

**CORRIGENDUM TO “OSCILLATORY MOTIONS IN  
RESTRICTED  $N$ -BODY PROBLEMS” [J.  
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ABSTRACT. At the beginning of paper [1] there is an error that spreads along the rest of the work and the conclusions are not correct in their present form. Precisely, in Section 2, page 783, there is a contradiction related to the scaling. In the paragraph before formula (6) it is said that  $t \rightarrow \varepsilon^3 t$  but Hamiltonian (6) is not scaled accordingly.

We have fixed the problem and, after performing due changes, the conclusions are obtained. The existence of the manifolds at infinity is guaranteed (Theorem 3.1) and the transversal intersection of them is concluded in Theorem 5.1. The applications in Section 6 are also valid after adapting them to the new version of the theorems.

1. LIST OF CHANGES

For the sake of clarity, our intention in this corrigendum has not been providing an exhaustive list of changes but only pointing out the error and its main consequences. Some misprints found in the original paper have been also fixed. We apologize for the inconveniences this mistake could have caused the reader. A full amended version of the original paper is available upon request to the authors. We appreciate the comments of Prof. M. Guardia, P. Martín and T.M. Seara who pointed out the crucial error in paper [1]. The authors have received partial support from Project 2017–88137–C2–1–P of the Ministry of Economy, Industry and Competitiveness of Spain. The facilities provided by Cinvestav – IPN (Mexico) are also acknowledged.

- In the last paragraph of page 782 change: “ $(Q, P) = e^{-it}(q, p)$ ” by “ $(Q, P) = e^{-tJ}(q, p)$ ”.

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*Key words and phrases.* Restricted  $N$ -body problem; central configurations; cometary case; symplectic scaling; invariant manifolds at infinity; McGehee’s coordinates; Melnikov function; transversality of manifolds; Smale’s horseshoe; oscillatory motions.

- In the paragraph before formula (6) eliminate  $t \rightarrow \varepsilon^3 t$ .
- Multiply by  $\varepsilon^3$  the right members of formulae (6), (7), (8), (9), (10), (16), (20), (22) except for the equation corresponding to  $\dot{s}$ , that should be  $\dot{s} = 1 - \varepsilon^3 x^4 \Theta$ . Accordingly, the expression in between (16) and (17) gets  $d\tau/dt = \varepsilon^3 x^3 / \sqrt{2}$ .
- Replace formula (12) by:

$$\Theta = \frac{1 \pm \sqrt{1 + 2\varepsilon^3 x^4 (C + \varepsilon^3 (x^2 - y^2))}}{\varepsilon^3 x^4} + O(\varepsilon^7).$$

- Substitute formula (13) by:

$$\begin{aligned} \dot{x} &= \frac{1}{\sqrt{2}} \varepsilon^3 x^3 y, \\ \dot{y} &= \frac{1}{\sqrt{2}} \varepsilon^3 x^4 + \varepsilon^3 x^6 f(x, y, \varepsilon, s, C), \\ \dot{s} &= 1 + C \varepsilon^3 x^4 + \varepsilon^6 x^6 - \varepsilon^6 x^4 y^2 + \varepsilon^6 x^8 g(x, y, \varepsilon, s, C). \end{aligned}$$

- The Poincaré map (15) reads now as:

$$\mathcal{P} : \begin{cases} x \rightarrow x + \sqrt{2} \pi \varepsilon^3 x^3 y + \varepsilon^6 x^7 y r_1(x, y, \varepsilon) \\ y \rightarrow y + \sqrt{2} \pi \varepsilon^3 x^4 (1 - C^2 x^2) + \varepsilon^6 x^6 r_2(x, y, \varepsilon) \end{cases}$$

where  $r_1$  and  $r_2$  are real analytic functions.

$$\tilde{\mathcal{P}} : \begin{cases} u \rightarrow u + \varepsilon^3 p_1(u, v) + \varepsilon^3 s_1(u, v, \varepsilon) \\ v \rightarrow v + \varepsilon^3 p_2(u, v) + \varepsilon^3 s_2(u, v, \varepsilon) \end{cases}$$

where  $p_1(u, v) = -\sqrt{2} \pi u(u+v)^3$ ,  $p_2(u, v) = \sqrt{2} \pi v(u+v)^3$  and  $s_1$  and  $s_2$  are real analytic functions starting with homogeneous polynomials of degree 6 in  $u$  and  $v$ .

- In McGehee's Theorem 3.1.:  $\delta = \delta(\varepsilon) > 0$  and  $\beta = \beta(\varepsilon) > 0$  and the manifolds vary uniform and smoothly for  $\varepsilon \geq \varepsilon_0 > 0$  small enough.
- The title of Section 4 should be "The main term of the Hamiltonian".
- In (19) change  $H_0$  by  $H_D$ .
- As  $\varepsilon = 0$  is nonsense, Section 5 starts as follows: In this section we compare equations (13) and (16). We observe that the Poincaré map of the last one has  $(0, 0)$  as a fixed point with a homoclinic orbit parametrized by (18). The fixed point is preserved by the Poincaré map of (13), however the homoclinic orbit is broken into the curves  $W_\varepsilon^s(0, 0)$  and  $W_\varepsilon^u(0, 0)$ . Indeed, the smooth dependence on  $\varepsilon$  implies that  $W_\varepsilon^s(\gamma)$  and  $W_\varepsilon^u(\gamma)$  are

