1 \prec y_2 \prec y_3 \prec \ldots \prec y_n \prec y_{n+1} \prec \ldots \prec y$ holds true. In other words, the given set X does not contain, with respect to $\prec$, neither any infinite up chain with an upper bound, nor any infinite down chain with a lower bound.

The following characterizations of representability are key facts encountered in this literature.[2, 22]

Proposition 9. On a nonempty set X the following statements hold true:

(a) A total preorder $\precsim$ is representable if and only if it is perfectly separable.
(b) An interval order $\prec$ is representable if and only if it is interval order separable.
(c) A semiorder $\prec$ is representable if and only if it is interval order separable and regular with respect to sequences.

5. Binary relations and functional equations in fuzzy contexts

In this section we analyze different aspects related to uncertainty and fuzziness that appear closely related to the ideas developed in the previous sections. In addition we shall also discuss some uses in various fuzzy contexts of the results already obtained.

5.1. Uncertainty in binary relations defined through solutions of functional equations

At this stage, we may wonder which is the relationship between the results obtained in previous sections and a general idea of uncertainty, or with a fuzzy setting of a certain kind.

Apparently, when we deal with a binary relation $\mathcal{R}_F$ on a nonempty set X, such that $\mathcal{R}_F$ comes from a bivariate map $F : X \times X \to \mathbb{R}$ as $x\mathcal{R}_F y \Leftrightarrow F(x, y) > 0 \quad (x, y \in X)$, nothing here seems to be imprecise or uncertain.

However, if we look deeper, we may realize that given a binary relation $\mathcal{R}$ on a set X, there are many possible bivariate functions $F : X \times X \to \mathbb{R}$ such that $\mathcal{R}$ and $\mathcal{R}_F$ coincide so that $x\mathcal{R}y \Leftrightarrow F(x, y) > 0 \quad (x, y \in X)$. A trivial example is the characteristic function of the binary relation $\mathcal{R}$, defined on the Cartesian product $X \times X$. But, indeed, any other function G whose domain is X^2 and $x\mathcal{R}y \Leftrightarrow G(x, y) > 0 \quad (x, y \in X)$ has the same effect. Consequently, we may say that the values of the function or, perhaps better, the selection of the function F leads us to a situation of uncertainty.

Let us come again to typical applications in which the binary relation $\mathcal{R}$ is understood as a "preference" relation among the elements of X. In this case, suppose that we know a particular function $F : X \times X \to \mathbb{R}$ such that $x\mathcal{R}y \Leftrightarrow F(x, y) > 0 \quad (x, y \in X)$. Thus, if the value $F(x, y)$ is strictly positive, we may interpret it not only as giving us the information about the fact "y is preferred to x" but also as measuring how much it is preferred. When F is a bounded function, so that $|F(x, y)| \leq k$ for some strictly positive constant k, we may observe that defining $G : X \times X \to \mathbb{R}$ as $G(x, y) = \frac{F(x,y)}{k} \quad (x, y \in X)$, this new function G satisfies that $|G(x, y)| \leq 1$ for every $x, y \in X$ and also $x\mathcal{R}y \Leftrightarrow G(x, y) > 0 \quad (x, y \in X)$. Consequently, the binary relation $\mathcal{R}$ could also be interpreted by means of a *fuzzy relation* (see Definitions 7 and 8 below) defined on X through the function G. In other words, G gives rise to the membership function $\mu_{\mathcal{R}}$ of a fuzzy subset of the Cartesian product $X \times X$, by declaring that, for any $x, y \in X$, $\mu_{\mathcal{R}}(x, y) = G(x, y)$ if $x\mathcal{R}y \Leftrightarrow G(x, y) > 0$ and $\mu_{\mathcal{R}}(x, y) = 0$ otherwise. Notice also that this can be made whenever F is bounded. In particular, this happens when X is finite, or when X is compact as regards some topology τ and the bivariate function $F : X \times X \to \mathbb{R}$ is continuous with respect to the product topology $\tau \times \tau$ on $X \times X$ and the usual topology on the real line.

To summarize, if a binary relation $\mathcal{R}$ on a nonempty set X comes from a bivariate function $F : X \times X \to \mathbb{R}$ such that $|F(x, y)| \leq 1$ and $x\mathcal{R}y \Leftrightarrow F(x, y) > 0 \quad (x, y \in X)$, the map F actually defines a fuzzy subset of the Cartesian product $X \times X$. Particular features of this fuzzy subset would obviously be related to some classical functional equations that F may accomplish. Moreover, this could also lead to the definition of other fuzzy subsets, in this case of the universe X instead of the Cartesian product $X \times X$, as analyzed in Subsection 5.3.

5.2. Fuzzy sets related to Sincov and Ferrers functional equations

Remember that, in a formal way, the standard definition of a fuzzy subset goes as follows.

Definition 7. Let X be a nonempty set, usually said to be a *universe*. A *fuzzy subset* A of X is defined as a function $\mu_A : X \to [0,1]$ from X to the unit real interval, so that the graph is a subset of the Cartesian product $X \times [0,1]$. The function μ_A is said to be the *membership function* (or *indicator*) of A. Given $\theta \in [0,1]$, the *crisp* (that is, non-fuzzy) subset of the universe X defined by $A_\theta = \{t \in X : \mu_A(t) \geq \theta\}$ is said to be the θ*-cut* of the fuzzy set A.[23]

When dealing with fuzzy sets, some of the functional equations already studied could also appear in a natural way.[15, 17] For instance, the following result appears:

Proposition 10. When X is a nonempty set, and $F : X^2 \to [-1,1]$ is a solution of the Sincov functional equation, it is always possible to find a function $u : X \to [0,1]$ that induces F as $F(x,y) = u(y) - u(x)$, for every $x, y \in X$.

Proof. Define $u : X \to \mathbb{R}$ by declaring that $u(x) = -\inf\{F(x,a) : a \in X\} = \sup\{-F(x,a) : a \in X\}$, for every $x \in X$. Notice that $0 = F(x,x)$ because F is a solution of the Sincov functional equation. Hence $0 = -F(x,x) \leq u(x)$ holds true for any $x \in X$. In addition, $u(x) \leq 1$ for every $x \in X$, because, by hypothesis, $F(x,a) \in [-1,1]$ for every $a \in X$. Thus u is actually the membership function of a fuzzy subset A of the universe X. Furthermore, taking again into account that F satisfies the Sincov functional equation, given $x, y \in X$ it follows that $u(y) - u(x) = \sup\{-F(y,a) : a \in X\} - \sup\{-F(x,b) : b \in X\} = \sup\{-F(x,a) - F(y,x) : a \in X\} - \sup\{-F(x,b) : b \in X\} = \sup\{-F(x,a) + F(x,y) : a \in X\} - \sup\{-F(x,b) : b \in X\} = \sup\{-F(x,a) : a \in X\} + F(x,y) - \sup\{-F(x,b) : b \in X\} = F(x,y)$. Therefore the function u generates F. (See also the proof of Th. 3.1 in Ref. 15.) $\square$

Accordingly, we may indeed interpret u as the *membership* (or *indicator*) function of a suitable fuzzy subset of the universe X.[15] Needless to say, the function u may or may not be unique. What is important is that we may associate fuzzy subsets, in a natural way, to some particular solutions of certain Sincov equations.

A similar result appears provided that $F : X^2 \to [-\frac{1}{2}, \frac{1}{2}]$ satisfies the Ferrers functional equation.

Proposition 11. Let X be a nonempty set. A bivariate function $F : X^2 \to [-\frac{1}{2}, \frac{1}{2}]$ satisfies the Ferrers functional equation if and only if there exist two functions $g, h : X \to [0,1]$ such that $\sup\{g(z) : z \in X\} - \inf\{h(z); z \in X\} \leq \frac{1}{2}$, as well as $\inf\{g(z); z \in X\} - \sup\{h(z); z \in X\} \geq -\frac{1}{2}$ and, in addition, $F(x,y) = g(y) - h(x)$ holds true for any $x, y \in X$.

Proof. Assume that $F : X^2 \to [-\frac{1}{2}, \frac{1}{2}]$ satisfies the Ferrers functional equation. By Proposition 2 there exist two functions $g, h : X \to \mathbb{R}$ such that $F(x,y) = g(y) - h(x)$ holds true for any $x, y \in X$. Since $-\frac{1}{2} \leq F(x,y) \leq \frac{1}{2}$ for any $x, y \in X$, it follows that $\sup\{g(z) : z \in X\} - \inf\{h(z); z \in X\} \leq \frac{1}{2}$ and also $\inf\{g(z); z \in X\} - \sup\{h(z); z \in X\} \geq -\frac{1}{2}$.

Call $a = \min\{\inf\{g(z); z \in X\}, \inf\{h(z); z \in X\}\}$. Notice that a is well defined: in fact, if for instance $\inf\{g(z); z \in X\} = -\infty$, then it could be impossible to find h with $\sup\{h(z); z \in X\} \leq \inf\{g(z); z \in X\} + \frac{1}{2}$. Define now the functions $g^*, h^* : X \to \mathbb{R}$ by declaring that $g^*(t) = g(t) - a$ and $h^*(t) = h(t) - a \ \ (t \in X)$. Obviously $F(x,y) = g^*(y) - h^*(x)$ holds true for any $x, y \in X$. Similarly, $\sup\{g^*(z); z \in X\} \leq \inf\{h^*(z); z \in X\} + \frac{1}{2}$ and $\sup\{h^*(z); z \in X\} \leq \inf\{g^*(z); z \in X\} + \frac{1}{2}$.

Let us see now that both functions g^* and h^* take values in the interval $[0,1]$. To do so, first notice that $\inf\{g^*(z); z \in X\} = \inf\{g(z); z \in X\} - a \geq 0$. On the one hand we already had that $\sup\{g(z); z \in X\} - \inf\{h(z); z \in X\} \leq \frac{1}{2}$, so that $\sup\{g(z); z \in X\} - \inf\{h(z); z \in X\} \leq 1$. On other hand we also have that $\sup\{g(z); z \in X\} \leq \inf\{h(z); z \in X\} + \frac{1}{2} \leq \sup\{h(z); z \in X\} + \frac{1}{2} \leq \inf\{g(z); z \in X\} + 1$. Therefore $\sup\{g(z); z \in X\} - \min\{\inf\{g(z); z \in X\}, \inf\{h(z); z \in X\}\} \leq 1$. Consequently $\sup\{g^*(z); z \in X\} = \sup\{g(z); z \in X\} - a \leq 1$. The fact $h^*(t) \in [0,1] \ \ (t \in X)$ is proved in an entirely analogous way. Furthermore, the pair (g^*, h^*) satisfies that $\sup\{g^*(z); z \in X\} \leq \inf\{h^*(z); z \in X\} + \frac{1}{2}$ as well as $\sup\{h^*(z); z \in X\} \leq \inf\{g^*(z); z \in X\} + \frac{1}{2}$.

To prove the converse, assume now that $g, h : X \to [0,1]$ are two functions such that $\sup\{g(z) : z \in X\} - \inf\{h(z);\, z \in X\} \leq \frac{1}{2}$, as well as $\inf\{g(z);\, z \in X\} - \sup\{h(z);\, z \in X\} \geq -\frac{1}{2}$ and $F(x,y) = g(y) - h(x)$ hold true for any $x, y \in X$. Then F satisfies the Ferrers functional equation by Proposition 2 and, in addition, $\sup\{F(x,y);\, x,y \in X\} \leq \sup\{g(z);\, z \in X\} - \inf\{h(z);\, z \in X\} \leq \frac{1}{2}$ and $\inf\{F(x,y);\, x,y \in X\} \geq \inf\{g(z);\, z \in X\} - \sup\{h(z);\, z \in X\} \geq -\frac{1}{2}$. Therefore F takes values in the interval $[-\frac{1}{2}, \frac{1}{2}]$.

This concludes the proof. □

Remark 9. Here again, we could interpret f, g as the indicator functions of suitable intertwined fuzzy subsets of the universe X.

Regarding the Ferrers and the semitransitivity functional equation, some other "'more sophisticated"' results of this kind have recently been obtained too.[17]

5.3. Fuzzy binary relations and type 2-fuzzy subsets

Apart from these relationships between fuzzy sets and functional equations in terms of the interpretation of some involved functions as the indicators of suitable fuzzy subsets of a universe, we may considered other different approaches, still related to fuzzy set theory and functional equations, as follows:

Definition 8. Let X be a nonempty set called universe. A *fuzzy binary relation* $\mathcal{A}$ on X is a fuzzy subset of the Cartesian product $X^2 = X \times X$. In other words, $\mathcal{A}$ is framed through a membership function $\mu_{\mathcal{A}} : X^2 \to [0,1]$.

Different kinds of fuzzy binary relations have been introduced in the literature to generalize and extend to the fuzzy setting different notions related to orderings on sets.[24–28] We may notice that given a fuzzy binary relation $\mathcal{A}$ on a universe X, for every $\theta \in [0,1]$ the corresponding θ-cut $\mathcal{A}_\theta$ is a (crisp) binary relation on X. Obviously, it may happen that these θ-cuts can have associated the solution of some functional equations as those analyzed in Section 3, so that $\mathcal{A}_\theta$ coincides with $\mathcal{R}_F$ for some suitable bivariate map $F : X \times X \to \mathbb{R}$. Furthermore, the given fuzzy binary relation $\mathcal{A}$ on X is actually a bivariate map from X^2 into $[0,1]$. Hence $\mathcal{A}$ could perhaps satisfy –directly, a priori– some functional equation as, e.g., the Sincov equation, the Ferrers equation or the semitransitivity functional equation. This could be used to study and classify fuzzy binary relations, on a universe X, that have additional particular features. As far as we know a study of this kind remains open and it would be a novelty in this framework.

In the same way, we could also consider functional equations and/or binary relations corresponding to the so-called *type* 2- *fuzzy sets*, that constitute a usual tool in applications as, for instance, Image Processing.

Definition 9. Let X be a nonempty set called universe. A *type* 2- *fuzzy subset of the universe* X $\mathcal{A}$ is a map $f : X \to [0,1]^{[0,1]}$, where $[0,1]^{[0,1]}$ stands for the set of all functions that map the unit interval $[0,1]$ into itself.[29–32]

Remark 10. Notice that a fuzzy subset of type 2 of the universe X can also be interpreted as a fuzzy subset of the Cartesian product $X \times [0,1]$. Indeed a map f from X into $[0,1]^{[0,1]}$ assigns to a generic element $x \in X$ a function $f_x : [0,1] \to [0,1]$. If we define $F : X \times [0,1] \to [0,1]$ by declaring that $F(x,t) = f_x(t)$ for any $x \in X$ and $t \in [0,1]$ we may accordingly identify F to f.

5.4. Fuzzy numbers, functional equations and interval orders

To conclude this section, we comment another appealing approach that could relate fuzzy numbers, functional equations and interval orders. It leans on the definition of a fuzzy number, that was introduced by Dubois and Prade.[31, 33] Other appealing studies on the concept of a fuzzy number appear in Ref. 39 and Ref. 40.

Definition 10. A *fuzzy number* is defined as a fuzzy subset of the real line $\mathbb{R}$ whose membership function $u : \mathbb{R} \to [0, 1]$ satisfies the following properties.

i) u is upper-semicontinuous,

ii) for every $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$ it holds that $u[\lambda x + (1 - \lambda)y] \geq \min\{u(x), u(y)\}$ (the map u, with this property, is said to be *convex*),

iii) there exists at least one point $a \in \mathbb{R}$ such that $u(a) = 1$,

iv) the closure –as regards the usual Euclidean topology of the real line– of the set $\{x \in \mathbb{R} : u(x) > 0\}$ is compact (the map u, with this property, is said to have *compact support*).

Whenever $u : \mathbb{R} \to [0, 1]$ is a fuzzy number in the sense of Definition 10, assume that for every $\lambda \in [0, 1]$ the λ-cut $u_\lambda = \{x \in \mathbb{R} : u(x) \geq \lambda\}$ is a closed interval of real numbers, say $[a(\lambda), b(\lambda)]$. In this case, an important result by Goetschel and Voxman proves that the pair of functions $a(\lambda)$ and $b(\lambda)$ satisfy the following properties:

i) $a(\lambda)$ is a bounded left continuous nondecreasing function on $(0, 1]$,

ii) $b(\lambda)$ is a bounded left continuous nonincreasing funtion on $(0, 1]$,

iii) $a(\lambda)$ and $b(\lambda)$ are right continuous at $\lambda = 0$,

iv) $a(1) \leq b(1)$.

Conversely, if a pair of functions $f(\lambda)$ and $g(\lambda)$ satisfy the above conditions (i)-(iv), then there exists a unique fuzzy number whose membership function is $u : \mathbb{R} \to [0, 1]$ such that the λ-cut u_λ is the interval $[f(\lambda), g(\lambda)]$ for every $\lambda \in [0, 1]$, the unit interval.[36]

At this stage, in this setting we may construct an interval order $\prec_u$ on $[0, 1]$ by declaring that, for every λ, μ in the unit interval, $\lambda \prec_u \mu \Leftrightarrow b(\mu) < a(\lambda)$ holds. Needless to say, that interval order would obviously be representable through the pair of functions $a(\lambda), b(\lambda)$. Furthermore, the real-valued bivariate map F defined on the unit square $[0, 1]^2$ as $F(s, t) = a(t) - b(s) \quad (s, t \in [0, 1])$ satisfies the Ferrers functional equation, takes values in $[-1, 1]$ and also $F(s, s) \leq 0$ holds true for every $0 \leq s \leq 1$.

In view of Propositions 7, 9 and 11 above, we may then conjecture that, perhaps, suitable interval orders on the unit interval $[0, 1]$ could always have a similar aspect, that is, they could be represented by an adequate pair of functions $a(\lambda), b(\lambda)$ accomplishing the properties (i)-(iv) of the statement of the previous important result due to Goetschel and Voxman, and therefore coming from the definition of a unique fuzzy number u with additional properties ad hoc.[33, 36] As far as we know, this possible connection between fuzzy numbers, interval orders on the unit interval and their semicontinuous representations (if any), and particular solutions of the Ferrers functional equation remains unexplored.

Remark 11. Concerning semicontinuous representations of interval orders some recent results have recently appeared in the specialised literature, but no relationship with fuzzy numbers in the sense of Dubois and Prade has been analyzed in those papers.[37, 38]

6. Further comments and concluding remarks

In our study concerning which binary relations can be understood as naturally associated to the solutions of a suitable functional equation in two variables, we have achieved some results that characterize, on the one hand, those that are reflexive, irreflexive or asymmetric, and, on the other hand, those that correspond to numerically representable total preorders, interval orders and semiorders. These are closely related to solutions of several classical equations, namely: Sincov, Ferrers or separability, and semitransitivity or restricted separability.

However, many intermediate situations still remain as an *open problem* to be analyzed in future research. We do not have at hand yet characterizations of properties as transitivity, or acyclicity through suitable functional equations. NNeither we have characterizations of those binary relations that give rise to an acyclic graph, nor to a tree –that is also a directed graph– or to a finite union of trees among others. In particular situations (e.g., on countable sets) some characterizations of acyclicity have already been introduced in the literature.[8, 39, 40] But, as far as we know, they have not been built in terms of functional equations but using other techniques as *pseudoutility functions* (also known as *order homomorphisms*).[8] In the same way, there are techniques that detect if a binary relation on a finite set is actually a tree, as the well-known Kruskal's algorithm.[41] But also in this situation, these techniques do not lean on functional equations.

Apart from this big jump from asymmetric binary relations to representable total preorders, where most intermediate properties need more research and new achievements and discoveries, another important fact has to do with *continuous solutions* of some of the classical functional equations involved in the numerical representability of several kinds of orderings, namely total preorders, interval orders and semiorders. Indeed, assuming that the nonempty set X is equipped with some topology, the results stated in Propositions 6, 7 and 8 do not say anything about the continuity of either the utility functions or their associated bivariate maps –solutions of some classical functional equation– that appear in their respective statements.

