CHAOS IN THE LIBRATION MOTION OF AN ASYMMETRIC NON-RIGID SPACECRAFT

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1. ABSTRACT

We study the libration motion dynamics of an asymmetric spacecraft in circular orbit under the influence of a gravity gradient torque [1].

The spacecraft is perturbed by a small aerodynamic drag torque proportional to the angular velocity of the body about its mass center. We also suppose that one of the moments of inertia of the spacecraft is a periodic function of time.

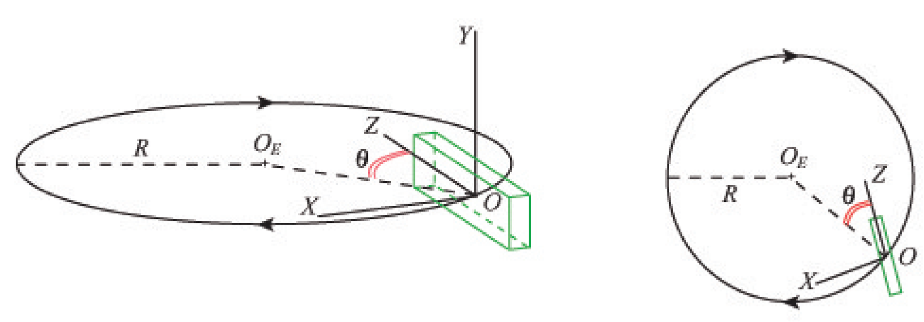
Under both perturbations, we show that the system exhibits a transient chaotic behavior by means of the Melnikov method [2]. This method give us an analytical criterion for heteroclinic chaos in terms of the system parameters.

The dynamical behavior of the libration motion is also numerically investigated by means of time histories, Poincaré maps[3], power spectra[4] and atraction basins[5]. These computer numerical simulations confirm the analytical results provided by the Melnikov method.

2. THE UNPERTURBED SYSTEM

• We consider an asymmetric spacecraft in a circular orbit of radius R with orbital angular velocity ω_o in the gravitational field of the Earth.

• We suppose the attitude motion only made of **planar librations** caused by the gravity gradient torque in the orbital plane[1].



• Applying the theorem of angular momentum about the mass center O of the spacecraft[1], the equation of the libration motion is

$$B\,\ddot{\theta} = 3\,\omega_o^2\,(C-A)\,\sin\theta\,\cos\theta$$

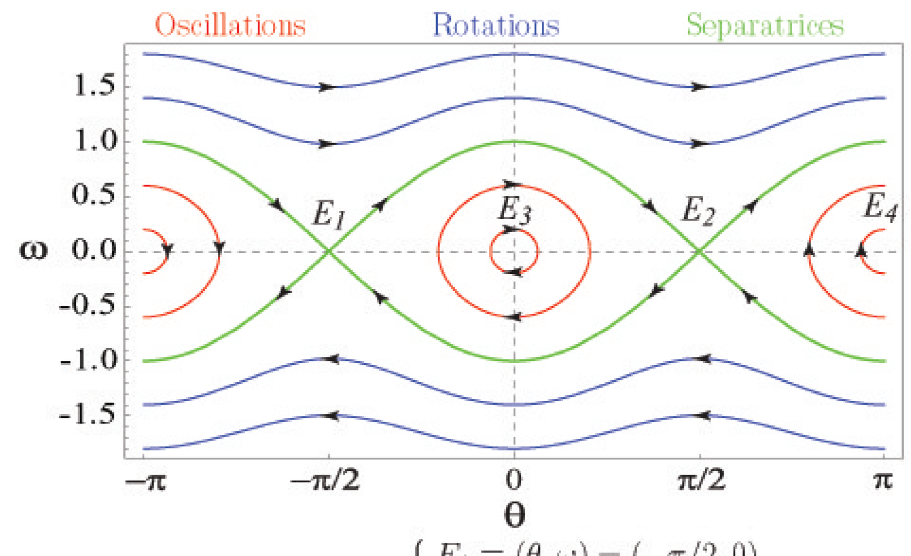
$\theta \equiv$ the libration angle
$A > B > C \equiv$ the moments of inertia of the spacecraft