parametrized by orbits of the form

$$\begin{aligned}\varphi_s(\tau, x_\varepsilon^s, y_\varepsilon^s) &= \xi(\tau - \tau_0) + \varepsilon^4 \varphi_s^1(\tau, \tau_0, \varepsilon) + \dots, \text{ for } \tau \geq \tau_0, \\ \varphi_u(\tau, x_\varepsilon^u, y_\varepsilon^u) &= \xi(\tau - \tau_0) + \varepsilon^4 \varphi_u^1(\tau, \tau_0, \varepsilon) + \dots, \text{ for } \tau \leq \tau_0,\end{aligned}$$

with  $\tau_0 = \tau(s_0)$ , the functions  $\varphi_s^1$  and  $\varphi_u^1$  are determined by the first variational equation of the  $\varepsilon^4$ -perturbation of the main term along the orbit  $\xi(\tau)$ , that is, the corrections to  $\xi(\tau - \tau_0)$  corresponding to the perturbation of the Duffing Hamiltonian of order  $\varepsilon^4$ , the next terms, i.e.  $\varphi_s^2, \varphi_u^2$ , correspond to the perturbation of order  $\varepsilon^7$ , etc. Besides,  $\varphi_s(\tau, x_\varepsilon^s, y_\varepsilon^s), \varphi_u(\tau, x_\varepsilon^u, y_\varepsilon^u)$  are taken so that  $\varphi_s(\tau_0, x_\varepsilon^s, y_\varepsilon^s), \varphi_u(\tau_0, x_\varepsilon^u, y_\varepsilon^u) \in \Sigma$ , guaranteeing the existence of a parametrization of the manifolds as above. Our purpose now is to determine the speed of breaking up of  $W_\varepsilon^s(0, 0)$  and  $W_\varepsilon^u(0, 0)$  under the perturbation.

- At the bottom of page 787 and top of page 788, it should be

$$\mathcal{M}_\nu(\tau_0, \varepsilon) = \int_{-\infty}^{\infty} \frac{dH_D}{d\tau}(\xi(\tau), \tau + \tau_0) d\tau$$

is the first term of the Melnikov function,  $x, y$  are evaluated in the unperturbed homoclinic and the derivative of  $H_D$  with respect to  $\tau$  is considered up to the perturbation of order  $\varepsilon^\nu$  in the Duffing Hamiltonian such that the function  $\mathcal{M}_\nu$  is not identically zero. We decompose

$$H_D(\varphi_s(\tau_0, x_\varepsilon^s, y_\varepsilon^s)) - H_D(\varphi_u(\tau_0, x_\varepsilon^u, y_\varepsilon^u)) = \mathcal{M}(\tau_0, \varepsilon) + \mathcal{R}(\tau_0, \varepsilon), \quad (1)$$

where

$$\mathcal{M}(\tau_0, \varepsilon) = \sum_{l=2}^{\infty} \varepsilon^{2l} \mathcal{M}_{2l}(\tau_0, \varepsilon)$$

is the so called Melnikov function and  $\mathcal{M}_{2l}$  contain the main terms of the total variation of  $H_D$ ; their computation is similar to that of  $\mathcal{M}_\nu$  above. In other words, the series contains the whole Hamiltonian  $H_\varepsilon$  with  $x, y$  replaced by their values in the Duffing equation while  $\mathcal{R}$  corresponds to the contribution to the separation provided by the higher order terms of the orbits on the stable and unstable manifolds given above. After changing from  $\tau_0$  to  $s_0$  the infinite series is interpreted as a Fourier series in  $s_0$  whose coefficients are given as asymptotic expressions of  $\varepsilon$ . In the Appendix we will provide the leading terms of the series and the corresponding estimates for the higher orders. Regarding  $\mathcal{R}$  we shall prove that it is smaller than the dominant terms of  $\mathcal{M}$  at least in regions of the phase space containing

portions of the stable and unstable manifolds of  $\gamma$  big enough where transversality is checked.

- In (23):  $s' = \sqrt{2}(\varepsilon^{-3} - \Theta x^4)x^{-3}$ .
- The  $\varepsilon^3$  factor provokes that in order to get the conclusions of Theorem 5.1 we need to consider the subsequent orders in the Legendre approximation.
- To obtain the Melnikov function we use the integral corresponding to  $H_D$  in (19), and compute the total derivative, assuming that  $\Theta_0$  in (19) is considered as  $\Theta$ , arriving at

$$\frac{dH_D}{d\tau} = \frac{\partial H_D}{\partial x}x' + \frac{\partial H_D}{\partial y}y' + \frac{\partial H_D}{\partial \Theta}\Theta'.$$

- At this point we define  $\tilde{\Theta}_0 = \Theta_0/\varepsilon$ .
- In the expression of  $s'(\tau)$  and  $s(\tau)$  in (24), replace  $\Theta_0$  by  $\tilde{\Theta}_0$ .
- We observe that the total derivative of the Duffing Hamiltonian has become:

$$\frac{dH_D}{d\tau} = \varepsilon^4 M_4 + \varepsilon^6 M_6 + O(\varepsilon^8).$$