Furthermore, some constructions of utility functions to represent total preorders, or suitable solutions of the Ferrers equation to represent interval orders are based on *numerical series*.[1, 8] And it is well-known that most of the functions so constructed through numerical series have discontinuities.

In this direction, perhaps it may be enlightening to include here a brief account concerning how the resolution of the functional equation of separability was used to characterize the numerical representability of interval orders.[1] Assume that we are looking for a characterization of the representability of an interval order $\prec$ on a set X (i.e.: an asymmetric Ferrers binary relation on X) by means of a pair of real-valued functions $u, v : X \to \mathbb{R}$ such that $x \prec y \Leftrightarrow v(x) < u(y) \ \ (x, y \in X)$. As stated in Proposition 7, this is equivalent to find a bivariate map $F : X \times X \to \mathbb{R}$ such that $x \prec y \Leftrightarrow F(x, y) > 0$ and F satisfies the separability equation $F(x, y) + F(y, z) = F(x, z) + F(y, y) \ \ (x, y, z \in X)$. Just by suitable solving –when this is possible– this separability equation associated or its equivalent Ferrers functional equation, a condition that characterizes the representability of interval orders, namely the *interval order separability* (see Definition 6), was discovered and introduced in this specialized literature.

Let us outline here the key steps in that procedure. Let X be a nonempty set and $\prec$ an interval order on X. Given $x, y \in X$, denote $A(x, y) = \{s \in X : x \precsim a \prec s \precsim y \text{ for some } a \in X\}$. It can be proved that:

(1) $A(x, y) \cap A(z, t) = A(x, t) \cap A(z, y) \ \ (x, y, z, t \in X)$.

(2) $A(x, y) \cup A(y, t) = A(x, z) \cap A(y, y) \ \ (x, y, z, t \in X; x \precsim y \precsim z \text{ and } x \precsim z)$.

(3) If $x \prec y$, then $A(y, x) = \emptyset; A(x, x) \cup A(y, y) \subsetneq A(x, y)$ and $A(x, x) \cap A(y, y) = \emptyset$.

It can also be proved that $\prec$ is interval order separable if and only if there exists a countable subset $D \subseteq X$ such that for every $x, y \in X$ with $x \prec y$ there exists $d \in D$ such that $d \in A(x, y)$, $d \notin A(x, x), d \notin A(y, y)$. Now define the bivariate function $F : X^2 \to \mathbb{R}$ by declaring that $F(x, y)$ is the sum of the series of terms $-(2^{-n})$ whenever $d_n \in D \cap A(x, y)$ and $x \precsim y$, and $F(x, y) = -F(y, x) + F(x, x) + F(y, y)$ otherwise. The map F satisfies the separability equation and $\prec$ is representable if and only if $x \prec y \Leftrightarrow F(x, y) > 0 \ \ (x, y \in X)$.

All this appears in Ref. 1, and was improved in Ref. 2.

However, since then, successive trials of doing something similar in order to characterize the Scott-Suppes representability of semiorders have been done with no avail. Also, attempts to get some similar results, but avoiding the use of numerical series, and searching for characterizations of the *continuous* representability of interval orders and semiorders have failed till now. It is true that some characterizations of continuous representability of interval orders and semiorders have been obtained for suitable particular cases.[9, 22, 42–45] But this was done with quite different techniques from the ones used in Ref 1. The corresponding associated Ferrers or semitransitivity functional equations in that continuous case are solved *indirectly* and after a long and difficult argument around, in which first it is almost imperative to build continuous functions that represent the given interval order or semiorder in the sense of Definition 3, and when and only when all this task has been completed, use then Propositions 7 or 8 to find the suitable continuous solution of the functional equation involved.

Another class of functional equations related to binary relations may appear when working with some *particular space with additional structure*, instead of just an abstract set X. Here, typical situations arise when dealing with the Euclidean space $\mathbb{R}^n$ as universe. For instance, we could be interested to analyze binary relations $\mathcal{R}$ on $\mathbb{R}^n$ such that for every $x, y \in \mathbb{R}^n$ and every positive real number λ it holds true that $x\mathcal{R}y \Leftrightarrow \lambda x \mathcal{R} \lambda y$. This property is usually called *homotheticity*.[46–50]

If $\mathcal{R}$ is the associated binary relation $\mathcal{R}_F$ of a bivariate function $F : \mathcal{R}^n \times \mathcal{R}^n \to \mathbb{R}$, there are several functional equations that, if satisfied by F, force the relation $\mathcal{R}$ to be homothetic. An *example*, provided that F is differentiable, is the functional equation $x \cdot \partial F/\partial x + y \cdot \partial F/\partial y = 0 \;\; (x, y \in \mathbb{R}^n)$, which characterizes the fact of F being homogeneous of degree zero.[52]

Another typical example corresponds to binary relations $\mathcal{R}$ on $\mathbb{R}^n$ such that for every $x, y, z \in \mathbb{R}^n$ it holds true that $x\mathcal{R}y \Leftrightarrow x + z \;\mathcal{R}\; y + z$. This property is usually called *translation-invariance* and it is related to the study of ordered algebraical structures.[53–56]

So we may conclude here by pointing out, once more, that there is a huge variety of open questions and ideas to be explored in this particular setting.

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Type-2 Fuzzy Entropy-Sets

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***Abstract*—The final goal of this study is to adapt the concept of fuzzy entropy of De Luca and Termini to deal with Type-2 Fuzzy Sets. We denote this concept Type-2 Fuzzy Entropy-Set. However, the construction of the notion of entropy measure on an infinite set, such us $[0,1]$, is not effortless. For this reason, we first introduce the concept of quasi-entropy of a Fuzzy Set on the universe $[0,1]$. Furthermore, whenever the membership function of the considered Fuzzy Set in the universe $[0,1]$ is continuous, we prove that the quasi-entropy of that set is a fuzzy entropy in the sense of De Luca y Termini. Finally, we present an illustrative example where we use Type-2 Fuzzy Entropy-Sets instead of fuzzy entropies in a classical fuzzy algorithm.**

***Index Terms*—Type-2 Fuzzy Sets; Quasi-entropy measure; Entropy measure.**

I. Introduction

The concept of fuzzy entropy measure was introduced by De Luca and Termini in [1] in order to measure how far a Fuzzy Set is from a crisp one. Since then, this concept has been adapted to the different extensions of Fuzzy Sets [2] and with different interpretations. All of them measure how far the considered extension is from a set of reference (which may be that of crisp sets, of Fuzzy Sets, etc).

In this sense, it is worth mentioning the following concepts: the Atanassov intuitionistic fuzzy entropy measure, given by Szmidt et al. [3] to measure how far an Atanassov Intuitionistic Fuzzy Set (AIFS) is from a crisp set; the entropy for Interval-Valued Fuzzy Sets (IVFS) defined by Burillo et al. [4], which measures how far an IVFS or AIFS is from a Fuzzy Set; and finally, the idea given by Pal et al. [5] which combines two concepts similar to those given by Szmidt et al. and Burillo et al. in one single bi-valuated measure. (We should recall that AIFSs, IVFSs and Fuzzy Sets are particular instances of Type-2 Fuzzy Sets (T2FS) (see Fig. 1) [2]).

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Furthermore, we know that, for a Fuzzy Set on the finite universe $U = \{u_1, \cdots, u_n\}$, the value $\mathcal{A}(u_i) \in [0,1]$ is a number which represents the membership degree of u_i to $\mathcal{A}$. From the beginning of fuzzy theory in 1965, many authors were very critical with it: if Fuzzy Sets are used to represent uncertainty associated to a fact, how can the membership values be an exact number $\mathcal{A}(u_i)$ without taking into account the uncertainty associated to the way these numbers are built? This fact led to the introduction in 1971 [6] of T2FS in the following sense: for a Type-2 Fuzzy Set A_2 defined on the finite universe U, the membership degree of each element to the set, i.e., $A_2(u_i)$, is a Fuzzy Set on the infinite universe $[0,1]$. With Zadeh's interpretation, in this paper we consider that the Fuzzy Set $A_2(u_i)$ represents the uncertainty associated to the building of $\mathcal{A}(u_i) \in [0,1]$.

In this setting, we understand De Luca and Termini fuzzy entropy E of the set $A_2(u_i)$, $E(A_2(u_i))$, as a measure of the *doubt (uncertainty)* associated to the value $\mathcal{A}(u_i) \in [0,1]$ given by the expert. In this way, if $E(A_2(u_i)) = 0$, we assume that there is no doubt associated with the value $\mathcal{A}(u_i)$; that is, there is no doubt associated with the numerical value given to represent the membership degree of u_i to the Fuzzy Set $\mathcal{A}$. However, if $E(A_2(u_i)) = 1$, then the doubt with respect to the value $\mathcal{A}(u_i)$ is maximal.

Taking into account the definition of fuzzy entropy, if the Fuzzy Set $A_2(u_i)$ on $[0,1]$ is

$$A_2(u_i)(x) = \begin{cases} 1 \text{ if } x = \mathcal{A}(u_i) \\ 0 \text{ otherwise} \end{cases}$$

then $E(A_2(u_i)) = 0$.

Similarly if $A_2(u_i)(x) = 0.5$ for all $x \in [0,1]$ then $E(A_2(u_i)) = 1$.

From these considerations, in this work we aim at the following objectives:

(A) To extend the concept of fuzzy entropy in the sense of De Luca and Termini to T2FSs.
(B) To provide a construction method of such entropies.
(C) To introduce an illustrative example where the notion of entropy that we propose for T2FSs is used in an algorithm that was originally developed using the concept of fuzzy entropy for Fuzzy Sets or for some extensions.

Regarding objective (A), it is important to remark the following: In the same spirit as in the work by Pal et al. [5], we consider that the entropy of a T2FS A_2 on a finite universe U must not be a number, but a Fuzzy Set (Type-1) $E_{T2}(A_2)$ on the same universe U. We call this Fuzzy Set, Type-2 Fuzzy Entropy-Set. The values $E_{T2}(A_2)(u_i) \in [0,1]$ are given by the fuzzy entropies of the Fuzzy Sets on the universe $[0,1]$ used to represent the membership of u_i to the set A_2. With our interpretation we have that each value of $E_{T2}(A_2)(u_i)$ represents the doubt associated to the membership degree of the element u_i to the Fuzzy Set $\mathcal{A}$ on the considered finite universe U.

We also introduce a measure call pointwise measure which assigns to each T2FS a numerical value obtained through an appropriate aggregation of the elements in the Fuzzy Set $E_{T2}(A_2)$. We see that this measure has properties similar to those of De Luca and Termini's fuzzy entropy.

Regarding objective (B): In order to build the Type-2 Fuzzy Entropy-Set the following problem arises: we should calculate the fuzzy entropy of Fuzzy Sets which are defined on non-finite universes (the interval $[0,1]$). This problem leads us to introduce the concept of quasi-entropy. The latter does not exactly match fuzzy entropy as defined by De Luca and Termini. However, if we consider Fuzzy Sets defined on the universe $[0,1]$ with a continuous membership function, then the concept of quasi-entropy and the concept of fuzzy entropy defined by De Luca and Termini are the same. We build Type-2 Fuzzy Entropy-Sets from the quasi-entropies.

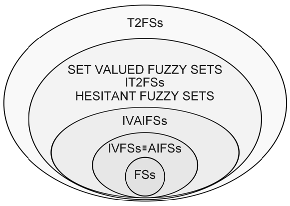

Fig. 1. Inclusion relationships of extensions of Fuzzy Sets in [2]

Regarding objective (C): As an illustrative example of the utility of our theoretical developments, we rewrite the algorithm for image segmentation which uses fuzzy techniques, i.e., Huang and Wang's algorithm [7], [8]. We consider an image as a Type-2 Fuzzy Set and we replace fuzzy entropy by our concept of Type-2 Fuzzy Entropy-Set. It is worth to note that *the purpose of this example is not to provide a new method*, but just to show how our theoretical developments can be used to understand an image as a Type-2 Fuzzy Set (over the universe of the intensity levels) and hence how a well-known algorithm can be extended to this setting, as it has already been extended to some other settings such as IVFSs or AIFSs [9], [10].

This paper is organized as follows. In the following section we recall some definitions and properties which will be used in the subsequent of this work. Then, in Section III we introduce the concept of fuzzy quasi-entropy measure for an infinite universe $[0,1]$ analyzing the particular case of continuous membership degrees. Sections IV and V present the Type-2 Fuzzy Entropy-Set together with some specific cases of these sets and the definition of pointwise measure. Section VI presents an illustrative example in image thresholding. Finally, in Section VII we include some conclusions and references.

II. Preliminary notions

In this paper, we denote by X a non-empty universe in a Fuzzy Set that can be either finite or infinite. When the universe is finite, it is denoted by U.

Definition 2.1: [11] A Fuzzy Set (FS) (or Type-1 Fuzzy Set) A is a mapping $A : X \mapsto [0,1]$ where the value $A(x)$ is referred to as the membership degree of the element x to the Fuzzy Set A.

The set of all FSs on X is denoted by $FS(X)$.

From the notions given by Zadeh in [12], a Type-2 Fuzzy Set (T2FS) can be defined as follows.

Definition 2.2: A Type-2 Fuzzy Set (T2FS) A_2 on X is a mapping $A_2 : X \mapsto FS([0,1])$ where the membership degree of an element of the universe X is a Fuzzy Set on the infinite universe $[0,1]$.

From Definition 2.2, it can be seen that, mathematically, a T2FS is a mapping $A_2 : X \mapsto [0,1]^{[0,1]}$, where

$$[0,1]^{[0,1]} = \{f \mid f : [0,1] \mapsto [0,1]\}.$$

We denote by $T2FS(X)$ the class of all T2FSs on the universe X.

Fuzzy entropy measure was formalized in terms of axiom construction by De Luca and Termini in [1] in order to assess the amount of vagueness within a FS. However, depending on the properties demanded, we can find in the literature different axiomatic definitions of the concept of fuzzy entropy measure, such as [13], [14], [15]. In particular, we base our definition on [14].

Definition 2.3: A function $E : FS(X) \mapsto [0,1]$ is called an entropy measure on $FS(X)$ if it satisfies the following properties:

(E1) $E(A) = 0$ if and only if A is crisp.
(E2) $E(A) = 1$ if and only if $A(x) = \frac{1}{2}$ for all $x \in X$.
(E3) If $A, B \in FS(X)$, and for all $x \in X$

$$\left.\begin{array}{c} A(x) \leq B(x) \leq \frac{1}{2} \\ \text{or} \\ A(x) \geq B(x) \geq \frac{1}{2} \end{array}\right\} \text{then } E(A) \leq E(B)$$

(E4) $E(A) = E(N(A))$ for all $A \in FS(X)$, where $N(A) = \{(x, 1 - A(x))\}$ for all $x \in X$.

It should be pointed out that $(E1) - (E3)$ generate De Luca and Termini axiomatic definition and $(E4)$ is a property frequently demanded in image processing.

Definition 2.3 is based on the standard negation $N(x) = 1-x$. In the case of another strong negation being considered, property $(E2)$ would be

$$E(A) = 1 \text{ if and only if } A(x) = e \text{ for all } x \in X,$$

where e is the equilibrium point of the strong negation considered.

Finally, in Definition 2.3 it does not matter whether the universe X is finite or infinite, but dealing with infinite universes requires a more complicated mathematical formalism. Thus, most of the works in the literature take into account only the finite case (universe U).

A construction method of entropies was given in [14], using aggregation functions and the concept of E_N function, which we recall now.

Definition 2.4: A function $E_N : [0,1] \to [0,1]$ is called a normal E_N-function associated with the strong negation N, if it satisfies the following conditions:

1) $E_N(x) = 1$ if and only if $x = e$ (where e is the equilibrium point of N; that is, $N(e) = e$.)
2) $E_N(x) = 0$ if and only if $x = 0$ or $x = 1$.
3) If $y \geq x \geq e$ or $y \leq x \leq e$, then $E_N(x) \geq E_N(y)$.
4) $E_N(x) = E_N(N(x))$ for all $x \in [0,1]$.

In particular, entropies of FSs on finite universes can be built from E_N-functions as follows.

Theorem 2.1: Let $M : [0,1]^n \to [0,1]$ be such that it fulfills

(M1) $M(x_1, \cdots, x_n) = 0$ if and only if $x_1 = \cdots = x_n = 0$;
(M2) $M(x_1, \cdots, x_n) = 1$ if and only if $x_1 = \cdots = x_n = 1$;
(M3) For any pair $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ of n-tuples such that $x_i, y_i \in [0,1]$ for all $i \in \{1, \cdots, n\}$, if $x_i \leq y_i$ for all $i \in \{1, \cdots, n\}$, then $M(x_1, \cdots, x_n) \leq M(y_1, \cdots, y_n)$;
(M4) M is a symmetric function in all its arguments.

Then $E(\mathcal{A}) = M_{i=1}^{n} E_N(\mathcal{A}(u_i))$ for all $\mathcal{A} \in FS(U)$ satisfies $(E1) - (E4)$ of Definition 2.4.

Example 2.2: If we take $E_N(x) = 1 - |2x - 1|$ and $M(x_1, \cdots, x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i$, then

$$E(\mathcal{A}) = \frac{1}{n}\sum_{i=1}^{n} 1 - |2\mathcal{A}(u_i) - 1|$$

is Yager's measure of fuzziness [16].

Restricted Equivalence Functions R are functions which satisfy frequently demanded properties for the comparison of images. They were introduced by Bustince et al. in [8], [14], [17].

Definition 2.5: A function $R : [0,1]^2 \to [0,1]$ is called a restricted equivalence function if it satisfies the following conditions:

(R1) $R(x,y) = R(y,x)$ for all $x, y \in [0,1]$;
(R2) $R(x,y) = 1$ if and only if $x = y$;
(R3) $R(x,y) = 0$ if and only if $\{x,y\} = \{0,1\}$;
(R4) $R(x,y) = R(N(x), N(y))$ for all $x, y \in [0,1]$, being N a strong negation on $[0,1]$;
(R5) For all $x, y, z \in [0,1]$ such that $x \leq y \leq z$ then $R(x,z) \leq R(x,y)$ and $R(x,z) \leq R(y,z)$.

III. Fuzzy quasi-entropy measure for an infinite universe

In order to develop our notion of entropy measure on T2FSs, we study some results about entropy measures on FSs whose universe X is infinite. In particular, we focus on the notion of an entropy measure on $FS([0,1])$. When the universe X is infinite some mathematical operations, such as the integration operation, yield the same value for different sets A_1, A_1' such that $A_1 = A_1'$ a.e. (almost everywhere). [1] To handle this situation in a suitable way, we adapt the concept of entropy measure given by De Luca and Termini [1].

As we have seen in Theorem 2.1, in the case of finite universes, entropy can be built aggregating appropriate functions (E_N-functions); in particular, the arithmetic mean can be used for the aggregation. If we try to extend this procedure to the universe $[0,1]$, it is natural to use an integral instead of the arithmetic mean. A problem arises, however, with axioms $(E1)$ and $(E2)$. For instance, consider the functions $f_1(t) = 0$ for all $t \in [0,1]$, $f_2(t) = 0.3$ if $t = 0.3$ or $t = 0.8$ and $f_2(t) = 0$ otherwise. These functions are different, but the integral of both on $[0,1]$ equals 0, since they differ in a zero-measure set (a finite set of points).

So we should modify axioms $(E1)$ and $(E2)$. This can be done in two different ways.

1) They can be kept as they stand in Definition 2.3. In this case, the value of the function in one single point would determine that the entropy was not zero or one, even if the function equals 0 or 0.5, respectively, in any other point. This would be too harsh.
2) We can rewrite axioms $(E1)$ and $(E2)$ considering that functions which are equal almost everywhere must have the same entropy. This is something which is usually done for many applications, and it is the approach that we choose in this work.

