

Topologies for semicontinuous multi-utilities

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Abstract

The present paper gives a topological solution to representability problems related to multi-utility, in the field of Decision Theory. Necessary and sufficient topologies for the existence of a semicontinuous and finite Richter-Peleg multi-utility for a preorder are studied. It is well known that, given a preorder on a topological space, if there is a lower (upper) semicontinuous Richter-Peleg multi-utility, then the topology of the space must be finer than the Upper (resp. Lower) topology. However, this condition fails to be sufficient. Instead of search for properties that must be satisfied by the preorder, we study finer topologies which are necessary or/and sufficient for the existence of semicontinuous representations. We prove that Scott topology must be contained in the topology of the space in case there exists a finite lower semicontinuous Richter-Peleg multi-utility. However, the existence of this representation cannot be guaranteed. A sufficient condition is given by means of Alexandroff's topology, for that, we prove that *more order* implies *less Alexandroff's topology*, as well as the converse. Finally, the paper is implemented with a topological study of the maximal elements.

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1 Introduction and motivation

In the present paper we study the existence of *lower-semicontinuous Richter-Peleg multi-utilities* for preorders on topological spaces. The existence of Richter-Peleg multi-utilities has been recently studied by Alcantud et al. [1] (see the introduction in this paper in order to find the motivations for adopting such a representation).

It was already observed that this kind of representations not always exists for preorders endowed with the *Upper topology* τ_u (see Theorem 3.1 in Alcantud et al. [1]). On the other hand, it is well known that the weak lower contour sets of the preorder have to be closed in the more general case when there exists a lower-semicontinuous multi-utility (see e.g. Proposition 2.1 in Bosi and Herden [5]). Negative conditions for the existence of a finite (Richter-Peleg) continuous multi-utility representations were presented in Alcantud et al. [1] and Kaminski [13].

The goal of this paper is to identify some other topologies related to the preorder, with respect to which it is possible to characterize the lower-semicontinuous Richter-Peleg multi-utility representability. Therefore, these topologies have to be finer than the *Upper topology*. Scott topologies have been used in computing in order to characterize the functions between lattices (in particular, dcpo-s) that preserve suprema of directed sets [7, 12]. In any case, the present paper study the more general case of preorders and their finite lower-semicontinuous Richter-Peleg multi-utilities, so we do not assume the existence of suprema and we search for a family of functions that fully characterize the order structure (i.e. a multi-utility instead of a single utility function).

In this line, we prove that if there exists a finite lower-semicontinuous Richter-Peleg multi-utility for a given preorder, then the topology of the space refines the Scott topology. Thus, we achieve a significant necessary condition for the existence of the desired representation: from now, if we search for finite lower semicontinuous Richter-Peleg multi-utilities we should start from a topological space that refines the Scott topology, and not the Upper.

Furthermore, we also present an example in order to show that this necessary condition is not sufficient for the general case. Hence, we continue in the study of the adequate topologies to guarantee the existence of the lower-semicontinuous Richter-Peleg multi-utility. For that, we prove that there always exists this kind of representation when the preorder is endowed with a topology that is finer than the Alexandroff topology. Thus, we achieved a sufficient condition. For this purpose, *order* and *topology* are interacted, and we prove that *more order* implies *less Alexandroff's topology*, as well as the converse.

Throughout the paper we also focus on some other results related to the topic in order to interact with our present results. For example, several authors have work under the hypothesis in which any linear extension of the preorder is lower-semicontinuous (see for example Mashburn [14] on *pliable spaces*). From a topological point of view, this is strongly related to the Alexandroff topology. Therefore, this kind of topologies cannot be considered strange at all. Moreover, this kind of topologies are quasimetrizable (see [17], for instance) and that could be an interesting tool to complement the lack of a metric. Finally, a topological study of maximal elements is included.

The structure of the paper goes as follows. Section 2 contains the notation and the preliminaries. Section 3 presents necessary conditions for the existence of a (finite) lower semicontinuous multi-utility representation of a preorder. Section 4 is devoted to the sufficient conditions for the existence of such representations of preorders. Finally, in Section 5, a topological study of maximal elements is given. Some final comments and conclusions end the paper.

2 Notation and preliminaries

From now on X will stand for a nonempty set.

Definition 2.1. A preorder \preceq on X is a binary relation on X which is reflexive and transitive. An antisymmetric preorder is said to be an order or a partial order. A total

preorder \preceq on a set X is a preorder such that if $x, y \in X$ then $[x \preceq y] \vee [y \preceq x]$. A total order is also called a *linear order*, and a totally ordered set (X, \preceq) is also said to be a *chain*.

If \preceq is a preorder on X , then as usual we denote the associated *asymmetric* relation by \prec and the associated *equivalence* relation by \sim and these are defined, respectively, by $[x \prec y \iff (x \preceq y) \wedge \neg(y \preceq x)]$ and $[x \sim y \iff (x \preceq y) \wedge (y \preceq x)]$. If it holds that $\neg(x \preceq y)$ as well as $\neg(y \preceq x)$ (that is, they are *incomparable*) we shall denote it by $x \bowtie y$. The asymmetric part of a linear order (respectively, of a total preorder) is said to be a *strict linear order* (respectively, a strict total preorder).