- The integrals corresponding to  $M_4$  and  $M_6$  are

$$\mathcal{M}_4(s_0; \Theta_0, \varepsilon) = \int_{-\infty}^{\infty} \tilde{M}_4 dz = \pm \frac{2}{\Theta_0^6} \mathcal{F}_4(\Theta_0, \varepsilon)(c_2 \sin 2s_0 - c_3 \cos 2s_0), \quad (25)$$

where

$$\begin{aligned} \mathcal{F}_4(\Theta_0, \varepsilon) = & \int_{-\infty}^{\infty} \frac{1}{(z^2 + 1)^6} \left( 2(7z^4 - 12z^2 + 1) \cos\left(\frac{\tilde{\Theta}_0^3}{3} z(z^2 + 3)\right) \right. \\ & \left. + z(3z^4 - 26z^2 + 11) \sin\left(\frac{\tilde{\Theta}_0^3}{3} z(z^2 + 3)\right) \right) dz \end{aligned} \quad (26)$$

and

$$\begin{aligned} \mathcal{M}_6(s_0; \Theta_0, \varepsilon) &= \int_{-\infty}^{\infty} \tilde{M}_6 dz \\ &= \pm \frac{2}{\Theta_0^8} \left( \mathcal{F}_{6,1}(\Theta_0, \varepsilon)(d_2 \cos s_0 - d_1 \sin s_0) \right. \\ & \quad \left. + \mathcal{F}_{6,2}(\Theta_0, \varepsilon)(d_4 \cos 3s_0 - d_3 \sin 3s_0) \right), \end{aligned} \quad (28)$$

with

$$\begin{aligned}
\mathcal{F}_{6,1}(\Theta_0, \varepsilon) &= \int_{-\infty}^{\infty} \frac{1}{(z^2 + 1)^6} \left( (9z^2 - 1) \cos\left(\frac{\tilde{\Theta}_0^3}{6} z(z^2 + 3)\right) \right. \\
&\quad \left. + 2z(2z^2 - 3) \sin\left(\frac{\tilde{\Theta}_0^3}{6} z(z^2 + 3)\right) \right) dz, \\
\mathcal{F}_{6,2}(\Theta_0, \varepsilon) &= \int_{-\infty}^{\infty} \frac{1}{(z^2 + 1)^8} \left( (27z^6 - 125z^4 + 69z^2 - 3) \times \right. \\
&\quad \times \cos\left(\frac{\tilde{\Theta}_0^3}{2} z(z^2 + 3)\right) \\
&\quad \left. + 2z(2z^6 - 39z^4 + 60z^2 - 11) \sin\left(\frac{\tilde{\Theta}_0^3}{2} z(z^2 + 3)\right) \right) dz. \tag{29}
\end{aligned}$$

- It is straightforward to check that the relevant factor of the terms related to  $\cos s_0$ ,  $\sin s_0$  in  $\varepsilon^{2(2l+1)} \mathcal{M}_{2(2l+1)}$ ,  $l \geq 2$  is

$$d_2^{(l)} \cos s_0 - d_1^{(l)} \sin s_0 \quad \text{with} \quad \begin{cases} d_1^{(l)} = \sum_{j=1}^{N-1} m_j a_{j1} (a_{j1}^2 + a_{j2}^2)^l, \\ d_2^{(l)} = -\sum_{j=1}^{N-1} m_j a_{j2} (a_{j1}^2 + a_{j2}^2)^l. \end{cases}$$

- Theorems 5.1 and 5.2 come together to establish the following:

**Theorem 5.1.** *There exists  $\varepsilon_0 > 0$  such that for any  $\varepsilon$  with  $\varepsilon_0 \leq \varepsilon \ll 1$  the stable and unstable manifolds of the periodic orbit  $\gamma$  related to Hamiltonian (6) intersect transversally if one of the following situations is given: i)  $d_1$  or  $d_2$  do not vanish; ii)  $d_1 = d_2 = 0$  and there exists  $l \geq 2$  such that  $d_1^{(l)}$  or  $d_2^{(l)}$  do not vanish; iii) all the previous terms are zero and  $c_2$  or  $c_3$  do not vanish; iv) all the preceding terms are zero and there is a non-null constant term accompanying  $\cos ks_0$  or  $\sin ks_0$  for some  $k \geq 2$ .*

*Proof.* In the Appendix, using an argument of Sanders [4] we prove that  $\mathcal{R}$  can be maintained small enough in regions of the phase space that include parts of the stable and unstable manifolds of  $\gamma$  such that transversality is satisfied. More precisely, we can control the size of  $\mathcal{R}$  for all  $|\tau| \geq K > 0$  and  $K$  a constant independent of  $\varepsilon$ , keeping it smaller than the dominant terms of  $\mathcal{M}$ . Then we prove that the leading terms of the Melnikov function are those factorizing  $\cos s_0$ ,  $\sin s_0$ , then those factorizing  $\cos 2s_0$ ,  $\sin 2s_0$  and so on. Therefore, the most influential term in the Melnikov function is the one appearing in  $\varepsilon^6 \mathcal{M}_6$ , that is  $(d_2 \cos s_0 - d_1 \sin s_0) \mathcal{F}_{6,1}(\Theta_0, \varepsilon)$ . As  $\mathcal{F}_{6,1}$  does not vanish for  $\Theta_0 \neq 0$  and  $\varepsilon \geq \varepsilon_0$  small enough, we focus on the analysis of the possible zeros of the factor  $f(s_0) = d_2 \cos s_0 - d_1 \sin s_0$ . Multiple roots of  $f(s_0) = 0$  occur only when  $d_1 = d_2 = 0$ . In this case we

consider the next most important term of the Melnikov function, that is,  $\varepsilon^{10}\mathcal{M}_{10}$ . Its corresponding constant terms are  $d_1^{(2)}$ ,  $d_2^{(2)}$  and they always appear in the form  $d_2^{(2)} \cos s_0 - d_1^{(2)} \sin s_0$ , and similarly for  $l > 2$ . Thus, it is enough that there is a non-null coefficient  $d_j^{(l)}$ ,  $j = 1, 2$ ,  $l \geq 2$ , to establish the transversality condition. When all terms related to  $\cos s_0$ ,  $\sin s_0$  vanish, we take into account the next main terms, that are those of  $\varepsilon^4\mathcal{M}_4$ . They appear in the form  $c_3 \cos 2s_0 - c_2 \sin 2s_0$ , thus it is enough that  $c_2$  or  $c_3$  do not vanish to get transversality. When  $c_2 = c_3 = 0$  we need to consider the next terms related to  $\cos 2s_0$ ,  $\sin 2s_0$ , and these terms appear in  $\varepsilon^8\mathcal{M}_8$ . We continue until we identify a non-null constant term accompanying  $\cos ks_0$  or  $\sin ks_0$ , concluding the transversality of the manifolds in that case.  $\square$

## 6. APPLICATIONS

**6.1. Restricted circular 3-body problem.** Now the parameters that determine the conclusion of Theorem 5.1 are  $d_1$  and  $d_2$ .