Taking into account these considerations, we propose the following definition (note axioms $E1^*$ and $E2^*$). We take the name of quasi-entropy because an exact copy of De Luca and Termini's definition of entropy would correspond to approach 1) above, which we have not followed.

Definition 3.1: Let $A \in FS([0,1])$, we define the set $H_A = \{x \mid A(x) \in]0,1[\}$.

Definition 3.2: A function $E^* : FS([0,1]) \mapsto [0,1]$ is called a quasi-entropy measure on $FS([0,1])$ if it satisfies the following properties:

$(E1^*)$ $E^*(A) = 0$ if and only if the Lebesgue measure of H_A is null, i.e., $m(H_A) = 0$, where m denotes the Lebesgue measure in $\mathbb{R}$.
$(E2^*)$ $E^*(A) = 1$ if and only if $A(x) = \frac{1}{2}$ a.e. in $[0,1]$.
$(E3^*)$ If $A, B \in FS([0,1])$, and for all $x \in [0,1]$

$$\left.\begin{array}{c} A(x) \leq B(x) \leq \frac{1}{2} \\ \text{or} \\ A(x) \geq B(x) \geq \frac{1}{2} \end{array}\right\} \text{then } E^*(A) \leq E^*(B).$$

$(E4^*)$ $E^*(A) = E^*(N(A))$ for all $A \in FS([0,1])$ where $N(A) = \{(x, 1 - A(x))\}$ for all $x \in [0,1]$.

Remark 1: Notice that properties $(E3^*)$ and $(E4^*)$ are exactly equal to the properties $(E3)$ and $(E4)$ of entropy measure in FSs given in Definition 2.3.

From here on, we only consider FSs in the universe $X = [0,1]$ and such that the function $A : X \mapsto [0,1]$ is a Lebesgue integrable function. Observe that since Lebesgue integrable functions are a large class of functions, even restricting to them is not a major concern.

In order to construct a quasi-entropy measure we start by defining a function Γ and we study under which conditions it fulfills properties $(E1^*) - (E4^*)$ individually.

Let $g :]0,1[\mapsto [0,1]$ be a Lebesgue integrable function. We define function $\Gamma : FS([0,1]) \mapsto [0,1]$ as

$$\Gamma(A) = \int_{H_A} g(A(y))dy. \quad (1)$$

[1] Given two functions f_1, f_2, we say $f_1 = f_2$ a.e. if $f_1(x) = f_2(x)$ for all x in the domain except for a set of null measure. Particularly, $f_1 = c$ a.e. where c is a constant if $f_1(x) = c$ except for a set of null measure.

Example 3.1: Let $g(x) = 2\min(x, 1-x)$ and consider the following FS on $[0,1]$: $A(x) = 1$ for all $x \in [0,1]$. Then, by Eq. (1) we have

$$\Gamma(A) = \int_{H_A} g(A(y))dy = \int_{H_A} 2\min(1,0) = 0.$$

In Theorem 3.2, we study those sets which have minimum entropy measure, i.e., property $(E1^*)$.

Theorem 3.2: Let $\Gamma : FS([0,1]) \mapsto [0,1]$ be a function given by Eq. (1). Then

Γ satisfies $(E1^*)$ if and only if $g(z) \neq 0$ for all $z \in]0,1[$.

Proof. See Appendix.

Example 3.3: Figure 2 shows $g_1(z) = 1 - z$, $g_2(z) = z^2$ and $g_3(z) = 0.3$ for $z \in]0,1[$ which satisfy the property of Theorem 3.2.

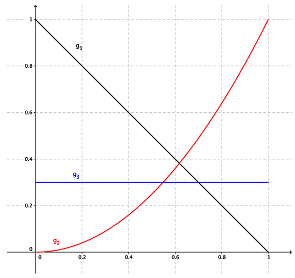

Fig. 2. Functions g_1, g_2, g_3 satisfying $E1^*$.

Example 3.4: Let $g(x) = 2\min(x, 1-x)$ and consider the following FS on $[0,1]$: $A(x) = 0.5$ for all $x \in [0,1]$. Then, by Eq.(1) we have

$$\Gamma(A) = \int_{H_A} g(A(y))dy = \int_{H_A} 2\min(0.5, 0.5) = 1$$

In Theorem 3.5 we focus on the sets with maximum entropy measure, namely, property $(E2^*)$.

Theorem 3.5: Let $\Gamma : FS([0,1]) \mapsto [0,1]$ be a function given by Eq. (1). Then,

$$\Gamma \text{ satisfies } (E2^*) \text{ if and only if } g^{-1}(1) = \left\{\frac{1}{2}\right\}$$

Proof. See Appendix.

Example 3.6: Figure 3 shows three functions which satisfy the property of Theorem 3.5.

$$g_1(z) = -\left(z - \frac{1}{2}\right)^2 + 1 \qquad \text{for } z \in]0,1[$$

$$g_2(z) = \begin{cases} 0 & \text{if } 0 < z \leq 0.1, \\ 2.5z - 0.25 & \text{if } 0.1 < z \leq 0.5, \\ 1.5 - z & \text{if } 0.5 < z < 1. \end{cases}$$

$$g_3(z) = \begin{cases} z & \text{if } 0 < z < 0.5, \\ -2z + 2 & \text{if } 0.5 \leq z < 1. \end{cases}$$

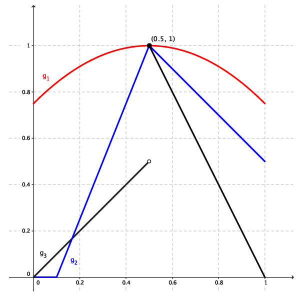

Fig. 3. Functions g_1, g_2, g_3 satisfying $E2^*$.

In Theorem 3.7, the monotonicity of quasi-entropy measure, property $(E3^*)$, is analyzed.

Theorem 3.7: Let $\Gamma : FS([0,1]) \mapsto [0,1]$ be a function given by Eq. (1). Then, Γ satisfies $(E3^*)$ if and only if g is increasing on $\left]0, \frac{1}{2}\right]$ and decreasing on $\left[\frac{1}{2}, 1\right[$.

Proof. See Appendix.

Example 3.8: Figure 4 shows functions which satisfy the property of Theorem 3.7.

$$g_1(z) = \begin{cases} 5z & \text{if } 0 < z < 0.2, \\ 1 & \text{if } 0.2 \leq z < 1, \end{cases}$$

$$g_2(z) = \begin{cases} z & \text{if } 0 < z < 0.5, \\ \\ 1 - z & \text{if } 0.5 \leq z < 1. \end{cases}$$

$$g_3(z) = \begin{cases} z + 0.3 & \text{if } 0 < z \leq 0.5, \\ 1.4 - 1.4z & \text{if } 0.5 < z < 1. \end{cases}$$

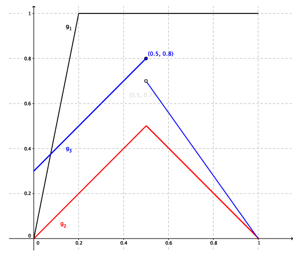

Fig. 4. Functions g_1, g_2, g_3 satisfying $E3^*$.

Finally, in Theorem 3.9 we study property $(E4^*)$, analyzing the symmetry of entropy measures.

Theorem 3.9: Let $\Gamma : FS([0,1]) \mapsto [0,1]$ be a function given by Eq. (1). Then,

Γ satisfies $(E4^*)$ if and only if g is a symmetric function with respect to $z = \frac{1}{2}$, i.e., $g(z) = g(1-z)$ for all $z \in]0,1[$.

Proof. See Appendix.

Example 3.10: Figure 5 shows functions g_1, g_2, g_3 which satisfy property of Theorem 3.9.

$$g_1(z) = 4(z - 0.5)^2 \qquad \text{for } z \in]0,1[$$

$$g_2(z) = \begin{cases} 0 & \text{if } 0 < z \leq 0.2, \\ z - 0.2 & \text{if } 0.2 < z \leq 0.5, \\ -z + 0.8 & \text{if } 0.5 < z \leq 0.8, \\ 0 & \text{if } 0.8 < z < 1. \end{cases}$$

$$g_3(z) = \min\{8z^3, 8(1-z)^3\} \qquad \text{for } z \in]0,1[$$

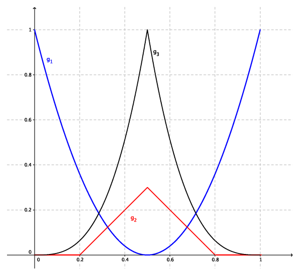

Fig. 5. Functions g_1, g_2, g_3 satisfying $E4^*$.

After studying each property separately, the following corollary holds true.

Corollary 3.11: Let Γ be given by Eq. (1). Then Γ is a quasi-entropy measure if and only if g satisfies the conditions demanded in Theorems 3.2, 3.5, 3.7 and 3.9.

Proposition 3.12: Let g be an E_N-function associated with the strong negation N given by $N(x) = 1-x$ for all $x \in [0,1]$. Then the function Γ given by Eq. (1) in terms of g is a quasi-entropy.

Proof. It follows from the Corollary 3.11 and properties of E_N-functions (see [14]).

In [14], it is proved that, from a restricted equivalence function R, we can build an E_N-function as follows: $E_N(x) = R(x, 1-x)$. So the following corollary is straight.

Corollary 3.13: Let R be a restricted equivalence function and let $g(x) = R(x, 1-x)$. Then, Γ given by Eq. (1) in terms of g is a quasi-entropy.

Example 3.14: Fig. 6 shows three functions g_1, g_2, g_3 which satisfy all the conditions of Theorems 3.2, 3.5, 3.7 and 3.9, so from Corollary 3.11 they generate quasi-entropy measures:

$$g_1(z) = -4z^2 + 4z \qquad \text{for } z \in]0,1[$$
$$g_2(z) = \min\{2z, 2-2z\} \qquad \text{for } z \in]0,1[$$
$$g_3(z) = \min\{8z^3, 8(1-z)^3\} \qquad \text{for } z \in]0,1[$$

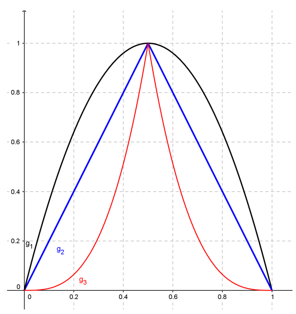

Fig. 6. Functions g_1, g_2, g_3 which generate a quasi-entropy measure.

In the following we compute an example of the calculation of a quasi-entropy.

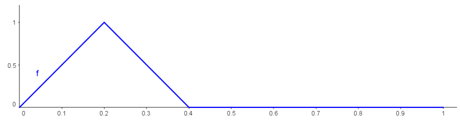

Fig. 7. Graph of function f of Example 3.15.

Example 3.15: Let $f \in FS([0,1])$ be given by

$$f(x) = \begin{cases} 5x & \text{if } 0 \le x \le 0.2, \\ 2-5x & \text{if } 0.2 < x \le 0.4, \\ 0 & \text{otherwise,} \end{cases}$$

displayed in Figure 3.15. Consider the quasi-entropy measure E^* generated as in Eq. (1) by $g(z) = \min\{2z, 2-2z\}$. Then:

$$E^*(f) = \int_0^{0.1} 10y\,dy + \int_{0.1}^{0.2} (2-10y)dy + \int_{0.2}^{0.3} (10y-2)dy + \int_{0.3}^{0.4} (4-10y)dy = 0.2$$

A. Quasi-entropy measure on Continuous functions

As we have said before, when we use integrals sets of zero measure are ignored. This has led us to modify in the previous section the first and second axioms of the definition of entropy 2.3 by De Luca and Termini. But in the case of continuous functions, if a function is constant almost everywhere, then it is constant everywhere, and this kind of technical problems may be ignored. That is, if we consider just those FSs on the universe $[0,1]$ with a continuous membership function, then our definition of entropy can be written as the one which was introduced by De Luca and Termini; i.e., Definition 2.3. For this reason in this section we study quasi-entropy measures restricted to the class of $FS([0,1])$ whose membership degree is a continuous function.

Definition 3.3: Let $FS_C([0,1])$ be the set of all FSs on the universe $X = [0,1]$ whose membership degree $A : X \mapsto [0,1]$ leads to a continuous function.

In the following theorem we introduce a method to build entropies in the sense of De Luca and Termini as long as the membership function of the considered FS on $[0,1]$ is continuous.

Theorem 3.16: Let be $g\ :]0,1[\mapsto [0,1]$ satisfying the properties of the Theorems 3.2, 3.5, 3.7 and 3.9 and let Γ be given as in Eq. (1). If we restrict to FS_C then $\Gamma|_{FS_C}$ is an entropy measure in the sense of De Luca and Termini [1]. Namely, the function Γ on $FS_C([0,1])$ satisfies:

(E1) $\Gamma(A) = 0$ if and only if A is crisp.
(E2) $\Gamma(A) = 1$ if and only if $A(x) = \frac{1}{2}$ in $[0,1]$.
(E3) If $A, B \in FS_C([0,1])$, and for all $x \in [0,1]$

$$\left.\begin{array}{l} A(x) \le B(x) \le \frac{1}{2} \\ \text{or} \\ A(x) \ge B(x) \ge \frac{1}{2} \end{array}\right\} \text{then } \Gamma(A) \le \Gamma(B)$$

(E4) $\Gamma(A) = \Gamma(N(A))$ for all $A \in FS([0,1])$, where $N(A) = \{(x, 1-A(x))\}$ for all $x \in X$.

Note that imposing continuity is not a too hard restriction, since, for instance, in many applications, in order to build linguistic labels, these are defined through continuous membership functions (triangular, trapezoidal, etc. [18]).

Corollary 3.17: Let g be an E_N-function associated with the strong negation N given by $N(x) = 1 - x$ for all $x \in [0,1]$. Then

$$\Gamma(A) = \int_{H_A} g(A(y))dy$$

is a fuzzy entropy in the sense of De Luca and Termini on $FS_C([0,1])$. In particular, if R is a restricted equivalence function, then

$$\Gamma(A) = \int_{H_A} R(A(x), 1 - A(x))dx$$

is also an entropy in the sense of De Luca and Termini.

IV. Type-2 Fuzzy Entropy-Set

De Luca and Termini introduced the notion of entropy measure as a function whose domain and codomain are a FS and $[0,1]$, respectively, i.e. a function $E : FS(X) \mapsto [0,1]$. In this way, the codomain of the entropy function and the codomain of the FS coincide. Due to the introduction of the concept of T2FS (by Zadeh [12]) as a function whose image is a FS, the proposal of this work is to define the entropy measure of a T2FS by means of a function whose domain is a T2FS and the codomain is a FS.

Given a T2FS (with universe X), each element $x \in X$ is associated with a $FS([0,1])$ where its quasi-entropy measure can be calculated. Observe that since the universe is infinite, most of the entropy measure constructions on the literature cannot be applied. By calculating the quasi-entropy measure, for each $x \in X$ we obtain a value in $[0,1]$, i.e., each element of the universe X is associated with a value in $[0,1]$. A reasonable way of expressing the entropy measure of a T2FS is by means of a function $E_{T2} : T2FS(X) \mapsto FS(X)$.

Definition 4.1: Let X be the universe of a T2FS A_2 and let $E^* : FS([0,1]) \mapsto [0,1]$ be a quasi-entropy measure. A Type-2 Fuzzy Entropy-Set is a function $E_{T2} : T2FS(X) \mapsto FS(X)$ given by

$$E_{T2}(A_2) = \{(x, E^*(A_2(x))) | \, x \in X\}. \qquad (2)$$

The given construction of Type-2 Fuzzy Entropy-Set on Definition 4.1 measures the lack of knowledge or uncertainty about the membership degrees. Thereby, any set with "crisp" membership degrees such as FSs or IVFSs has entropy measure 0.

Next, we present an example where the Type-2 Fuzzy Entropy-Set is calculated.

Example 4.1: Let $U = \{u_1, u_2, u_3, u_4\}$ be the universe and $A_2 : T2FS(U) \mapsto FS(U)$ be the T2FS given by $A_2 = \{(u_i, A_2(u_i) = f_i) \mid i \in \{1,2,3,4\}\}$ where

$$f_1(x) = \begin{cases} 0.5 & \text{if } x = 0.3, \\ 0.25 & \text{if } x = 0.5, \\ 1 & \text{if } x = 0.8, \\ 0 & \text{otherwise.} \end{cases}$$

$$f_2(x) = \begin{cases} 1 & \text{if } x \in [0.2; 0.4] \cup [0.7; 1[, \\ 0 & \text{otherwise.} \end{cases}$$

$$f_3(x) = \begin{cases} 0 & \text{if } 0 < x \leq 0.5, \\ 2.5x - 1.25 & \text{if } 0.5 < x \leq 0.9, \\ 1 & \text{if } 0.9 < x \leq 1 \end{cases}$$

$$f_4(x) = \begin{cases} 5x & \text{if } 0 < x \leq 0.2, \\ -1.25x + 1.25 & \text{if } 0.2 < x \leq 1. \end{cases}$$

as in Figure 8. Consider the quasi-entropy measure E^* generated as in Eq. (1) by $g(z) = -4z^2 + 4z$. Then:

$E^*(f_1) = 0$, $E^*(f_2) = 0$, $E^*(f_3) = \frac{4}{15}$ and $E^*(f_4) = \frac{2}{3}$,

and consequently the Type-2 Fuzzy Entropy-Set is given by

$$E_{T2}(A_2) = \left\{ (u_1, 0), (u_2, 0), \left(u_3, \frac{4}{15}\right), \left(u_4, \frac{2}{3}\right) \right\}$$

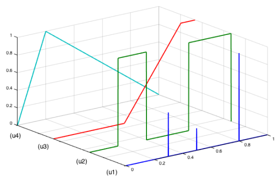

Fig. 8. Graph of the Type-2 Fuzzy Set A_2.

V. Specific cases. Pointwise measure

A. Some specific cases

In this section we show how we can recover Fuzzy Sets and extensions from T2FSs such that its Type-2 Fuzzy Entropy-Set is null.

Let $A_2 \in T2FS(U)$ such that

$$E_{T2}(A_2) = \{(u_i, 0) | u_i \in U\} \, ;$$

that is,

$$E^*(A_2(u_i)) = 0 \text{ for every } u_i \in U$$

where E^* is the quasi-entropy associated to E_{T2}.

Then:

- If the Fuzzy Sets $A_2(u_i)$ on the universe $[0,1]$, (built to represent the doubt associated to the membership degrees of the elements u_i to the Fuzzy Set $\mathcal{A}$ on the universe U), are crisp sets as the following:

  $$A_2(u_i)(x) = \begin{cases} 1 \text{ if } x = a_{0_i} \\ 0 \text{ otherwise,} \end{cases}$$

  then, taking into account the interpretation discussed in the introduction, we do not have any doubt about the membership degrees of the elements to the Fuzzy Set $\mathcal{A} \in FS(U)$ and it is the ideal case. In this setting, we can take as Fuzzy Set $\mathcal{A}$:

  $$\mathcal{A} = \{(u_i, \mathcal{A}(u_i) = a_{0_i}) | i \in \{1, \cdots, n\}\}$$

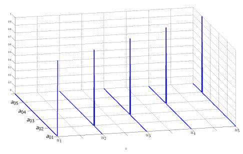

Fig. 9. Example of a Fuzzy Set.