Next Definition 2.2 introduces the notion of representability for preorders.

Definition 2.2. A total preorder \preceq on X is called representable if there is a real-valued function $u: X \rightarrow \mathbb{R}$ that is order-preserving, so that, for every $x, y \in X$, it holds that $[x \preceq y \iff u(x) \leq u(y)]$. The map u is said to be a *utility function*.

In case of not necessarily total preorder, a real-valued function $u: X \rightarrow \mathbb{R}$ is said to be a *Richter-Peleg representation* if it satisfies that $[x \preceq y \implies u(x) \leq u(y)]$ (i.e. u is *isotonic*) as well as $[x \prec y \implies u(x) < u(y)]$. In case of a total preorder, this definition coincides with the previous one.

A (not necessarily total) preorder \preceq on a set X is said to have a *multi-utility representation* if there exists a family \mathcal{U} of isotonic real functions such that for all points $x, y \in X$ the equivalence

$$x \preceq y \iff \forall u \in \mathcal{U} (u(x) \leq u(y)) \tag{1}$$

holds.

A particular case of the previous representation is the so called *Richter-Peleg multi-utility representation* ([15]), which holds when all the functions of the family \mathcal{U} in representation (1) are *order-preserving* with respect to the preorder \preceq (i.e., for all $u \in$

\mathcal{U} , and $x, y \in X$, $x \prec y$ implies that $u(x) < u(y)$). It is well known that in this case the family \mathcal{U} also represents the *strict part* \prec of \succsim , in the sense that, for all $x, y \in X$, $x \prec y$ if and only if $u(x) < u(y)$ for all $u \in \mathcal{U}$.

It is known that a multi-utility representation exists for every not necessarily total preorder \succsim on X (see Evren and Ok Proposition 1 in [10]). However, there are preorders that fails to be Richter-Peleg multi-utility representable (see [1], see also [4]).

Definition 2.3. A total preorder \succsim defined on X is said to be perfectly separable if there exists a countable subset $D \subseteq X$ such that for every $x, y \in X$ with $x \prec y$ there exists $d \in D$ such that $x \succsim d \succsim y$.

A preorder \succsim is said to be *order separable in the sense of Debreu* (see [6]) if there exists a countable subset $D \subseteq X$ such that for every $x, y \in X$ with $x \prec y$ there exists $d \in D$ such that $x \succsim d \succsim y$.

Theorem 2.4 on representability for total preorders is well known [6].

Theorem 2.4. A total preorder \succsim on X is representable if and only if it is perfectly separable.

Definition 2.5. Let \prec denote an asymmetric binary relation on (X, τ) . Given $a \in X$ the sets $L_{\prec}(a) = \{t \in X : t \prec a\}$ and $U_{\prec}(a) = \{t \in X : a \prec t\}$ are called, respectively, the strict lower and upper contours of a relative to \prec . We say that \prec is τ -continuous (or just *continuous*) if for each $a \in X$ the sets $L_{\prec}(a)$ and $U_{\prec}(a)$ are τ -open.

We will denote the *order topology* generated by \prec as τ_{\prec} , and it is defined by means of the subbasis provided by the lower and upper contour sets.

Let \succsim denote a reflexive binary relation on (X, τ) . Given $a \in X$ the sets $L_{\succsim}(a) = \{t \in X : t \succsim a\}$ and $U_{\succsim}(a) = \{t \in X : a \succsim t\}$ are called, respectively, the *weak lower and upper contours* of a relative to \succsim . We say that \succsim is τ -lower semicontinuous (τ -upper semicontinuous) if for each $a \in X$ the sets $L_{\succsim}(a)$ (resp. $U_{\succsim}(a)$) are τ -closed.

Definition 2.6. A preorder \succsim on a set X is said to be near-complete if every subset of X consisting of mutually incomparable elements is finite.

The following result was presented by Evren and Ok [10, Theorem 3].

Theorem 2.7. Let X be a topological space with a countable basis. If \succsim is a near-complete upper (lower) semicontinuous preorder on X , then it has an upper (lower) semicontinuous finite multi-utility representation.

The theorem above presents a sufficient condition for the existence of an upper (lower) semicontinuous finite multi-utility; however, there is not a similar result for the case of an upper (lower) Richter-Peleg multi-utility.

Definition 2.8. Let \succsim be a preorder defined on X . The Upper topology τ_U is obtained by choosing the closed sets to be the weak lower contour sets (as well as their finite unions and infinite intersections).

Definition 2.9. We say that $f: (X, \tau) \rightarrow \mathbb{R}$ is lower semi-continuous at x_0 if for every $\varepsilon > 0$ there exists a neighborhood U of x_0 such that $f(x) > f(x_0) - \varepsilon$ for all $x \in U$.