For  $\mu = 1/2$  we have to consider higher-order terms, starting with the harmonics  $\cos s_0$ ,  $\sin s_0$ . In this case we note that the parameters  $d_1^{(l)}$ ,  $d_2^{(l)}$  vanish for all  $l \geq 2$ , then we calculate  $c_2 = 3/4$  and  $c_3 = 0$ .

**6.2. Equilateral restricted 4-body problem.** Again, the essential coefficients are  $d_i$  instead of  $c_i$ :

$$\begin{aligned} d_1 &= \frac{3}{2}(m_1 + 2m_2 - 1)(2m_1^2 + 2m_2^2 + 2m_1m_2 - m_1 - 2m_2), \\ d_2 &= -\frac{3\sqrt{3}}{2}m_1(2m_1^2 + 2m_2^2 + 2m_1m_2 - 3m_1 - 2m_2 + 1). \end{aligned}$$

When  $m_1 = m_2 = 1/3$  we check that the coefficients  $d_1^{(l)}$ ,  $d_2^{(l)}$  vanish for all  $l \geq 2$ . Moreover,  $c_2 = c_3 = 0$  and all terms accompanying  $\cos 2s_0$ ,  $\sin 2s_0$  vanish as well. Thus, we consider the leading terms of  $\cos 3s_0$ ,  $\sin 3s_0$  and get  $d_3 = 0$ ,  $d_4 = 5/(3\sqrt{3})$ .

**6.3. Restricted rhomboidal 5-body problem.** In terms of  $x$ ,  $y$ , the perturbation parameters  $d_i$  are identically zero. Moreover the coefficients  $d_1^{(l)}$ ,  $d_2^{(l)}$  also vanish for all  $l \geq 2$ , thus we need to obtain  $c_2 = -3y^2 + 6\mu(x^2 + y^2)$  and  $c_3 = 0$ . When  $c_2$  is non-zero Theorem 5.1 applies. For  $c_2 = 0$  we should go to higher orders in  $\varepsilon$ . According to

Theorem 5.1, the next terms that have to be checked are the coefficients of  $\cos 2s_0$ ,  $\sin 2s_0$  that appear in the function  $\varepsilon^8 \mathcal{M}_8$ . Specifically, we have calculated

$$\mp 10\varepsilon^8 \Theta_0^{-10} \mathcal{F}_{8,1}(\Theta_0, \varepsilon) (c_2^{(2)} \sin 2s_0 - c_3^{(2)} \cos 2s_0),$$

where  $\mathcal{F}_{8,1}$  is a function of order  $\varepsilon^{-5/2} \exp(-2\tilde{\Theta}_0^3/3)$  when  $\Theta_0 > 0$  and of order  $\varepsilon^{-1} \exp(2\tilde{\Theta}_0^3/3)$  when  $\Theta_0 < 0$ . We get  $c_3^{(2)} = 0$  and  $c_2^{(2)} = 0.20447308\dots$  for  $a = 1.32018439\dots b$  and  $c_2^{(2)} = -0.20447308\dots$  for  $a = 0.75746994\dots b$ .

#### 6.4. Collinear restricted $N$ -body problem.

6.4.1. *Collinear restricted 8-body problem.* The parameters  $d_1$ ,  $d_2$ ,  $d_1^{(l)}$  and  $d_2^{(l)}$  are zero for all  $l \geq 2$ . Then, the essential perturbation parameters that lead to the conclusion of Theorem 5.1 are  $c_2 = 1.76876487\dots$ , and  $c_3 = 0$ .

6.4.2. *Collinear restricted 11-body problem.* As in the previous case, the parameters  $d_1$ ,  $d_2$ ,  $d_1^{(l)}$  and  $d_2^{(l)}$  also vanish. Then we need to obtain the leading terms of the harmonics  $\cos 2s_0$ ,  $\sin 2s_0$ . We calculate the coefficients  $c_i$  and get their values also exactly, an approximation of them accurate up to eight decimal places being  $c_2 = 1.95579995\dots$  and  $c_3 = 0$ .

6.5. **Polygonal restricted  $N$ -body problem.** The right member of Formula (33) is multiplied by  $\varepsilon^3$ . An important feature of this Hamiltonian is that terms of order higher than  $\varepsilon^{2N+1}$  can depend on  $\cos(N-1)(t-\theta)$  but they are of smaller influence than the one of order  $\varepsilon^{2N+1}$ .

In this case the total derivative is:

$$\begin{aligned} \frac{dH_D}{d\tau} &= -\frac{1}{4}\Theta_0^2 \sinh 2\tau \sum_{j=1}^{2N-3} \varepsilon^j U_j \frac{2^{j/2+1}(j+2)}{(|\Theta_0| \cosh \tau)^{j+4}} \\ &\quad - \varepsilon^{2N-2} \frac{2^N}{\Theta_0^{2N} \cosh^{2N+1} \tau} (NV_{N-1} \sinh \tau \\ &\quad \quad \quad + W_{N-1} (N \sinh \tau \cos q(s_0, \tau) \\ &\quad \quad \quad \mp (N-1) \sin q(s_0, \tau))) \\ &\quad + O(\varepsilon^{2N-1}), \end{aligned}$$

where  $\Theta_0$  is replaced by  $\tilde{\Theta}_0$  in the expression of  $q(s_0, \tau)$ .

