ON PARAFREE RESIDUALLY FREE GROUPS

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Abstract

Parafree groups introduced by Gilbert Baumslag in the 60s share many properties with free groups in an attempt to resolve the conjecture that a group of cohomological dimension one is free. Formally, a group is said to be parafree if its quotients by the terms of its lower central series are the same as those of a free group and if it is residually nilpotent.

Residually free and fully residually free groups are a natural class of groups that appear naturally in various contexts in group theory. Most recently, the theory of limit groups and finitely generated residually free groups came into prominence in the works of Kharlampovich-Miasnikov and Sela on the elementary theory of free groups.

In this work we will explore the structure of parafree residually free groups. In particular, we prove that any parafree residually free group is a limit group and then we classify limit groups which are parafree.

Resumen

Los grupos Parafree introducidos por Gilbert Baumslag en los años 60 comparten muchas propiedades con los grupos libres en un intento de resolver la conjetura de que un grupo de dimensión cohomológica uno es libre. Formalmente, se dice que un grupo es Parafree si los cocientes por los términos de su serie central inferior son los mismos que los de un grupo libre y si es residualmente nilpotente.

Los grupos residualmente libres y totalmente residuales libres son una clase de grupos que aparecen de forma natural en varios contextos de la teoría de grupos. Recientemente, la teoría de los grupos límite y de los grupos residualmente libres finitamente generados ha cobrado importancia gracias a los trabajos de Kharlampovich-Miasnikov y Sela sobre la teoría elemental de los grupos libres.

En este trabajo exploraremos la estructura de los grupos parafree y residualmente libres. En particular, demostramos que cualquier grupo residualmente libre parafree es un grupo límite y, a continuación, clasificamos los grupos límite que son parafree.

Contents

Introduction.

This work aims to study residually free parafree groups. Both of these families of groups have residual properties and they share a lot of properties of free groups.

The first family we will study are residually free groups, especially an important subclass of residually free groups called fully residually free groups. Of course, free groups are fully residually free groups but also there are non-free groups that are fully residually groups, for example, fundamental groups of Riemann surfaces, surface groups in short. An important property of fully residually free groups is that they have the same first-order theory as a free group.

What we call first-order theory is the set of true sentences in a group constructed using logic operators. For example, the following sentence expresses that a group is abelian.

$$\forall x \forall y \; [x, y] = 1$$

The theory of fully residually free groups play an important role in the solution to the conjectures that Alfred Tarski proposed about the first-order theory of free groups.

Tarski Conjecture. *Any two non-abelian free groups satisfy the same first-order theory*

Tarski Conjecture. *If the non-abelian free group H is a free factor in the free group G then the inclusion map $i : H \to G$ is an elementary embedding.*

Tarski Conjecture. *The elementary theory of the countable non-Abelian free groups is decidable.*

These questions then became well-known conjectures but remained open for 60 years. They were proven in the period 1996-2006 independently by Olga Kharlampovich and Alexei Myasnikov and by Zlil Sela. The proofs, by both sets of authors, were monumental and involved the development of several new areas of infinite group theory.

Kharlampovich and Myasnikov use techniques of algebraic geometry to solve equations over free groups and prove Tarski's conjectures meanwhile Sela gives a geometric approach. He defines limit groups as the quotient of a group and the stable kernel of a stable sequence and these are precisely the finitely generated fully residually free groups.

In his work Sela describes limit groups as a graph of groups with a nice JSJ decomposition. Using this, we construct limit groups inductively.

The second family of groups we will study is parafree groups, groups that are residually free and have the same isomorphism types of nilpotent quotients as some free group. The possibility that free groups can be characterized by their lower central series was originally raised by Hanna Neumann.

Hanna Neumann Conjecture. *Suppose that G is a finitely generated residually nilpotent group with the same lower central sequence as a free group. Then G is free*

Gilbert Baumslag answered this question with a negative example, i.e., there exist non–free parafree groups. An unsolved conjecture about parafree groups is the following.

Baumslag Conjecture. *The second homology of an Eilenberg-Maclane space for a parafree group vanishes.*

In this thesis we do not address this question. We are concerned with the construction of parafree limit groups. The main results of this thesis are the following.

Theorem. *Parafree groups that are universally-existentially equivalent to a non-abelian free group are free groups.*

Theorem. *Constructible parafree limit groups are parafree and limit groups.*

§1. Free groups and free products.

Let G be a group with generating non-empty set X and relations R then we can present G as

$$G = \langle X \mid R \rangle$$

We allow R to be empty.

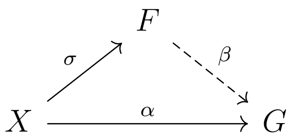
1.1 Free groups.

Definition 1.1. *Let X be a set, then the group F is free on X if X is a set of generators for F and there are no non-trivial relations. In particular, F has a presentation of the form.*

$$F = \langle X \mid \rangle$$

We can define free groups in terms of category theory.

Definition 1.2. *Let F be a group, X a non-empty set, and $\sigma : X \mapsto F$ an injective function, then F is said to be a* ***free group on*** $\mathbf{X}$ *if to each function $\alpha : X \to G$, where G is a group, there corresponds a unique homomorphism $\beta : F \to G$ such that $\alpha = \beta\sigma$, that is, the following diagram commute.*



It is easy to prove that σ is necessarily injective, so we can say that β is the unique extension of α to F. With this definition, we can prove the following.

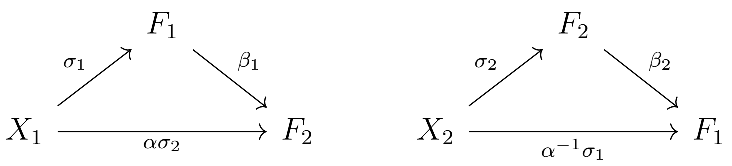
Proposition 1.3. *Every group is an image of a free group.*

Proof. Let G be a group with generating set X and let F be a free group on a set Y, then if we take $f : Y \to X$ a surjective map we can extend f to a homomorphism from F to X and it will be an epimorphism as G is generated by X. ■

Also, free groups are characterized by his cardinally.

Proposition 1.4. *Let F_1, F_2 be free groups on the sets X_1, X_2 respectively and $|X_1| = |X_2|$ then $F_1 \cong F_2$.*

Proof. Let $\sigma_1 : X_1 \to F_1$ and $\sigma_2 : X_2 \to F_2$ be the given injections and let $\alpha : X_1 \to X_2$ be a bijection, this is possible as X_1 and X_2 have the same cardinal, then



Hence $\sigma_1\beta_1\beta_2 = \alpha\sigma_2\beta_2 = \alpha\alpha^{-1}\sigma_1 = \sigma_1$, so then $\beta_1\beta_2 = \mathrm{id}_{F_1}$, and with the same argument $\beta_2\beta_1 = \mathrm{id}_{F_2}$ so β_1 is an isomorphism and $F_1 \cong F_2$ ■

With this, we can define the rank of a free group.

Definition 1.5. *Let F be a free group, the **rank** of F is the cardinality of X.*

Now we give a few examples.

Example 1.6. *The free group of rank 2 has the following presentation.*

$$F =< a, b >$$

Example 1.7. *The group $(\mathbb{Z}, +)$ is a free group on the set $\{1\}$, this group is not only free as the elements of the group commute, this kind of free groups are known as free abelian groups.*

Definition 1.8. *Let F be a group, X a non-empty set, and $\sigma : F \mapsto X$ an injective function. Then F is said to be a **free abelian group on X** if to each function $\alpha : X \to H$, where H is an abelian group, there corresponds a unique homomorphism $\beta : F \to H$ such that $\alpha = \sigma\beta$.*

Free abelian groups are characterized by the cardinal of the generator set.

Proposition 1.9. *All free groups of rank 2 or more can't be abelian.*

This gives us the following corollary.

Corollary 1.10. *A free group is abelian if and only if it is cyclic.*

Theorem 1.11. *A free group of rank n contains $2^n - 1$ subgroups of index 2.*

Proof. Let F be the free group on the set $X = \{x_1, \ldots, x_n\}$ and $\sigma F \to X$ injective. Take the map $f : X \to C_2 = \langle c \rangle$ defined as

$$f(x_1) = c \quad f(x_i) = 1 \; i = 2, \ldots, n$$

By the universal property of free groups there exists a unique homomorphism $\phi : F \to C_2$ such that $f = \phi\sigma$ then we have that $f(x_i) = \phi(x_i)$ for all $i = 1 \ldots, n$. Taking $\ker\phi$ we have that

$$F/\ker\phi = \{\ker\phi, x_1 \ker\phi\}$$

Then $\ker \phi$ is a subgroup of index 2. As we can take f in a $2^n - 1$ different ways we have that F contains $2^n - 1$ different subgroups of index 2. ■

1.2 Free products.

Definition 1.12. *Let $A = \langle X_1 \mid R \rangle$ and $B = \langle X_2 \mid S \rangle$ two groups then the* ***free product*** *of A and B is the group, denoted as $A * B$, with presentation.*

$$A * B = \langle X_1, X_2 \mid R, S \rangle$$

This definition can be extended to a family of groups.

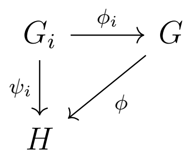
Definition 1.13. *Let G_i be a family of groups with $i \in I$ a set of subindexes, then the free product of the family, denoted as $*G_i$ is the group with the presentation.*

$$*G_i = \langle \text{gens}\{G_i\}_{i \in I} \mid \text{rels}\{G_i\}_{i \in I} \rangle$$

this is the group whose generators consist of the disjoint union of the generators of G_i for every $i \in I$ and whose relations are the disjoint union of the generators of G_i for every $i \in I$.

Also, we can write these definitions with universal properties.

Definition 1.14. *Let G_i a collection of groups with $i \in I$ a set of subindex. A* ***free product*** *consists in a group G and a collection of homomorphisms $\phi_i : G_i \to G$ such that for any homomorphism ψ_i between G_i and any group H there is a unique homomorphism $\psi : G \to H$ such that $\psi_i = \psi \phi_i$. In other words, the following diagram commutes.*



Theorem 1.15 (Kurosh subgroup theorem)**.** *Every subgroup of a free product is itself a free product. Explicitly if $G = A * B$ and H is a subset of G then.*

$$H = F(X) * \left(*_i g_i A_i g_i^{-1} \right) * \left(*_j h_j B_j h_j^{-1} \right)$$

Where $F(X)$ is a free group generated by $X \subset G$, A_i is a family of subgroups of A, B_j is a family of subgroups of B, and g_i and h_j are elements of G.