Remark 2.10. It is known that $f: (X, \tau) \rightarrow \mathbb{R}$ is lower semi-continuous at x_0 if and only if f is continuous with respect to the Upper topology on the real line associated with the natural (total) order \leq on \mathbb{R} (i.e., $f: (X, \tau) \rightarrow (\mathbb{R}, \tau_u)$ is continuous). Equivalently, f is lower semi-continuous at x_0 if $f^{-1}((-\infty, f(x_0)])$ is closed. This can be expressed too as

$$\liminf_{x \rightarrow x_0} f(x) \geq f(x_0).$$

Definition 2.11. Let \succsim be a binary relation on X . A subset $G \subseteq X$ is said to be an up-set if $\forall x, y \in X$, $x \in G$ and $x \succsim y$ implies that $y \in G$.

Dually, a subset $G \subseteq X$ is said to be a down-set if $\forall x, y \in X$, $x \in G$ and $y \succsim x$ implies that $y \in G$.

Theorem 2.12. *A total preorder \lesssim on a topological space is representable through a continuous utility function if and only if \lesssim is perfectly separable and τ -continuous.*

Theorem 2.12 on continuous representability is also well-known in this literature [8, 9, 6].

Corollary 2.13. *A preorder \lesssim on (X, τ) is τ -lower semicontinuous if and only if the topology τ is finer than the Upper topology τ_u .*

Definition 2.14. *A topological space (X, τ) is an Alexandroff space [19] if arbitrary intersections of open sets are open.*

Now, we include another definition of Alexandroff spaces but now related to a preordered set.

Definition 2.15. *Let (X, \lesssim) be a preordered set. The corresponding Alexandroff's topology τ_A related to \lesssim on X is defined by choosing the open sets to be the up-sets:*

$$\tau_A = \{ G \subseteq X : \forall x, y \in X (x \in G \wedge x \lesssim y) \Rightarrow y \in G \}$$

The corresponding closed sets are the down-sets:

$$\{ S \subseteq X : \forall x, y \in X (x \in S \wedge y \lesssim x) \Rightarrow y \in S \}$$

Remark 2.16. *This relation between topology and preorder is made by means of the specialisation preorder [2].*

Given any topological space (X, τ) , the specialization preorder \lesssim on X is defined by

$$x \lesssim y \iff x \in \bar{y}.$$

Dually, starting from a preorder on X , a topology arise through the implication before. This topology is an Alexandroff topology as regards Definition 2.14, and it coincides with the topology of Definition 2.15.[2]

Definition 2.17. Let (X, \preceq) be an ordered set. The Scott topology τ_S on X is defined by choosing the open sets to be the up-sets that satisfy the following condition (for any directed set $(x_i)_{i \in I} \subseteq X$):

$$\sup(x_i)_{i \in I} = s \in U \Rightarrow (x_i)_{i \in I} \cap U \neq \emptyset.$$

The definition is analogous in the case of preordered sets, taking into account that now the supremum is unique (in case it exists) except indifference (i.e. equivalence). Hence, equivalent elements are topologically indistinguishable (i.e. they share the same open neighborhoods, see [3]) in the Scott topology.

It is straightforward to see that the Upper topology is contained in the Alexandroff topology. The following example shows that this inclusion may be strict.

Example 2.18. Let X be the infinite union $\bigcup_{n \in \mathbb{N}} X_n$ where $X_n = [0, +\infty)$ for each $n \in \mathbb{N}$ (we denote X_n by $[0, \infty)_n$ and by x_n any element of X_n). Now we define the preorder \sqsubseteq on X by $x \sqsubseteq y$ if and only if $x, y \in X_k$ and $x \leq y$, with $k \in \mathbb{N}$. Hence, x and y are incomparable for any $x \in X_m$ and $y \in X_n$, for any $n \neq m$.

On this preordered set, notice that the subset $A = [1, +\infty)_1 \cup X_2 \cup \dots \cup X_n \cup \dots$ (since it is an up-set) is open in the Alexandroff topology, whereas it fails to be open in the Upper topology (there is no open neighbourhood of the element 1_1 contained in A). Notice too that the subset $B = \bigcup_{n \in \mathbb{N}} (0, +\infty)_n$, for example, is also open in the Alexandroff topology, whereas it fails to be open in the Upper topology.

The reason that makes A fail to be open in the Upper topology is that not every up-set can be open, only those of the kind $X \setminus \bigcup_{i=1}^n L_{\sqsubseteq}(a^i)$ ($a^i \in X$) are open in the Upper topology. So, in particular, notice that $A|_{X_1} = [1, +\infty)$ is not open even if we are just working on the set $X_1 = [0, +\infty)$ with the usual order \leq .

On the other hand, the reason that makes B fail to be open in the Upper topology is that the arbitrary intersection of open sets fails to be open (this property is satisfied

by the Alexandroff topology, but not by the Upper). Thus, for any open set $U \in \tau_u$ there are an infinite number of bottom elements 0_k .

3 Necessary topology: Scott

It is known that the τ -lower semicontinuity of the preorder is not enough in order to warrant the existence of a (lower-semicontinuous) Richter-Peleg multi-utility (see e.g. Alcantud et al. [1]). In other words, the Upper topology is not enough in order to guarantee the existence of a lower-semicontinuous representation.