- If the Fuzzy Sets $A_2(u_i)$ on the universe $[0,1]$ are crisp sets as follows:

$$A_2(u_i)(x) = \begin{cases} 1 \text{ if } x = a_{0_i^1} \text{ or } x = a_{0_i^2} \text{ or } x = a_{0_i^{m_i}} \\ 0 \text{ otherwise,} \end{cases}$$

then we can take as set $\mathcal{A}$ the following Typical Fuzzy Multiset $\mathcal{A}$ (on the universe U) [2] for which there is no doubt on the numerical values taken for representing the membership degrees:

$$\mathcal{A} = \{(u_i, a_{0_i^1}, a_{0_i^2}, a_{0_i^{m_i}}) | i \in \{1, \cdots, n\}\}$$

where m_1 denotes the cardinal of the Fuzzy Multiset associated with u_i.

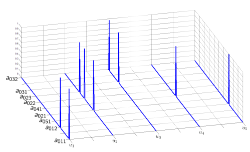

Fig. 10. Example of a Typical Fuzzy Multiset.

- If the Fuzzy Sets $A_2(u_i)$ on the universe $[0,1]$ are crisp sets as follows:

$$A_2(u_i)(x) = \begin{cases} 1 \text{ if } x \in [\underline{a}_{0_i}, \overline{a}_{0_i}] \\ 0 \text{ otherwise} \end{cases}$$

then we can take as $\mathcal{A}$ the following Interval-Valued Fuzzy Set:

$$\mathcal{A} = \{(u_i, [\underline{a}_{0_i}, \overline{a}_{0_i}]) | i \in \{1, \cdots, n\}\}$$

Notice that with our interpretation, it comes out that we have no doubt about the values for the intervals given in order to represent the membership values of the elements to the set.

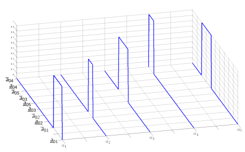

Fig. 11. Example of an Interval-valued Type-2 Fuzzy Set.

In the three considered cases, we recover Fuzzy Sets (in the first case) or well-known extensions of Fuzzy Sets (in the other two cases) whose Type-2 Fuzzy Entropy-Set is always a null Fuzzy Set. In any case, for each of the considered cases (Fuzzy Sets, Fuzzy Multisets and Interval-Valued Fuzzy Sets) there exist ad hoc definitions to calculate their entropy. For instance, De Luca and Termini's for Fuzzy Sets, Szmidt et al.'s or Burillo et al.'s for Interval-Valued Fuzzy Sets, etc.

Although we are not recovering a fuzzy extension, it is worth to mention that if

$$A_2(u_i) = \{(x, A_2(u_i)(x) = 0.5) | \, x \in [0,1]\} \text{ for all } u_i \in U$$

then $E_{T2}(A_2) = \{(u_i, 1) | u_i \in U\}$.

B. Pointwise measure

In this section, we introduce the concept of pointwise measure. With this measure we assign to each $A_2 \in T2FS(U)$ a numerical value which is obtained aggregating the values in the corresponding Type-2 Fuzzy Entropy-Set $E_{T2}(A_2)$ built as explained in Section IV.

Proposition 5.1: Let $M : [0,1]^n \to [0,1]$ be a function such that it satisfies $(M1) - (M3)$ of Theorem 2.1. Let $A_2 \in T2FS(U)$ and its corresponding $E_{T2}(A_2) \in FS(U)$ constructed with the method developed in Section IV. Under these conditions the function

$$Pm : T2FS(U) \to [0,1] \text{ given by}$$
$$Pm(A_2) = \underset{i=1}{\overset{n}{M}} E_{T2}(A_2)(u_i)$$

satisfies the following properties:

(Pm1) $Pm(A_2) = 0$ if and only if for every $u_i \in U$, $E^*(A_2(u_i)) = 0$; namely, for every $u_i \in U$, $H_{A_2(u_i)}$ has null Lebesgue measure;

(Pm2) $Pm(A_2) = 1$ if and only if for every $u_i \in U$, $E^*(A_2(u_i)) = 1$; namely, for every $u_i \in U$, $A_2(u_i)(x) = 0.5$ a.e. in $[0,1]$;

(Pm3) If $A_2, B_2 \in T2FS(U)$, satisfy that for every $u_i \in U$: for all $x \in [0,1]$

$$\left. \begin{array}{l} A_2(u_i)(x) \leq B_2(u_i)(x) \leq \frac{1}{2} \\ \text{or} \\ A_2(u_i)(x) \geq B_2(u_i)(x) \geq \frac{1}{2} \end{array} \right\} \text{then}$$
$$Pm(A_2) \leq Pm(B_2);$$

(Pm4) $Pm(A_2) = Pm(N(A_2))$ for all $A_2 \in T2FS$, where $N(A_2) = \{(u_i, N(A(u_i)))\}$.

Proof. It is just a straight calculation.

Remark 2: In this way, Pm does not measure the classical concept of entropy, in the sense that it does not measure how far a T2FS is from a crisp one. However, it gives a global value of the uncertainty associated with which values should represent the membership degrees of u_i for all $u_i \in U$. In particular, if there is no doubt about the membership degrees of any element $u_i \in U$ independently if they are crisp, Fuzzy Set, IVFS, etc, then the punctual measure Pm returns 0.

VI. An Illustrative Example in Image Thresholding

In this section we develop an example of application of Type-2 Fuzzy Entropy-Set. We present an adaptation of Huang and Wang's method [7] to segment images in grayscale. To do so, we build a T2FS associated with the image and we calculate its Type-2 Fuzzy Entropy-Set.

Image segmentation consists of dividing an image into regions (objects) that compound it [19]. More specifically, it consists of assigning a label to each pixel of the image, so that all the pixels which share certain properties have the same label. One of the most used techniques in image segmentation is thresholding or segmentation by gray levels [20], [21], [22]. It is based on the assumption that the objects of the image are only characterized by the intensity of their pixels. When the image has only two objects (called object and background), this thresholding technique consists of finding an intensity value (t) to be considered the threshold. Using that value, we label all the pixels whose intensities are lower or equal than t as background and all the pixels whose intensities are greater than t as object (or vice versa). When there are more than two objects in the image, we need more thresholds, in such a way that all the pixels whose intensities are between two consecutive thresholds belong to the same object.

The results of thresholding are limited when comparing with other segmentation techniques, because the single characteristic they take into account is the intensity of every pixel. However, its advantages are the simplicity and low computational cost. This is why this procedure is commonly used as a first step of more complex segmentation algorithms.

We consider an image as a set of elements arranged in N rows and M columns. Each element of a grayscale image has a value of intensity q between 0 and $L-1$ (usually $L = 256$). However, we work with normalized images $\frac{q}{L-1}$ in such a way that $q \in [0, 1]$.

As we have said in the introduction, we rewrite Huang and Wang's algorithm [7] using T2FSs and Type-2 Fuzzy Entropy-Sets (see Algorithm 1).

Algorithm 1 Thresholding algorithm

INPUT: Image to segment
OUTPUT: t the best threshold

```
{Construction of the T2FS}
for each intensity level t ∈ {0, 1/255, ..., 254/255} (For every possible threshold) do
    Construct a FS on the universe [0, 1] associated with the intensity level t
end for
Calculate the Type-2 Fuzzy Entropy-Set of the resulting T2FS
Select as best threshold t the one associated with the lowest element in the Type-2 Fuzzy Entropy-Set
```

The main idea of this procedure consists in creating a T2FS associated with the image and calculating its entropy set. One of the most difficult tasks is the construction of the T2FS. It should represent the information of how would be the image if we segment it with every possible threshold. For this purpose, we start by fixing the referential set of the T2FS as the set of all possible thresholds in the image: $U = \{0/255, 1/255, \ldots, 254/255\}$ (remember the image is normalized). For every element in U, its membership degree is given by a Fuzzy Set. This set has a continuous referential set from 0 to 1. In Figure 12 we show a T2FS that fulfills our conditions.

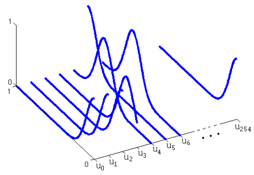

Fig. 12. Example of a Type-2 Fuzzy Set.

Each of these functions represents, for a fixed threshold, the membership degree of every possible intensity either to the object or to the background. To construct each of these sets, following [8], we start by calculating the average intensity of the pixels lower or equal than the studied threshold (denoted as $m_B(t)$) and the average intensity of the pixels greater than the studied threshold (denoted as $m_O(t)$).

The membership function quantifies how close is every posible value (q) to the average of the background or to the average of the object, by means of restricted equivalence functions:

$$\mathcal{A}(u_i)(q) = \begin{cases} R(q, m_B(u_i)) & \text{if } q \le u_i \\ R(q, m_O(u_i)) & \text{if } q > u_i. \end{cases} \quad (3)$$

We linearly interpolate between every pair of consecutive points $(q_i, (\mathcal{A}(u_i)(q_i)))$ and $(q_{i+1}, (\mathcal{A}(u_i)(q_{i+1})))$ with $i \in \{0, \cdots, 254\}$. That is, we take the points $(0, \mathcal{A}(u_i)(0))$ and $(1/255, \mathcal{A}(u_i)(1/255))$ and, for each $s \in [0, 1/255]$, we define its membership as:

$$A_2(s) = 255(\mathcal{A}(u_i)(1/255) - \mathcal{A}(u_i)(0))s + \mathcal{A}(u_i)(0) .$$

Next, we repeat this procedure for each interval$[j/255, (j+1)/255]$, $(j = 0, \ldots, 254)$, calculating in each case the equation of the line which passes through the points $(j/255, \mathcal{A}(u_i)(j/255))$ and $((j+1)/255, \mathcal{A}(u_i)((j+1)/255))$.

In this way, we get a continuous membership function defined over the whole universe $[0, 1]$. This membership function is piecewise linear and it has only two points where its value is 1: the average of the background ($m_B(t)$) and the average of the object ($m_O(t)$).

To select the best threshold from the T2FS we use its Type-2 Fuzzy Entropy-Set. We are looking for the threshold whose membership function is as higher as possible for all the pixels in the image. The entropy is minimum when the membership is 0 or 1, and maximum in the middle point. To adapt this concept to our problem, we scale our membership function to $[0.5, 1]$, in such a way that the minimum entropy is only achieved when the membership degree is 1.

With our membership construction, the calculation of our entropies is simple, since we can divide the area in 255 trapezoids and we just need to sum the entropy measure of each of these parts multiplied by the proportion of pixels with that intensity. That is, we calculate the entropy of each FS as

$E(A_2(u_i)) = \int g(A_2(u_i)(x))dx$ where $A_2(u_i)$ is the FS associated with u_i on the universe $[0,1]$ and $g(x) = R(x, 1-x)$.

In this way we obtain a set of entropies, each one associated with an element of the universe (possible thresholds based on our construction) and we can build the Type-2 Fuzzy Entropy-Set. Finally, we select as the best threshold, the one associated with the lowest entropy measure.

With an illustrative aim, we use this algorithm for thresholding the image in Figure 13.

Fig. 13. Original image to segment.

After constructing the T2FS for this image, we use $g(x) = R(x, 1-x) = 1 - |2x-1|$ to get its associated Type-2 Fuzzy Entropy-Set. The resulting set is as follows:

$$E_{T2} = \{(u_0, 0.6014), (u_1, 0.6010), (u_2, 0.6004), \ldots, (u_{254}, 0.5935)\}$$

For a better visualization of this set, in Figure 14 we show it graphically.

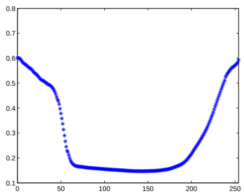

Fig. 14. Fuzzy Entropy-Set for thresholding the image of Figure 13.

The minimum of this Type-2 Fuzzy Entropy-Set corresponds to the element $(u_{143}, 0.1467)$. So the threshold used to segment the image is 143/255 and we get the image shown in Figure 15.

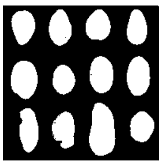

Fig. 15. Image of Figure 13 segmented with threshold 143.

To further extend this illustrative example, we consider now a set of 8 standard images for thresholding and their ideal segmentations; that is, the segmentation provided by an expert. For each of them (see Figure 16) we show the original image, the ideal segmentation and the segmented image obtained with our method using the function $E(A_2(u_i)) = \int g(A_2(u_i)(x))dx$ with $g(x) = R(x, 1-x) = 1 - |2x-1|$.

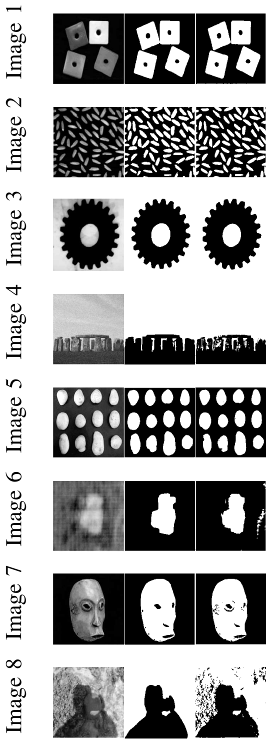

Fig. 16. Original images (first column), ideal segmentations (second column) and segmentations obtained by our proposal using the Type-2 Fuzzy Entropy-Set (third column)

Our proposed Algorithm 1 uses the Type-2 Fuzzy Entropy-Set to calculate the threshold for segmenting an image. In the fuzzy literature, there exist several fuzzy algorithms which use extensions of FSs (for instance [8], [23], [14]) for thresholding images. All of them, including our proposal, are based on Huang and Wang's algorithm [7], which is an adaptation of the classical method by Otsu [20]. It is important to notice that none of these algorithms is better than the others for every image. For this reason, we propose to use a combination of the results obtained with different algorithms, including our Algorithm 1. To show the goodness of this proposal, we use the following 5 thresholding algorithms.

- Otsu's algorithm [20];
- area algorithm [8] with $\varphi_1(x) = x^2$ and $\varphi_2(x) = x$;
- ignorance functions based algorithm [23] with $G_u(x,y) = 2\sqrt{(1-x)(1-y)}$ if $(1-x)(1-y) \leq 0.25$ and $G_u(x,y) = 1/(2\sqrt{(1-x)(1-y)})$ otherwise
- Algorithm 1 with $E(A_2(u_i)) = \int g(A_2(u_i)(x))dx$ and $g(x) = R(x, 1-x) = 1 - |2x-1|$
- Algorithm 1 with $E(A_2(u_i)) = \int g(A_2(u_i)(x))dx$ and $g(x) = R(x, 1-x) = 1 - (2x-1)^2$

In Table I we study the obtained thresholds as well as the percentage of pixels correctly segmented with respect to the ideal segmentation for each of the algorithms and each of the 8 images shown in Figure 16. For the sake of simplicity, thresholds have been multiplied by 255. Moreover, we consider the combination of all the obtained thresholds using the arithmetic mean and we also calculate for the latter the percentage of well segmented pixels.

As we can see in Table I, it does not exist one single method which is the best for every possible image. However, when we take the mean of several methods we get good results, which are even the best ones for 4 of the 8 images. So, after combining the results of several algorithms (including Algorithm 1), we see that the obtained segmentations are very good. These segmentations can be taken as a first step in the calculation of segmentations which take into account more properties of the images, apart from the intensity of the pixels.

VII. Conclusions and Future Work

The construction of entropy measures for Fuzzy Sets with infinite universes results intricate. In this direction one of the main novelties of this study is the introduction of the concept of quasi-entropy. Defined slightly different than the fuzzy entropy given by De Luca and Termini it is proven that both concepts are equivalent if we restrict to continuous membership functions. The quasi-entropy measure has been applied to a T2FS (whose membership degree for an element x in the universe X is a $FS([0,1])$), generating the novel concepts of Type-2 Fuzzy Entropy-Set and pointwise measure. Finally, we have shown the usefulness of the Type-2 Fuzzy Entropy-Set in an illustrative example in Huang and Wang's algorithm for image thresholding.

Due to the relevance of a theoretical method to calculate the entropy of T2FSs we leave for a future work the deeper study of the application, i.e. we leave for future work the deep analysis of the conditions under which the algorithms considered in the illustrative example (Algorithm 1) can improve the thresholds usually calculated.

Appendix
Proofs of the Theorems

Theorem 3.2 Let $\Gamma : FS([0,1]) \mapsto [0,1]$ be a function given by Eq. (1). *Then*

$$\Gamma \text{ satisfies } (E1^*) \text{ if and only if } g(z) \neq 0 \text{ for all } z \in]0,1[.$$

Proof.

$\Rightarrow$) *Let Γ satisfy $(E1^*)$.*

Suppose that $g(z_0) = 0$ for some $z_0 \in]0,1[$. Let $A \in FS([0,1])$ be given by $A(z) = z_0$ for all $z \in [0,1]$. Then, $\Gamma(A) = \int_{H_A} g(A(y))dy = \int_0^1 g(z_0) = 0$ and Γ does not satisfy $(E1^*)$.

$\Leftarrow$) *Take $g(z) \neq 0$ for all $z \in]0,1[$.*

- *If H_A has Lebesgue measure 0 then $\Gamma(A) = \int_{H_A} g(A(y))dy = 0$.*
- *If $\Gamma(A) = \int_{H_A} g(A(y))dy = 0$, since $g(z) \neq 0$ for all $z \in]0,1[$, then $g(A(y)) \neq 0$ for all $y \in H_A$. Consequently, $\Gamma(A) = \int_{H_A} g(A(y))dy = 0$ can only hold if $m(H_A) = 0$.*

Thus, Γ satisfies $(E1^*)$.

Theorem 3.5 Let $\Gamma : FS([0,1]) \mapsto [0,1]$ be a function given by Eq. (1). *Then,*

$$\Gamma \text{ satisfies } (E2^*) \text{ if and only if } g^{-1}(1) = \left\{\frac{1}{2}\right\}$$

Proof.

$\Rightarrow$) *Let Γ satisfy $(E2^*)$.*

- *Suppose that $g(\frac{1}{2}) \neq 1$. Let the FS A be given by $A(x) = \frac{1}{2}$ for all $x \in [0,1]$. Then $\Gamma(A) = \int_{H_A} g(A(y))dy = \int_0^1 g(\frac{1}{2}) = g(\frac{1}{2}) \neq 1$, which is in contradiction with $(E2^*)$.*
- *Suppose $g(z_0) = 1$ for some $z_0 \neq \frac{1}{2}$. Given $A(x) = z_0$ for all $x \in [0,1]$ we have $\Gamma(A) = \int_{H_A} g(A(y))dy = \int_0^1 g(z_0) = g(z_0) = 1$, which is again in contradiction with $(E2^*)$.*

$\Leftarrow$) *Let g satisfy $g^{-1}(1) = \{\frac{1}{2}\}$.*

- *If $A(x) = \dfrac{1}{2}$ a.e. in $[0,1]$, then $m(\{x \in H_A \mid A(x) \neq \dfrac{1}{2}\}) \leq m(\{x \mid A(x) \neq \dfrac{1}{2}\}) = 0$ and $m(\{x \mid A(x) = \dfrac{1}{2})\}) = 1$. Thus, $\Gamma(A) = \int_{H_A} g(A(y))dy = \int_{\{x \in H_A | A(x) \neq \frac{1}{2}\}} g(A(y))dy + \int_{\{x | A(x) = \frac{1}{2}\}} g(A(y))dy = 0 + \int_{\{x | A(x) = \frac{1}{2}\}} g(\frac{1}{2})dy = g(\frac{1}{2}) = 1$.*
- *Now take $\Gamma(A) = \int_{H_A} g(A(y))dy = 1$. Since $m(H_A) \leq 1$ and $g(z) \leq 1$ then $\Gamma(A) = 1$ can only hold if $m(H_A) = 1$ and $g(A(y)) = 1$ for all $y \in H_A$. But given $y \in H_A$, $g(A(y)) = 1$ only if $A(y) = \frac{1}{2}$. Since the measure of H_A is 1, this means that $A = \frac{1}{2}$ a.e. in $[0,1]$.*

Consequently, Γ satisfies $(E2^*)$.