The Melnikov function is:

$$\mathcal{M}_{2N-2}(s_0; \Theta_0, \varepsilon) = -\frac{2^N W_{N-1}}{\Theta_0^{2N}} \int_{-\infty}^{\infty} \frac{1}{(z^2 + 1)^{N+1}} (Nz \cos \tilde{q}(s_0, z) \mp (N-1) \sin \tilde{q}(s_0, z)) dz, \quad (34)$$

where  $\Theta_0$  is replaced by  $\tilde{\Theta}_0$  in the expression of  $\tilde{q}(s_0, z)$ .

When  $N = 7$  we get

$$\mathcal{M}_{12}(s_0; \Theta_0, \varepsilon) = \pm \frac{231}{4\Theta_0^{14}} \mathcal{F}_{12}(\Theta_0, \varepsilon) \sin 6s_0$$

where

$$\mathcal{F}_{12}(\Theta_0, \varepsilon) = \int_{-\infty}^{\infty} \frac{1}{(z^2 + 1)^{14}} \left( p_1(z) \cos(\tilde{\Theta}_0^3 z(z^2 + 3)) + p_2(z) \sin(\tilde{\Theta}_0^3 z(z^2 + 3)) \right) dz,$$

with

$$\begin{aligned} p_1(z) &= 2(45z^{12} - 968z^{10} + 4257z^8 - 5544z^6 + 2255z^4 - 240z^2 + 3), \\ p_2(z) &= z(7z^{12} - 534z^{10} + 4785z^8 - 11220z^6 + 8217z^4 - 1782z^2 + 79). \end{aligned}$$

For  $N = 8$  the corresponding Melnikov function reads as

$$\mathcal{M}_{14}(s_0; \Theta_0, \varepsilon) = \pm \frac{429}{\Theta_0^{16}} \mathcal{F}_{14}(\Theta_0, \varepsilon) \sin 7s_0$$

where

$$\mathcal{F}_{14}(\Theta_0, \varepsilon) = \int_{-\infty}^{\infty} \frac{-1}{(z^2 + 1)^{16}} \left( p_3(z) \cos(\frac{7}{6}\tilde{\Theta}_0^3 z(z^2 + 3)) + p_4(z) \sin(\frac{7}{6}\tilde{\Theta}_0^3 z(z^2 + 3)) \right) dz,$$

with

$$\begin{aligned} p_3(z) &= 119z^{14} - 3549z^{12} + 23023z^{10} - 48477z^8 + 37037z^6 - 9919z^4 \\ &\quad + 749z^2 - 7, \\ p_4(z) &= 2z(4z^{14} - 413z^{12} + 5278z^{10} - 19019z^8 + 24024z^6 - 11011z^4 \\ &\quad + 1638z^2 - 53). \end{aligned}$$

The graphs of  $\mathcal{F}_{12}$ ,  $\mathcal{F}_{14}$  are given in Fig. 2.

The asymptotic estimates of the Appendix hold provided  $\varepsilon$  is small enough so that  $\mathcal{F}_{12}$  and  $\mathcal{F}_{14}$  do not vanish.

Examining (34) carefully, it is not difficult to infer that the integrals appearing in the Melnikov function  $\mathcal{M}_{2N-2}$  are of the same type as  $\mathcal{F}_4$ ,  $\mathcal{F}_{6,1}$  and so on. We also take into account that the smallest harmonic



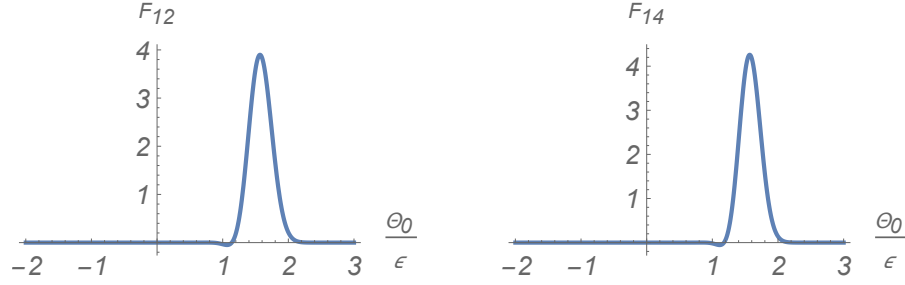


FIGURE 2. On the left: graph of  $\mathcal{F}_{12}$ ; on the right, graph of  $\mathcal{F}_{14}$ .

appearing in the Melnikov function is  $\sin(N-1)s_0$ , thus we end up with an expression like

$$\mathcal{M}_{2N-2}(s_0; \Theta_0, \varepsilon) = \pm \frac{K}{\Theta_0^{2N}} \mathcal{F}_{2N-2}(\Theta_0, \varepsilon) \sin(N-1)s_0,$$