The following example shows that even dealing with a preorder which is semi-continuous multi-utility representable finitely, the corresponding finite semicontinuous Richter-Peleg multi-utility representation fails to exist.

Example 3.1. Let $\mathbb{R}_1, \mathbb{R}_2$ be two copies of the real line \mathbb{R} , and consider the set $X = \mathbb{R}_1 \cup \mathbb{R}_2$ endowed with the Upper topology associated to the partial order \lesssim defined as follows

$$x \lesssim y \iff \{x \leq y, x, y \in \mathbb{R}_i, i = 1, 2 \quad \text{or} \quad x \leq y, x \in (-\infty, 0)_i, y \in [0, +\infty)_j, i \neq j.\}$$

It is easy to check that this preorder can be represented by means of a finite lower semicontinuous multi-utility, for instance through the following functions:

$$u_1(x) = \begin{cases} x & ; \quad x \in \mathbb{R}_1 \cup [0, +\infty)_2 \\ 0 & ; \quad x \in (-\infty, 0)_2 \end{cases} \quad v_1(x) = \begin{cases} x & ; \quad x \in \mathbb{R}_2 \cup [0, +\infty)_1 \\ 0 & ; \quad x \in (-\infty, 0)_1 \end{cases}$$

$$u_2(x) = \begin{cases} \arctan(x) & ; x \in \mathbb{R}_1 \cup (-\infty, 0)_2 \\ \frac{\pi}{2} & ; x \in [0, +\infty)_1 \end{cases} \quad v_2(x) = \begin{cases} \arctan(x) & ; x \in \mathbb{R}_2 \cup (-\infty, 0)_1 \\ \frac{\pi}{2} & ; x \in [0, +\infty)_2 \end{cases}$$

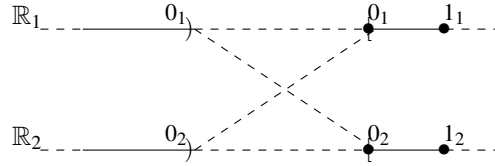


Figure 1: Preorder defined on $\mathbb{R} \times \{1, 2\}$.

However, there is not a finite lower semicontinuous Richter-Peleg multi-utility for this preordered set and with respect to the Upper topology. To see that, first notice that for any $x_i \in (-\infty, 0)_i$, the sequence $(-\frac{1}{n})_{n \in \mathbb{N}}$ contained in $(-\infty, 0)_j$ converges to x_i , as well as $-\frac{1}{n} \bowtie x_i$ and $-\frac{1}{n} \bowtie x_i - \varepsilon$, for some $\varepsilon > 0$ and $i \neq j$. Hence, for any $n \in \mathbb{N}$, there is a function u of the multi-utility such that $u(-\frac{1}{n}) < u(x_i - \varepsilon) < u(x_i)$. Furthermore, since the amount of functions is finite, it actually holds that there is a function u and an infinite subset $M \subseteq \mathbb{N}$ such that $u(-\frac{1}{n}) < u(x_i - \varepsilon) < u(x_i)$ for any $n \in M \subseteq \mathbb{N}$. Thus, $\liminf u(\frac{1}{n}) < u(x_i)$, so u fails to be lower semicontinuous.

From the examples above, we are able to extract the following conditions that must be satisfied for the existence of the desired representation.

Proposition 3.2. *Let \preceq be a preorder on a topological space (X, τ) . Assume that there exists a lower-semicontinuous Richter-Peleg multi-utility. Let $(x_i)_{i \in I}$ be a net in X . If $(x_i)_{i \in I}$ converges to a and there is b such that $b \bowtie x_i$ ($\forall i > i_0$), then $\neg(b \prec a)$ or the multi-utility is infinite.*

Proof. By reduction to the absurd, if there is $b \in X$ such that $b \prec a$, then $u(b) < u(a)$ is satisfied for any function u of the multi-utility. On one hand, if $(x_i)_{i \in I}$ converges to

a and u is a lower semicontinuous function, then it holds that $\liminf u(x_i) \geq u(a)$. On the other hand, if $x_i \bowtie b$ for any $i \in I$ then, for each $i \in I$ there must be two functions u_i and v_i in the multi-utility such that $u_i(x_i) < u_i(b)$ as well as $v_i(x_i) > v_i(b)$.

Thus, if the amount of functions is finite, then there is a subnet $(x_j)_{j \in J \subseteq I}$ and two functions u and v such that $u(x_j) < u(b)$ and $v(x_j) > v(b)$, as well as $u(b) < u(a)$ and $v(b) < v(a)$. Hence, $\liminf u(x_i) \leq u(b) < u(a)$, so u fails to be lower semicontinuous at a , arriving to the desired contradiction. □

Since it is necessary to ask for some properties to the Upper topology in order to achieve a lower-semicontinuous Richter-Peleg multi-utility, we decide to study some other topologies (finer than the Upper). Due to this deliberation, we achieve the following result.