Theorem 3.7 Let $\Gamma : FS([0,1]) \mapsto [0,1]$ be a function given by Eq. (1). *Then, Γ satisfies $(E3^*)$ if and only if g is increasing on $\left]0,\frac{1}{2}\right]$ and decreasing on $\left[\frac{1}{2},1\right[$.*

Proof.

$\Rightarrow$) *Let Γ satisfy $(E3^*)$.*

1) *Suppose g is not increasing in $]0,\frac{1}{2}]$. Then, there exist z_1, z_2 such that $0 < z_1 < z_2 \leq \frac{1}{2}$ and $g(z_1) > g(z_2)$. Let $A(x) = z_1$ for all $x \in [0,1]$ and $B(x) = z_2$ for all $x \in [0,1]$. As $A(x) \leq B(x) \leq \frac{1}{2}$ for all $x \in [0,1]$, by $(E3^*)$ it must be satisfied that $\Gamma(A) \leq \Gamma(B)$. But $\Gamma(A) = \int_{H_A} g(A(y))dy = \int_0^1 g(z_1)dy = g(z_1)$ and $\Gamma(B) = \int_{H_B} g(B(y))dy = \int_0^1 g(z_2)dy = g(z_2)$, which is in contradiction with $g(z_1) > g(z_2)$.*
2) *Suppose that g is not decreasing in $[\frac{1}{2},1[$. Then, there exist z_1, z_2 such that $\frac{1}{2} \leq z_1 < z_2 < 1$ and $g(z_1) < g(z_2)$. Let $A(x) = z_2$ for all $x \in [0,1]$ and $B(x) = z_1$ for all $x \in [0,1]$. Since $\frac{1}{2} \leq B(x) \leq A(x)$ for all $x \in [0,1]$, by $(E3^*)$ $\Gamma(A) \leq \Gamma(B)$ must be satisfied. But $\Gamma(A) = \int_{H_A} g(A(y))dy = \int_0^1 g(z_2)dy = g(z_2)$ and $\Gamma(B) = \int_{H_B} g(B(y))dy = \int_0^1 g(z_1)dy = g(z_1)$, which is in contradiction with $g(z_1) < g(z_2)$.*

$\Leftarrow$) *Let g be increasing in $]0,\frac{1}{2}]$ and decreasing in $[\frac{1}{2},1[$.*

	Otsu		Area		Ignorance		Alg1v1		Alg1v2		Average	
	u	%	u	%	u	%	u	%	u	%	u	%
Im. 1	79	93.6614	50	97.3064	13	96.6738	50	97.3064	29	97.2375	44	**97.4059**
Im. 2	74	92.2227	56	92.7227	11	90.8861	58	92.7074	47	92.7099	49	**92.7762**
Im. 3	104	98.0148	87	98.2887	13	97.6454	96	98.1731	88	98.2887	77	**98.3741**
Im. 4	136	**95.8283**	135	95.7278	135	95.7278	135	95.7278	134	95.5912	135	95.7278
Im. 5	127	95.8474	140	95.9545	177	93.3757	143	**95.9621**	157	95.2479	148	95.7224
Im. 6	134	95.6408	138	96.4085	97	64.4245	138	96.4085	141	**96.7835**	129	94.9316
Im. 7	71	95.9748	50	96.6721	3	92.2469	52	96.6337	49	96.7208	45	**96.8029**
Im. 8	123	89.0935	121	**89.5726**	121	**89.5726**	121	**89.5726**	121	**89.5726**	121	**89.5726**

TABLE I

THRESHOLDS (MULTIPLIED BY 255) AND PERCENTAGE OF WELL CLASSIFIED PIXELS. (OTSU) RESULTS OBTAINED WITH OTSU'S METHOD. (AREA) RESULTS OBTAINED AREA ALGORITHM AND $\varphi_1(x) = x^2$ AND $\varphi_2(x) = x$. (IGNORANCE) RESULTS OBTAINED WITH WITH THE ALGORITHM BASED ON THE IGNORANCE AND $G_u(x, y) = 2\sqrt{(1-x)(1-y)}$ IF $(1-x)(1-y) \leq 0.25$ AND $G_u(x, y) = 1/(2\sqrt{(1-x)(1-y)})$ OTHERWISE. (ALG2V1) RESULTS OBTAINED WITH OUR PROPOSAL, USING $E = \int g(A(x))dx$ WITH $g(x) = R(x, 1-x) = 1 - |2x - 1|$. (ALG2V2) RESULTS OBTAINED WITH OUR PROPOSAL, USING $E = \int g(A(x))dx$ WITH $g(x) = R(x, 1-x) = 1 - (2x-1)^2$.

First of all, notice that g has a maximum on $\frac{1}{2}$.

Suppose that $A, B \in FS([0,1])$ satisfy that for all $x \in [0,1]$

$$\left.\begin{array}{l} A(x) \leq B(x) \leq \frac{1}{2} \\ \text{or} \\ A(x) \geq B(x) \geq \frac{1}{2} \end{array}\right\} \qquad (4)$$

and let us see that $E^*(A) \leq E^*(B)$.

First, we prove $H_A \subseteq H_B$. Take $x \in H_A$, by the Definition of H_A then $A(x) \neq 0$ and $A(x) \neq 1$. There are three different cases:

- *If $A(x) < \frac{1}{2}$ then $0 < A(x) \leq B(x) \leq \frac{1}{2}$, so $0 < B(x) < 1$ and $x \in H_B$.*
- *If $A(x) > \frac{1}{2}$ then $1 > A(x) \geq B(x) \geq \frac{1}{2}$, so $0 < B(x) < 1$ and $x \in H_B$.*
- *If $A(x) = \frac{1}{2}$ then $\frac{1}{2} \leq B(x) \leq \frac{1}{2}$, so $0 < B(x) = \frac{1}{2} < 1$ and $x \in H_B$.*

Thus, $H_A \subseteq H_B$. Thereby,

$$\begin{aligned} \Gamma(A) &= \int_{H_A} g(A(y))dy \leq \int_{H_B} g(A(y))dy \\ &= \int_{\{x \mid 0 < B(x) < \frac{1}{2}\}} g(A(y))dy + \int_{\{x \mid B(x) = \frac{1}{2}\}} g(A(y))dy \\ &+ \int_{\{x \mid \frac{1}{2} < B(x) < 1\}} g(A(y))dy \leq \int_{\{x \mid 0 < B(x) < \frac{1}{2}\}} g(B(y))dy \\ &+ \int_{\{x \mid B(x) = \frac{1}{2}\}} g(B(y))dy + \int_{\{x \mid \frac{1}{2} < B(x) < 1\}} g(B(y))dy \\ &= \int_{H_B} g(B(y))dy = \Gamma(B) \end{aligned}$$

where the first inequality holds due to $H_A \subseteq H_B$ and the second one because g is an increasing function on $]0, \frac{1}{2}]$, because g has a maximum on $\frac{1}{2}$ and because g is decreasing on $[\frac{1}{2}, 1[$, respectively.

Theorem 3.9 Let $\Gamma : FS([0,1]) \mapsto [0,1]$ be a function given by Eq. (1). Then,

Γ satisfies $(E4^*)$ if and only if g is a symmetric function with respect to $z = \frac{1}{2}$, i.e., $g(z) = g(1-z)$ for all $z \in]0,1[$.

Proof. *First of all, notice that $H_{N(A)} = \{x \mid N(A(x)) \in]0,1[\} = \{x \mid 1 - A(x) \in]0,1[\} = \{x \mid A(x) \in]0,1[\} = H_A$.*

$\Rightarrow$) Let Γ satisfy $(E4^*)$.

Suppose that g is not symmetric, then there exists $z_0 \in]0,1[$ such that $g(z_0) \neq g(1 - z_0)$. Let $A(x) = z_0$ for all $x \in [0,1]$, then $N(A(x)) = 1 - z_0$ for all $x \in [0,1]$. However, function Γ yields

$\Gamma(A) = \int_{H_A} g(A(y))dy = \int_0^1 g(z_0)dy = g(z_0)$ *and*

$$\begin{aligned} \Gamma(N(A)) &= \int_{H_{N(A)}} g(N(A(y)))dy \\ &= \int_{H_{N(A)}} g(1 - z_0)dy = g(1 - z_0), \end{aligned}$$

which is in contradiction with $(E4^*)$.

$\Leftarrow$) Let g be a symmetric function with respect to $z = \frac{1}{2}$. Then

$$\begin{aligned} \Gamma(A) &= \int_{H_A} g(A(y))dy \\ &= \int_{H_{N(A)}} g(A(y))dy = \int_{H_{N(A)}} g(1 - A(y))dy \\ &= \int_{H_{N(A)}} g(N(A(y)))dy = \Gamma(N(A)) \end{aligned}$$

where the second equality holds because $H_A = H_{N(A)}$, the third one holds because g is symmetric and the fourth one by the expression of negation.

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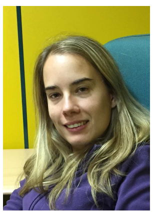

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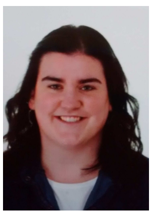

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A framework for radial data comparison and its application to fingerprint analysis

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Abstract

This work tackles the comparison of radial data, and proposes comparison measures that are further applied to fingerprint analysis. First, we study the similarity of scalar and non-scalar radial data, elaborated on previous works in fuzzy set theory. This study leads to the concepts of Restricted Radial Equivalence Function and Radial Similarity Measure, which model the perceived similarity between scalar and vectorial pieces of radial data, respectively. Second, the utility of these functions is tested in the context of fingerprint analysis, and more specifically, in the singular point detection. With this aim, a novel template-based singular point detection method is proposed, which takes advantage of these functions. Finally, their suitability is tested in different fingerprint databases. Different similarity measures are considered to show the flexibility offered by these measures and the behavior of the new method is compared with well-known singular point detection methods.

Keywords: Radial data, Restricted equivalence function, similarity measure, fingerprint singular point detection

1. Introduction

The ability to quantify the similarity between two objects in a given universe is a pillar in applied fields of research. Historically, this quantification has been based on metrics, which are able to capture, in a sensible (and coherent) manner, the proximity of any two objects in a measurable universe. Metrics hold very interesting properties, specifically triangular inequality, which preserves the notion that the shortest path between two objects is the straight one. However, they also impose the need for the representation of the objects in a metric space, as well as notions (e.g. transitivity), which are not natural in certain scenarios [1].

When it comes to measuring dissimilarity between multivalued data, L_p metrics often come as a straightforward option; the most relevant case is $p = 2$, which recovers the Euclidean metric. The L_p metric has been long criticized, specially regarding its low accuracy in capturing perceptual dissimilarities. For example, Attneave stated that the assumption that the *psychological space is Euclidean in its character* is *exceedingly precarious* [2]. Obviously, there exist other metrics yielding more (perceptually) accurate measurements of dissimilarity, specially when they are designed for well-defined scenarios [3, 4]. The debate about the restrictivity of the requisites imposed by metrics is still open [5]. Literature contains both practical [6], and theoretical criticisms. Authors as Tversky [7] or Santini and Jain [5] criticized the necessity of imposing metric conditions to similarity measures, as well as the representation of objects in metric spaces, given that they are often missing in human understanding. Tversky [7, 8] also revisited the necessity of

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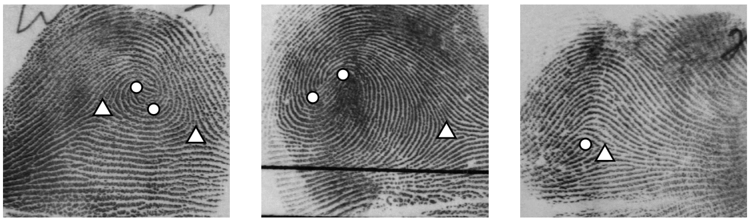


Figure 1: Examples of singular points detected on fingerprints extracted from the NIST-4 dataset [19]. The *deltas*, represented as triangles, are triangular-like ridge confluences, while *cores*, represented as circles, take place at curly ridge structures.

symmetry and the directional nature of comparisons in certain scenarios. Finally, the low representativity of the values given by metrics for large-range comparisons has also been under debate [9, 10].

Different mathematical theories have tackled the modelling of similarity with tools other than metrics, leading to what Zadeh referred to as *a vast armamentarium* of techniques for comparison [11]. In fact, even axiomatic representations of non-metric comparison frameworks have appeared in the literature (e.g. [7] for set-based similarity, or [12, 13] for T-indistinguishability). In the context of fuzzy set theory, a range of authors have elaborated on the semantic interpretation of similarities and dissimilarities [5], since Zadeh introduced similarity as an extension of equivalence [11]. This is natural, considering that the concepts of proximity and similarity (as well as ordering or clustering) are strongly related to human interpretation, and hence prone to be tackled in fuzzy terms. A large variety of proposals have appeared for modelling both similarity and dissimilarity; in this work we focus our interest on two of them: Restricted Equivalence Functions (REFs) for the comparison of membership degrees and Similarity Measures (SMs) for the comparison of fuzzy sets on discrete universes [14].

In this paper we propose a definition of the concept of REFs and SMs for radial data. This study is motivated by the increasing relevance of radial data in real applications, especially in those demanding the extraction of information by means of computer vision techniques. Very often, computer vision handles radial data in different flavours (e.g., angular, vector or tensorial data [15]) and consequently demands well-defined operators for different tasks, including data comparison. Typically, the study of radial data has been restricted to radial statistics, which mostly study the fitting and analysis of well-known distributions on radial set-ups. To the best of our knowledge, there are no studies on the quantification of the similarity of elements in a radial universe. This situation has led many researchers to use *ad-hoc* operators to deal with the special conditions of the data, instead of creating a framework in which different operators can be encompassed. For this reason in this work we develop a framework aiming at easing the comparison of radial data. More specifically, we define Restricted Radial Equivalence Functions (RREFs), as well as Restricted Similarity Measures (RSMs), which attempt to mimic the behaviour of REFs and SMs in radial universes.

As a case of study, we present an application of RREFs and RSMs to biometric identification, specifically to singular point detection in fingerprint recognition [16]. Fingerprints can be seen as a set of ridges (lines) that represent the relief of the skin in the fingertip surface. Hence, their analysis is often based on studying the line patterns in a local or semi-local basis. Within fingerprint analysis, a fundamental operation is the detection and localization of the so-called *Singular Points* (SPs), which are structural singularities in the ridges (see Fig. 1). SP detection is often related to specific occurrences in the orientation of the ridges of neighbouring regions, which are usually found using semi-local analysis [17] or complex convolution filters [18].

On this account, a simple yet effective framework for SP detection is presented in this paper by means of RREFs and RSMs, which shows the usefulness and flexibility of these new measures. Furthermore, other well-known SP detection algorithms have been used as a baseline for performance evaluation [20, 21]. In this comparative analysis we have considered two different types of databases: NIST-4 database [19], the most commonly used fingerprint database and synthetic fingerprint databases generated by SFinGe[1].

The remainder of the work is as follows. In Section 2 we review the concepts of REF and SM, as well as some standard notation on radial data. Section 3 is devoted to introduce the concepts of RREF and RSM. Both RREF and RSM are used in Section 4, in which we present our proposal for SP detection in fingerprints. Section 5 includes an experimental study in which we illustrate the performance of our SP detection method, compared to other well-known methods in the literature. Finally, Section 6 gathers some conclusions and a brief discussion on potential future evolutions of our method.

2. Preliminaries

Among the areas in which fuzzy set theory has played a relevant role, data similarity modelling is one of the most prominent. The reason is that the natural concepts of similarity, closeness or likeliness are inherently bounded to human interpretation. Hence, different proposals have appeared to effectively model the comparison of pieces of information. Among these, we find fuzzy metric spaces [6], with interesting advantages over classical metric spaces in terms of interpretability [22] or equivalence and similarity measures [14], which we take as inspiration to develop measures that can handle radial data. Next, we recall the concepts of REF and SM.

Definition 1. *A continuous, strictly decreasing function $n : [0,1] \to [0,1]$ such that $n(0) = 1$, $n(1) = 0$ and $n(n(x)) = x$ for all $x \in [0,1]$ (involutive property) is called strong negation.*

Definition 2. *[14] A mapping $r : [0,1]^2 \to [0,1]$ is said to be a Restricted Equivalence Function (REF) associated with the strong negation n if it satisfies the following:*

(R1) $r(x,y) = r(y,x)$ *for all* $x, y \in [0,1]$*;*

(R2) $r(x,y) = 1$ *if and only if* $x = y$*;*

(R3) $r(x,y) = 0$ *if and only if* $\{x,y\} = \{0,1\}$*;*

(R4) $r(x,y) = r(n(x), n(y))$ *for all* $x, y \in [0,1]$*;*

(R5) For all $x, y, z, t \in [0,1]$, *such that* $x \leq y \leq z \leq t$ *then* $r(y,z) \geq r(x,t)$.

Note that (R5) means that, for all $x, y, z \in [0,1]$, if $x \leq y \leq z$ then $r(x,y) \geq r(x,z)$ and $r(y,z) \geq r(x,z)$.

REFs attempt to capture the perceived similarity between two values in $[0,1]$, which in fuzzy set theory usually represent membership degrees. It is usual to construct REFs from a pair of automorphisms of the unit interval, as proposed in [14], although alternative methods have also been studied [23].

Definition 3. *A continuous, strictly increasing function $\varphi : [a,b] \to [a,b]$ such that $\varphi(a) = a$ and $\varphi(b) = b$ is called automorphism of the interval $[a,b] \subset \mathbb{R}$.*

Proposition 1. *[14] Let φ_1, φ_2 be two automorphisms of the interval $[0,1]$. Then*

$$r(x,y) = \varphi_1^{-1}(1 - |\varphi_2(x) - \varphi_2(y)|)$$

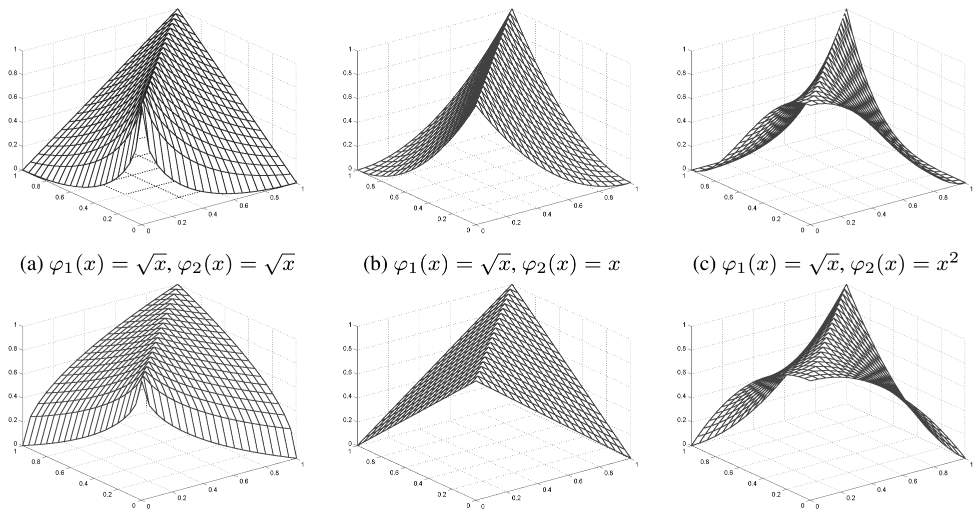
is a REF associated with the strong negation $n(x) = \varphi_2^{-1}(1 - \varphi_2(x))$.

Example. Let $\varphi_1(x) = x$ and $\varphi_2(x) = \sqrt{x}$, then

$$r(x,y) = 1 - |\sqrt{x} - \sqrt{y}| \tag{1}$$

is a REF associated with $n(x) = (1 - \sqrt{x})^2$.