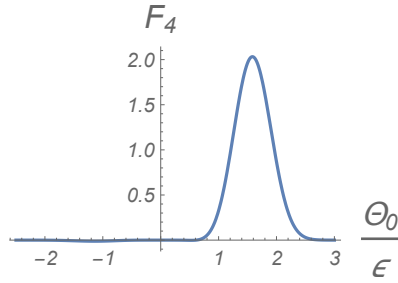
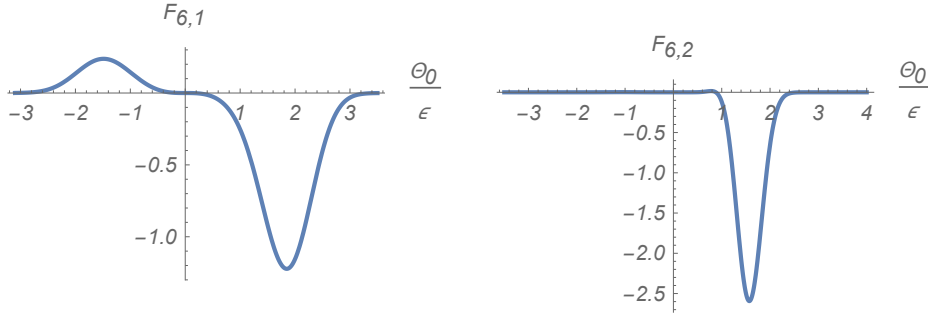
with  $K$  a non-null constant. Applying the estimates provided in the Appendix and taking  $\varepsilon$  small enough, we conclude that  $\varepsilon^{2N-2} \mathcal{M}_{2N-2}$  behaves like  $\varepsilon^{-N-1/2} \exp(-(N-1)\tilde{\Theta}_0^3/3)(1+O(\varepsilon))$  when  $\Theta_0 > 0$  and  $\varepsilon^{-N+1} \exp((N-1)\tilde{\Theta}_0^3/3)(1+O(\varepsilon))$  for negative  $\Theta_0$ . Thus, we can apply Theorem 5.1, achieving the transversality of the manifolds of  $\gamma$  in the polygonal restricted  $N$ -body problem for all  $N \geq 4$ .

Qualitative study of functions  $\mathcal{F}_4(\Theta_0, \varepsilon)$ ,  $\mathcal{F}_{6,1}(\Theta_0, \varepsilon)$  and  $\mathcal{F}_{6,2}(\Theta_0, \varepsilon)$

The graph of function  $\mathcal{F}_4$  is given in Fig. 3. The function has two zeroes (instead of one), namely  $\tilde{\Theta}'_0 = 0$  and  $\tilde{\Theta}^*_0 = 0.61078210\dots$ , and  $\mathcal{F}_4(\tilde{\Theta}_0) < 0$  for  $\tilde{\Theta}_0 < \tilde{\Theta}^*_0$  (excepting at  $\tilde{\Theta}'_0$ ), while  $\mathcal{F}_4(\tilde{\Theta}_0) > 0$  for  $\tilde{\Theta}_0 > \tilde{\Theta}^*_0$ .

The graphs of functions  $\mathcal{F}_{6,1}$  and  $\mathcal{F}_{6,2}$  can be seen in Fig. 4. Function  $\mathcal{F}_{6,1} = 0$  has its unique root at  $\tilde{\Theta}_0 = 0$  whereas the roots of  $\mathcal{F}_{6,2} = 0$  occur at  $\tilde{\Theta}_0 = 0$ ,  $\tilde{\Theta}_0 = 0.15745028\dots$ ,  $\tilde{\Theta}_0 = 0.87685728\dots$ . Besides,  $\mathcal{F}_{6,1} > 0$  when  $\tilde{\Theta}_0 < 0$ ,  $\mathcal{F}_{6,1} < 0$  when  $\tilde{\Theta}_0 > 0$  while  $\mathcal{F}_{6,2} > 0$  when  $\tilde{\Theta}_0 < 0$  and  $0.15745028\dots < \tilde{\Theta}_0 < 0.87685728\dots$ ,  $\mathcal{F}_{6,2} < 0$  when  $0 < \tilde{\Theta}_0 < 0.15745028\dots$  and  $\tilde{\Theta}_0 > 0.87685728\dots$ .

Due to the highly oscillatory character of the integrals, an asymptotic analysis of  $\mathcal{F}_4$ ,  $\mathcal{F}_{6,1}$ ,  $\mathcal{F}_{6,2}$  and other related functions involved in the Melnikov functions obtained in Section 5 is due.

FIGURE 3. The graph of the function  $\mathcal{F}_4(\Theta_0, \varepsilon)$ .FIGURE 4. The graphs of the functions  $\mathcal{F}_{6,1}(\Theta_0, \varepsilon)$  and  $\mathcal{F}_{6,2}(\Theta_0, \varepsilon)$ .

Following [3] we introduce the improper integrals

$$I_k(\delta) = \int_0^\infty \frac{\cos(\delta(z + z^3/3))}{(1+z^2)^k} dz, \quad J_k(\delta) = \int_0^\infty \frac{z \sin(\delta(z + z^3/3))}{(1+z^2)^k} dz.$$

In [2] it is proved that  $J_k$  can be written in terms of  $I_k$  through

$$J_{k+2}(\delta) = \frac{\delta}{2(k+1)} I_k(\delta),$$

whereas for  $\delta > 0$  big enough, the following estimates hold:

$$\begin{aligned} I_{2n-1}(\delta) &= \exp(-2\delta/3) \left( \frac{\pi}{2^{n+1}(2n-2)!!} \delta^{n-1} + O(\delta^{n-3/2}) \right), \\ I_{2n}(\delta) &= \exp(-2\delta/3) \left( \frac{\sqrt{\pi}}{2^{n+1}(2n-1)!!} \delta^{n-1/2} + O(\delta^{n-1}) \right). \end{aligned} \quad (35)$$

The functions  $\mathcal{F}_4$ ,  $\mathcal{F}_{6,1}$ ,  $\mathcal{F}_{6,2}$  as well as the rest of the functions appearing in  $\mathcal{M}_{2k}$  with  $k > 3$  can be cast in terms of  $I_k$  and  $J_k$ , after performing a partial fraction decomposition.