Proof. One can see a proof of a version of this theorem in [Rob96]. ■

We give the following trivial property of free products of free groups:

Theorem 1.16. *Free product of free groups is free.*

Also, we see how we can write a free group as a free product.

Theorem 1.17. *Let F be a free group and let ϕ be a surjective homomorphism from F onto the free product $*A_i$. Then $F = *F_i$ where $\phi(F_i) = A_i$*

Proof. Take the inverse map of ϕ, ϕ^{-1} and denote by $\phi^{-1}(A_i) = F_i$ then

$$*_i F_i = *_i \phi^{-1}(A_i) = \phi^{-1}(*_i A_i) = F$$

■

1.2.1 Amalgamated product.

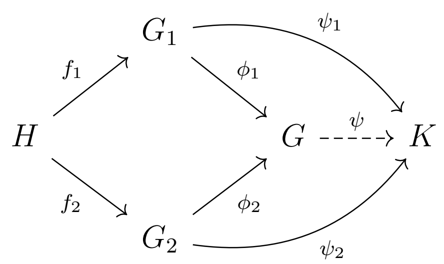
The main idea of amalgamation of groups is trying to "glue" two groups G_1 and G_2 with two isomorphic subgroups H and K, respectively, in such a way that these two subgroups become the same subgroup for the generated group.

Definition 1.18. *Let $G_1 = \langle X_1 \mid R \rangle$ and $G_2 = \langle X_2 \mid S \rangle$ with H a subgroup of G_1 and K a subgroup of G_2 with $f : H \to K$ and isomorphism, then the* ***free product of $\mathbf{G_1}$ and $\mathbf{G_2}$ amalgamated by H*** *is the group denoted by $A *_H B$ with presentation.*

$$A *_H B = \langle X_1, X_2 \mid R, S, H = f(H) \rangle$$

In particular, if we take $H = \{1\}$ we have the free product of A and B. As with the free product, we can write this definition in terms of universal properties.

Definition 1.19. *Let G_1 and G_2 be groups, and H another group such that $f_i : H \to G_i$ is a monomorphism for $i = 1, 2$. Then the* ***free product of $\mathbf{G_1}$ and $\mathbf{G_2}$ with H amalgamated*** *is the group $G = A *_H B$ if exists a pair of homomorphisms $\phi_i : G_1 \to G$ for $i = 1, 2$ with $\phi_1 f_1 = \phi_2 f_2$ such that for any pair of homomorphisms $\psi_1 : G_1 \to K$, $\psi_2 : G_2 \to K$ into a group K with $\psi_1 f_1 = \psi_2 f_2$ there exists a unique homomorphism $\psi : G \to K$ such that $\psi_1 = \psi\phi_1$ and $\psi_2 = \psi\phi_2$, i.e., the following diagram commutes.*



Example 1.20. *Let $G_1 = \langle x \rangle$ and $G_2 = \langle y \rangle$ be groups and take the subgroups $H = \langle x^2 \rangle$ and $K = \langle y^2 \rangle$ with the isomorphism $f : H \to K$ then the free product of G_1 and G_2 with H amalgamated is*

$$G_1 *_H G_2 = \langle x, y \mid x^2 = y^3 \rangle$$

1.2.2 HNN extensions.

The HNN-extension is a kind of a free product with the idea that if G is a group with isomorphic subgroups we extend G into a group such that every isomorphic subgroup is conjugated.

Definition 1.21. *Let $G = \langle X \mid R \rangle$ be a group with H_i proper subgroups and a collection of isomorphisms $\phi_i : H_i \to K$ then the* ***HNN extension of*** **G** *is the group, denoted by $G*_K$, of the form.*

$$G*_K = \langle X, \{t_i\}_{i\in I} \mid R, \{t_i H_i t_i^{-1} = \phi_i(H_i)\}_{i\in I} \rangle$$

With I being a set of subindexes. We call G the base, $\{t_i\}_{i\in I}$ is the free part and $\{H_i, \phi_i(H_i)\}_{i\in I}$ the associated subgroups of $G*_K$.

As before we write this definition in terms of category theory.

Definition 1.22. *Let G be a group with H and K proper subgroups with an isomorphism $\phi : H \to K$. Then the* ***HNN extension of*** **G** *consists of a group $G*_K$, a family $\{t_i\}_{i\in I}$ of elements of G and a homomorphism $\psi : G \to G*_K$ with $t(\psi(H))t^{-1} = \psi\phi(H)$ such that for any group A, any $a \in A$ and any homomorphism $f : G \to A$ with $af(G)a = f\phi(A)$ there is a unique homomorphism $\omega : G*_K \to A$ with $\omega(t) = a$ such that $f = \omega\psi$, i.e., the following diagram commutes.*

$$\begin{array}{ccc} G & \xrightarrow{\psi} & G*_K \\ & \searrow^{f} & \downarrow{\scriptstyle \omega} \\ & & A \end{array}$$

Example 1.23. *Take $G = \mathbb{Z} = \langle a \rangle$ we want to make the HNN extension of G for this we take $\alpha : \langle a \rangle \to \langle a^2 \rangle$ this is clearly an isomorphism therefore the HNN extension of G is*

$$G*_{\langle a^2 \rangle} = \langle a, t \mid tat^{-1} = a^2 \rangle$$

This group is known as the Baumslag-Solitar group and is denoted by $BS(1,2)$. The Baumslag-Solitar groups, $BS(m,n)$ are the groups

$$BS(m,n) = \langle a, b \mid ba^m b^{-1} = a^n \rangle$$

In fact, all this groups are HNN extensions with infinite cyclic associated subgroups.

Definition 1.24. *Let G be an HNN extension of a group H with associated subgroups A and B. G is called a **separated HNN extension** if for any $h \in H$*

$$A^h \cap B = 1$$

1.3 Bass-Serre Theory.

Definition 1.25. *A **graph** X is a pair of sets V and E termed the vertices and the edges of X, equipped with three maps*

$$o : E \to V, \quad t : E \to V, \quad ^- : E \to E$$

Satisfying the following:

1. *For every $e \in E$, $e \neq \overline{e}$*
2. *$o(e) = t(\overline{e})$*

We term $t(e)$ the terminus of e, $o(e)$ the origin of e and $\overline{e}$ the inverse of e.

Definition 1.26. *A **graph of groups** is the pair (G, X) given by*

1. *An oriented connected graph X.*
2. *For each vertex $v \in V(X)$, a vertex group G_v*
3. *For each edge $e \in E(X)$, an edge group G_e, equipped with a monomorphism $G_e \to G_{t(e)}$ and $G_e = G_{\overline{e}}$*

Definition 1.27. *A graph that is connected and contains no cycles is called a **tree**.*

Proposition 1.28. *Let X be a graph, then there exists a subgraph that is a tree, moreover, the set of subgraphs that are trees has a non-unique maximal element called **maximal tree**.*

Proof. A direct consequence of Zorn's lemma. ■

Proposition 1.29. *The maximal tree of a connected graph X contains all the vertices of X.*

Proof. Suppose that Γ is the maximal tree of X, as X is connected if $v \in V(X)$ is not in Γ we can extend Γ by adjunction of the vertex v but this is a contradiction with the maximality of Γ, therefore, $v \in V(\Gamma)$. ■

Definition 1.30. *We say that a group G* ***acts on a graph*** *X if it comes equipped with a homomorphism*

$$\phi : G \to \mathrm{Aut}\,(X)$$

If we denote by $g \cdot h$ the action of G over the set of vertex and edges of X, we have that G acts on X in the following way

1. Every vertex v_h is sent to $v_{g \cdot h}$.

2. Every edge e between v_h and v_{hs} is sent to the edge $g \cdot e$ between $v_{g \cdot h}$ and $v_{g \cdot hs}$.

Example 1.31. *Let G be a group, X the set of generators of G. We define the* ***Cayley graph*** *of G relative to X by*

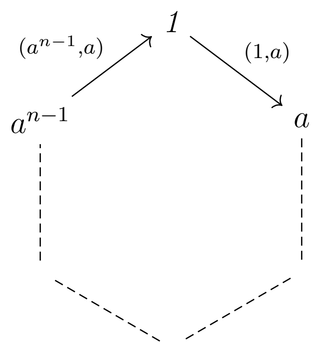
1. *The set of vertexes is G.*

2. *The set of edges is the disjoint union of the sets $G \times X$ and $S \times G$.*

3. *$o(g, s) = g$, $t(g, s) = gs$, $\overline{(g, s)} = (s, g)$ and $\overline{(s, g)} = (g, s)$.*

Notice that G acts on the graph by left multiplication. A few examples of a Cayley graph are the following:

