The evolution of the notion of overlap functions

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1 The origin of the notion of overlap function

In many different real situations it is necessary to determine to which one of a set of classes a considered instance belongs to. In the case of disjoint crisp classes, it may be assumed that the instance belongs to one or another class, but not to more than one class at the same time. However, it may happen that, even for crisp classes, the boundary separating one class from another is not clearly defined for the expert, due to the lack of information or the imprecision on the available data. Finally, it also happens commonly that classes are not clearly defined by their own nature, and hence several or even all the instances may belong up to some extent to two or more of the classes [1, 4, 15].

This situation arises, for instance, in some image processing problems (see [21]). Consider an image where we want to separate an object from the background. A possible approach to deal with this problem is to represent the object by means of an appropriate fuzzy set and the background by means of another fuzzy sets [5]. These

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fuzzy sets can be built , for instance, on the referential set of the considered possible intensities and need not be disjoint in the sense of Ruspini. Obviously, the result of the separation procedure will be worse or better according to the accuracy of the fuzzy sets used to represent the background and the object. To build the membership functions that define these fuzzy sets it is necessary to know the exact property that provides a characterization for those pixels which belong to the object (or the background). This property determines the expression of the membership function associated to the fuzzy set representing the object (background)(see [8, 9]). But, in most of the real cases, the expert is not able to provide a specific expression for this function. Furthermore, there exist some pixels that the expert assign to the object or to the background without any doubt, but there are other pixels which the expert can not assign with certitude either to the object or to the background. For these last pixels the value of the membership function is not accurately known, since the pixels belong, up to some extent, to both classes (object and background).

Overlap functions were first introduced to deal with this kind of problems [6]. Overlap functions provide a mathematical model in such a way that the outcome of the function can be interpreted as the representation of up to what extent the considered instance belongs to both classes simultaneously. In this way, overlap functions are also related to the notion of overlap index and in fact they can be used to build the latter, see [23]. Moreover, the concept of overlap has also been extended to a more general situation in [20], with three or more classes involved. In this chapter, we start reviewing results for the bivariate case and later we also consider the general n-dimensional case.

In this chapter we make an overview of some aspects of the history of the notion of overlap functions and their applications. The structure of the chapter is as follows. We start recalling the original definition of overlap function. Then, we comment the link between overlap functions and triangular norms. We also discuss the extension to any dimension of the definition of overlap functions and we finish with some applications.

2 The mathematical definition of overlap functions

As we have said, the concept of overlap as a bivariate aggregation operator was first introduced in [6] to deal with the problem of measuring up to what extent a given object (instance) belongs simultaneously to two classes. The original definition reads as follows.

Definition 1. A mapping $G_O : [0,1]^2 \to [0,1]$ is an overlap function if it satisfies the following conditions:
(G_O1).- G_O is symmetric.
(G_O2).- $G_O(x,y) = 0$ if and only if $xy = 0$.
(G_O3).- $G_O(x,y) = 1$ if and only if $xy = 1$.
(G_O4).- G_O is increasing in both variables.
(G_O5).- G_O is continuous.

Regarding this definition, it is worth to mention that continuity arises precisely from the applicability of the notion of overlap functions in image processing, since slight variations in the intensity of the pixels should not in principle lead to large variations in the final outcome.

There are many possible examples of overlap functions. For instance, $G_O(x,y) = (\min(x,y))^p$ for $p > 0$ or $G_O(x,y) = xy$ are overlap functions. Moreover, the class of bivariate overlap functions, $\mathscr{G}$, is convex and overlap functions can be characterized by the following result.

Theorem 1. *The mapping $G_O : [0,1]^2 \to [0,1]$ is an overlap function if and only if*

$$G_O(x,y) = \frac{f(x,y)}{f(x,y)+h(x,y)}$$

for some $f,h : [0,1]^2 \to [0,1]$ such that
1) *f and h are symmetric;*
2) *f is non decreasing and h is non increasing;*
3) *$f(x,y) = 0$ if and only if $xy = 0$;*
4) *$h(x,y) = 0$ if and only if $xy = 1$;*
5) *f and h are continuous functions;*

Theorem 1 allows the definition of interesting families of overlap functions.

Corollary 1. *Let f and h be two functions in the setting of the previous theorem. Then, for $k_1,k_2 \in]0,\infty[$, the mappings*

$$G_S^{k_1,k_2}(x,y) = \frac{f^{k_1}(x,y)}{f^{k_1}(x,y)+h^{k_2}(x,y)}$$

define a parametric family of overlap functions.