• Using the dimensionless time $\tau = \omega_o t$ the equation of motion results

$$\ddot{\theta} = -K\,\sin\theta\,\cos\theta \iff \begin{cases} \dot{\theta} = \omega = f_1 \\ \dot{\omega} = -K\,\sin\theta\,\cos\theta = f_2 \end{cases}$$

where $\begin{cases} \omega \equiv \text{the angular velocity} \\ K = 3(A-C)/B \end{cases}$

• **The phase space of the unperturbed system**



• Two unstable equilibria $\Rightarrow \begin{cases} E_1 \equiv (\theta,\omega) = (-\pi/2, 0) \\ E_2 \equiv (\theta,\omega) = (\pi/2, 0) \end{cases}$

• Two stable equilibria $\Rightarrow \begin{cases} E_3 \equiv (\theta,\omega) = (0, 0) \\ E_4 \equiv (\theta,\omega) = (\pi, 0) \end{cases}$

• Two kinds of libration motions $\Rightarrow$ { Oscillations: Inside the separatrices; Rotations: Outside the separatrices }

• The Separatrices are made of the stable $W_s(E_i)$ and unstable $W_u(E_i)$ manifolds of the unstable equilibria E_1, E_2 and take the form

$$[\theta^{\pm}(\tau),\ \omega^{\pm}(\tau)] = \{\pm \arcsin[\tanh(\sqrt{K}\ \tau)],\ \pm\sqrt{K}\ \mathrm{sech}(\sqrt{K}\ \tau)\} \quad (1)$$

• In the unperturbed problem, the stable $W_s(E_i)$ and unstable $W_u(E_i)$ manifolds of the saddle points E_{12}, **join smoothly and coincide**.

$$W_u(E_1) \equiv W_s(E_2) \qquad\qquad W_s(E_1) \equiv W_u(E_2)$$

3. THE PERTURBED SYSTEM

• First Perturbation. The spacecraft is a **non perfectly rigid** body. The greatest moment of inertia varies periodically with time as

$$A(t) = A_o + A_1 \cos\nu t \qquad \text{with} \quad A_1 \ll A_o$$

• Second Perturbation. **Small aerodynamic viscous drag**. The drag torque N_d is opposite and proportional to the angular velocity

$$N_d = -\gamma\,\dot{\theta} = -\gamma\,\omega \qquad \text{with} \quad \gamma \ll 1$$

• The main effect of the viscous drag is to **decrease the libration motion** until the spacecraft reaches an stable equilibrium E_3 or E_4.

• In this case the first order system of equations of motion is

$$\begin{cases} \dot{\theta} = \omega = f_1 + g_1 & (g_1 = 0) \\ \dot{\omega} = -K\ \sin\theta\cos\theta\ -\ \epsilon\ \sin\theta\cos\theta\cos(\eta\tau)\ +\ \delta\,\omega = f_2 + g_2 \end{cases}$$

• The perturbed system depends on these **four parameters**:

$$K = \frac{3(A_o - C)}{B} \qquad \epsilon = \frac{3A_1}{B} \ll 1 \qquad \eta = \frac{\nu}{\omega_o} \qquad \delta = \frac{\gamma}{B\omega_o} \ll 1$$

$K \equiv$ parameter describing the asymmetry of the spacecraft
$\epsilon \equiv$ amplitude of the variation of the moment of inertia $A(t)$
$\eta \equiv$ frequency of the variation of the moment of inertia $A(t)$
$\delta \equiv$ viscous drag parameter

• Under perturbations, the invariant manifolds $W_s(E_i)$ and $W_u(E_i)$ of the saddle points E_{12}, are not forced to coincide in the Poincaré map.

• If the invariant manifolds **intersect transversely** each other in the Poincaré map, a heteroclinic tangle is generated giving rise **transient chaotic motions**.