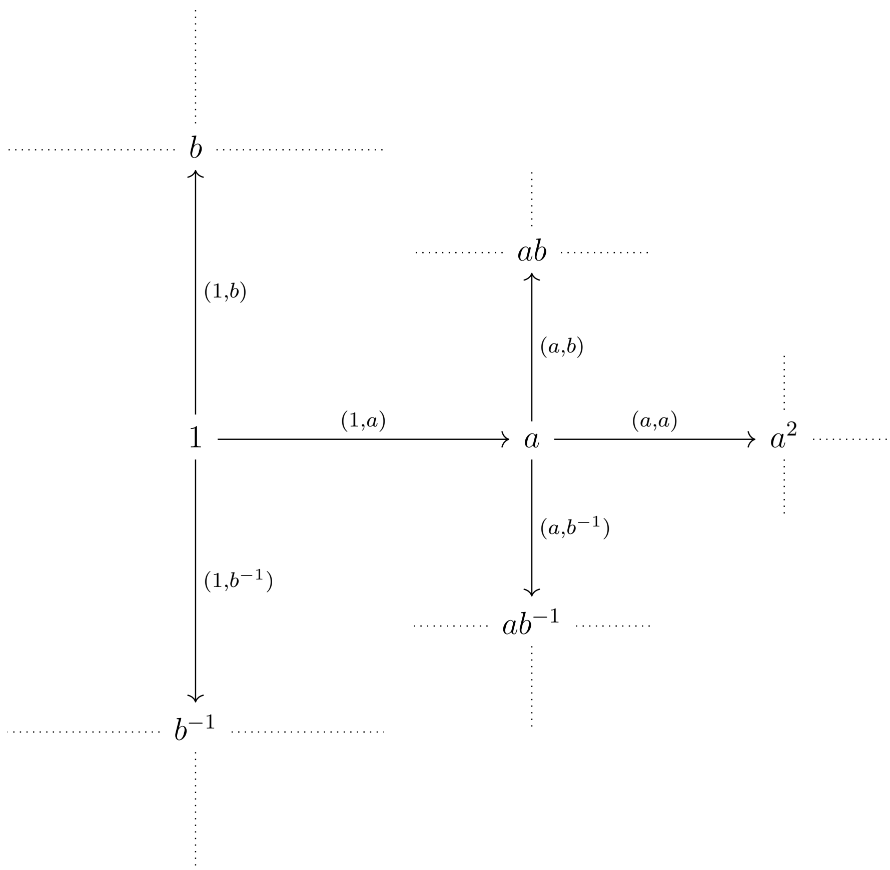
- $G = \langle a \mid a^n = 1 \rangle$



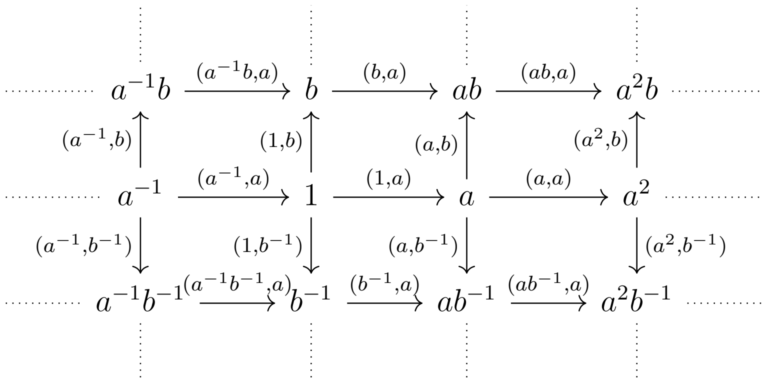
- $G = \langle a \rangle$

$$\cdots\cdots \; a^{-1} \xrightarrow{(a^{-1},a)} 1 \xrightarrow{(1,a)} a \xrightarrow{(a,a)} a^2 \; \cdots\cdots$$

- $G = \langle a, b \rangle$



- $G = \langle a, b \mid ab = ba \rangle$



Definition 1.32. *Let X a graph on which a group G acts. An inversion is a pair of an element $g \in G$ and an edge y of X such that $gy = \overline{y}$. If there is no such pair we say that G* ***acts without inversion****.*

Definition 1.33. *If G acts on a graph without inversion we can define the* ***quotient graph*** *${}^{G}\!/_{X}$ in an obvious way; the vertex sex of ${}^{G}\!/_{X}$ is the quotient of the vertexes of*

X under the action of G and the edge set of ${}^{G}\!/_{X}$ is the quotient of the edges of X under the action of G.

The point here is that since G acts without inversion on X the application $^{-}$ acts as involution on the orbits of the edges therefore we can partition the set of edges, $E(X)$ into $E_+ \cup E_-$. As a consequence, the map

$$p : X \to {}^{G}\!/_{X}$$

defined by $p(e) = G \cdot e = \langle e \rangle$ and $p(v) = G \cdot v = \langle v \rangle$ for each edge e and each vertex v is a morphism of graphs. If $\tilde{T}$ is the maximal tree of the quotient ${}^{G}\!/_{X}$ we say that $\tilde{T}$ lifts to a tree T in X if $p|_T$ is an isomorphism between T and $\tilde{T}$.

Proposition 1.34. *Let G be a group acting without inversion on a graph X, if $\tilde{T}$ is the maximal tree of ${}^{G}\!/_{X}$ then there exists a tree T in X such that $\tilde{T}$ lifts to T.*

Proof. Let T_1 be a tree in X which is maximal subject to p mapping T_1 injectively into $\tilde{T}$ and suppose $p(T_1) \neq \tilde{T}$, then there is a vertex $\langle v \rangle \in V(\tilde{T})$ such that $\langle v \rangle \notin V(p(T_1))$, assume that $\langle w \rangle \in V(p(T_1))$ is the adjacent vertex of $\langle v \rangle$ and let $\langle e \rangle$ be the edge of $\tilde{T}$ between them.

As $\langle w \rangle \in V(p(T_1))$ then we have $g \cdot w \in T_1$ for some $g \in G$ and consider $g \cdot e \in X$ then $g \cdot e \notin T_1$ because $p(g \cdot e) = \langle e \rangle$. If we adjoin the edges $g \cdot e$, $g \cdot \overline{e}$ and the vertex $t(g \cdot e)$ to T_1 the resultant graph T_2 is a tree with $t(g \cdot e) \notin T_1$ as $p(t(g \cdot e)) = t(\langle e \rangle) = \langle v \rangle$ but then p is injective on T_2 contradicting with the maximality of T_1, therefore $p(T_1) = \tilde{T}$. ■

Definition 1.35. *Let Y be a connected graph, T the maximal tree of Y and (G, Y) the graph of groups. The **fundamental group** $\pi_1(G, Y, T)$ of (G, Y) at T is the group generated by the vertex groups and the edge elements g_y subject to the relations*

$$\begin{aligned} g_y a^y g_y^{-1} = a^{\overline{y}} \quad g_y g_{\overline{y}} = 1 \quad &\textit{if} \quad y \in E(Y),\ a \in G_y \\ g_y = 1 \quad &\textit{if} \quad y \in E(T) \end{aligned}$$

We can write the presentation of the fundamental group in the following way, if $y \in E(Y)$ and $y \notin E(T)$, $p \in V(Y)$ and $a \in G$ then

$$\pi_1(G, Y, T) = \langle G_p, g_y \mid g_y a^y g_y^{-1} = a^{\overline{y}}, g_y g_{\overline{y}} = 1 \rangle$$

In [SS02] and [Bau93] it is proven that the fundamental group is independent on the choice of T.

Theorem 1.36. *Let G be a group acting without inversion on a tree X then*

$$\pi_1(G, Y, T) \cong G$$

where $Y = {}^{G}\!/_{X}$.

One of the great consequences of the Bass-Serre theory is that we can characterize free groups with the action on a tree.

Definition 1.37. *A group G is said to act freely on a tree if it acts without inversion and only the identity element fixes a vertex.*

Theorem 1.38. *G acts freely on a tree, then G is free.*

We can prove the first result of Kurosh theorem.

Theorem 1.39 (Subgroups of a free group)**.** *Let F be a free group, the every subgroup H of F is itself free.*

Proof. As F is free it acts freely on a tree T, if H is a subgroup of F then acts freely on a subtree of T, then itself is free. ■

Also using Bass-Serre theory we can give equivalent theorems to the subgroups of free groups in the case of HNN extensions and amalgamated products.

Definition 1.40. *Let Γ be a tree which is a graph of groups with $\{G_i\}_{i\in I}$ vertex groups. Suppose that each for pair of vertexes G_i and G_j that are joined there is an associated isomorphism ϕ_{ij} from a subgroup U_{ij} of G_i onto a subgroup U_{ji} of G_j, such that $\phi_{ij} = \phi_{ji}^{-1}$. The associated group G to the tree Γ is called* ***tree product*** *of the factors and is denoted by*

$$\left(\prod *A_i : \ U_{ij} = \phi_{ij}(U_{ij})\right)$$

Theorem 1.41 (Subgroups of a free group with amalgamation)**.** *Suppose that*

$$G = A *_C B$$

is the amalgamated product of two groups, then if H is a subgroup of G it is itself an HNN extension of a tree product in which the vertex groups are conjugates of subgroups of either A or B and edge groups are conjugates of subgroups of C. The associated subgroups are conjugates of subgroups of C.

Theorem 1.42. *Let*

$$G = A *_C B$$

where A and B are free and C is cyclic, then every finitely generated subgroup of G is finitely presented.

Theorem 1.43 (Subgroups of an HNN extension)**.** *Let* $G*_{\phi_i}$ *be the HNN extension of a group* G *with presentation.*

$$G*_{\phi_i} = \langle X, \{t_i\}_{i\in I} \mid R, \{t_i H_i t_i^{-1} = \phi_i(H_i)\}_{i\in I}\rangle$$

If K *is a subgroup of* $G*_{\phi_i}$ *then* K *is a treed HNN group whose base is a tree product* S *with vertices of the form* $gXg^{-1}\cap K$ *with* $g \in G$*, amalgamated subgroups in the base are either trivial or* $t_iH_it_i^{-1} \cap K$ *and the associated subgroups are contained in a vertex of* S *and equals this vertex or has the form* $t_iH_it_i^{-1} \cap K$*.*

The proof of this theorems can be found in [KS70] and in [KS71] respectively.

§2. Residually free and fully residually free groups.

Let $\mathcal{P}$ be a property of a group inherited by subgroups. We say that a group G is **residually $\mathcal{P}$** if for all $g \in G$ non-trivial there exists a group H_g having the property $\mathcal{P}$ and an epimorphism $\phi_g : G \to H_g$ such that $\phi_g(g) \neq 1$. Equivalently G is **residually $\mathcal{P}$** if given any non-trivial element $g \in G$, there exists a normal subgroup N of G such that $g \notin N$ with G/N having the property $\mathcal{P}$.

There are considerable residual properties studied, but now we want to focus on residual freeness and later on residually nilpotency.

Definition 2.1. *A group* G *is **residually free** if for each* $g \in G$ *non-trivial exists a free group* F_g *and an epimorphism* $h_g : G \to F_g$ *such that* $h_g(g) \neq 1$*. Equivalently for each* $g \in G$ *there is a normal subgroup* N *such that* G/N *is free and* $g \notin N$*.*

Proposition 2.2. *Every subgroup of a residually free group is residually free.*

G. Baumslag showed the following property of subgroups of residually free groups

Theorem 2.3 ([Bau62])**.** *Every* 2 *generated subgroup of a residually free group is free.*

Now we need to extend residual freeness to a set of elements of G.