Corollary 2. *In the same setting of Theorem 1, let us assume that G_O can be expressed in two different ways:*

$$G_O(x,y) = \frac{f_1(x,y)}{f_1(x,y)+h_1(x,y)} = \frac{f_2(x,y)}{f_2(x,y)+h_2(x,y)}$$

for any $x,y \in [0,1]$ and let M be a bivariate continuous aggregation function that is homogeneous of order one. Then, if we define $f(x,y) = M(f_1(x,y), f_2(x,y))$ and $h(x,y) = M(h_1(x,y), h_2(x,y))$ it also holds that

$$G_O(x,y) = \frac{f(x,y)}{f(x,y)+h(x,y)} \ .$$

3 Overlap functions vs.triangular norms

Note that from Definition 1 it follows straightforwardly that overlap functions are particular instances of aggregation functions without divisors of zero or one. In fact, there exists a close relation between overlap functions and the well-known class of triangular norms [2, 6]. Recall that the latter are commonly used in order to represent the intersection between fuzzy sets and one of their crucial properties is that of associativity.

Theorem 2. *Let G_O be an associative overlap function. Then G_O is a t-norm.*

Theorem 3. *Let T be a continuous triangular norm without divisors of zero. Then T is an overlap function.*

However, when we are dealing with only two classes, associativity is not a natural requirement, as only two variables are involved. In this sense, overlap functions can be considered as an alternative tool to triangular norms. In particular, this approach has been succesfully developed to build new classes of residual implication functions replacing the t-norm by an overlap function. Furthermore, in the same way as t-conorms can be defined by considering the dual of t-norms, it is possible to define the concept of grouping function as the dual (with respect to some strong negation N) of an overlap function.

Definition 2. A mapping $G_G : [0,1]^2 \to [0,1]$ is a grouping function if it satisfies the following conditions:
(G_O1).- G_G is symmetric.
(G_O2).- $G_G(x,y) = 0$ if and only if $x = y = 0$.
(G_O3).- $G_G(x,y) = 1$ if and only if $x = 1$ or $y = 1$.
(G_O4).- G_G is increasing in both variables.
(G_O5).- G_G is continuous.

Combining the notions of overlap and grouping functions, it is possible for instance to define preference structures in the same way as Fodor and Roubens, but replacing t-norms and t-conorms by overlap and grouping functions, respectively [7]. More specifically, given a fuzzy preference relation R, we can define the degree of indifference between alternatives i and j as:

$$I_{ij} = G_O(R_{ij}, R_{ji})$$

and the degree of incomparability between the same alternatives as

$$J_{ij} = 1 - G_G(R_{ij}, R_{ji})$$

where G_O is an overlap function and G_G is a grouping function. Note that in this case, the considered strong negation is Zadeh's negation. This approach was further analyzed in [7], where its application to decision making problems was widely discussed.

4 n-dimensional overlap functions

Since overlap functions are not assumed to be associative, its extensions to the n-dimensional case with $n > 2$ requires a specific definition. On the other hand, in many problems it is natural to deal with more than two classes. These considerations led the authors in [20] to propose the following definition of n-dimensional overlap function

Definition 3. An n-dimensional aggregation function $G_O : [0,1]^n \longrightarrow [0,1]$ is an n-dimensional overlap function if and only if:

1. G_O is symmetric.
2. $G_O(x_1, \ldots, x_n) = 0$ if and only if $\prod_{i=1}^{n} x_i = 0$.
3. $G_O(x_1, \ldots, x_n) = 1$ if and only if $x_i = 1$ for all $i \in \{1, \ldots, n\}$.
4. G_O is increasing.
5. G_O is continuous.

Note that, taking into account this definition, an object c that belongs to three classes C_1, C_2 and C_3 with degrees $x_1 = 1$, $x_2 = 1$ and $x_3 = 0.5$ will not have the maximum degree of overlap since condition (3) of the previous definition is not satisfied. Even more, if the degrees are $x_1 = 1$, $x_2 = 1$ and $x_3 = 0$, from the second condition we will conclude that the n-dimensional degree of overlapping of this object into the classification system given by the classes C_1, C_2 and C_3 will be zero. This is the reason why this first extension of the original idea of overlap proposed in [6] has been called *n-dimensional* overlap. Note that this definition is closely related to the idea of intersection of n classes, and in this sense, it follows also one of the inspirations behind the original idea of overlap function.

Example 1. It is easy to see that the following aggregation functions are n-dimensional overlap functions:

- The minimum powered by p. $G_O(x_1, \ldots, x_n) = \min\limits_{1 \le i \le n} \{x_i^p\} = \left[\min\limits_{1 \le i \le n} \{x_i\}\right]^p$ with $p > 0$. Note that for $p = 1$ we recover an averaging aggregation function.
- The product. $G_O(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i$.
- The Einstein product aggregation operator. $EP(x_1, \ldots, x_n) = \dfrac{\prod_{i=1}^{n} x_i}{1 + \prod_{i=1}^{n}(1 - x_i)}$
- The sinus induced overlap $G_O(x_1, \ldots, x_n) = \sin \dfrac{\pi}{2} (\prod_{i=1}^{n} x_i)^p$ with $p > 0$.