4. THE MELNIKOV FUNCTION. ANALYTICAL CRITERION FOR CHAOS

• The Melnikov function $M^{\pm}(\tau_0)$ give us a **measure of the distance** between the stable and unstable manifolds of the perturbed system[2]

• The **condition for intersections** between the invariant manifolds, and therefore for **chaos** is that $M^{\pm}(\tau_0)$ **has simple zeroes.**

• The Melnikov function, $M^{\pm}(\tau_0)$, for the system is given by

$$M^{\pm}(\tau_0) = \int_{-\infty}^{\infty} \{f_1[\bar{z}^{\pm}(\tau)]g_2[\bar{z}^{\pm}(\tau), \tau+\tau_0] - f_2[\bar{z}^{\pm}(\tau)]g_1[\bar{z}^{\pm}(\tau), \tau+\tau_0]\}d\tau$$

$$M^{\pm}(\tau_0) = -\int_{-\infty}^{\infty} \epsilon\omega^{\pm}(\tau)\{\sin[\theta^{\pm}(\tau)]\cos[\theta^{\pm}(\tau)]\cos[\eta(\tau+\tau_0)] + \delta\omega^{\pm}(\tau)\}d\tau$$

$\bar{z}^{\pm}(\tau) = (\theta^{\pm}(\tau)\omega^{\pm}(\tau)) \equiv$ solutions (1) of the unperturbed separatrices.

• After calculations, the Melnikov function $M(\tau_0)$ finally results in

$$M(\tau_0) = \epsilon\,\frac{\pi\,\eta^2}{2\,K}\ \mathrm{cosech}\left(\frac{\pi\,\eta}{2\sqrt{K}}\right)\sin(\eta\tau_0) - 2\,\delta\,\sqrt{K}$$

• From last equation we get a **critical value** δ_c for the drag parameter

$$\delta_c = \frac{\pi\,\epsilon\,\eta^2}{4\,\sqrt{K^3}}\ \mathrm{cosech}\left(\frac{\pi\,\eta}{2\sqrt{K}}\right) \qquad (2)$$

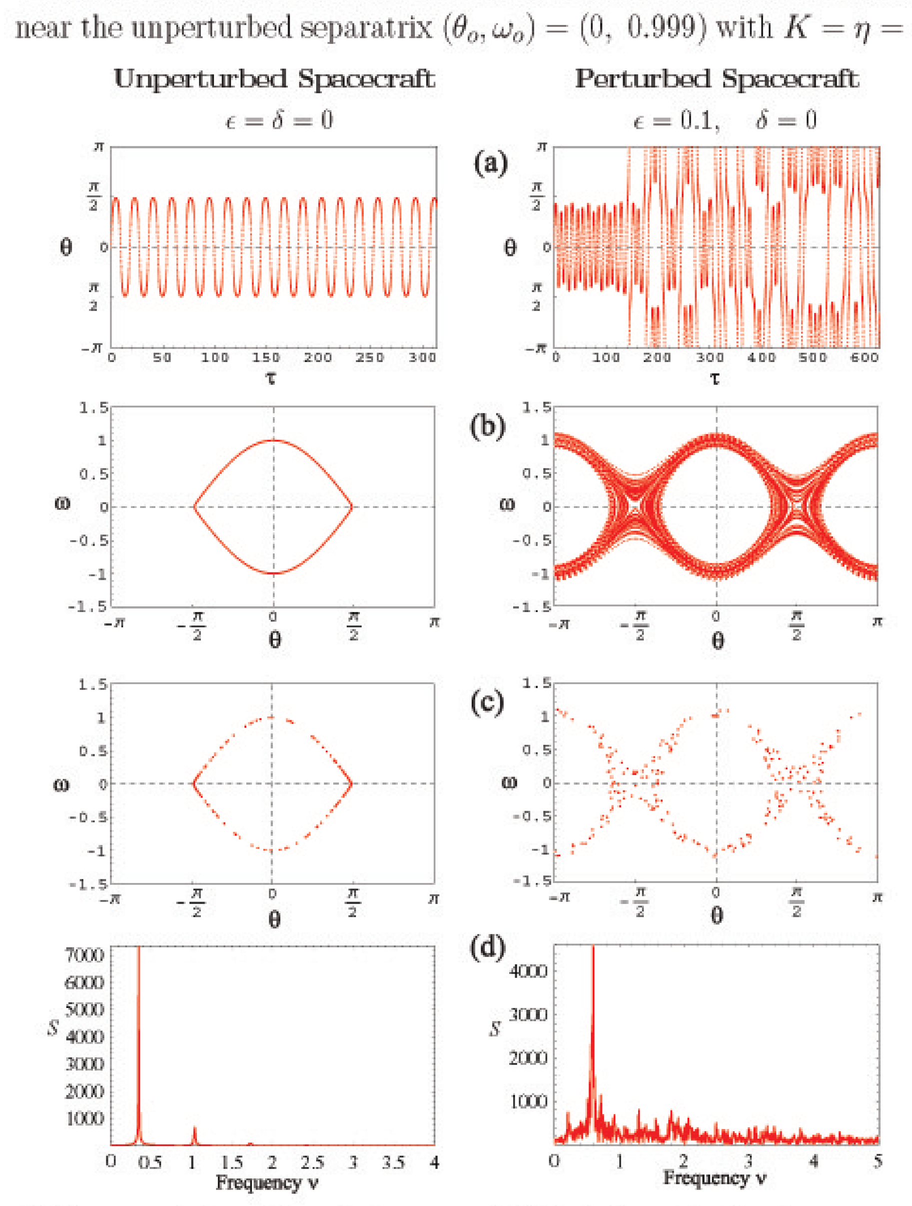
• With this critical value δ_c is easy to establish the following **analytical criterion for the existence of heteroclinic chaos**