Definition 2.4. *A group* G *is* **n-*residually free***, *for a natural number* n*, provided to* *every ordered n-tuple* $(g_1, \ldots, g_n) \in (G\backslash\{1\})^n$ *there is a free group* F *and an epimorphism* $h : G \to F$ *such that* $h(g_i) \neq 1$ *for every* $i \in \{1, \ldots, n\}$*. If* G *is n-residually free for every* $n \in \mathbb{N}$ G *is* ***fully residually free****.*

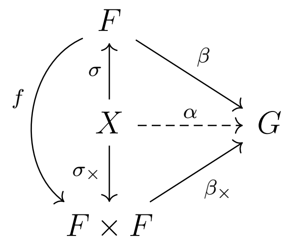
Example 2.5. *Trivially every free group is residually free furthermore is fully residually free. But not every residually free group is a free group as we will see later.*

Example 2.6. *It is easy to prove that every free abelian group is fully residually free as every subgroup is normal.*

Example 2.7. *Let* F *be a non-abelian free group on a set* X*, we want to show that* $F \times F$ *is not free. Using the definition in terms of category theory we have that if* $\sigma : F \mapsto X$ *is an injective function then for every* $\alpha : X \to G$*, where* G *is a group, there is a unique homomorphism* $\beta : F \to G$ *such that* $\alpha = \beta\sigma$*.*

Suppose that $F \times F$ *is free also then if* $\sigma_\times : X \to F \times F$ *is injective then for every* $\alpha : X \to G$*, where* G *is a group, there is a unique homomorphism* $\beta_\times : F \times F \to G$ *such that* $\alpha = \beta_\times\sigma_\times$*.*

We can take $f : F \mapsto F \times F$ *being an epimorphism then we have the following diagram,*



As we can see we have $\alpha = \beta_\times f\sigma$ *but this is a contradiction with the unicity of* β *so* $F \times F$ *can't be free on* X*.*

Now we want to show that $F \times F$ *is residually free, taking the* f *mentioned before we got that* $F \times F$ *is residually free. Before we show that this group is not fully residually free we have to introduce some new concepts.*

2.0.1 Commutative transitive.

Definition 2.8. *A group* G *is* ***commutative transitive****, CT to shorten, provided the relation of commutativity is transitive on non-identity elements, i.e. for every* $x, y, z \in G\backslash\{1\}$ *if* $[x, y] = 1$ *and* $[y, z] = 1$ *then* $[x, z] = 1$*.*

This property holds on every free group as the centralizer of every element is cyclic. The following result gives equivalent formulations of the CT property.

Lemma 2.9. *Let G be a group. The following statements are pairwise equivalent.*

1. *G is commutative transitive.*

2. *The centralizer $C_G(x)$ of every non-trivial element $x \in G$ is abelian.*

3. *Every pair of distinct maximal abelian subgroups in G has trivial intersection.*

Proof. (1. $\Longrightarrow$ 2.) Take $z, y \in C_G(x)$ for a non-trivial element $x \in G$ then $[z, x] = 1$ and $[x, y] = 1$ therefore $[z, y] = 1$ so $C_G(x)$ is abelian.

(2. $\Longrightarrow$ 3.) Suppose that every centralizer is abelian and $M_1 \cap M_2 \neq \{1\}$ for two maximal abelian subgroups. Then there exists $x \in M_1 \cap M_2$ non-trivial, let $y \in M_1 \backslash (M_1 \cap M_2)$. If y commutes with every $z \in M_2$ then the subgroup generated by M_2 and y would be abelian, but this is a contradiction with the maximality of M_2 therefore we can suppose that y and z do not commute. Now we have that x and z commute as both are in M_2 but also x and y commute as both are in M_1 and in $C_G(x)$ and by hypothesis $C_G(x)$ is abelian then y and z must commute, which is a contradiction so $M_1 \cap M_2 \neq \{1\}$.

(3. $\Longrightarrow$ 1.) Suppose that $[x, y] = 1$ and $[y, z] = 1$ then x, y, z are in the same maximal abelian subgroup by hypothesis therefore $[x, z] = 1$ so G is CT.

∎

Lemma 2.10. *Any fully residually free group is commutative transitive.*

Proof. Take $a, b, c \in G$ non-trivial elements in a fully residually free group such that $[a, b] = 1 = [b, c]$. As G is fully residually free exists a homomorphism ϕ between G and a free group F such that $\phi(a), \phi(b), \phi(c)$ are non-trivial. Since ϕ is a homomorphism $\phi([a, b]) = 1 = \phi([b, c])$ and as F is free is commutative transitive then $\phi([a, c]) = 1$ then $[a, c] = 1$ so G is commutative transitive. ∎

Example 2.11. *Now we can show that $F \times F$ with F a free group is not fully residually free as it is not commutative transitive.*

With this result we have.

Proposition 2.12. *The class of fully residually free groups is a proper subclass of the residually free groups class, i.e. all fully residually free groups are contained in the class of residually free groups while the inverse inclusion is not true.*

Example 2.13. *The free product of residually free groups is not necessarily a residually free group, For example take A as the direct product of a free group of rank 1 and a free group of rank 2.*

$$A = \langle a_1, a_2, a_3 \mid [a_1, a_2] = 1,\ [a_1, a_3] = 1 \rangle$$

and $B = \langle b_1, b_2 \mid \ \rangle$ which is free therefore residually free. Let $A * B$ be the free product of A and B and consider

$$w = [[[b, a_1], [b_2, a_1]], [a_2, a_3]]$$

Clearly $w \in A * B$, if $A * B$ were residually free there would be a free group F and a map $\phi_w : A * B \to F$ such that $\phi(w) \neq 1$. Then $\phi(a_1) \neq 1$ and $[\phi(a_2), \phi(a_3)] \neq 1$ and by the relations of the group A, $[\phi(a_1), \phi(a_2)] = 1 = [\phi(a_2), \phi(a_3)]$ but then $\phi(a_2)$ and $\phi(a_3)$ are contained in a cyclic subgroup of F containing $\phi(a_1)$ because F is free. This forces $\phi(a_2)$ and $\phi(a_3)$ to commute which is a contradiction with $[\phi(a_2), \phi(a_3)] \neq 1$. Therefore $A * B$ can't be residually free.

In [Bau67a] B. Baumslag demonstrated a characterization of residually free non-fully residually free

Theorem 2.14. *Let G be a residually free group. Then G is non-fully residually free if and only if it does contain an isomorphic subgroup to $F_2 \times \mathbb{Z}$.*

Proof. Suppose G is non-fully residually free then is non-commutative transitive so there are elements $g_1, g_2 \in G$ such that commute with $x \in G$ but do not commute with each other. Let $H = \langle g_1, g_2 \rangle$ and $K = \langle c \rangle$. By the properties of residually free groups H is free then centerless so $H \cap K = 1$ then $H \times K$ is an isomorphic subgroup to $F_2 \times \mathbb{Z}$.

Conversely if G contains an isomorphic subgroup to $F_2 \times \mathbb{Z}$, then the two generators of F_2 make G non-CT therefore non-fully residually free. ■

We want to see under what conditions the free product of two groups is fully residually free. To achieve this, we prove that a residually free group that is commutative transitive must be fully residually free.

2.0.2 Conjugately separated abelian.

Definition 2.15. *Let G be a group and H a subgroup of G. H is* ***conjugately separated*** *in G provided $gHg^{-1} \cap H = \{1\}$ for $g \in G \backslash H$.*

Definition 2.16. *A group G is said to be* ***conjugately separated abelian****, CSA group, if every maximal abelian subgroup is conjugately separated.*

Lemma 2.17. *Let G be a fully residually free group then G is a CSA group.*

Proof. Take u a non-trivial element of G and as G is fully residually free the maximal abelian subgroups are the same as the centralizers so we take $C_g(u)$ and we prove that is conjugately separated. Take $w = gzg^{-1} \neq 1$ in $gC_G(u)g^{-1} \cap C_G(u)$ with $g \notin C_G(u)$ then $[g, u] \neq 1$. As G is fully residually free there is a free group F and an epimorphism $\phi : G \to F$ such that $\phi(w) \neq 1$ and $\phi([g, u]) \neq 1$ then $\phi(w) \in \phi(g)C_F(\phi(u))\phi(g)^{-1} \cap C_F(\phi(u))$ however as every free group is CTA then $\phi(g) \in C_F(\phi(u))$ contradicting $\phi([g, u]) \neq 1$ this shows that if $gC_G(u)g^{-1} \cap C_G(u) \neq 1$ then $g \in C_G(u)$ hence all maximal abelian subgroups are conjugately separated and G is a CSA group. ■

Lemma 2.18. *Let G be a non-abelian CSA group, normal abelian subgroups are trivial.*

Proof. Let G be a non-abelian CSA group with N a normal abelian subgroup contained in the maximal M abelian subgroup. Let $g \notin M$ then

$$N = gNg^{-1} \subset gMg^{-1}$$

Then necessarily $N = \{1\}$. ■

Lemma 2.19. *The class of CSA groups is a proper subclass of CT groups.*

Proof. First we show that CSA groups are CT. Suppose that G is a CSA group, then if M_1 and M_2 are maximal abelian subgroups of G then by definition they are conjugately separated. Assume that exists $z \in M_1 \cap M_2$ and $w \in M_1 \backslash M_2$ then $wzw^{-1} = z$ is a non-trivial element of $wM_1w^{-1} \cap M_1$ but as M_1 is conjugately separated this is impossible then $M_1 \subset M_2$ and by maximality $M_1 = M_2$ and G is a CT group.

Now let's see that exists a CT group that is not CSA. Let p, q two distinct primes such that $p|q-1$ and let G be a non-abelian group of order pq then the centralizer of every element in G is of order p or q and hence G is CT. The q-Sylow subgroup of G is normal and by the above lemma, G is necessarily non-CSA. ■

Proposition 2.20. *Let G be a residually free group and commutative transitive group. If $g_1, \ldots, g_m$ a set of non-trivial elements of G then if there exists $g \in G$ such that $g \notin N$ for every N normal subgroup of G then $g_1, \ldots, g_m \notin N$.*

Proof. Assume that for $m \geq 1$ given non-trivial set $g_1, \ldots, g_m$ in G there exists a non-trivial element $g \in G$ such that for all normal subgroups N of G if $g \notin G$ then $g_i \notin N$ for all $i = 1, \ldots, m$. As G is residually free this is true for $m = 1$. We show that given $g_1, \ldots, g_m, g_{m+1}$ we can find a $h \neq 1$ such that if $h \notin N$ for any normal subgroup N of G then $g_i \notin N$ for $i = 1, \ldots, m, m+1$. Let g be the assumed element for $g_1, \ldots, g_m$ end for each $x \in G$ let

$$c(x) = [g, x g_{m+1} x^{-1}]$$

If $c(x) = 1$ for all x by commutative transitivity the normal closure $N_{g_{m+1}}$ is abelian and hence here trivial but g_{m+1} is non-trivial then $c(x) \neq 1$ for some x, we choose $h = c(x)$ then if $h \notin N$ for some normal subgroup N of G it follows that $g_1, \ldots, g_m, g_{m+1} \notin N$. ■

Lemma 2.21. *Let G be a residually free group then G is fully residually free if and only if G is CT.*

Proof. We already proved the direct implication, let's see the inverse. Suppose that G is a residually free group which also is CT, let $g_1, \ldots, g_n$ a set of non-trivial elements of G, we want to find a normal subgroup N such that G/N is free and $g_i \notin N$ for every $i = 1, \ldots, n$. By the last proposition we have that if exists $g \notin N$ for every normal subgroup then $g_i \notin N$ for every $i = 1, \ldots, n$ but as G is residually finite there is such g and G/N is free hence G is fully residually finite. ■

With can summarize these results with the following corollary.