The characterization results already introduced for bivariate overlap functions may be extended in a straight way for n-dimensional overlap functions. In particular, we remark the following result.

Proposition 1. *Let $A_n : [0,1]^n \longrightarrow [0,1]$ be an aggregation function. If A_n is averaging, then A_n is an n-dimensional overlap function if and only if it is symmetric, continuous, has zero as absorbing element and satisfies $A_n(x,1,\dots,1) \neq 1$ for any $x \neq 1$.*

In fact, we have the following theorem.

Theorem 4. *Let $G_1,\dots G_m$ be n-dimensional overlap functions and let $M : [0,1]^m \longrightarrow [0,1]$ be a continuous aggregation function such that if $M(x) = 0$ then $x_i = 0$ for some i and $M(x) = 1$ only if $x_i = 1$ for some i. Then the aggregation function $G : [0,1]^n \longrightarrow [0,1]$ defined as $G(x) = M(G_1(x),\dots,G_m(x))$ is an n-dimensional overlap function.*

Notice that, since any averaging aggregation function M being s continuous satisfies the conditions of the previous Theorem, it is possible to conclude that any continuous averaging aggregation of n-dimensional overlap functions is also an n-dimensional overlap function. Also note thatit is also possible to consider $m = 1$, so, for instance, the power of a 2-dimensional overlap function is again an overlap function.

Example 2. As an illustrative case, consider that of OWA operators (see [3]. Let $W = (w_1,\dots,w_n) \in [0,1]^n$ be a weighting vector. Then, the only OWA operator which is also an n-dimensional overlap function is that for which $w_n = 1$. That is, the minimum. On the other hand, for any continuous t-norm T with no zero divisors, its n-ary form is always an n-dimensional overlap function.

For the case of Kolmogorov-Nagumo means the following holds.

Proposition 2. *Let $f : [0,1] \to [-\infty,0]$ be a continuous increasing bijection, that is, $f :]0,1] \to]-\infty,0]$ is an increasing bijection such that $\lim_{x\to 0} f(x) = -\infty$, so by abuse of notation we define $f(0) = -\infty$. Then the function: $G_O(x_1,\dots,x_n) = f^{-1}(\frac{1}{n}f(x_1) + \dots + \frac{1}{n}f(x_n))$ is an n-dimensional overlap function.*

5 Applications of overlap functions

As we have already said, the first application of overlap functions was in image processing, and more specifically, in image thresholding problems. However, along the last years, the notion of overlap function has displayed a strong versatility and have been applied to many different situations and problems. Among them we can mention:

- Decision making problems [17].
- Forest fire detection [19].
- Classification problems [16, 22, 23].
- Approximate reasoning [18].

The number of different applications is growing very fast, as these functions have shown themselves very useful in order to build new aggregation functions which are well adapted to specific data. In this sense, it is worth to mention the construction of fuzzy measures (used as the basis to obtain Choquet integrals and hence averaging aggregation functions) using overlap functions and overlap indexes obtained from them. See in particular, [23] for more details.

Besides, recently there have been discussed relevant results on the development of implication functions defined in terms of overlap functions [9, 10, 12, 13, 24]. Also other classes of aggregation functions have been defined in terms of overlap functions [25]. Finally, also extensions of the notions of overlap function from a theoretical point of view have been considered [8]. Some important properties of overlap and grouping functions, such as migrativity, homogeinity or Archimedeanity, were studied in [2, 11, 26]. Additive generators of overlap functions were introduced in [14].

6 Akcnowledgements

This work has been partially supported by research project TIN2016-77356-P of the Spanish Government, by Brazilian Funding Agency CNPq (Proc. 307781/2016-0), and by project APVV-14-0013.

References

References

1. A. Amo, J. Montero, G. Biging, V. Cutello, Fuzzy classification systems, European Journal of Operational Research 156 (2004) 459–507.
2. B. C. Bedregal, G. P. Dimuro, H. Bustince, E. Barrenechea, New results on overlap and grouping functions, Information Sciences 249 (2013) 148–170.
3. G. Beliakov, H. Bustince, T. Calvo, A practical guide to averaging functions, Springer, 2016.
4. H. Bustince, M. Pagola, E. Barrenechea, Construction of fuzzy indices from fuzzy DI-subsethood measures: Application to the global comparison of images, Information Sciences 177 (3) (2007) 906–929.
5. H. Bustince, E. Barrenechea, M. Pagola, Image thresholding using restricted equivalence functions and maximizing the measures of similarity, Fuzzy Sets and Systems 158 (5) (2007) 496–516.
6. H. Bustince, J. Fernandez, R. Mesiar, J. Montero, R. Orduna, Overlap functions, Nonlinear Analysis: Theory Methods & Applications 72 (3–4) (2010) 1488–1499.
7. H. Bustince, M. Pagola, R. Mesiar, E. Hullermeier, F. Herrera,Grouping, overlap, and generalized bientropic functions for fuzzy modeling of pairwise comparisons, IEEE Transactions on Fuzzy Systems 20 (3) (2012) 405-415
8. L. De Miguel, D. Gómez, J T. Rodríguez, J. Montero, H. Bustince, G. P Dimuro, J.A. Sanz, General overlap functions, Fuzzy Sets and Systems, in press.