For $\delta < \delta_c \Rightarrow M(\tau_0)$ has simple zeroes $\Rightarrow$ Transient chaotic motions

For $\delta \geq \delta_c \Rightarrow M(\tau_0)$ has not simple zeroes $\Rightarrow$ Regular motions

5. NUMERICAL SIMULATIONS. EFFECT OF THE VARIABLE MOMENT OF INERTIA

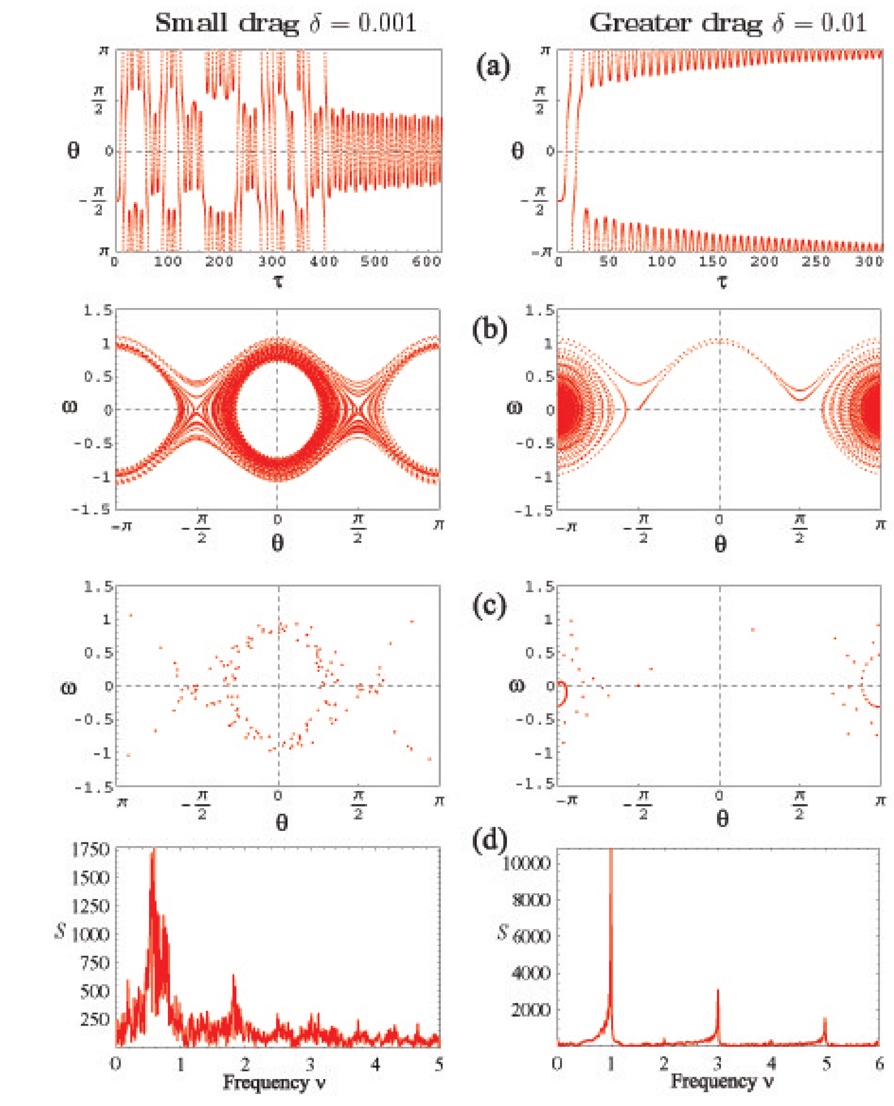
• Numerical simulation of the libration motion for a initial condition near the unperturbed separatrix $(\theta_o, \omega_o) = (0,\ 0.999)$ with $K = \eta = 1$