[1] Synthetic Fingerprint Generator: `http://biolab.csr.unibo.it/sfinge.html`



(a) $\varphi_1(x) = \sqrt{x}, \varphi_2(x) = \sqrt{x}$ (b) $\varphi_1(x) = \sqrt{x}, \varphi_2(x) = x$ (c) $\varphi_1(x) = \sqrt{x}, \varphi_2(x) = x^2$

(d) $\varphi_1(x) = x, \varphi_2(x) = \sqrt{x}$ (e) $\varphi_1(x) = x, \varphi_2(x) = x$ (f) $\varphi_1(x) = x, \varphi_2(x) = x^2$

Figure 2: Restricted Equivalence Functions (REFs) created from automorphisms in the unit interval as in Proposition 1.

Figure 2 contains the visual representation of several REFs constructed following Prop. 1. Note that in this work we are referring to REFs in the original sense, although the concept has been exported to, e.g. interval data [24]. While REFs are useful to compare scalar data (membership degrees), similarity measures were developed to compare non-scalar data. Even though the measures were initially designed to compare fuzzy sets on discrete universes, they can be further applied to many other objects (e.g. vectors or matrices). Similarity measures were originally proposed by Liu [25] and its definition is trivially applied to $[0,1]^k$.

Definition 4. *[25] A mapping $s : [0,1]^k \times [0,1]^k \to \mathbb{R}^+$ is called k-ary Similarity Measure (SM) associated with the strong negation n if it satisfies the following:*

(S1) $s(\boldsymbol{x},\boldsymbol{y}) = s(\boldsymbol{y},\boldsymbol{x})$ *for all* $\boldsymbol{x},\boldsymbol{y} \in [0,1]^k$*;*

(S2) $s(\boldsymbol{x}, n(\boldsymbol{x})) = 0$ *if and only if* $x_i \in \{0,1\}$ *for all* $i \in \{1,\dots,k\}$ *and* $n(\boldsymbol{x}) = (n(x_1),\dots,n(x_k))$*;*

(S3) $s(\boldsymbol{z},\boldsymbol{z}) = \max_{(\boldsymbol{x},\boldsymbol{y}\in[0,1]^k)} s(\boldsymbol{x},\boldsymbol{y})$ *for all* $z \in [0,1]^k$*;*

(S4) For all $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{t} \in [0,1]^k$*, if* $\boldsymbol{x} \le \boldsymbol{y} \le \boldsymbol{z} \le \boldsymbol{t}$ *then* $s(\boldsymbol{y},\boldsymbol{z}) \ge s(\boldsymbol{x},\boldsymbol{t})$ *where* $\boldsymbol{x} \le \boldsymbol{y}$ *implies that* $x_i \le y_i$ *for all* $i \in \{1,\dots,k\}$.

Similarly to what happens for REFs, the property (S4) is equivalent to: For all $\mathbf{x},\mathbf{y},\mathbf{z} \in [0,1]^k$, if $\mathbf{x} \le \mathbf{y} \le \mathbf{z}$ then $s(\mathbf{x},\mathbf{y}) \ge s(\mathbf{x},\mathbf{z})$ and $s(\mathbf{y},\mathbf{z}) \ge s(\mathbf{x},\mathbf{z})$.

Similarity Measures (SMs) can be constructed in different ways, although the most popular method originates from the combination of REFs and aggregation functions [26].

Definition 5. *A mapping $f : [0,1]^k \to [0,1]$ is called k-ary aggregation operator if it satisfies the following:*

(AO1) If $x_i = 0$ *for all* $i \in \{1,\dots,k\}$*, then* $f(\mathbf{x}) = 0$*;*

(AO2) If $x_i = 1$ *for all* $i \in \{1,\dots,k\}$*, then* $f(\mathbf{x}) = 1$*;*

(AO3) f *is increasing in all of its arguments.*

Proposition 2. *[14] Let r be a REF and let f be a k-ary aggregation function such that $f(\boldsymbol{x}) = 0$ if and only if $x_i = 0$ for all $i \in \{1,\dots,k\}$ and $f(\boldsymbol{x}) = 1$ if and only if $x_i = 1$ for all $i \in \{1,\dots,k\}$. The function $s_{[f,r]} : [0,1]^k \times [0,1]^k \mapsto [0,1]$, given by*

$$s_{[f,r]}(\boldsymbol{x},\boldsymbol{y}) = f(r(x_1,y_1),\dots,r(x_k,y_k)) \tag{2}$$

is a k-ary similarity measure which satisfies the following:

- $s_{[f,r]}(\boldsymbol{x},\boldsymbol{y}) = s_{[f,r]}(\boldsymbol{y},\boldsymbol{x})$ *for all* $\boldsymbol{x},\boldsymbol{y} \in [0,1]^k$*;*
- $s_{[f,r]}(\boldsymbol{x}, n(\boldsymbol{x})) = 0$ *if and only if* $x_i \in \{0,1\}$ *for all* $i \in \{1,\dots,k\}$ *and* $n(\boldsymbol{x}) = (n(x_1),\dots,n(x_k))$*;*
- $s_{[f,r]}(\boldsymbol{x},\boldsymbol{y}) = 1$ *if and only if* $x_i = y_i$ *for all* $i \in \{1,\dots,k\}$*;*
- *For all* $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{t} \in [0,1]^k$*, if* $\boldsymbol{x} \leq \boldsymbol{y} \leq \boldsymbol{z} \leq \boldsymbol{t}$ *then* $s_{[f,r]}(\boldsymbol{y},\boldsymbol{z}) \geq s_{[f,r]}(\boldsymbol{x},\boldsymbol{t})$ *where* $\boldsymbol{x} \leq \boldsymbol{y}$ *implies that* $x_i \leq y_i$ *for all* $i \in \{1,\dots,k\}$*;*
- $s_{[f,r]}(\boldsymbol{x},\boldsymbol{y}) = s_{[f,r]}(n(\boldsymbol{x}), n(\boldsymbol{y}))$ *for all* $\boldsymbol{x},\boldsymbol{y} \in [0,1]^k$.

Next, we include some conventions on the use of radial data that will hold in the remainder of this work. In order to face those cases in which two angles represent the same direction (e.g. $\frac{\pi}{2}$ and $\frac{5\pi}{2}$), an equivalence relation R is defined.

Let $a, b \in \mathbb{R}$. We say that aRb if and only if $a = b + 2k\pi$, where $k \in \mathbb{Z}$. In this way, the equivalence class $[a] = \{b \mid bRa\}$ is the set containing all the data associated with the same direction. In the same way, any semiopen interval whose width is 2π (i.e., an interval of the form $[\omega, \omega + 2\pi[$) is the quotient set (a set which contains one element and only one of each equivalence class). In particular, the intervals $[0, 2\pi[$ and $[-\pi, \pi[$ are the most frequently used quotient sets in radial data.

In this work we consider the quotient set $\Omega = [0, 2\pi[$ and we define on Ω the classical operations sum ($\oplus$) and difference ($\ominus$), given by $a \oplus b = [a + b]$ and $a \ominus b = [a - b]$ where $[t]$ denotes the only element $z \in \Omega$ such that zRt. Under these conditions, we refer to *mirroring* as the mapping: $m : \Omega \longrightarrow \Omega$ such that $m(a) = a \oplus \pi$.

In order to properly explain our incoming proposal on the quotient set Ω, we present the general definition of metric.

Definition 6. *A function* $d : U \to \mathbb{R}^+$ *is called metric on* U *if it satisfies the following:*

(D1) $d(a,b) = 0$ *if and only if* $a = b$*;*

(D2) $d(a,b) = d(b,a)$ *for all* $a, b \in U$*;*

(D3) $d(a,c) \leq d(a,b) + d(b,c)$ *for all* $a, b, c \in U$.

3. Comparison of radial data

As reported by Fisher [27], radial data has been a subject of analysis since mid-18th century. However, most of the literature on radial data is based on adapting the usage of distributions to the circular set-up, probably because data analysis for natural sciences was the field in which radial data was first studied [28, 29].

One of the open problems in radial data is data comparison. In fact, to the best of our knowledge, no explicit mention to the quantification of similarity between two angles has been performed in the literature. There have been concepts such as the *sample median direction* [27] or the *sample modal direction* [30], and metrics such as the angular metric on $[0, 2\pi[$ ($d^*(a,b) = \min(|b-a|, 2\pi - |b-a|)$), which represents the amplitude of the shortest arc encompassing two angles. However, no development has been made on interpretable measures able to adapt to human perception or evaluation. This section is devoted to develop functions that are able to measure the perceived similarity between scalar and vector angular data. Section 3.1 covers the comparison of scalar radial data, whereas Section 3.2 covers the comparison of vector radial data[2].

[2]There exist in the literature certain controversy w.r.t. the most adequate name of radial data, including angular data or radial data. In this manuscript we adhere to *radial*.

3.1. Restricted Radial Equivalence Functions

The comparison of linear data has produced a vast amount of literature, despite coming from a relatively simple concept. The concept of similarity becomes much more intricate when applied to radial data. In this section we define operators that model the comparison of elements in a radial context Ω, all inspired by the operators in Section 2.

Definition 7. *A mapping $r_\theta : \Omega^2 \to [0,1]$ is called a Restricted Radial Equivalence Function (RREF) associated with the metric d if it satisfies the following:*

(RR1) $r_\theta(a,b) = r_\theta(b,a)$ *for all* $a, b \in \Omega$*;*

(RR2) $r_\theta(a,b) = 1$ *if and only if* $d(a,b) = 0$*;*

(RR3) $r_\theta(a,b) = 0$ *if and only if* $d(a,b)$ *is maximum;*

(RR4) $r_\theta(a,b) = r_\theta(m(a), m(b))$ *for all* $a, b \in \Omega$*;*

(RR5) For all $a, b, c, d \in \Omega$*, if* $d(b,c) \leq d(a,d)$*, then* $r_\theta(b,c) \geq r_\theta(a,d)$*.*

Definition 7 is not a direct extension of Definition 2 to radial data. The differences arise from the use of distances in (RR5) instead of orders (as in (R5)). Nevertheless, this change is due to the difficulties in the interpretation of orders in radial universes. Despite this modification, we believe that the spirit and semantics of RREFs are those of REFs.

In this work, we only consider RREFs associated with the angular metric $d^*(a,b) = \min(|b-a|, 2\pi - |b-a|)$ but many other metrics are also eligible, e.g.:

$$d(a,b) = \begin{cases} 0 & \text{if } a = b, \\ \pi & \text{if } d^*(a,b) = \pi, \\ \pi/2 & \text{otherwise.} \end{cases}$$

Proposition 3. *Let r_θ be a RREF associated with the metric d^*. For all $a_1, b_1, a_2, b_2 \in \Omega$, if $d^*(a_1,b_1) = d^*(a_2,b_2)$ then $r_\theta(a_1,b_1) = r_\theta(a_2,b_2)$.*

Proof. *Let $a_1, b_1, a_2, b_2 \in \Omega$ such that $d^*(a_1,b_1) = d^*(a_2,b_2)$. According to (RR5), $d^*(a_1,b_1) \leq d^*(a_2,b_2)$ implies $r_\theta(a_1,b_1) \geq r_\theta(a_2,b_2)$. Analogously, $d^*(a_2,b_2) \leq d^*(a_1,b_1)$ implies $r_\theta(a_2,b_2) \geq r_\theta(a_1,b_1)$ so the equality holds.*

Corollary 1. *Let $h : \Omega^2 \to [0,1]$. If h satisfies (RR5) with respect to the metric d^* then it also satisfies (RR4).*

Proof. *Trivial by Proposition 3 since $d^*(a,b) = d^*(m(a), m(b))$.*

Following the construction of REFs, we also define a new possible construction method for RREFs, which is also based on automorphisms.

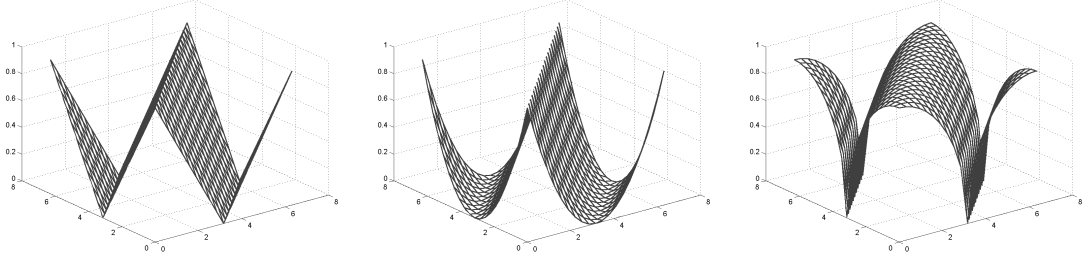
Proposition 4. *Let φ and ψ be automorphisms of the intervals $[0,1]$ and $[0,\pi]$, respectively. The mapping $t : \Omega^2 \to [0,1]$ given by*

$$t(a,b) = \varphi^{-1}\left(1 - \left(\frac{1}{\pi}\psi\left(d^*(a,b)\right)\right)\right) \tag{3}$$

is a RREF.

Proof. *Direct by the properties of the metric d^*.*

Some examples of RREFs constructed as in Proposition 4 are included in Fig. 3.



(a) $\varphi(x) = x$, $\psi(x) = x$ (b) $\varphi(x) = x$, $\psi(x) = \sqrt{\pi x}$ (c) $\varphi(x) = x^2$, $\psi(x) = \frac{x^2}{\pi}$

Figure 3: Restricted radial equivalence functions created as in Proposition 4 from automorphisms in the unit interval (φ) and in $[0, \pi]$ (ψ).

3.2. Radial Similarity Measures

Definition 8. *A mapping $s_\theta : \Omega^k \times \Omega^k \to \mathbb{R}^+$ is said to be a k-ary Radial Similarity Measure (RSM) associated with the metric d^* if it satisfies the following:*

(SR1) $s_\theta(\boldsymbol{a}, \boldsymbol{b}) = s_\theta(\boldsymbol{b}, \boldsymbol{a})$ *for all* $\boldsymbol{a}, \boldsymbol{b} \in \Omega^k$*;*

(SR2) $s_\theta(\boldsymbol{a}, \boldsymbol{b}) = 0$ *if and only if* $d^*(a_i, b_i) = \pi$ *for all* $i \in \{1, \ldots, k\}$*;*

(SR3) $s_\theta(\boldsymbol{c}, \boldsymbol{c}) = Max_{\boldsymbol{a},\boldsymbol{b} \in \Omega^k} s_\theta(\boldsymbol{a}, \boldsymbol{b})$ *for all* $c \in \Omega^k$*;*

(SR4) For all $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \Omega^k$*, if* $d^*(\boldsymbol{a}, \boldsymbol{d}) \geq d^*(\boldsymbol{b}, \boldsymbol{c})$ *then* $s_\theta(\boldsymbol{a}, \boldsymbol{d}) \leq s_\theta(\boldsymbol{b}, \boldsymbol{c})$*, where* $d^*(\boldsymbol{a}, \boldsymbol{d}) \geq d^*(\boldsymbol{b}, \boldsymbol{c})$ *implies that* $d^*(a_i, d_i) \geq d^*(b_i, c_i)$ *for all* $i \in \{1, \ldots, k\}$.

RSMs can be constructed from RREFs, aggregating their results over each element, as it is done for SMs.

Proposition 5. *Let r_θ be a RREF and let f be a k-ary aggregation function such that $f(\boldsymbol{x}) = 0$ if and only if $x_i = 0$ for all $i \in \{1, \ldots, k\}$ and $f(\boldsymbol{x}) = 1$ if and only if $x_i = 1$ for all $i \in \{1, \ldots, k\}$. The function $s_{\theta_{[f, r_\theta]}} : \Omega^k \times \Omega^k$, given by*

$$s_{\theta_{[f, r_\theta]}}(\boldsymbol{a}, \boldsymbol{b}) = f(r_\theta(a_1, b_1), \ldots, r_\theta(a_k, b_k)) \tag{4}$$

is a k-ary radial similarity measure that satisfies

- $s_{\theta_{[f, r_\theta]}}(\boldsymbol{a}, \boldsymbol{b}) = s_{\theta_{[f, r_\theta]}}(\boldsymbol{b}, \boldsymbol{a})$ *for all* $\boldsymbol{a}, \boldsymbol{b} \in \Omega^k$*;*
- $s_{\theta_{[f, r_\theta]}}(\boldsymbol{a}, \boldsymbol{b}) = 0$ *if and only if* $d^*(a_i, b_i) = \pi$ *for all* $i \in \{1, \ldots, k\}$*;*
- $s_{\theta_{[f, r_\theta]}}(\boldsymbol{a}, \boldsymbol{b}) = 1$ *if and only if* $a_i = b_i$ *for all* $i \in \{1, \ldots, k\}$*;*
- *For all* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \Omega^k$*, if* $d^*(\boldsymbol{a}, \boldsymbol{d}) \geq d^*(\boldsymbol{b}, \boldsymbol{c})$ *then* $s_{\theta_{[f, r_\theta]}}(\boldsymbol{a}, \boldsymbol{d}) \leq s_{\theta_{[f, r_\theta]}}(\boldsymbol{b}, \boldsymbol{c})$*, where* $d^*(\boldsymbol{a}, \boldsymbol{d}) \geq d^*(\boldsymbol{b}, \boldsymbol{c})$ *implies that* $d^*(a_i, d_i) \geq d^*(b_i, c_i)$ *for all* $i \in \{1, \ldots, k\}$.
- $s_{\theta_{[f, r_\theta]}}(\boldsymbol{a}, \boldsymbol{b}) = s_{\theta_{[f, r_\theta]}}(m(\boldsymbol{a}), m(\boldsymbol{b}))$ *for all* $\boldsymbol{a}, \boldsymbol{b} \in \Omega^k$ *where* $m(\boldsymbol{a}) = (m(a_1), \ldots, m(a_k))$.

4. Template-based singular point detection

In this section we present a SP detection method based on RSMs and RREFs. Section 4.1 introduces the problem of SP detection, while Section 4.2 presents the template based SP detection method and Section 4.3 outlines the resulting algorithm.

(a) Arch (b) Tented Arch (c) Left loop (d) Right loop (e) Whorl

Figure 4: Examples of fingerprint for each of the five classes in the Henry classification system. These fingerprints have been created using the SFinGe tool [39, 40].

4.1. Fingerprint classification and singular point detection

Fingerprint-based identification is the most popular type of biometrical identity authentication systems. These systems carry out the authentication analysing the ridge patterns in the surface of the fingertips. There are two main tasks that can be performed in the context of fingerprint analysis, namely *identification* and *verification*. The former term refers to the localization of an individual in a database by the usage of an input fingerprint. The latter term refers to confirming whether an input fingerprint corresponds to a certain individual in the database. In both cases, having an accurate way to perform one-to-one fingerprint comparisons is critical [31].

The most common approach to decide whether two fingerprints are produced from the same fingertip is to compare local ridge features, usually, the so-called *minutiae*, which are local discontinuities or anomalies in the ridge pattern [32, 33]. Examples of minutiae are ridge breaks and bifurcations. The whole process of deciding whether two fingerprints belong to the same individual is known as *matching* [31, 34–36].

Fingerprint matching is not trivial, and very often demands a certain computational effort. This is not a problem in fingerprint verification (since the input fingerprint is only compared with those corresponding to the claimed identity), but it becomes critical in fingerprint identification. For this reason, several strategies have been developed to minimize the number of comparisons to be performed. Among them, the most used one is *classification* [16, 37, 38], which consists of classifying each fingerprint according to the general structure of its ridges. In this way, when an input fingerprint has to be matched, it only need to be compared with those belonging to the same class. Although different classification schemes for fingerprints have been proposed, most of the authors in fingerprint analysis use the five major classes in the Henry system [39], namely *arch*, *tented arch*, *left loop*, *right loop*, *whorl*. Examples of these five classes can be found in Fig. 4. As a result of this division, the number of comparisons in an identification process can be drastically reduced. Nonetheless, this task also holds great responsibility. If a fingerprint is misclassified, the system will not be able to perform a correct identification or it may lead to an increase in the computational effort due to the greater number of comparisons that must be carried out.