Corollary 2.22. *Let G be a residually free group, then the following are equivalent*

1. *G is fully residually free.*
2. *G is a commutative transitive group.*
3. *G is a conjugately separated abelian group.*

2.0.3 Centralizer extensions.

Definition 2.23. *Let G be an abelian group, then a non-trivial element $g \in G$ is a **torsion element** if $g^n = 1$ for some $n \in \mathbb{N}$. If there are no torsion elements in G then G is a torsion-free abelian group.*

Example 2.24. *Trivially free abelian groups are torsion-free abelian groups.*

Definition 2.25. *Let $G = \langle X \mid R \rangle$ be a commutative transitive group and $B = \langle Y \mid S \rangle$ a torsion-free abelian group. Take $u \in G$ a non-trivial element and let $C_G(u)$ be his centralizer. Then*

$$G(u, B) = \langle X, Y \mid R, S, [B, C_G(u)] = 1 \rangle$$

is a ***centralizer extension of*** $\mathbf{G}$ ***by*** $\mathbf{B}$*. If B is infinite cyclic then $G(a,t)$ is the HNN extension*

$$G(u,t) = \langle G, t \mid R, z^t = z \; \forall z \in C_G(u) \rangle$$

and is called the ***free rank one extension of the centralizer of*** $\mathbf{u}$ ***in*** $\mathbf{G}$*.*

We want to prove that if the base G group is fully residually free then the free rank one extension of the centralizer is also fully residually free. First, we need the following lemma.

Lemma 2.26 ([Bau62] Big powers lemma)**.** *Let $b, a_1, a_2, \ldots, a_k$ be elements of a free group. If*

$$a_1 b^{n_1} a_2 b^{n_2} \ldots a_k b^{n_k} = 1$$

for infinitely many integral values of n_i for every $i = 1, 2, \ldots, k$ then there exists an $1 \leq i \leq k$ such that

$$a_i b = b a_i$$

Theorem 2.27. *Let G be a fully residually free group. Let $a \in G$ a non-trivial element. Then the free rank one extension of the centralizer of a in G is also fully residually free.*

Proof. We can view the free rank one extension of the centralizer as the free product with amalgamation

$$G(a,t) = G *_{C_G(a)} (C_G(a) \times \langle t \rangle)$$

Now let $g_1, g_2, \ldots, g_k$ be finitely many non-trivial elements of $G(a,t)$ we may write

$$g_j = a_{0,j} t^{m_{1,j}} a_{1,j} t^{m_{2,j}} \ldots a_{N(j)-1,j} t^{m_{N(j),j}} z_j$$

where $N(j) \geq 0$, $a_{i,j} \in G \backslash C_G(a)$, $m_{i,j} \in \mathbb{Z} \backslash \{0\}$ and $z_j \in C_G(a)$. As $a_{i,j} \in G \backslash C_G(a)$ we have $[a_{i,j}, a] \neq 1$. Since G is fully residually free there is a free group F and an epimorphism $\phi : G \to F$ such that

$$[\phi(a_{i,j}), \phi(a)] \neq 1$$

Let $C_F(\phi(a)) = \langle f \rangle$ be the centralizer of $\phi(a)$ in F. Suppose $\phi(z_j) = f^{e_j}$. We may define an extension of ϕ as $\psi_{:} G(a,t) \to F$ by $\psi_n|_g = \phi$, $\psi_n(t) = f^n$.

As we have $\psi_n(g_j) = 1$ for infinitely many $n \in \mathbb{N}$ we can write

$$\phi(a_{0,j}) f^{m_{1,j} n} \phi(a_{1,j}) f^{m_{2,j} n} \ldots \phi(a_{N(j)-1,j}) f^{m_{N(j),j} n} f^{e_j}$$

for infinitely many values of n.

Applying the big powers lemma we have $\phi(a_i, j)f = f\phi(a_{i,j})$ but this means $[\phi(a_{i,j}), \phi(a)] = 1$ which contradicts our choice of ϕ. This shows that the set

$$S_j = \{n \in \mathbb{N} \ : \ \psi_n(g_j) \neq 1\}$$

is a cofinite set for the natural numbers, then its complement $S_j' = \mathbb{N} \backslash S_j$ is finite. Notice that if the finite intersection $S_1 \cap S_2 \cap \cdots \cap S_k$ is empty then

$$(S_1 \cap S_2 \cap \cdots \cap S_k)' = S_1' \cup S_2' \cup \cdots \cup S_k' = \mathbb{N}$$

which is impossible since it is a finite union of finite sets then we can take $n \in S_1 \cap S_2 \cap \cdots \cap S_k$ and then $\psi_n(g_j) \neq 1$ for all $j = 1, \ldots k$. Therefore $G(a, t)$ is fully residually free. ■

Now we can extend this for centralizer extensions of G by an abelian fully residually free group A.

Theorem 2.28. *Let G be a fully residually free group and A an abelian free group. Then the centralizer extension of G by A is also fully residually free.*

2.1 Limit groups.

Definition 2.29. *Let G be a finitely generated group and a free group F. A sequence $\{f_i\}_{i \in I} \in \operatorname{Hom}(G, F)$ is* ***stable*** *if, for all $g \in G$, the sequence $\{f_i(g)\}_{i \in I}$ is eventually always $\{1\}$ or never $\{1\}$.* ***Stable kernel*** *of $\{f_i\}_{i \in I}$, denoted $\underrightarrow{\operatorname{Ker}} f_i$ is*

$$\underrightarrow{\operatorname{Ker}} f_i = \{g \in G \mid f_i(g) = 1 \textit{ for almost all } i\}$$

A finitely generated group Γ is a ***limit group*** *if there is a finitely generated group G and a stable sequence $\{f_i\}_{i \in I}$ in $\operatorname{Hom}(G, F)$ so that*

$$\Gamma \cong G / \underrightarrow{\operatorname{Ker}} f_i$$

Now we can show that for finitely generated groups being fully residually free is equivalent to being a limit group.

Theorem 2.30. *Let Γ be a finitely generated group, then Γ is a limit group if and only if it is fully residually free.*

Proof. $\Longrightarrow$ Let Γ be a limit group finitely generated, and let G and $\{f_i\}_{i \in I}$ be a group and a stable sequence in $\operatorname{Hom}(G, F)$ such that

$$\Gamma \cong G / \underrightarrow{\operatorname{Ker}} f_i$$

Now consider the sequence of quotients

$$G \to G_1 \to G_2 \to \cdots \to \Gamma$$

obtained by adjoining one relation at a time. As Γ is finitely generated the sequence terminates. Let $H = G_j$ such that $\mathrm{Hom}\,(H, F) = \mathrm{Hom}\,(\Gamma, F)$.

$\Longleftarrow$ Suppose that G is a finitely generated fully residually free group. Let

$$S_1 \subset S_2 \subset \cdots \subset G$$

be a covering of G by an increasing sequence of finite sets of elements of G. Then since G is fully residually free for each i there is a homomorphism $f_i : G \to F$ which is injective on S_i. Since the S_i sequence covers G this is stable with trivial stable kernel then

$$G \cong {}^{G}\!\big/_{\underset{\longrightarrow}{\mathrm{Ker}} f_i}$$

Therefore G is a limit group.

■

Of course, limit groups have the same properties as fully residually free groups.

Proposition 2.31. *Limit groups satisfy the following properties:*

1. *A limit group is commutative transitive and CSA.*
2. *Any finitely generated subgroup of a limit group is a limit group.*
3. *Two elements of a limit group generate a free abelian group or a non-abelian free group of rank 2.*

Definition 2.32. *Let G be a group and A a commutative ring with unity. Then G is an A-group if we can define an action, called an A-action, $G \times A \to G$ defined as $(g, \alpha) = g^\alpha$ such that*

- $g^1 = g$, $g^0 = 1$, $1^\alpha = 1$.
- $g^{\alpha+\beta} = g^\alpha g^\beta$, $g^{\alpha\beta} = (g^\alpha)^\beta$.
- $(g^h)^\alpha = (g^\alpha)^h$.
- *If* $[g, h] = 1$ *then* $(gh)^\alpha = g^\alpha h^\alpha$.

For all $g, h \in G$ and $\alpha, \beta \in A$

In this context we define the free exponential group $F^{\mathbb{Z}[x]}$ from the polynomial ring $\mathbb{Z}[x]$. Lyndon [LS15] proved that given finitely many non-trivial elements $f_1, \ldots, f_n$ in the group $F^{\mathbb{Z}[x]}$ there is a homomorphism $\phi : F^{\mathbb{Z}[x]} \to F$ which is the identity on F and for which $\phi(f_1), \ldots, \phi(f_n)$ are all non-trivial. In particular since F is a free group it follows that $F^{\mathbb{Z}[x]}$ is fully residually free.

In [KM98] Kharlampovich showed that every limit group is isomorphic to a subgroup of $F^{\mathbb{Z}[x]}$ from this we have the following theorems.

Theorem 2.33. *A finitely generated group is a limit group if and only if it is a subgroup of an iterated free rank one extensions of centralizers of a free group.*

Theorem 2.34. *Every finitely generated residually free group G is a subgroup of a direct product of finitely many limit groups.*

2.2 Splittings and JSJ decomposition.

Definition 2.35. *We call a* ***splitting*** *the decomposition of a group G as a fundamental group of a graph of groups. A splitting is a $\mathbb{Z}-$*splitting *(abelian splitting) if every edge group is infinite cyclic (abelian). Splittings of the type $G = A *_C B$ or $G = A*_C$ are called elementary $\mathbb{Z}-$*splitting *(abelian).*

Definition 2.36. *A group G is* ***freely decomposable*** *if it is isomorphic to a non-trivial free product. Otherwise G is called* ***freely indecomposable****.*

Proposition 2.37. *Finitely generated free abelian groups are decomposable.*

Theorem 2.38. *Every freely indecomposable non-abelian limit group has an abelian splitting.*

Proof. Let G be a limit group then in theorem 2.33 we have that G is a subgroup formed from a free group by finitely many free rank one extensions of centralizers. Since each of these is an HNN group with abelian associated subgroups we have that G has an abelian splitting. ■

Moreover we have

Theorem 2.39 ([KM05])**.** *Every freely indecomposable non-abelian limit group has an* $\mathbb{Z}-$splitting.