We start with  $\Theta_0 > 0$  and take  $\varepsilon$  small enough with  $\varepsilon \geq \varepsilon_0 > 0$  to avoid the possible zeroes of the functions  $\mathcal{F}_4$ ,  $\mathcal{F}_{6,1}$ , etc. Concerning

$\mathcal{M}_4$  we apply the estimates (35) to  $\mathcal{F}_4$ , and after arranging the function conveniently, we conclude that

$$\varepsilon^4 \mathcal{M}_4 = \frac{4\sqrt{\pi}}{3} \varepsilon^{-7/2} \Theta_0^{3/2} e^{-\frac{2}{3}\tilde{\Theta}_0^3} (c_2 \sin 2s_0 - c_3 \cos 2s_0)(1 + O(\varepsilon)).$$

Similarly, for  $\mathcal{M}_6$  we get

$$\begin{aligned} \varepsilon^6 \mathcal{M}_6 &= -\frac{\sqrt{\pi}}{12\sqrt{2}} \varepsilon^{-3/2} \Theta_0^{-1/2} e^{-\frac{1}{3}\tilde{\Theta}_0^3} (d_2 \cos s_0 - d_1 \sin s_0)(1 + O(\varepsilon)) \\ &\quad - \frac{9\sqrt{3\pi}}{5\sqrt{2}} \varepsilon^{-9/2} \Theta_0^{5/2} e^{-\tilde{\Theta}_0^3} (d_4 \cos 3s_0 - d_3 \sin 3s_0)(1 + O(\varepsilon)). \end{aligned}$$

From the above calculations it is clear that for  $\varepsilon$  small enough the most important terms are those related to  $\exp(-\tilde{\Theta}_0^3/3)$ , then those related to  $\exp(-2\tilde{\Theta}_0^3/3)$ , next those with  $\exp(-\tilde{\Theta}_0^3)$  and so on. Furthermore, we observe that the terms with asymptotic estimates having the factor  $\exp(-\tilde{\Theta}_0^3/3)$  correspond to the harmonics  $\cos s_0$ ,  $\sin s_0$ , and in general, the asymptotic expressions with  $\exp(-k\tilde{\Theta}_0^3/3)$  are related to the harmonics  $\cos ks_0$ ,  $\sin ks_0$ . In addition to this, for  $k \geq 2$  the terms of  $\cos ks_0$ ,  $\sin ks_0$  are of order  $\varepsilon^{-k-3/2} \exp(-k\tilde{\Theta}_0^3/3)(1 + O(\varepsilon))$  while for  $k = 1$  the terms of  $\cos s_0$ ,  $\sin s_0$  have the estimate  $\varepsilon^{-3/2} \exp(-\tilde{\Theta}_0^3/3)(1 + O(\varepsilon))$ . This in turn implies that the leading terms in the Melnikov function are the ones depending on the coefficients  $d_1$ ,  $d_2$ , the next ones those with estimate  $\varepsilon^{-1/2} \exp(-\tilde{\Theta}_0^3/3)$  to which follows the rest of terms with harmonics  $\cos s_0$ ,  $\sin s_0$ . Next we consider the main terms factorized by  $\cos 2s_0$ ,  $\sin 2s_0$ , they are the ones depending on  $c_2$ ,  $c_3$ . We continue with the higher order terms and so on. The Melnikov function becomes the formal Fourier series

$$\mathcal{M}(s_0; \Theta_0, \varepsilon) = \sum_{k=1}^{\infty} \alpha_k(\varepsilon) \cos ks_0 + \beta_k(\varepsilon) \sin ks_0$$

with

$$\begin{aligned} \alpha_1(\varepsilon) &= \varepsilon^{-3/2} e^{-\frac{1}{3}\tilde{\Theta}_0^3} (A_1 + O(\varepsilon)), & \beta_1(\varepsilon) &= \varepsilon^{-3/2} e^{-\frac{1}{3}\tilde{\Theta}_0^3} (B_1 + O(\varepsilon)), \\ \alpha_k(\varepsilon) &= \varepsilon^{-k-3/2} e^{-\frac{k}{3}\tilde{\Theta}_0^3} (A_k + O(\varepsilon)), & \beta_k(\varepsilon) &= \varepsilon^{-k-3/2} e^{-\frac{k}{3}\tilde{\Theta}_0^3} (B_k + O(\varepsilon)), \end{aligned}$$

for  $k \geq 2$  and

$$\begin{aligned} A_1 &= -\frac{\sqrt{\pi}}{12\sqrt{2}} \Theta_0^{-1/2} d_2, & B_1 &= \frac{\sqrt{\pi}}{12\sqrt{2}} \Theta_0^{-1/2} d_1, \\ A_2 &= \frac{4\sqrt{\pi}}{3} \Theta_0^{3/2} c_3, & B_2 &= -\frac{4\sqrt{\pi}}{3} \Theta_0^{3/2} c_2, \end{aligned}$$

$$A_k, B_k, k \geq 3 \text{ constants independent of } \varepsilon.$$

Regarding the estimate of  $\mathcal{R}$  we use the ideas of Sanders [4] for the case of exponentially small estimates. Given a vector field of the form  $\dot{x} = f_0(x) + \varepsilon f_1(x, t, \varepsilon)$  with  $x \in D \subset \mathbb{R}^2$  and  $\varepsilon$  a small parameter, he