Fingerprint classification is often defined as the problem of learning a classifier able to determine the class to which a (previously unseen) fingerprint belongs to. In order to do so, the classifier is usually learned from a set of labelled fingerprints. Fingerprints are classified using global features from the ridge flow, instead of local ones as it is done in fingerprint matching. Hence, fingerprint classification consists of two well-differentiated steps[3]: (a) feature extraction, where fingerprints are processed to obtain their feature vector, and (b) classification, where a classifier associates such vector to one of the classes. In this work we focus on the first one; more specifically, on the detection of the so-called singular points (SPs), which are the most commonly used feature for classification.

SPs are locations of the fingerprint in which abnormal ridge patterns occur. In a fingerprint, two types of SPs can be found: *cores* (where ridges tend to converge) and *deltas* (where the ridge flow diverges). The

[3]We refer to [16, 37] for a detailed review on the topic.

importance of these features for classification is clear, given that the classes in the Henry system can indeed be described in terms of SPs:

- *Arch*: There are no SPs, since the ridges flow horizontally producing a small bump in the center of the fingerprint.
- *Tented Arch*: There is one core and one delta, and the delta is under the core. The ridge flow is similar to that of the Arch type, but at least one ridge shows high curvature.
- *Left Loop*: One core and one delta, and the delta is underneath and on the right of the core. One or more ridges flow from the left side, curve back, and disappear again to the left margin of the fingertip.
- *Right Loop*: There is one core and one delta, and the delta is underneath and to the left of the core. One or more ridges flow from the right side, curve back, and disappear again on the right margin of the fingertip.
- *Whorl*: There are two cores and two deltas, and at least one of its ridges makes a full turn around the center of the fingerprint.

A proper description of SPs might be sufficient for the classification of a fingerprint, by using a fixed rule-based strategy [17, 41]. Still, learning-based approaches to classification have also been proposed [21, 42, 43]. In general, it is accepted that accurate SP detection is required in order to reach the highest possible accuracy in the posterior classification process. Note that, apart from their relevance in fingerprint classification, SPs are also used for some other processes on fingerprints, e.g. fingerprint alignment with respect to a reference point [44] (usually a core point).

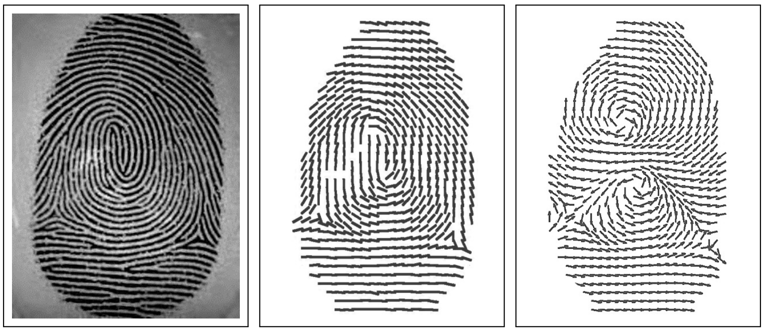
Despite the importance of SPs, their extraction is still an open problem for which proposals are constantly being presented [16, 45, 46]. Most of such proposals are based on the semi-local analysis of the so-called Orientation Map (OM), which is a block-based description of the ridge flow in a fingerprint [47] (see Fig. 5(b) for an example). The most relevant proposal for SP detection is the Poincarè method [20]. In this method, each of the blocks in the orientation map is assigned a Poincarè index, which is computed as the total rotation of the orientations around it. This index determines the presence of a SP, as well as its type (*i.e.*, either core or delta). A popular approach, also based on OMs, is the one proposed by Nilsson and Bigun [18]. Additionally to the usage of complex filters, an interesting novelty in [18] resides in the use of *Squared OMs* (SqOM) [48], which are obtained by multiplying by 2 the orientation at each block of an OM. This simple representation of the OM has the key advantage of producing rotation-invariant patterns at SP locations. Fig. 5 illustrates how SPs look in both conventional and squared orientation maps.

In our SP detection framework we aim at exploiting the high visibility of SPs in squared OMs. More specifically, we propose to use a template-based approach to SP detection which consists of comparing templates of SPs with the actual occurrences of SqOMs. In this context, RREFs and RSMs become crucial, allowing the comparison of the directions in the SqOMs with those in the templates. Forthcoming sections provide details on our method.

4.2. Template-based singular point detection

Template matching procedures are recurrent solutions in digital image processing. The reason is that the only a priori information needed for such procedures is an expression of the goal (which stems from the definition of the problem) and a comparison measure able to quantify the similarity between the input data and the template. Examples of template based methods for image processing are some low-level feature detectors [49–51], or composite object detectors (e.g. the eye detector in [52]). Moreover, despite template matching is conceptually simple, it has also evolved into rather complex theories, among which we can list, for example, mathematical morphology [53].

In this section we present a framework for SP detection based on templates, which is referred to as *Template-based SP Detection* method (TSPD method). To the best of our knowledge, no author has proposed to use templates to represent SPs, probably due to the lack of reliable comparison methods that can handle the matching score. The most similar approach is the usage of complex filters [21], which are convolved with the complex representation of the OM. Notice that we also include this method as baseline performer



(a) (b) (c)

Figure 5: Whorl image generated with SFinGe (a), together with its Orientation Map (OM) (b) and Squared Orientation Map (SqOM) (c). We have considered a size of block of 12×12 pixels to optimize the visibility of each block.

in the experiments. From our point of view, template-matching is a natural way to search for SPs, as long as the definition of SP is vague and based on human perception.

Any template matching-based framework is composed of three nuclear components: (a) an appropriate representation of the input data, (b) templates describing the patterns to be searched in terms of the input data, and (c) a reliable tool to quantify the similarity between both representations. Since our framework is deeply based on mimicking human perception, our aim is to maintain all three components as faithful as possible to the human comprehension of the problem. Consequently, we elaborate on the ridge-like representation of fingerprints (a) and templates (b), while employing RSMs for (c).

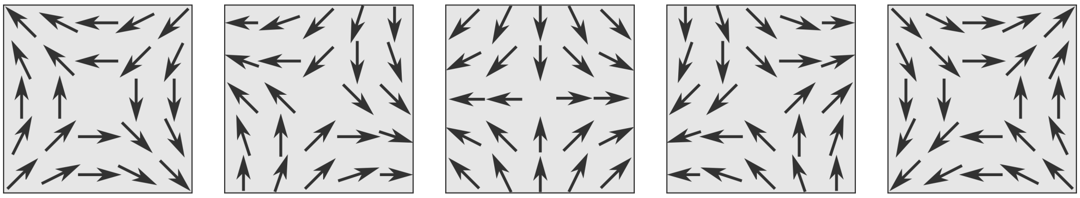
(a) *Fingerprint representation using orientation maps.*- The most obvious representation for SP detection is the fingerprint image itself, since that is all the information humans need to locate SPs. However, this representation is inconvenient, and would dramatically hinder the representation of the template. The reason is that, despite the fact that humans take as input the original image, the location of SPs is based on the analysis of the ridges. That is, the humans automatically convert the tone-based representation of the fingerprint into a ridge-based one.

The representation of the ridges in an image has been often studied, and most of the authors agree on using OMs [47]. These maps divide the fingerprint images into disjoint blocks, assigning to each of them a unique orientation given by the majority ridge orientation of its pixels. The best-known approach to OM extraction is the gradient method [45]. In this method, the orientation of the ridges is computed pixel-wise as the perpendicular to the gradient direction. In Fig. 5 we display the OM of a whorl type fingerprint. We observe how cores take oriented cup-like patterns, which are dependent on the specific orientation of each SP, and deltas produce triangular orientation patterns.

In this work we use the SqOMs, which are better fitted than OMs to our goals. This representation, as shown in Fig. 5, produces interesting changes in the representation of SPs. More specifically, it creates a rotation-invariant representation of cores, which are represented as either clockwise or anticlockwise streams. Regarding deltas, the situation is not as positive, since their appearance does not become rotation-invariant. In any case, using SqOMs simplifies the design of the templates, and is kept as standard representation in our framework.

(b) *Templates for SP representation.*- The templates in our framework must be a minimal set such that it completely captures the way in which SPs appear (or are perceived) in a SqOM. Cores manifest themselves as either clockwise or anticlockwise sequences of orientations, so there is only need for two templates. Moreover, these templates can be functionally represented in a very simple manner.

Let the origin $(0, 0)$ represent the center of a template T of size $(2n + 1) \times (2n + 1)$. The orientation at



(a) $T_{\alpha=0}$ (b) $T_{\alpha=\frac{\pi}{8}}$ (c) $T_{\alpha=\frac{\pi}{4}}$ (d) $T_{\alpha=3\frac{\pi}{8}}$ (e) $T_{\alpha=\frac{\pi}{2}}$

Figure 6: Examples of delta SP templates generated as in Eq. (6) with different values of α.

a position $(x, y) \in [-n, n]^2$ of a core template is given by

$$T(x,y) = \begin{cases} \operatorname{atan2}(y,x) & \text{if it is a clockwise core, and} \\ \operatorname{atan2}(-y,-x) & \text{if it is an anticlockwise core,} \end{cases} \tag{5}$$

where $\operatorname{atan2}(y, x)$ is the well-known sign-sensitive version of the arctangent of $\frac{y}{x}$, *i.e.*, the angle of the vector (x, y) with respect to the positive x-axis. Note that the center of the template has no value, and hence contains no information for the matching process.

With respect to deltas, the problem becomes trickier. In a general manner, a delta is represented as a triangular pattern in the OM, and becomes a symmetric pattern with vectors opposing each other in two orthonormal directions in the SqOM (see Fig. 5). None of those representations is rotation invariant, and consequently an orientation-dependent template must be created to represent delta SPs. The orientation at a position $(x, y) \in [-n, n]^2$ of a delta SP template with orientation $\alpha \in [0, \pi]$ is given by

$$T_\alpha(x,y) = \operatorname{atan2}(-(\cos(\alpha)y - \sin(\alpha)x), \sin(\alpha)y + \cos(\alpha)x)\ . \tag{6}$$

Figure 6 displays the delta SP template for different values of α. In such images we can observe how the pattern is composed of two orthonormal axis, one acting as an *attractor* to the origin, the other one being a *repeller* to it.

According to the previous template definitions, there are two decisions to be made on the set of templates. The first decision affects the number of delta SP templates to be used, *i.e.* the number of different values of α to produce a pattern. One can foresee that a greater number of templates will lead to more accurate detections, although it might also lead to a better fitting of abnormal ridge occurrences that do not correspond to SPs as well as a higher computational effort. The second decision relates to the size of the templates. Indeed the size of the templates must be dependent upon the size of the blocks in the SqOM, as well as upon the expected granularity of the fingerprint capturing process. These parameters are discussed in Section 5.3.

(c) *Comparison of SqOMs and templates.-* The comparison of SqOMs and templates is done in the simplest possible manner. For each template we produce a similarity map with the same dimensions as the SqOM. Each position of such similarity maps corresponds to the value yielded by the RSM between the template and the neighbourhood of the block. Finally, all the similarity maps corresponding to the same type of SP are fused using the $\max$ operator. In this way, we obtain two graded representations of the presence of SPs, one for each type of SP.

4.3. Proposed algorithm

The ideas in Section 4.2 outline the complete algorithm for the detection of SPs in fingerprints.

Note that the algorithm is presented without giving the specific parameters used to make it as general as possible. Furthermore, these parameters must be chosen depending on the characteristics of the images in which it is applied. For this reason, the complete parameter specification used in this paper is shown in Section 5.3. This algorithm is composed of the following phases, which are also schematically represented in Fig. 7:

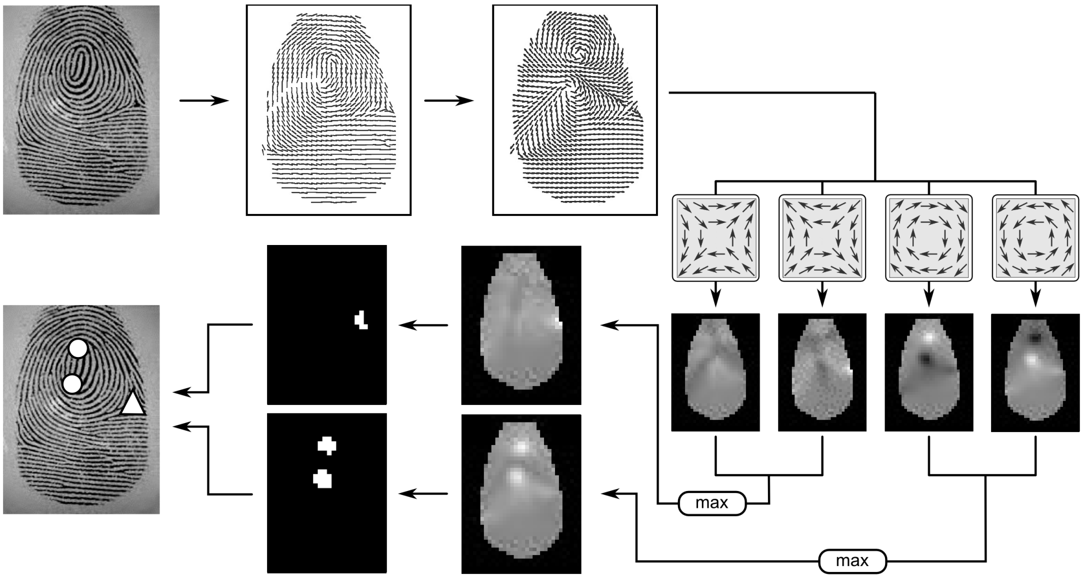


Figure 7: Schematic representation of the proposed framework for singular point detection using orientation templates and radial similarity measures, namely Template-based singular point detection (TSPD).

1. Dividing the image into non-overlapping blocks.
2. Segmenting the image using the previous calculated blocks.
   2.1.- Normalizing the image to a desired mean and variance, 100 and 1000, respectively [33].
   2.2.- Segmenting the fingerprint and the background, by assigning to the latter those blocks for which the variance of the pixel intensities is greater than 30 [33].
3. Calculating the orientation map over the segmented image.
   3.1.- Computing the gradient at each pixel of the image (e.g., using Sobel masks) [45].
   3.2.- Since the gradients are computed for each pixel and the result of a single pixel may not be reliable enough, the OM is smoothed to get more accurate orientations. In order to do so, the technique by Kass and Witkin [48] is used.
   3.3.- Creating the OM from the regularized orientations.
4. Creating the SqOM by multiplying by two the values in the OM.
5. Detecting singular points.
   5.1.- Computing the similarity map for each template. This is done by comparing the elements of the SqOM and those in each templates using the RSMs in Section 3.2.
   5.2.- Fusing the similarity maps corresponding to each type of SP. This is done by obtaining, at each block, the maximum response for the cores and, in parallel, for the deltas.
   5.3.- Selecting cores and deltas. This is done by taking the two points with the highest local response for core and delta similarity map in parallel. They are considered as a SP if they overall a threshold (Table 2).

Regarding computational times of the new technique, it is worth mentioning that it is comparable to the others methods we have used, Liu and Poincarè. All the methods compared share the same computational process. They go trough the fingerprint image executing a series of operations to get local maximums (SPs).

One of the differences between the new method and Liu's method is that it does not use multi-scale image processing. This leads to a speed increase in execution time, but we consider it negligible. The segmentation step and the creation of the OM are common to all the methods and the complexity of the singular point extraction is equivalent to Liu's method. It may be a slight increase in complexity but it is imperceptible.

5. Experiments

The TSPD method has qualitative advantages compared to other traditional methods, e.g., the simplicity of the process and its visualization. However, it also demands a quantitative verification. In this section we check the quantitative performance of our method compared to that of the most relevant SP detection methods in the literature. In Section 5.1, we review the datasets we have used in the comparison. Section 5.2 covers the details on the quantification of the results, whereas Section 5.3 contains a detailed review of the setting and parametrization of the SP detection methods. The results, as well as a brief discussion, are included in Section 5.4.

5.1. Datasets

This experiment uses fingerprints from two different sources. The first one is National Institute of Standards and Technology Special Database 4 (NIST-4) [19]. This database contains rolled-ink fingerprints from the FBI (Federal Bureau of Investigation) and is, historically, the most used benchmark in the fingerprint classification literature. NIST-4 contains 4000 fingerprints (of 512×480 pixels) taken from 2000 fingertips. Hence, there are two captions of each fingerprint. Note that in NIST-4, as well as in most of the available databases, there is no ground truth with respect to the position and type of the SPs. On this account, we have manually labelled the first 1000 fingerprints from NIST-4 database to evaluate the proposed method. Labelling has been carried out according to the specifications given in the specialized literature and has been thoroughly revised by multiple reviewers. In order to ease the process of evaluating further proposals, our ground truth is available at [54]. For illustrative purposes, Fig. 1 shows three fingerprints from NIST-4, together with the corresponding ground truth data.

Even though NIST-4 is widely accepted, it should also be mentioned that in this dataset fingerprint classes are evenly represented, contrary to the reality in real-world databases, in which their distribution is skewed. Moreover, the representation of rolled-ink captions is limited, since most of the current applications collect fingerprint images using optical sensors. For these reasons, we also consider three additional databases generated with SFinGe synthetic fingerprint generator [39, 40]. Although synthetic, SFinGe-generated fingerprints reflect in a faithful manner the difficulties in real scenarios. Moreover, the class distribution can be adjusted to that in reality, that is, 3.7%, 2.9%, 31.7%, 33.8% and 27.9% for arch, tented arch, right loop, left loop and whorl, respectively. Finally, it is relevant that SFinGe itself provides the ground truth data for the SPs, and hence the evaluation process becomes completely objective, whereas in NIST-4 there may be a certain error due to the manual labelling. The validity of the fingerprint images produced by SFinGe is, in any case, widely accepted, to the point that it has already been used in several editions of the Fingerprint Verification Competition (FVC) [55–59] with results similar to those obtained with real fingerprint databases.

Aiming at simulating different scenarios, we have used three different quality profiles in the generation of fingerprints with SFinGe. For each quality profile, we generate 1000 fingerprint images. The following profiles are considered in our experiments (the rest of the parameters used in SFinGe tool are presented in Table 1):

- *High Quality No Perturbations* (HQNoPert): High quality fingerprints without any kind of perturbation;
- *Default*: Middle quality fingerprints with slight localization and rotation perturbations;
- *Varying Quality and Perturbations* (VQandPert): Fingerprints with different qualities are included, which are perturbed in location, rotation and geometric distortions.

Scanner parameters
Acquisition area: 0.58" × 0.77" (14.6mm × 19.6mm) Resolution: 500 dpi, Image size: 288 × 384 Background type: Optical Background noise: Default Crop borders: 0 x 0
Generation parameters
Seed: 1 Impression per finger: 25 (only the first one is used) Class distribution: Natural Generate pores: enabled Save ISO templates: enabled
Output settings
Output file type: WSQ

Table 1: Setting of SFinGe for the generation of the three datasets used in the experimental validation.

(a) Profile *HQNoPert* (b) Profile *Default* (c) Profile *VQandPert*

Figure 8: Fingerprints generated using SFinGe with different quality profiles .

The fingerprints generated for each quality profile are rather different, and also significantly different from those in NIST-4. Fig. 8 includes one fingerprint for each of the above mentioned quality profiles. In the remainder of this work, we refer to the datasets created with profiles HQNoPert, Default and VQandPert as SFinGe Dataset 1, 2 and 3, respectively.