Example 2.40. *The converse of this theorem is not true. We define the* ***Generalised Baumslag-Solitar Groups****. GBS in short, as the fundamental groups of a finite graph of groups in which all vertex and edge groups are infinite cyclic. Clearly, the Baumslag-Solitar are contained inside this family. Later we will prove that these groups are not limit groups.*

Definition 2.41. *Let G be a group with H and K subgroups, we say H can be* ***conjugated into*** *K if is a conjugate of a subgroup of K.*

Definition 2.42. *A subgroup H of a group G is called* ***elliptic*** *in a given splitting of G if H can be conjugated into a vertex group. Otherwise, H is hyperbolic.*

By Bass-Serre theory, the stabilizer of any vertex in a tree is a conjugate of a vertex group therefore the elliptic elements are just the stabilizers of a vertex.

Theorem 2.43. *Let G be a limit group and M the non-cyclic maximal abelian subgroup of G. Then*

1. *If $G = A *_C B$ is an abelian splitting then M is elliptic*
2. *If $G = A*_C$ is an abelian splitting then either M is elliptic or either there is a conjugate M^g such that*
$$G = A *_C M^g$$

Definition 2.44. *Let G be a group with an abelian splitting, this splitting is called normal if all maximal abelian non-cyclic subgroups of G are elliptic. By $\mathcal{D}(G)$ we denote the set of all normal splittings of G.*

Definition 2.45. *Let G be a group with an abelian splitting of G, a vertex group G_v is called quadratically hanging, in short QH, if the following conditions hold:*

1. *Admits one of the following presentations*
$$\left\langle p_1, \ldots, p_m, a_1, \ldots a_g, b_1, \ldots, b_g \,\middle|\, \prod_{j=1}^{g} [a_i, b_i] \prod_{k=1}^{m} p_k = 1 \right\rangle$$
with $g \geq 0, m \geq 1$
$$\left\langle p_1, \ldots, p_m, a_1, \ldots a_g \,\middle|\, \prod_{j=1}^{g} a_i^2 \prod_{k=1}^{m} p_k = 1 \right\rangle$$
with $g \geq 1, m \geq 1$

2. *For every edge $e \in E(\Lambda)$ outgoing from v, the edge group G_e is conjugated to one of the subgroups $\langle p_i \rangle$, $i = 1, \ldots, m$*

3. *For each p_i there is an edge $e_i \in E(\Lambda)$ outgoing from v such that G_{e_i} is a conjugate of $\langle p_i \rangle$*

From this, we see that every surface subgroup is a QH subgroup.

Definition 2.46. *A QH-subgroup Q is called a **maximal QH-subgroup**, MQH in short, if for every elementary abelian splitting of G either Q is elliptic or the edge group C can be conjugated into Q.*

Definition 2.47. *Let G be a group with a splitting. We distinguish three kinds of vertexes:*

- *QH vertexes if it is a QH subgroup.*
- *Abelian vertexes if it is a non-cyclic abelian subgroup.*
- *Otherwise rigid vertexes.*

Proposition 2.48 (JSJ decomposition)**.** *Let G be a freely indecomposable limit group. There exists a cyclic splitting $D \in \mathcal{D}(G)$ of G with the following properties*

1. *Every MQH-subgroup can be conjugated to a vertex in D; every QH-subgroup can be conjugated into one of the MQH-subgroups; non-MQH-subgroups are of two types: maximal abelian and non-abelian; every non-MQH vertex group in D is elliptic in every splitting in $\mathcal{D}(G)$.*

2. *If an elementary cyclic splitting is hyperbolic in another elementary cyclic splitting, then the edge group can be conjugated into some maximal QH-subgroup.*

*We call this splitting a **cyclic JSJ decomposition** of G.*

The relevant fact for limit groups is the following result

Theorem 2.49. *Let G be a limit group freely indecomposable then:*

1. *If G is indecomposable relative to a JSJ decomposition is either a surface group, a free group, or a free abelian group.*

2. *If G is a non-abelian and non-surface group then admits a non-trivial cyclic JSJ decomposition.*

2.3 Constructible Limit Groups.

In this section, we will see that limit groups can be built up inductively from simpler limit groups. But first, we need the following definitions

Definition 2.50. *Let A be an abelian non-cyclic vertex group and denote by $P(A)$ the subgroup of A generated by the incident edge groups then we define the* ***peripheral subgroup*** $\overline{P(A)}$ *as*

$$\overline{P(A)} = \bigcap_{\substack{f \in \mathrm{Hom}(A,\mathbb{Z}) \\ P(A) \subset \ker f}} \ker f$$

Definition 2.51. *Let B be a rigid vertex group, the* ***envelope*** *of B, $\tilde{B}$, is the group defined by first replacing each abelian vertex with the peripheral subgroup and then letting $\tilde{B}$ be the subgroup of the resulting group generated by B and by the centralizers of the incident edge-groups.*

Definition 2.52. *The class of* ***constructible limit groups****, CLG, is defined inductively as follows*

1. *Level 0 of the class are finitely generated free groups.*
2. *A group G is of level n if and only if either*
   (a) *$G = G_1 * G_2$ with G_1 and G_1 groups of level lower than n.*
   (b) *There exists a homomorphism $\rho : G \to G'$ with G' of level lower than n and G has a generalized abelian decomposition such that*
      - *ρ is injective on the peripheral subgroup of each abelian vertex.*
      - *ρ is injective on each edge group G_e and at least one of the images of G_e in a vertex group of the one-edged splitting induced by G_e is a maximal abelian subgroup.*
      - *The image of each QH-vertex group is a non-abelian subgroup of G'.*
      - *For every rigid vertex group B, ρ is injective on the envelope of B*

Before proving that constructible limit groups, CLG in short, are the same as limit groups we need to introduce some new concepts.

Definition 2.53. *Let G be a group with one abelian edge split, that is G splits either as an amalgamated product or as an HNN extension. The* ***Dehn twist*** *obtained from the* *corresponding splitting of G is the automorphism defined as:*

$$\begin{aligned} \tau_c(a) &= a && \text{if } a \in A \\ \tau_c(b) &= c^{-1}bc && \text{if } b \in B \end{aligned}$$

If G splits as an amalgamated product $A *_C B$ with $c \in C_B(C)$. And if G splits as a HNN extension $A*_C$ with $c \in C_A(C)$ then the Dehn twist is defined as:

$$\begin{aligned} \tau_c(a) &= a \quad \text{if } a \in A \\ \tau_c(t) &= ct \end{aligned}$$

Definition 2.54. *Let $D \in \mathcal{D}(G)$ be a group splitting of a group G. The associated modular group* $\mathrm{Mod}(D)$ *is the subgroup of* $\mathrm{Aut}(G)$ *generated by*

1. *Inner automorphisms.*
2. *Dehn twists of edges of D*
3. *Dehn twist corresponding to some essential $\mathbb{Z}-$splitting of G along a cyclic subgroup of a QH vertex.*
4. *Unimodular automorphisms of abelian vertices which are the identity on the peripheral subgroup.*

*The **modular group** of G,* $\mathrm{Mod}(G)$*, is the subgroup of* $\mathrm{Aut}((G))$ *generated by* $\mathrm{Mod}(D)$ *for all splittings $D \in \mathcal{D}(G)$ of G.*

Definition 2.55. *A **generalized Dehn twist** on a one-edge splitting is a Dehn twist or if A is an abelian vertex an automorphism of A which fixes all the edge subgroups of G_v.*

Theorem 2.56. *The modular group* $\mathrm{Mod}(G)$ *is generated by generalized Dehn Twists.*

Proof. It only remains to show the case of abelian vertexes because the other cases are Dehn Twists. Let A be an abelian vertex then we have

$$G = A \underset{P(A)}{*} B$$

For some subgroup B of G. Any unimodular automorphism of A in this splitting is a generalized Dehn twist. ■

Definition 2.57. *Let G be a group with generating set S. A morphism $h : G \to F$ is called **short** if*

$$\max_{g\in S} |h(g)| \leq \max_{g\in S} |i_c \circ h \circ \sigma(g)|$$

Where $\sigma \in \mathrm{Mod}(G)$ and i_c is the conjugation by $c \in F$ and $|\cdot|$ denotes the word length in F.

Theorem 2.58. *Let G be a freely indecomposable group. Let $f_i : G \to F$ a convergent sequence of short homomorphisms. Then*

$$\underset{\longrightarrow}{\mathrm{Ker}} f_i \neq 1$$

Theorem 2.59. *The class of constructible limit groups coincides with the class of limit groups.*

Proof. Let G be a limit group and D his cyclic JSJ decomposition, we want to prove that this decomposition satisfies the second part of the definition of CLG.

Let $\{f_i\}$ be a sequence of homomorphisms from G to a free group with f_i injective on elements of length at most i in the word metric relative to the generating set of G. Then the stable kernel of the $\{f_i\}$ is trivial. Choose $\{\hat{f}_i\}$ to be short maps equivalent to $\{f_i\}$ so then the map

$$\rho : G \to G' = G/\underset{\longrightarrow}{\mathrm{Ker}} \hat{f}_i$$

is an proper epimorphism by theorem 2.58. By induction assume that G' is a CLG.

Let G_e be an edge group of D therefore is maximal abelian in this decomposition and elliptic, this means that all generalized Dehn twists are inner automorphisms. Take g be a non-trivial element of G_e then $\hat{f}_i(g)$ is conjugate to $f_i(g)$ which is non-trivial for all sufficiently large i this means that $\rho|_{G_e}$ is injective.

As the peripheral group $\overline{P(A)}$ is elliptic in every one-edge splitting, by definition of the JSJ decomposition, $\mathrm{Mod}(G)$ acts as inner automorphism as in the case of edge groups and therefore the restriction of ρ to the peripheral group is injective. Similarly, the restriction of ρ to the envelope of a rigid vertex is injective as the envelope is also elliptic in every splitting.

Let Q be a QH-vertex and suppose that $\rho(Q)$ is abelian, then $\hat{f}_i$ is abelian for sufficiently large i but every element of $\mathrm{Mod}(G)$ maps Q to a conjugate of itself so eventually, $f_i(Q)$ is abelian contradicting the triviality of the stable kernel of f_i. ■

2.4 Universally Free and Elementary Free groups.

We say that a set L is a **first order language** provided with an equality $(=)$, a binary operation symbol $(\cdot)$, unary operation symbol $(^{-1})$ and a constant symbol (1).

A **formula** ϕ in L is a logical expression containing a string of variables $\overline{x} = (x_1, \ldots, x_n)$, logical connectives $\wedge, \vee$ and $\neg$ and the quantifiers $\forall$ and $\exists$. Whenever a formula ϕ is true in a group G we write $G \models \phi$.