Theorem 3.3. *Let \preceq be a preorder on a topological space (X, τ) . If there exists a finite lower-semicontinuous Richter-Peleg multi-utility, then τ is finer than the Scott topology τ_{Scott} . However, this latter condition is not sufficient in order to guarantee the existence of a finite lower-semicontinuous Richter-Peleg multi-utility.*

Proof. Let's see that any open set U in the Scott topology is also open in τ . That is, let's see that any up-set U satisfying that $\{\sup(x_i)_{i \in I} = s \in U \Rightarrow (x_i)_{i \in I} \cap U \neq \emptyset\}$ (for any directed set $(x_i)_{i \in I}$) is contained in τ . To see that, we shall prove that U is an open neighbourhood of any of its points.

Let x be any point of U . Since each function u_k of the multi-utility $\mathcal{U} = \{u_k\}_{k=1}^N$ is lower semicontinuous, then for any $\varepsilon > 0$ there exists an open neighbourhood V_k^ε of x such that $u_k(V_k) \subseteq (u_k(x) - \varepsilon, +\infty)$. Now, we define the open set $V^\varepsilon = \bigcap_{k=1}^N u_k^{-1}((u_k(x) - \varepsilon, +\infty))$. Notice that for any $y \in \bigcap_{k=1}^N u_k^{-1}([u_k(x), +\infty))$ it holds that $x \preceq y$. Dually, for any $y \in \bigcap_{k=1}^N u_k^{-1}((-\infty, u_k(x)])$ it holds that $y \preceq x$.

We distinguish two cases:

- (i) If there is one $\varepsilon_0 > 0$ such that $V^{\varepsilon_0} \subseteq U$, then we conclude that U is an open neighbourhood of x , finishing our proof.
- (ii) If case (i) does not hold, then for any $\varepsilon > 0$ it holds that $V^\varepsilon \not\subseteq U$. Hence, for each $\varepsilon = \frac{1}{n}$ ($n \in \mathbb{N}$) we can construct an increasing sequence $(x_n)_{n \in \mathbb{N}}$ such that each x_n is in $\bigcap_{k=1}^N u_k^{-1}((u_k(x) - \frac{1}{n}, +\infty)) \setminus u_k^{-1}((u_k(x) - \frac{1}{n+1}, +\infty))$. Notice that $x_n \prec x$ for any $n \in \mathbb{N}$, so x is an upper bound of the sequence. Observe too that $\sup(u_k(x_n))_{n \in \mathbb{N}} = u_k(x)$.

Now, we distinguish the following cases:

- (a) If $\sup(x_n)_{n \in \mathbb{N}} = \bar{x} \in U$, then we arrive to the absurd $(x_n)_{n \in \mathbb{N}} \cap U \neq \emptyset$. That is, this case cannot hold.
- (b) If $\sup(x_n)_{n \in \mathbb{N}} = \bar{b}$, then $b \preceq x$. If $b \prec x$, then $u_k(b) < u_k(x)$ for any $k = 1, \dots, n$. Thus, there is an $\varepsilon_0 = \min\{u_k(x) - u_k(b)\}_{k=1}^N > 0$ such that $u_k(x_n) < u_k(x) - \varepsilon_0$ for any $n \in \mathbb{N}$, which is absurd. That is, there is no element b such that $x_n \prec b \prec x$. If $b \sim x$, then $b \in U$ as in case (a) (remember that, according to Definition 2.17, in that case b and x are indistinguishable, so they share the open neighborhoods).
- (c) If $\sup(x_n)_{n \in \mathbb{N}}$ does not exist, then (since there is no element b such that $x_n \prec b \prec x$) there must be element b such that $x_n \prec b$ for any $n \in \mathbb{N}$ as well as $b \bowtie x$ (otherwise x would be the supremum and that would be the aforementioned case (a)). So, there is a function $u_j \in \mathcal{U}$ such that $u_j(b) < u_j(x)$. Hence, there is an $\varepsilon_0 = \{u_j(x) - u_j(b)\}_{k=1}^n > 0$ such that $u_j(x_n) < u_j(x) - \varepsilon_0$ for any $n \in \mathbb{N}$, so $\liminf u_j(x_n) < u_j(x)$. Hence, u_j fails to be lower semicontinuous, arriving to a contradiction.

To conclude the proof, we show in the following example that, even if the topology τ is the Scott topology, that does not guarantee the existence of a finite lower semicontinuous Richter-Peleg multi-utility. \square

Example 3.4. Let $X = \{(-\infty, 0) \cup \{1\} \cup [2, +\infty)\}$ be a set endowed with the Scott topology associated to the partial order \lesssim defined as follows:

$$x \lesssim y \iff x \leq y, \forall y \in X \setminus \{1\}, \forall x \in X, \quad \text{and} \quad 1 \bowtie y, \forall y < 0.$$

Let's see that there is no finite lower semicontinuous Richter-Peleg multi-utility for this preordered set.

Let \mathcal{U} be a finite Richter-Peleg multi-utility. First, notice that the sequence $(-\frac{1}{n})_{n \in \mathbb{N}}$ converges to 2, as well as $1 \prec 2$ and $1 \bowtie -\frac{1}{n}$ for any $n \in \mathbb{N}$. Hence, for any $n \in \mathbb{N}$, there is a function u_n of the multi-utility such that $u(-\frac{1}{n}) < u(1) < u(2)$. Furthermore, since the amount of functions is finite, it actually holds that there is a function u and an infinite subset $J \subseteq \mathbb{N}$ such that $u(-\frac{1}{n}) < u(1) < u(2)$ for any $n \in J \subseteq \mathbb{N}$. Thus, $\liminf u(-\frac{1}{n}) < u(2)$, so u fails to be lower semicontinuous at 2.