a) Time evolution of angle θ. b) Trajectory in the phase space.
c) Poincarè surface of section. d) Power spectrum.

6. NUMERICAL SIMULATIONS. EFFECT OF THE AERODYNAMIC DRAG

• Numerical simulation of the libration motion of the spacecraft under both perturbations ($K = \eta = 1$, $\epsilon = 0.1$, $\delta \neq 0$) for a initial condition close to the unperturbed separatrix $(\theta_o, \omega_o) = (-\pi/2, 0.001)$



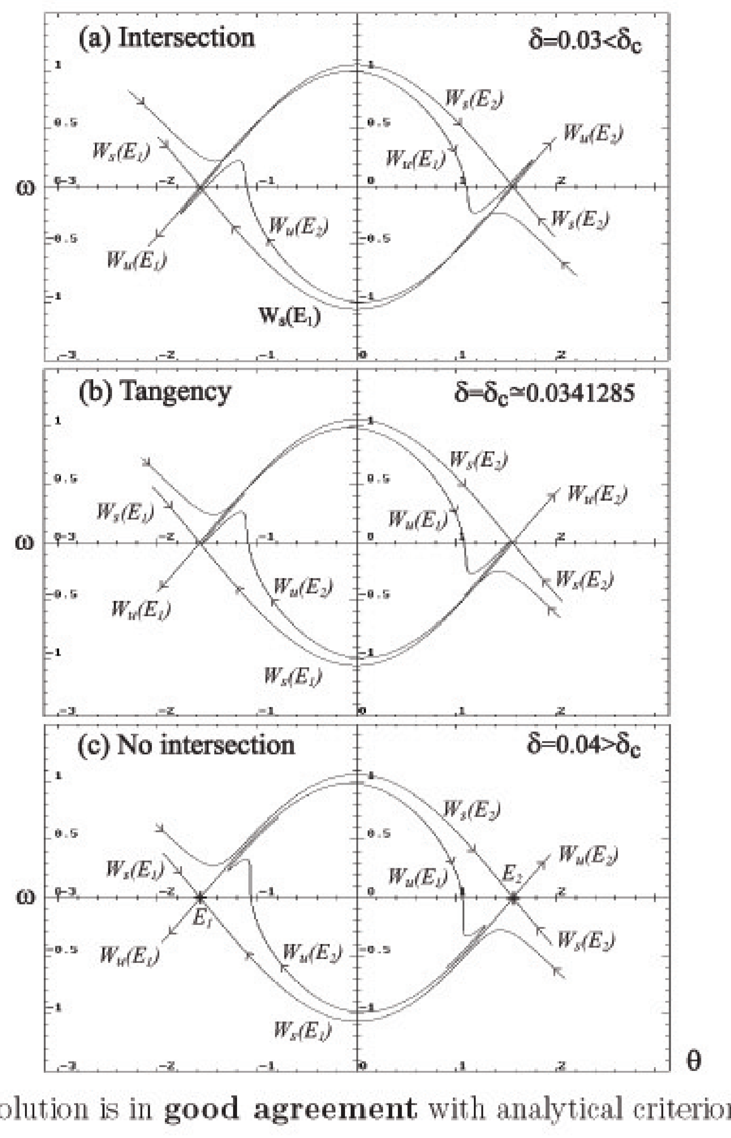
a) Time evolution of angle θ. b) Trajectory in the phase space.
c) Poincarè surface of section. d) Power spectrum.