A variable in a formula is called **bounded** if it is restricted by a quantifier, otherwise is called free. A **sentence** in L is a formula in which all variables are bounded. We write by $\mathrm{Th}(G)$ the set of all sentences true in a group G.

Definition 2.60. *We can differentiate three types of sentences in a first order theory:*

- *A **universal sentence** in L is one of the form $\forall \overline{x}\,(\phi(\overline{x}))$.*
- *An **existential sentence** is one of the form $\exists \overline{x}\,(\phi(\overline{x}))$.*
- *An **existential-universal sentence** is one of the form $\exists \overline{x} \forall \overline{x}\,(\phi(\overline{x}))$.*

If G is a group, we write $\mathrm{Th}_{\forall}(G), \mathrm{Th}_{\exists}(G), \mathrm{Th}_{\forall\exists}(G)$ for the set of all universal sentences, existential sentences, universal-existential sentences, respectively, in the group G.

Definition 2.61. *Let G and H be groups*

- *G and H are elementary equivalent, provided* $\mathrm{Th}(G) = \mathrm{Th}(H)$.
- *G and H are universally equivalent, provided* $\mathrm{Th}_{\forall}(G) = \mathrm{Th}_{\forall}(H)$. *In short we write $\forall$-equivalent.*
- *G and H are existentially equivalent, provided* $\mathrm{Th}_{\exists}(G) = \mathrm{Th}_{\exists}(H)$.*In short we write $\exists$-equivalent.*
- *G and H are universally-existentially equivalent, provided* $\mathrm{Th}_{\forall\exists}(G) = \mathrm{Th}_{\forall\exists}(H)$.*In short we write $\forall\exists$-equivalent.*

Definition 2.62. *If G and H are groups and $f : H \to G$ is a monomorphism then f is an **elementary embedding** whenever $\phi(\overline{h})$ is true in H if and only if $\phi(f(\overline{x}))$ is true in G with $\overline{h} = (h_1, \ldots, h_n) \in H^n$. If H is a subgroup of G and $i : H \to G$ is an elementary embedding then H is an **elementary subgroup**.*

Definition 2.63. *Let G be a group*

- *G is **universally free** provided G is $\forall$-equivalent to a non-abelian free group.*
- *G is **elementary free** provided G is elementary equivalent to a non-abelian free group.*

Separately, Kharlampovich and Myasnikov [KM06] and Sela [Sel06] have discovered that the Tarski conjectures are true.

Theorem 2.64. *Let $\mathbb{F}_n, \mathbb{F}_k$ two free groups with $2 \leq k \leq n$ then the standard embedding $i : \mathbb{F}_k \to \mathbb{F}_n$ is an elementary embedding.*

Theorem 2.65. *Let $\mathbb{F}_n, \mathbb{F}_k$ two free groups with $2 \leq k \leq n$ then $\mathbb{F}_n$ and $\mathbb{F}_k$ are elementary equivalents, moreover $\mathrm{Th}(\mathbb{F}_2) = \mathrm{Th}(\mathbb{F}_n)$ for every $n \geq 2$.*

Theorem 2.66. *The elementary theory of the countable non-Abelian free groups is decidable.*

Now we state that universally free groups are precisely limit groups.

Theorem 2.67. *A finitely generated group G is universally free if and only if G is a non-abelian limit group.*

Proof. As the free group of rank 2 is contained in every non-abelian limit group, $\mathrm{Th}_\forall(\mathbb{F}_n) = \mathrm{Th}_\forall(\mathbb{F}_2) \subseteq \mathrm{Th}_\forall(G)$.

To see the direct implication we have that being commutative transitive is given by the universal sentence

$$\forall x, y, z\, ((y \neq 1) \wedge (xy = yx) \wedge (yz = zy)) \to (xz = zx)$$

As every free group is CT if G has the same universal theory as a non-abelian free group, G is CT now we have to show that G is residually free.

Let

$$G = \langle x_1, \ldots, x_n \mid R_1 = \cdots = R_s = 1\rangle$$

Where $R_i = R_i(x_1, \ldots, x_n)$ and suppose that w is a non-trivial element of G given by $w = W(x_1, \ldots, x_n)$. Consider now the existential sentence

$$\exists x_1, \ldots, x_n \left(\left(\bigwedge_{i=1}^{m} R_i(x_1, \ldots, x_n) = 1 \right) \wedge (W(x_1, \ldots, x_n) = 1 \right)$$

Clearly this sentence is true in G so this sentence must be true in all non-abelian free groups. Therefore in any non-abelian free group F there exists elements $a_1, \ldots, a_n$ such that $R_i(a_1, \ldots, a_n) = 1$ and $W(a_1, \ldots, a_n) \neq 1$. Take the map from G to F given by $x_i \to a_1$ for $i = 1, \ldots, n$ defines a homomorphism where the image of w is non-trivial then G is residually free and CT hence limit group. ■

Within the proof of the Tarski conjectures Sela [Sel06] discovered the following relevant fact of groups elementary free groups

Theorem 2.68. *A finitely generated group that is elementary free, must be a limit group that contains no non-cyclic free abelian subgroups*

Note that this theorem does not imply, that all limit group that contains no non-cyclic free abelian subgroups are elementary free.

We also have the following fact that implies that elementary free is the same as being $\forall\exists$-equivalent to a non-abelian free group

Theorem 2.69. *Every formula in the language of a free group is equivalent to a boolean combination of* $\forall\exists$*-formulas.*

In [KM98] we obtain a characterization about groups $\forall\exists$-equivalent to a non-abelian free group,

Theorem 2.70. *Let G be finitely generated group $\forall\exists$-equivalent to a non-abelian free group F. Then G is fully residually free and can be obtained from infinite cyclic groups by finitely many operations of the following type:*

- *Free products.*
- *Amalgamated products with infinite cyclic amalgamated subgroups at least one of which is maximal abelian.*
- *Separated HNN-extensions with infinite cyclic associated subgroups at least one of which is maximal abelian.*

2.5 Cyclic and Conjugacy Pinched one relator group.

In this section we want to construct new groups from free groups, suppose that F, F' are two free groups, not necessarily isomorphic, take $u \in F$ and $v \in F'$ such that $\alpha : \langle u \rangle \to \langle v \rangle$ is an isomorphism then we call **cyclically pinched one-relator group** the class of amalgamated products

$$F *_{u=v} F'$$

This is a group with presentation

$$\langle a_1, \dots a_n, b_1, \dots, b_m \mid u = v \rangle$$

Where $F = \langle a_1, \dots, a_g \rangle$ and $F' = \langle b_1, \dots, b_g \rangle$. Similarly if F is a free group with $u, v \in F$ such that $\alpha : \langle u \rangle \to \langle v \rangle$ is a monomorphism then we call **conjugacy pinched one-**

relator group the class of HNN extensions

$$F*_\alpha$$

This is a group with presentation

$$\langle a_1, \dots a_n, t \mid tut^{-1} = v \rangle$$

Where $F = \langle a_1, \dots a_n \mid \rangle$. Recall that an element in a free group is **primitive** if it is in some basis of the free group.

Theorem 2.71 ([Hou10])**.** *Let G be a cyclic pinched one-relator group having a presentation*

$$G = \langle x_1, \dots, x_n, y_1, \dots, y_n \mid u = v \rangle$$

Where $\{x_1, \dots, x_n\}$ is the generating set of a free group F with u a non-trivial element of F and $\{y_1, \dots, y_m\}$ is the generating set of a free group F' with v a non-trivial element of F'. Then G is free if and only if either u is primitive in F' or v is primitive in F.

Theorem 2.72 ([Hou10])**.** *Let G be a conjugacy pinched one-relator group having a presentation*

$$G = \langle x_1, \dots, x_n, t \mid u^t = v \rangle$$

Where $\{x_1, \dots, x_n\}$ is the generating set of a free group F with u and v non-trivial elements of F. Then G is free if and only if one of the following cases holds:

1. *F has a basis $\{u, y_1, \dots, y_{n-1}\}$ such that v is conjugates to $v' \in \langle y_1, \dots, y_{n-1} \rangle$.*

2. *F has a basis $\{v, z_1, \dots, z_{n-1}\}$ such that u is conjugates to $u' \in \langle z_1, \dots, z_{n-1} \rangle$.*

Definition 2.73. *We say that a group G is **n-free** if any set of n or fewer elements of G generates a free group.*

Theorem 2.74. *Let G be a cyclically pinched one-relator non-free group. Then*

1. *G is 2-free.*

2. *G is 3-free.*

3. *For all subgroups H of rank 4, one of the following occurs:*

   (a) *H is free of rank 4.*

   (b) *H has a one-relator presentation.*

Theorem 2.75. *Let G be a conjugacy pinched one-relator non-free group. If H is a two generator subgroup of G, then one of the following holds:*

1. *H is free of rank 2.*
2. *H is abelian.*
3. *H has a presentation $\langle a, b \mid b^a = b^{-1} \rangle$*

Theorem 2.76. *Every 2-free residually free group is fully residually free and 3-free.*

Proof. First, we demonstrate that every 2-free residually free is commutative transitive, hence fully residually free. Let G be a 2-free residually free group, and take M as the centralizer of a non-trivial element $x \in G$. Suppose $a, b \in M$, since G is 2-free a, x generate a free group but a, u commute this must be both powers of a single element g, $x = g^\gamma$ and $a = g^\alpha$. Similarly since x and b commute, we have an element $h \in G$ such that $x = h^\delta$ and $b = h^\beta$. Now consider the subgroup generated by h, g this is free because G is 2-free but we have $g^\gamma = h^\delta$ then it is cyclic. With this we have that any subgroup of M generated by two elements is cyclic, a straightforward induction shows that finitely generated subgroups of M are cyclic then M is locally cyclic and abelian. Applying theorem 2.9 we have that G is commutative transitive.

The second part is proven in [FGM+98] and presents the following classification of fully residually free groups. Here by rank, we mean the minimum number of generators of the group. ■

Theorem 2.77. *Let G be a fully residually free group. Then:*

1. *If* $\text{Rank}\,(G) = 1$ *then G is infinite cyclic.*
2. *If* $\text{Rank}\,(G) = 2$ *then G is free of rank 2 or free abelian of rank 2.*
3. *If* $\text{Rank}\,(G) = 3$ *then either G is free of rank 3, free abelian of rank 3 or a free rank one extension of centralizes of free group rank 2.*

Theorem 2.78. *Suppose that F is a free group with ϕ an isomorphism and let $u \in F$ be neither primitive nor a proper power in F. Then the cyclically pinched one-relator group*

$$F \underset{u=\phi(u)}{*} \phi(F)$$

is fully residually free. A group of this form is called a ***Baumslag double****.*

Proof. We will prove that if G is a Baumslag Double then is a CLG of level 2. Taking the retraction $\rho : G \to F$ we have that ρ is a homomorphism to a CLG of level 0.