4 Sufficient topology: Alexandroff

It was shown in the section before that the Scott topology is not enough in order to guarantee the existence of a lower-semicontinuous representation. Hence, now we search for topologies finer than the Scott one for which it is possible to guarantee the existence of a lower-semicontinuous Richter-Peleg multi-utility. Thus, we focus on Alexandroff topology.

First, let's see an interesting property satisfied by Alexandroff topologies, but not by the Upper nor by the Scott topologies.

Lemma 4.1. Let \sqsubseteq and \lesssim two preorders on X and τ_A^\sqsubseteq and τ_A^\lesssim the corresponding Alexandroff topologies. Then, \lesssim refines \sqsubseteq (i.e. $\sqsubseteq \subseteq \lesssim$) if and only if $\tau_A^\lesssim \subseteq \tau_A^\sqsubseteq$.

Proof. \Rightarrow : The inclusion $\tau_A^\lesssim \subseteq \tau_A^\sqsubseteq$ holds true if and only if any convergent net $(x_i)_{i \in I}$ on (X, τ_A^\sqsubseteq) also converges on (X, τ_A^\lesssim) . By reduction to the absurd, suppose there is a net $(x_i)_{i \in I}$ that converges to x on (X, τ_A^\sqsubseteq) but that fails to converge on (X, τ_A^\lesssim) . Thus,

there exists an open neighbourhood $U \in \tau_A^{\approx}$ with $x \in U$ such that $(x_i)_{i \in I}$ is not cofinally in U . Since the open sets are the up-sets, that means that there is a subnet $(x_j)_{j \in J \subseteq I}$ of $(x_i)_{i \in I}$ such that $\neg(x \approx x_j)$. Therefore, it also holds true that $\neg(x \sqsubseteq x_j)$ so, we have that $x \in U_{\sqsubseteq}(x) \in \tau_A^{\sqsubseteq}$ as well as $x_j \notin U_{\sqsubseteq}(x)$ (for any $j \in J$). Thus, the subnet $(x_j)_{j \in J}$ fails to converge to x on $(X, \tau_A^{\sqsubseteq})$, which contradicts the hypothesis¹.

\Leftarrow : Suppose that $\tau_A^{\approx} \subseteq \tau_A^{\sqsubseteq}$. By the specialisation order, $x \sqsubseteq y$ if and only if $x \in \bar{y}$ (in τ_A^{\sqsubseteq}). Since $\tau_A^{\approx} \subseteq \tau_A^{\sqsubseteq}$, it also holds that $x \in \bar{y}$, but now τ_A^{\approx} . Hence, we conclude that $x \approx y$. Thus, \approx refines \sqsubseteq . \square

The result above can be interpreted such as that *more order implies less Alexandroff's topology*, or *the more order the coarser Alexandroff topology*. This property is not satisfied in general by the Upper nor the Scott topologies. The following example is devoted to see that.

Example 4.2. Let X be the union between $X_1 = [0, +\infty)_1$ and $X_2 = [0, +\infty)_2$. As in Example 2.18, we define the preorder \sqsubseteq on X by $x \sqsubseteq y$ if and only if $x, y \in X_k$ and $x \leq y$, with $k = 1, 2$. Hence, x and y are incomparable for any $x \in X_m$ and $y \in X_n$ with $n \neq m$.

Now we define a preorder \approx which refines the previous one as follows:

$$x \approx y \iff \begin{cases} x \sqsubseteq y & ; \quad x, y \in X, \\ x \in [0, 1)_1 & ; \quad y \in [1, +\infty)_2, \\ x \in [0, 1)_2 & ; \quad y \in X_1, \end{cases}$$

Then, the lower set $L_{\approx}(2_2)$ is closed on τ_u^{\approx} (in other words, $X \setminus L_{\approx}(2_2)$ is open) whereas it is not in τ_u^{\sqsubseteq} . Hence, τ_u^{\sqsubseteq} cannot be finer than τ_u^{\approx} . Notice too that $1_1 \notin L_{\approx}(2_2)$, but it is contained in the closure $\overline{L_{\approx}(2_2)}$ with respect to τ_u^{\sqsubseteq} , thus, $L_{\approx}(2_2)$ is not closed on τ_u^{\sqsubseteq} .

Therefore, from Lemma 4.1 the following corollary arises:

¹Here, it is used that any subnet of a convergent net converges to the same point.

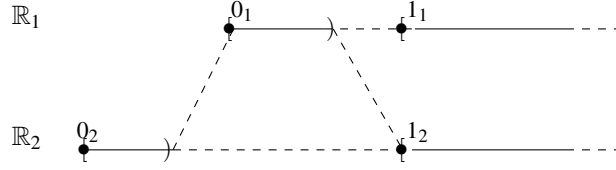


Figure 2: Preorder defined on $[0, +\infty)_1 \cup [0, +\infty)_2$.