5.2. Quantification of the results

In this experiment we have quantified the performance of each procedure in correctly and accurately detecting SPs. For each dataset we have created a confusion matrix which accounts for the success and fallout in SP detection. That is, given a dataset, a unique confusion matrix is completed from the confrontation of the SPs detected by the automatic method at each image and those in the ground truth.

After extracting the SPs for a fingerprint, we first compute the best-possible matching between the cores in the automatic solution to those in the ground truth, forcing a one-to-one correspondence. Each matched core in the automatic solution accounts for as True Positive (TP). Then, each unmatched core in the automatic solution and in the ground truth are tagged as False Positive (FP) and False Negative (FN), respectively. Finally, in case both the automatic solution and the ground truth contain less than two cores, the missing SPs are taken as correct predictions, and consequently are accounted for as True Negatives (TN). The process is analogous for the deltas, whose results are stored in a separate matrix. Note that each fingerprint can generate more than two hits in the confusion matrix, if SPs are both missed (FNs) and misdetected (FPs).

It should be considered that fingerprint analysis methods do not necessarily locate a SP at the exact location a human does. This is due to the discrete nature of data in an image and the scope of the semi-local analysis of the image needed to locate the SPs. Consequently, we consider some tolerance in the correspondence of SPs tagged by the automatic method to those in the ground truth. For the present experiment, this spatial tolerance is equivalent to 5% of the image diagonal for SFinGe databases and 10% of the image diagonal for NIST-4 database. The percentage difference between databases arises from the size of SFinGe and NIST-4 fingerprints (SFinGe ones are rectangular images, whereas NIST-4 ones are squared, but also the thickness of the ridges vary due to their different nature).

The results generated with the above-mentioned procedure lead to two confusion matrices for each dataset, one for cores and one for deltas. From such matrices, we have generated different scalar interpretations of the quality of the results. More specifically, we consider precision (PREC) and recall (REC), given by

$$\text{PREC} = \frac{\text{TP}}{\text{TP}+\text{FP}} \text{ and } \text{REC} = \frac{\text{TP}}{\text{TP}+\text{FN}}, \tag{7}$$

respectively. Precision and recall quantify the ability of the automatic method to obtain a reliable and complete collection of SPs, respectively. They can be combined to produce a scalar representation of the overall quality of the process. In this work we adhere to the so-called F-measure, given by

$$\text{F} = \frac{\text{PREC} \cdot \text{REC}}{0.5\,\text{PREC} + 0.5\,\text{REC}} \,. \tag{8}$$

Moreover, we also measure the percentage of fingerprints in which all the SPs (cores and deltas) have been correctly detected.

5.3. Experimental procedure

In this experiment, the results of the TSPD method have been compared with those of the Poincarè method [20], as well as to those of the one proposed by Liu [21]. The former method has been selected because it is the most used SP detection method in literature, whereas the latter method is included because it holds strong similarities to ours. Aiming at carrying out a fair comparison, the techniques used for OM computation, smoothing and segmentation are identical for each of the three SP detection methods.

Firstly, the image is divided into non-overlapping blocks of 5×5 pixels (for SFinGe databases) or 10×10 pixels (for the NIST-4 database). The different size between SFinGe and NIST-4 fingerprints makes it necessary to use different block sizes, since the ridges of the NIST-4 fingerprints are much thicker than SFinGe ones. Secondly, the image is segmented (as explained in Section 4.3) to avoid false SP detections in the ridge abnormalities occurring at the fingertip boundaries. Thirdly, to compute the gradients for the OM we use the well-known Sobel operators [60, 61], which is the most common option in fingerprint analysis. The resulting matrix is the OM, which is further regularized using a flat mask of 5×5 blocks [48]. Notice that we do not use a different size of mask for NIST-4 fingerprints since the block size used to compute the OM produces blocks with equivalent information regardless of the database.

Once the OM is generated, each of the methods needs to be customized, the details being as follows:

- *TSPD*– Regarding the templates, we need to set their size and the number of delta SP templates to use. In order to preserve the fairness of the comparison, we have considered a very basic setup, which is the baseline configuration of the method. This configuration involves only 4 templates (two for each type of SP), all of them of 5×5 blocks. In the case of the delta SP templates, we take $\alpha \in \{0, 90\}$. This is, objectively, the minimum set of templates to be used.

  As for the RSMs, we consider 9 measures, in order to shed light on the impact the RSMs have on the final results. This way, we are able to show their flexibility, allowing one to define different perceived similarities. The RREFs are constructed as in Prop. 4 from pairs of automorphisms (φ, ψ) given by

  $$\varphi(x) = x^{e_1} \text{ and } \psi(x) = \frac{x^{e_2}}{\pi^{e_2-1}}$$

  where $e_1, e_2 \in \{0.5, 1, 2\}$. This leads to 9 different pairs of automorphisms.

  Each combination of automorphisms, together with the thresholds used for the discrimination of the SPs, is shown in Table 2. Although some authors have studied the automatic determination of thresholds [62, 63], we avoid this step in order to preserve the clarity and reproducibility of the experiments. Several thresholds have been tested before selecting one that has a positive behaviour in all datasets (see Table 2).

Name	e_1	e_2	Threshold
C1	0.5	0.5	0.70
C2	0.5	1.0	0.85
C3	0.5	2.0	0.95
C4	1.0	0.5	0.55
C5	1.0	1.0	0.70
C6	1.0	2.0	0.85
C7	2.0	0.5	0.35
C8	2.0	1.0	0.60
C9	2.0	2.0	0.75

Table 2: List of configurations of the TSPD used in the experimental validation. For each of the configurations, we list the exponents (e_1, e_2) of the automorphisms used in the construction of the RREF, as well as the threshold used for SP discrimination.

- *Poincarè method*– This method consists of computing the difference between each orientation in a 3×3 neighbourhood and its clockwise successor. Those differences are further summed up to produce the Poincarè index in each block. This index takes value 0, $\frac{1}{2}$ or $-\frac{1}{2}$, indicating the absence of a SP, the presence of a core or the presence of a delta, respectively. Although other authors have used other configurations of the neighbourhood [17, 64], specially regarding its size, we maintain the widely accepted 3×3 size.
- *Liu's method*– In this method the SqOM is filtered with first order complex filters at different scales. More specifically, the large scale filters are used to discriminate the real SPs from spurious responses, while the fine scale ones determine their precise location. The threshold used for discrimination of SPs is set to 0.7 (this threshold is manually set to measure the performance of the method, as those in Table 2). Regarding the scales we consider, as in [21], filters of $s \times s$ blocks, with $s \in \{3, 5, 7, 9\}$.

5.4. Results

The results obtained for each method and dataset are listed in Tables 3-6, including:

- The values at each position of the confusion matrix (as explained in Section 5.2), namely PREC, REC and F. This information is displayed for cores and deltas separately.
- The average distance in pixels from the position at which the matched SPs where located and their position at the ground truth. This information is also listed individually for cores and deltas.
- The arithmetic mean between the F value for cores and deltas, namely Combined F (Comb. F).
- The percentage of fingerprints for which the method achieved a perfect detection (Perfect Detection Percentage, PDP). That is, the rate of fingerprints for which each method gathered the exact number of SPs, all of them being located within the tolerance ratio of 5% and 10% of the length of the image diagonal, for SFinGe and NIST-4 fingerprints, respectively.

For each dataset, the best performer at each statistic is boldfaced.

The first fact to be noticed from the results of the experiment is the great variability of performance across datasets, especially between NIST-4 and SFinGe datasets. This is due to the low quality of NIST-4 fingerprints, which often include damaged fingertips, hand-written annotations on the fingerprint margins, etc. This does not reduce the representativity of the datasets generated with SFinGe, since modern sensors for fingerprint recording produce images that are closer to those by SFinGe than to those in NIST-4. This variable behaviour has also been shown in previous studies on the topic [37].

Regarding the NIST-4 dataset, we find that TSPD-C3 is the best performer, obtaining the greatest PDP (69.90%). Although TSPD-C2 and TSPD-C8 stay close to this result, configurations such as TSPD-C7, TSPD-C9 and TSPD-C5 lead to the worst outcome. The relevance of the RSMs is clearly illustrated with these results. Besides, TSPD-C5 (when φ and ψ are the identity function) is not the best performer. From

	Template-based SP detection									Poincaré	Liu
Quant.	**C1**	**C2**	**C3**	**C4**	**C5**	**C6**	**C7**	**C8**	**C9**		
Cores											
TP	897	875	868	853	911	912	867	850	915	759	**922**
FP	273	147	132	161	266	241	292	138	261	**40**	329
FN	93	115	122	137	79	78	123	140	75	231	**68**
TN	776	889	905	878	777	799	756	902	782	**982**	721
PREC	.767	.856	.868	.841	.774	.791	.748	.860	.778	**.950**	.737
REC	.906	.884	.877	.862	.920	.921	.876	.859	.924	.767	**.931**
F	.831	.870	**.872**	.851	.841	.851	.807	.859	.845	.849	.823
Avg. dist.	14.4	**13.7**	13.8	14.9	14.2	14	15.6	14.3	13.9	13.8	16.4
Deltas											
TP	871	844	839	830	874	877	814	837	**881**	728	767
FP	340	176	159	188	393	351	335	163	426	53	**29**
FN	102	129	134	143	99	96	159	136	**92**	245	206
TN	731	884	898	879	682	722	754	897	648	1006	**1011**
PREC	.719	.827	.841	.815	.690	.714	.708	.837	.674	.932	**.964**
REC	.895	.867	.862	.853	.898	.901	.837	.860	**.905**	.748	.788
F	.798	.847	.851	.834	.780	.797	.767	.848	.773	.830	**.867**
Avg. dist.	11.2	10.4	**10.3**	11.3	11.1	10.6	12.1	11	10.7	13.5	11.5
Total											
Comb. F	.815	.859	**.861**	.843	.811	.824	.787	.853	.809	.839	.845
PDP	56.70	68.10	**69.90**	63.00	54.50	58.70	48.10	66.90	52.80	66.40	54.50

Table 3: Results gathered by each SP detection method on the NIST-4 dataset (1000 fingerprints).

this fact, we infer that choosing suitable automorphisms on the construction of the RREFs significantly improves the result of the TSPD method. In Table 3 we also observe that Liu's method obtains a PDP similar to our worst configurations (54.50%). Despite being the best method detecting cores (922), Liu's method also produces 329 false cores detections, significantly more than the Poincarè method (40) and TSPD-C3 (132). The behaviour is opposed regarding deltas, where the precision of Liu's method is very high (29 FPs and 1011 TNs), at the cost of very little recall (767 TPs)[4]. Otherwise, the method of Poincarè presents a high PDP, but is not the best performer because of its difficulties in detecting cores (231 FNs) and deltas (245 FNs). From this results, we understand that TSPD-C3 obtains the best results in general terms, showing the most equilibrated behaviour between successes (TPs, TNs) and failures (FPs, FNs).

For the SFinGe datasets, we observe that in Table 4 and Table 5, TSPD-C3 obtains the best PDP results (95.90% and 88.50% respectively), although in Table 6 the best one is TSPD-C6 (90.00%). The general trend observed in Tables 4-6 is that the TSPD method is usually able to outperform both the Poincarè and Liu's methods, although certain configurations fail to do so. A remarkable fact is the absolute absence of core FPs when using the TSPD method, which hardly ever account for more than 30 of such mistakes over 1000 fingerprints. This leads to high PREC and, as a consequence, to high F. The situation with the delta SPs is similar as it is for core SPs, but not as positive for the TSPD method. In Dataset 1, the results are similar to those of the cores, but the situation changes in Dataset 2 and is accentuated in Dataset 3.

Summing up, from the results in the present experiment we consider the TSPD method to be competitive with the contending methods. Although the TSPD method requires setting the parameters of the RSMs and thresholds, similar situations occurs with most of the SP detection methods (including Liu's method). Interestingly, the RREF leading to the best results in the TSPD method is not that constructed with the pair of automorphisms C5, indicating that non-linear modelling of dissimilarity can play a role in real applications. Specifically, the best-performing version is that using the pair of automorphisms C3, since it generally

[4]This behaviour of Liu's method is consistent with that observed by Galar *et al.* [37]

Quant.	C1	C2	C3	C4	C5	C6	C7	C8	C9	Poincaré	Liu
					Template-based SP detection						
Cores											
TP	1209	1215	1228	1157	1216	1230	1151	1165	1228	1187	**1232**
FP	1	**0**	1	**0**	**0**	**0**	16	**0**	3	**0**	58
FN	33	27	14	85	26	12	91	77	14	55	**10**
TN	757	**758**	757	**758**	**758**	**758**	742	**758**	756	**758**	702
PREC	.999	**1**	.999	**1**	**1**	**1**	.986	**1**	.998	**1**	.955
REC	.973	.978	.989	.932	.979	.990	.927	.938	.989	.956	**.992**
F	.986	.989	.994	.965	.989	**.995**	.956	.968	.993	.977	.973
Avg. dist.	7	6.7	6.4	7	6.8	6.4	7.2	6.9	6.5	**4.7**	7.2
Deltas											
TP	736	725	723	709	747	753	705	714	**754**	700	717
FP	17	1	3	13	22	17	44	8	28	**0**	12
FN	32	43	45	59	21	15	63	54	**14**	68	51
TN	1216	1231	1229	1219	1211	1215	1192	1224	1204	**1232**	1220
PREC	.977	.999	.996	.982	.971	.978	.941	.989	.964	**1**	.984
REC	.958	.944	.941	.923	.973	.980	.918	.930	**.982**	.911	.934
F	.968	.971	.968	.952	.972	**.979**	.929	.958	.973	.954	.958
Avg. dist.	4.4	4.1	**4**	4.2	4.3	4.3	4.5	4.1	4.3	4.5	4.9
Total											
Comb. F	.977	.980	.981	.958	.980	**.987**	.943	.963	.983	.966	.966
PDP	92.40	93.50	94.30	85.20	93.60	**95.90**	80.50	87.00	94.60	88.80	89.90

Table 4: Results gathered by each SP detection method on the Sfinge Dataset 1 (1000 fingerprints, profile HQNoPert).

Quant.	C1	C2	C3	C4	C5	C6	C7	C8	C9	Poincaré	Liu
					Template-based SP detection						
Cores											
TP	1295	1299	1312	1256	1305	**1319**	1254	1264	1315	1230	1317
FP	11	9	9	**6**	11	14	26	**6**	20	16	76
FN	37	33	20	76	27	**13**	78	68	17	102	15
TN	659	660	659	**663**	659	654	648	**663**	649	655	594
PREC	.992	.993	.993	**.995**	.992	.989	.980	**.995**	.985	.987	.945
REC	.972	.975	.985	.943	.980	**.990**	.941	.949	.987	.923	.989
F	.982	.984	.989	.968	.986	**.990**	.960	.972	.986	.954	.967
Avg. dist.	7.3	6.9	6.7	7.4	7.1	6.7	7.7	7.2	6.7	**4.9**	7.6
Deltas											
TP	713	705	703	685	720	**727**	670	693	**727**	651	698
FP	73	31	36	44	87	80	105	38	117	29	**18**
FN	44	52	54	72	37	**30**	87	64	**30**	106	59
TN	1172	1213	1209	1202	1159	1164	1151	1208	1132	1218	**1225**
PREC	.907	.958	.951	.940	.892	.901	.865	.948	.861	.957	**.975**
REC	.942	.931	.929	.905	.951	**.960**	.885	.915	**.960**	.860	.922
F	.924	.944	.940	.922	.921	.930	.875	.931	.908	.906	**.948**
Avg. dist.	4.8	4.6	**4.5**	4.9	4.7	4.7	5.2	4.7	4.6	4.8	5.6
Total											
Comb. F	.953	.964	**.964**	.945	.954	.960	.918	.952	.947	.930	.958
PDP	85.60	89.20	**90.00**	82.70	86.00	88.50	74.50	84.50	84.40	79.30	87.20

Table 5: Results gathered by each SP detection method on the Sfinge Dataset 2 (1000 fingerprints, profile Default).

Quant.	C1	C2	C3	C4	C5	C6	C7	C8	C9	Poincaré	Liu
	Template-based SP detection										
Cores											
TP	1140	1141	1167	1080	1148	**1176**	1080	1082	1167	1019	1172
FP	27	20	21	15	30	37	30	**12**	42	26	107
FN	50	49	23	110	42	**14**	110	108	23	171	18
TN	786	792	791	798	784	776	786	**801**	772	790	705
PREC	.977	.983	.982	.986	.975	.969	.973	**.989**	.965	.975	.916
REC	.958	.959	.981	.908	.965	**.988**	.908	.909	.981	.856	.985
F	.967	.971	**.981**	.945	.970	.979	.939	.947	.973	.912	.949
Avg. dist.	7.1	6.6	6.4	7.1	6.8	6.4	7.3	6.9	6.4	**5.1**	7.4
Deltas											
TP	656	651	649	629	663	669	620	633	**670**	612	639
FP	96	55	60	67	114	107	111	61	131	34	**13**
FN	42	47	49	69	35	29	78	65	**28**	86	59
TN	1212	1250	1245	1237	1195	1200	1197	1244	1176	1274	**1290**
PREC	.872	.922	.915	.904	.853	.862	.848	.912	.836	.947	**.980**
REC	.940	.933	.930	.901	.950	.958	.888	.907	**.960**	.877	.915
F	.905	.927	.923	.902	.899	.908	.868	.909	.894	.911	**.947**
Avg. dist.	4.9	4.6	**4.5**	4.8	4.8	4.7	5.3	4.7	4.7	4.8	5.4
Total											
Comb. F	.936	.949	**.952**	.924	.935	.944	.904	.928	.934	.912	.948
PDP	83.10	86.60	**88.50**	77.70	83.10	86.70	72.60	79.30	83.50	74.60	84.70

Table 6: Results gathered by each SP detection method on the Sfinge Dataset 3 (1000 fingerprints, profile VQandPert).

outperforms all of the other versions of the TSPD method in terms of Combined F and PDP, the SFinGe Dataset 1 being the sole exception to this fact.

It is worth noting that the TSPD methods have advantages over its counterparts other than pure performance. For example, it holds interesting visualization properties when it comes to error correction, partly derived from the simplicity of the method. Indeed, we have not exploited the potential use of multi-scale templates yet as Liu's method does.

Attending at the results obtained by the RREFs and RSMs, we can state that this extension of the REF and SM concepts considered in Fuzzy Sets theory is appropriate to deal with radial data. Even though radial data may be different from scalar data to some extent, vagueness and imprecision are inherent to both types of data in real applications. Hence, concepts from Fuzzy Sets are also interesting to deal with radial data, as we have shown in this paper. Moreover, the parametrizable construction proposed in Section 5.3 allows us to provide a flexible and configurable model, whose results can be adapted to each application.

6. Conclusions

This work has two main contributions. First, we have adapted the concepts of Restricted Equivalence Function (REF) and Similarity Measure (SM) to radial environments. The resulting operators, namely Restricted Radial Equivalence Function (RREF) and Radial Similarity Measure (RSM), capture the expected behaviour and semantics of the original operators, but at the same time embrace the cyclic nature of radial data. In both cases, we have analysed its properties and proposed construction methods. Second, we have proved the validity of the operators in a complex scenario, such as fingerprint analysis. In order to do so, we have presented a framework for Singular Point (SP) detection based on templates, which requires the use of RSMs at the template matching stage. This framework, namely Template-based Singular Point Detection (TSPD) method, shows promising results and illustrate the usefulness of RSMs for the comparison of radial data in scenarios in which imprecision and ambiguities occur.

We expect to expand the present work in two different lines of research. As for the theoretical aspects, we aim at adapting to radial data several other operators with special relevance in fuzzy set theory, e.g. aggregation operators or dissimilarity functions. Regarding the TSPD method, we intend to improve it by incorporating notions from multi-scale image processing, as well as by designing self-adapting RREFs which are able to modify their behaviour depending on the characteristics of the fingerprint image and/or the ridge orientation map.

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