defines the Melnikov integral  $\Delta_\varepsilon(t, x) = \varepsilon^{-1} f_0(x_0^{s,u}(t)) \wedge (x_\varepsilon^u(t) - x_\varepsilon^s(t))$  where  $\wedge$  is the wedge product in  $\mathbb{R}^2$ ,  $x_0^{s,u}$  refers to the parametrization of the stable and unstable manifolds of the unperturbed system and  $x_\varepsilon^u$ ,  $x_\varepsilon^s$  denote solutions on the unstable and stable manifolds, respectively. After some assumptions on the smoothness of the vector field, and a lemma regarding the relationships between the unperturbed and perturbed manifolds, Sanders arrives at an expression of the form

$$\Delta_\varepsilon(t, x) = \Delta_0(x) + O(\varepsilon(1 + e^{-\mu|t|})^2 \min\{1, e^{-\mu|t|}\}),$$

where  $\Delta_0$  stands for the usual Melnikov function and  $\mu$  is the Lipschitz constant associated to  $f_0$ . Systems of the type  $\dot{x} = \varepsilon f(x, t, \varepsilon)$ , after applying averaging and rescaling time by  $\tau = \varepsilon t$ , are transformed into  $y' = dy/d\tau = f_0(y) + \varepsilon f_1(y, \tau/\varepsilon, \varepsilon)$ , thus admitting the estimate  $O(\varepsilon(1 + \exp(-\mu|\tau|/\varepsilon))^2 \min\{1, \exp(-\mu|\tau|/\varepsilon)\}) = O(\varepsilon \exp(-\mu|\tau|/\varepsilon))$  for  $\tau \neq 0$ .

We adopt Sanders' point of view in our setting as follows. First, we notice that the hypotheses on the Hamiltonian and the existence of the manifolds are fulfilled. Then we realize that  $t$  and  $\tau$  are related through the change of time, assuming that  $x(\tau) = \sqrt{2}/\Theta_0 \operatorname{sech} \tau + O(\varepsilon^3)$  we get  $t = \Theta_0^3 \tau / (2\varepsilon^3) + O(1)$ . The Lipschitz constant of  $H_D$  along the homoclinic  $\xi(\tau)$  is calculated, obtaining  $\mu = \sqrt{2}/\Theta_0$ . Finally, after adjusting the factor  $\varepsilon^4$  in the whole expression of  $\Delta_\varepsilon$  so that we identify  $\Delta_0$  with  $\mathcal{M}$ , we get an upper bound on  $\mathcal{R}$  as  $O(\exp(-\Theta_0^2 |\tau| / (\sqrt{2}\varepsilon^3)))$ . To control the size of  $\mathcal{R}$  we compare the estimate with the dominant term of  $\mathcal{M}$ . When  $d_1$  or  $d_2$  are not zero, due to the presence of the factor  $\varepsilon^{-3/2}$  in  $\alpha_1$ ,  $\beta_1$ , it is enough that  $\exp(-\tilde{\Theta}_0^3/3) > \exp(-\Theta_0^2 |\tau| / (\sqrt{2}\varepsilon^3))$  from where it is deduced that  $|\tau| \geq \frac{\sqrt{2}}{3} \Theta_0$ , which is true in big portions of the manifolds as  $\Theta_0$  is of moderate size. Next, the transversality condition is verified in the part of phase space where this restriction on  $\tau$  holds, but it implies that transversality is satisfied for every  $\tau$  as this property is preserved through diffeomorphisms. When  $d_1 = d_2 = 0$  one compares  $\exp(-k\tilde{\Theta}_0^3/3)$  with  $\exp(-\Theta_0^2 |\tau| / (\sqrt{2}\varepsilon^3))$ , starting with  $k = 2$ .

When  $\Theta_0 < 0$  we notice that to apply the estimates given above we should consider the integrals  $I_k, J_k$  with  $\delta < 0$ . Then we realise that  $I_k(\delta) = I_k(-\delta)$ ,  $J_k(\delta) = -J_k(-\delta)$  and  $J_{k+2}(\delta) = \delta/(2(k+1))I_k(-\delta)$  and the estimates for  $I_k$  given in (35) apply replacing  $\delta$  by  $-\delta$  in the expressions. Proceeding similarly to the case  $\Theta_0 > 0$  we arrive at

$$\begin{aligned} \varepsilon^4 \mathcal{M}_4 &= \frac{5\pi}{8} \varepsilon^{-2} e^{\frac{2}{3}\tilde{\Theta}_0^3} (c_2 \sin 2s_0 - c_3 \cos 2s_0) (1 + O(\varepsilon)), \\ \varepsilon^6 \mathcal{M}_6 &= -\frac{5\pi}{128} \Theta_0^{-2} e^{\frac{1}{3}\tilde{\Theta}_0^3} (d_2 \cos s_0 - d_1 \sin s_0) (1 + O(\varepsilon)) \\ &\quad + \frac{63\pi}{64} \varepsilon^{-3} \Theta_0 e^{\tilde{\Theta}_0^3} (d_4 \cos 3s_0 - d_3 \sin 3s_0) (1 + O(\varepsilon)). \end{aligned}$$

Reasoning as in the positive case we realize that the main term in the Melnikov function is that of  $\varepsilon^6 \mathcal{M}_6$  having coefficients  $d_1, d_2$ , then the rest of terms related to  $\cos s_0, \sin s_0$ , next the term associated to  $\cos 2s_0, \sin 2s_0$  beginning with those whose coefficients are  $c_2, c_3$ , and so on. Moreover, when  $k \geq 2$  the terms of  $\cos ks_0, \sin ks_0$  behave like  $\varepsilon^{-k} \exp(k\tilde{\Theta}_0^3/3)(1 + O(\varepsilon))$ , thus we obtain analogous results to the case  $\Theta$  positive though with estimates of different order in  $\varepsilon$ .

The estimate analysis of  $\mathcal{R}$  is alike the procedure for  $\Theta_0 > 0$ .

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