9. M. Caoa, B. Q. Hua, J. Qiao, On interval (G,N)-implications and (O,G,N)-implications derived from interval overlap and grouping functions, International Journal of Approximate Reasoning 100 (2018) 135–160.
10. G. P. Dimuro, B. Bedregal, R. H. N. Santiago, On (G,N)-implications derived from grouping functions, Information Sciences 279 (2014) 1 – 17.
11. G. P. Dimuro, B. Bedregal, Archimedean overlap functions: The ordinal sum and the cancellation, idempotency and limiting properties, Fuzzy Sets and Systems 252 (2014) 39–54.
12. G. P. Dimuro, B. Bedregal, On residual implications derived from overlap functions, Information Sciences 312 (2015) 78 – 88.
13. G. P. Dimuro, B. Bedregal, H. Bustince, A. Jurio, M. Baczyński, K. Miś, QL-operations and QL-implication functions constructed from tuples (O,G,N) and the generation of fuzzy subsethood and entropy measures, International Journal of Approximate Reasoning 82 (2017) 170 – 192.
14. G. P. Dimuro, B. Bedregal, H. Bustince, M. J. Asiain, R. Mesiar, On additive generators of overlap functions, Fuzzy Sets and Systems 287 (2016) 76–96.
15. D. Dubois, W. Ostasiewicz, H. Prade, Fuzzy sets: History and basic notions, in: Fundamentals of Fuzzy Sets, Kluwer, Boston, MA, 2000.
16. M. Elkano, M. Galar, J. Sanz, H. Bustince, Fuzzy Rule-Based Classification Systems for multi-class problems using binary decomposition strategies: On the influence of n-dimensional overlap functions in the Fuzzy Reasoning Method, Information Sciences 332 (2016) 94–114.
17. M. Elkano, M. Galar, J. Sanz, P. F. Schiavo, S. Pereira, G. P. Dimuro, E. N. Borges, H. Bustince, Consensus via penalty functions for decision making in ensembles in fuzzy rule-based classification systems, Applied Soft Computing 67 (2018) 728–740.
18. S. Garcia-Jimenez, H. Bustince, E. Huellermeier, R. Mesiar, N.R. Pal, A. Pradera, Overlap indices: construction of and application to interpolative fuzzy systems, IEEE Transactions on Fuzzy Systems 23 (4) (2015) 1259-1273
19. S. Garcia-Jimenez, A. Jurio, M. Pagola, L. De Miguel, E. Barrenechea, H. Bustince, Forest fire detection: A fuzzy system approach based on overlap indices Applied Soft Computing 52 (2017) 834–842
20. D. Gómez, J.T. Rodriguez, J. Montero, H. Bustince, E. Barrenechea, n-Dimensional overlap functions, Fuzzy Sets and Systems 287 (2016) 57-75
21. A. Jurio, H. Bustince, M. Pagola, A. Pradera, R.R. Yager, Some properties of overlap and grouping functions and their application to image thresholding, Fuzzy Sets and Systems, 229, (2013) 69–90.
22. G. Lucca, J. Sanz, G.P. Dimuro, B. Bedregal, H. Bustince, A proposal for tuning the α parameter in C_αC-integrals for application in fuzzy rule-based classification systems, Natural Computing, in press, available online, DOI 10.1007/s11047-018-9678-x.
23. D. Paternain, H. Bustince, M. Pagola, P. Sussner, A Kolesárová, R Mesiar, Capacities and overlap indexes with an application in fuzzy rule-based classification systems, Fuzzy Sets and Systems 305 (2016) 70–94.
24. J. Qiao, B. Q. Hu, The distributive laws of fuzzy implications over overlap and grouping functions, Information Sciences 438 (2018) 107–126.
25. J. Qiao, B. Q. Hu, On the migrativity of uninorms and nullnorms over overlap and grouping functions, Fuzzy Sets and Systems 346 (2018) 1–54.
26. J. Qiao, B. Q. Hu, On homogeneous, quasi-homogeneous and pseudo-homogeneous overlap and grouping functions, Fuzzy Sets and Systems, in press.