7. STABLE AND UNSTABLE MANIFOLDS. HETEROCLINIC INTERSECTIONS

• Fixing the parameters $K = \eta = 1$, $\epsilon = 0.1$, the analytical criterion (2) gives a **critical drag parameter** of $\delta_c \approx 0.0341285$.

• We have numerically calculated [5] the invariant manifolds for those values of K, η, ϵ, tuning δ around that δ_c in order to detect the heteroclinic intersections.

Evolution of the manifolds as function of δ around the critical δ_c.



This evolution is in **good agreement** with analytical criterion (2).

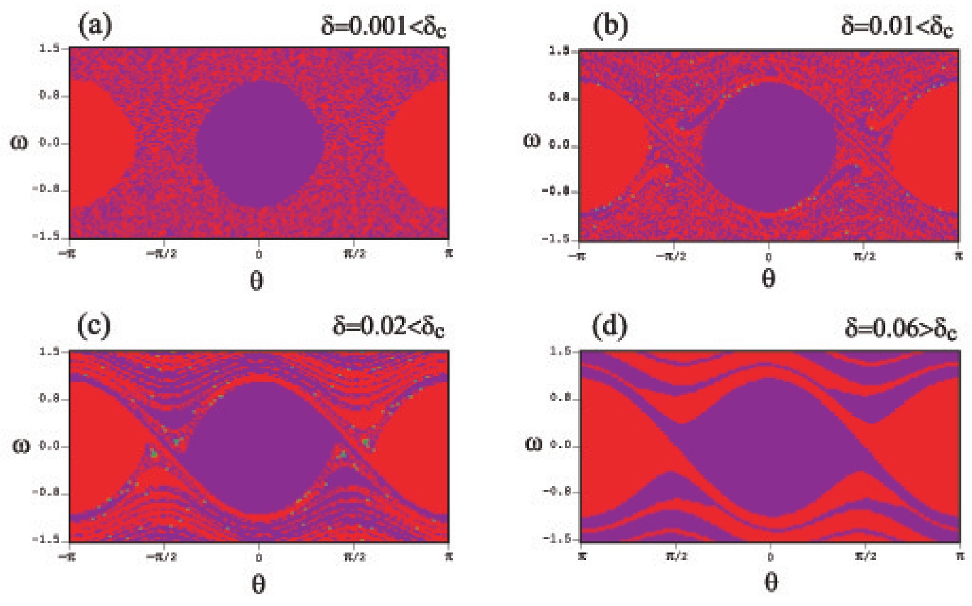
8. ATTRACTION BASINS

• The chaotic dynamical feature of the system is also reflected, in a very random **asymptotic behavior**.

• The main effect of the **drag** is opposing the libration motion. So, it is expected that the motion will decay, and the **final state** of the spacecraft will be at rest and with a **constant angle** $\theta = 0$ or $\theta = \pi$.

• That is, the two stable equilibria $E_3 \equiv (0,\ 0)$ and $E_4 \equiv (\pi,\ 0)$ are **two sinks for the system**.

• For fixed values of K, ϵ and η, we have calculated the attraction basins[5] of the system, tuning δ from the chaotic regime ($\delta < \delta_c$) to the regular one ($\delta > \delta_c$) to detect changes in the basins geometry.



Evolution of the attraction basins as a function of δ ($K = \eta = 1$, $\epsilon = 0.1$).
In red, initial conditions tending to $E_4 \equiv (\pi, 0)$.
Blue color stands for initial conditions tending to $E_3 \equiv (0, 0)$.

CONCLUSIONS

• We have studied the libration motion of an asymmetric spacecraft in circular orbit under a gravity gradient torque. The system is perturbed by a viscous drag and a time–dependent periodic moment of inertia.

• We have established the existence of transient heteroclinic chaos by means of the Melnikov method.

• This method has provided an analytical criterion for the existence of chaotic behavior in terms of the system parameters.

• We have found a transition from chaotic to regular regime, as the heteroclinic chaos can be removed by increasing the viscous drag.

• The analytical results given by the Melnikov method have been confirmed with very good agreement by numerical simulations.

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