Corollary 4.3. *Let \preceq be a preorder defined on a topological space (X, τ) . If τ is finer than the corresponding Alexandroff's topology τ_A^{\preceq} , then any linear extension of the preorder is lower-semicontinuous on τ .*

The following corollaries are consequence of the corollary before and Debreu's Open Gap Lemma [8] (see also [6]) and Theorem 3.1 of [1].

Corollary 4.4. *Let \preceq be a preorder defined on a second countable topological space (X, τ) . If τ is finer than the corresponding Alexandroff's topology τ_A^{\preceq} , then there exists a (not necessarily finite) lower-semicontinuous Richter-Peleg multi-utility representation.*

Corollary 4.5. *Let \preceq be a preorder on X which is order separable in the sense of Debreu. Let τ_A be the corresponding Alexandroff's topology. Then there exists a (not necessarily finite) lower-semicontinuous Richter-Peleg multi-utility representation.*

In order to show that the Alexandroff's topology is not strange at all, we include the following result that shows that some authors (see, for example, [14] on *pliable spaces*) have already work on this spaces (at least in a subset of the corresponding set, maybe unconsciously) when they worked under the assumption that any linear extension is lower semicontinuous.

Theorem 4.6. *Let \preceq be a preorder defined on a topological space (X, τ) . Assume there is chain (C, \preceq) included in (X, \preceq) and an element $x \in X$ such that $x \bowtie c$ for any $c \in C$. If any linear extension of the preorder is lower-semicontinuous in τ , then the reduction*

of τ to C (that is, $\tau|_C$) is finer than the corresponding Alexandroff's topology τ_A^{\succsim} on C .

Proof. First, since $x \bowtie c$ for any $c \in C$, for a given $c_0 \in C$ we can define an extension \succsim_1 of the preorder but now imposing that $c_0 \prec_1 x$ and including the corresponding transitive closure. Dually, we define the extension \succsim_2 of the preorder imposing that $x \prec_2 c_0$.

By Szpilrajn extension theorem [20], there exists a linear extension $\leq_1^{c_0}$ such that $c_0 <_1^{c_0} x$ and $x <_1^{c_0} c'$ for any $c' \in C$ with $c_0 \prec c'$. Analogously, there exists another linear extension $\leq_2^{c_0}$ such that $x <_2^{c_0} c_0$ and $c' <_2^{c_0} x$ for any $c' \in C$ with $c' \prec c_0$. That is, we can embed x in any desired point c of C , achieving two linear orders on $C \cup \{x\}$: \leq_1^c and \leq_2^c .

Since, by hypothesis, any linear extension of the preorder is lower-semicontinuous in τ , we deduce that the subsets $L_{\leq_1}(x) = L_{\leq_1}(c_0) \cup \{x\}$ and $L_{\leq_2}(x) = L_{\leq_2}(c_0)$ are closed in τ . Hence, restricting to C , it holds that both $L_{\succsim}(c_0)$ and $L_{\prec}(c_0)$ are closed in $\tau|_C$, and that will hold for any $c_0 \in C$. Thus, any down set is closed in $(C, \tau|_C)$, so we conclude that $\tau|_C$ is finer than the corresponding Alexandroff topology. \square

5 Some topological issues on maximal elements

Throughout the present paper, necessary and sufficient conditions for the existence of a lower semicontinuous Richter-Peleg multi-utility have been studied, from a topological point of view, focusing on the topological space instead of on the preorder relation.

Dealing with binary relations, in particular with preorders, the study of maximal elements plays a key role in optimization problems. In this section we include some topological descriptions related to these maximal elements.

First, we introduce some concepts (see [19]) related to Alexandroff topologies, as defined in Definition 2.14.

Definition 5.1. Let $S(x)$ denote the minimal open neighborhood of a point x in (X, τ_A) .

$S(x)$ is called irreducible if $S(y) \subseteq S(x)$ implies $S(x) = S(y)$. $S(x)$ is called basic if for any $S(y)$, any of the following two cases hold:

(i) If $S(x) \subseteq S(y)$ as well as $S(z) \subseteq S(y)$, then $S(x) \subseteq S(z)$.

(ii) If $S(x)$ is not contained in $S(y)$, then $S(x)$ and $S(y)$ are disjoint.

It is known [19] that basic subsets are irreducible.

The number $\min\{|V| : V \text{ is a finite cover of } X \text{ by minimal open neighborhoods}\}$ shall be denoted by $\min(X)$. The following theorem is also known [19].

Theorem 5.2. *Let (X, τ) be a compact Alexandroff space. Then, the number of basic sets is less than or equal to $\min(X)$.*

Proposition 5.3. *Let \preceq be a preorder on X and τ_A the corresponding Alexandroff topology related to \preceq . Then, $S(x) = U_{\preceq}(x)$, $x \in X$.*

Proof. $U_{\preceq}(x)$ is open and it contains x , thus, $S(x) \subseteq U_{\preceq}(x)$. On the other hand, by Definition 2.15, the open sets are the upsets so, since $x \in S(x)$ it holds that $U_{\preceq}(x) \subseteq S(x)$. □

Definition 5.4. *Let \preceq be a preorder on X . An element x is said to be maximal (dually, minimal) if there is no $z \in X$ such that $x \prec z$ (respectively, $z \prec x$).*

Proposition 5.5. *Let \preceq be a preorder on X and τ_A the corresponding Alexandroff topology related to \preceq . Then, $S(x)$ is irreducible if and only if x is a maximal element. In addition, $S(x) = \bar{x} = \{y \in X : y \sim x\}$.*

Proof. \Rightarrow : Let $S(x)$ be an irreducible subset. By contradiction, suppose that x is not maximal, i.e. there exists an element $z \in X$ such that $x \prec z$. Then, $S(z) = U_{\preceq}(z) \subseteq S(x) = U_{\preceq}(x)$ as well as $S(z) \neq S(x)$, arriving to the desired contradiction.

\Leftarrow : If x is a maximal element, then $S(x) = U_{\preceq}(x) = \bar{x}$. Hence, it is trivial that $S(x)$ is irreducible. □

We said that basic subsets are irreducible. Since irreducible sets are just the maximal elements, let's see now which kind of maximal elements are also basic.

Lemma 5.6. *Let \preceq be a preorder on X and τ_A the corresponding Alexandroff topology related to \preceq . An irreducible subset $S(x)$ is basic if and only if $U_{\preceq}(y)$ is a totally preordered set, for any $y \preceq x$.*

Proof. Let's see the double implication.

Let $y \in X$ such that $S(x) \subseteq S(y)$. Since $S(x)$ is basic, for any $S(z) \subseteq S(y)$ it holds that $S(x) \subseteq S(z)$. By Proposition 5.3, this means (i.e. it is equivalent) that for any $y, z \in X$ such that $y \preceq x$ and $y \preceq z$, it holds that $z \preceq x$, which is equivalent to fact that $U_{\preceq}(y)$ is a totally preordered set, for any $y \preceq x$. If $S(x)$ is not contained in $S(y)$ then, by Proposition 5.3, this is equivalent to the fact that $\neg(y \preceq x)$. \square

Due to the lemma before, we shall say that a maximal element x is *basic* if $S(x)$ it is (in the corresponding Alexandroff topology).

From lemma before, next corollary is deduced.

Corollary 5.7. *Let (X, \preceq) be a preordered set and τ_A the corresponding Alexandroff topology. A maximal element x is basic if and only if for any element y it holds that*

$$L_{\preceq}(x) \cap L_{\preceq}(y) \neq \emptyset \implies y \preceq x.$$

In other words, a maximal element x is basic if and only if there is no 'V', that is, there is no y and z such that $z \prec x$, $z \prec y$ as well as $x \bowtie y$.

Remark 5.8. *So, Lemma 5.6 says that an irreducible subset $S(x)$ is basic if and only if there is no bifurcation from below to above. Hence, in that case x is the supremum of its connected component.*

Related to optimization problems or decision theory, we may say that *basic elements are the best (so unique) choice in their connected component*, that is not the case

of (non-basic) maximal elements.

Finally, using Proposition 5.5 and Lemma 5.6, we translate Theorem 5.2 into the following corollary.

Corollary 5.9. *Let (X, \preceq) be a preordered set and τ_A the corresponding Alexandroff topology. If (X, τ_A) is compact, then the following inequalities are satisfied:*

$$M \geq B \leq \min(X) \leq m \leq M,$$

where M is the number of maximal elements, B the number of basic elements and m the number of minimal elements.

6 Further comments

In the present paper the authors have focused on semicontinuous finite Richter-Peleg multi-utility. As we said in Section 2, there is a theorem of Evren and Ok [10] that characterizes the existence of a semicontinuous finite multi-utility under the assumption of near-completeness and second countability.

In a previous paper, some of the authors of the present work studied the idea of *partial representability*. In order to illustrate some of these ideas, they introduced some examples. One of these examples was commented in a final remark as a possible counterexample for the aforementioned Theorem 2.7, however, that was not correct at all. The mistake lies in the fact that the example fails to satisfy the hypothesis of Theorem 2.7, hence, it is not a counterexample. In particular, the lower contour set $L_{\preceq}(0'5)$ of the example fails to be closed. Therefore, (and after a deeper study of the proof) the authors believe that the mentioned theorem given by Evren and Ok is correct.

7 Conclusions

After this work, we conclude that if we want to search for lower semicontinuous and finite Richter-Peleg multi-utilities, it is necessary to start the study from topological spaces that refine the corresponding Scott topology, and not –as usual– from Upper topologies. We also prove that more order implies less Alexandroff topology and, hence, it is deduced (as a sufficient condition) that the Alexandroff topology guarantees the existence of a lower-semicontinuous Richter-Peleg multi-utility representation. Finally, we show that under some hypothesis assumed in the literature (precisely, under the assumption that any linear extension is lower semicontinuous, so related to *pliable spaces*) the topology of the space has a strong relation with the Alexandroff topology.

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