As both vertexes are rigid we have that ρ has to be injective on the envelope of every vertex, but this is true since the envelope of every vertex is the vertex itself and both vertexes are isomorphic. So we have that ρ is injective in the envelope. ■

Theorem 2.79. *Suppose that F is a free group with ϕ an automorphism. Let $F_1 = F * F_2$, where F_2 is a free group. Let $u \in F$. Then the cyclically pinched one-relator group*

$$F \underset{u=\phi(u)}{*} F_1$$

is fully residually free. A group of this form is called a ***Disguised Baumslag double****.*

Proof. In this case, if G is a disguised Baumslag Double then is a CLG of level 3. Taking the retraction $\rho : G \to F_1$ we have that ρ is a homomorphism to a CLG of level 1 as it is the free product of two free groups.

Again as both vertexes are rigid we have that ρ has to be injective on the envelope of every vertex. The envelope of every vertex is the vertex itself so in the case of F_1, ρ is trivially injective for the envelope of F_1.

Now as $F_1 = F * F_2$ if we restrict the map ρ to F it is necessarily injective. ■

2.6 Surface groups.

Let G be the fundamental group of a compact surface of genus g. Then G has a one-relator presentation

$$\left\langle a_1, \ldots a_g, b_1, \ldots, b_g \;\middle|\; \prod_{j=1}^{g} [a_i, b_i] = 1 \right\rangle$$

in the orientable case and

$$\left\langle a_1, \ldots a_g \;\middle|\; \prod_{j=1}^{g} a_i^2 = 1 \right\rangle$$

in the non-orientable case. The groups with such presentation are **surface groups**.

Theorem 2.80. *The surface group of an orientable surface of genus $g \geq 2$ is a cyclically pinched one-relator group and a conjugacy pinched one-relator group.*

Proof. Suppose that G is the surface group of an orientable surface of genus $g \geq 2$ with the previous presentation, taking $u = [a_1, b_1][a_2, b_2] \ldots [a_{g_1}, b_{g-1}]$ and $v = [a_g, b_g]$ then

$$G = \langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid u = v^{-1} \rangle$$

Hence G is a conjugacy pinched one-relator group.
Now taking $u = a_g$, $v = [a_1, b_1][a_2, b_2] \dots [a_{g_1}, b_{g-1}]a_g$, $t = b_g$ we have that

$$G = \langle a_1, \dots, a_g, b_1, \dots, b_{g_1}, t \mid tut^{-1} = v \rangle$$

Hence G is also a conjugacy pinched one-relator group. ■

Theorem 2.81. *The surface group of a non-orientable surface group of genus $g \geq 2$ is a cyclically pinched group one-relator group.*

Proof. Suppose that G is the surface group of a non-orientable surface of genus $g \geq 2$ with the previous presentation, taking $u = a_1^2 \dots a_{g-1}^2$ and $v = a_g^2$ then

$$G = \langle a_1, \dots, a_g \mid u = v^{-1} \rangle$$

Hence G is a conjugacy pinched one-relator group. ■

A consequence of this is the following.

Corollary 2.82. *Every orientable surface group is fully residually free and every non-orientable non-exceptional surface group is fully residually free.*

Proof. Let G be the surface group of an orientable surface with the standard presentation then

$$G = F \underset{u=v}{*} F'$$

Where $F = \langle a_1, \dots, a_g \rangle$ and $F' = \langle b_1, \dots, b_n \rangle$. As F and F' have the same rank they are isomorphic and G is a Baumslag double hence G is residually free.

If G is a non-orientable surface group we have that we cant split G into two free groups of the same rank except if g is even and more than 3. In [Bau67a] is proven that the restriction of k being even can be removed. The case of g being three or less is the case of exceptional surface groups, that is G being the projective plane (is not torsion-free), the Klein bottle (is not commutative transitive) or $G = \langle a, b, c \mid a^2b^2c^2 \rangle = 1$ since three elements in a free group satisfying $a^2b^2c^2 = 1$ must commute. ■

Definition 2.83. *A group G has property **IF** if every subgroup of infinite index in G is free.*

Theorem 2.84. *Let G be a fully residually free group with property IF then G is either a cyclically pinched one-relator group or a conjugacy pinched one-relator group.*

Proof. We differentiate two cases, G is indecomposable relative to a JSJ decomposition or G admits a non-trivial cyclic JSJ decomposition.

In the first case, G is a surface group, a free group, or a free abelian group. The theorem covers the two first cases, suppose that G is free abelian. By property, IF G cannot have a rank great than two, then G is infinite cyclic or has the following presentation

$$G = \langle x, y \mid [x, y] = 1 \rangle$$

and G can be considered a conjugacy pinched one relator group.

Now suppose, that G is non-abelian or non-surface then G admits a non-trivial cyclic JSJ decomposition Γ. As before, we can assume that Γ is a one-edge splitting.

Suppose $G = A *_C B$ with A, B non-trivial. By property, IF both factors have to be free, then G is a cyclically pinched one relator group. Now suppose, that B is trivial, then G is an HNN extension of A bu a cyclic associated subgroup, and again by property IF, A has to be free and hence G is a conjugacy pinched one relator group. ■

Corollary 2.85. *Let G be a non-free fully residually free group with property IF then G is a surface group.*

§3. Parafree groups.

As usual we denote the conjugate of the element x by the element y, where x and y are elements in a group G, by x^y and the commutator of x and y by $[x, y]$. The lower central series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \dots \geq \gamma_n(G) \geq \dots$$

is defined inductively by

$$\gamma_{n+1} = [\gamma_n, G]$$

A group G is termed nilpotent if there some integer i such that $\gamma_i(G) = 1$. We define the abelianization of a group as the quotient

$$G^{\mathrm{ab}} = {}^{G}\!/_{\gamma_2(G)} = {}^{G}\!/_{[G,G]}$$

The group $[G, G]$ is called the derived subgroup.

Definition 3.1. *A group G is called **residually nilpotent** if for all $g \in G$ non-trivial element there exists a nilpotent group H_g and an epimorphism $\phi_g : G \to H_g$ such that $\phi(g) \neq 1$. Equivalently, G is residually nilpotent if given any non-trivial element $g \in G$ there exists a normal subgroup N if G such that $g \notin N$ with ${}^{G}\!/_{N}$ nilpotent.*

We also have a strong residual property called residually torsion-free nilpotent.

Definition 3.2. *A group G is called* ***residually torsion-free nilpotent*** *if for all $g \in G$ non-trivial element there exists a torsion free nilpotent group H_g and an epimorphism $\phi_g : G \to H_g$ such that $\phi(g) \neq 1$. Equivalently, G is residually nilpotent if given any non-trivial element $g \in G$ there exists a normal subgroup N if G such that $g \notin N$ with G/N torsion-free nilpotent.*

We can give another equivalent definition of residually nilpotent groups

Definition 3.3. *A group G is residually nilpotent if*

$$\bigcap_{i\in\mathbb{N}} \gamma_i(G) = \{1\}$$

In the case of residually free nilpotent groups, we have to define the isolator subgroup.

Definition 3.4. *For a subgroup of a group G, the isolator of H denoted as $\overline{H}$ is the subgroup generated by all $g \in G$ for which there is a positive integer n such that $g^n \in H$*

$$\overline{H} = \{g \in G \ / \ \exists n \in \mathbb{N} : \ g^n \in G\}$$

Definition 3.5. *With the anterior definition it makes sense to define*

$$\overline{\gamma_n(G)}/\gamma_n(G) = \operatorname{tor}\left(G/\gamma_n(G)\right)$$

Then a group G is residually torsion-free nilpotent if

$$\bigcap_{n=0}^{\infty} \overline{\gamma_n(G)} = 1$$

Theorem 3.6. *Limit groups are residually torsion-free nilpotent.*

Proof. A direct consequence of free groups being residually torsion-free nilpotent. ■

Definition 3.7. *A group G is called* ***parafree*** *if it is residually nilpotent and there exists a free group F such that*

$$G/\gamma_n(G) \cong F/\gamma_n(F)$$

for every n.

Theorem 3.8 ([JZM21])**.** *Parafree groups are residually torsion-free nilpotent.*

Lemma 3.9. *Free groups are parafree.*

Now we present a family of groups discovered by G. Baumslag that give a negative solution to the Hanna Neumann conjecture.

Example 3.10. *Given the non-zero integers i, j, we define the family of groups given by the presentation*

$$H_{i,j} = \langle a, b, c \mid a = [c^i, a][c^j, b] \rangle$$

We want to see that this family of groups consists on groups that are parafree non-free groups.

Proof. First of all we see that $H_{i,j}$ is not free. We can give $H_{i,j}$ the following presentation

$$H_{i,j} = \langle a, c, b \mid a^{c^i} = a[b, c^j]a \rangle$$

Then $H_{i,j}$ is a conjugacy pinched one relator group and using theorem 2.72 we have that $H_{i,j}$ is not free.

We are left with the proof of $H_{i,j}$ is parafree. Take K be the free group generated by a, b, c, then we have that ${}^{H_{i,j}}\!/_{\gamma_n(H_{i,j})}$ is the result of adding to ${}^{K}\!/_{\gamma_n(K)}$ the given relation. As ${}^{K}\!/_{\gamma_n(K)}$ is a free nilpotent group freely generated by a, b, c modulo $\gamma_n(K)$. Then it follows that ${}^{K}\!/_{\gamma_n(K)}$ is generated by

$$a^{-1}[c^i, a][c^j, b], b, c$$

modulo $\gamma_n(K)$. This means that killing the first generator maps us to the free rank 2 nilpotent group of class k.

${}^{G}\!/_{\gamma_n(F)}$ is the freest nilpotent group of class k which satisfies $a^{-1}[c^i, a][c^j, b] = 1$ therefore

$${}^{G}\!/_{\gamma_n(G)} \cong {}^{K}\!/_{\gamma_n(K)}$$

Now the hardest part is to prove that G is residually nilpotent. Recall that G is an HNN-extension of a free group generated by a, b, N, with associated subgroup the infinite cyclic group generated by c. In [KSW66] is given a procedure for obtaining generators and defining relations using the Reidemeister-Schreier procedure, in our case we have

$$N = \langle \ldots, a_{-1}, a_0, a_1, \ldots, b_{-1}, b_0, b_1, \ldots \mid a_n = a_{n+1}^{-1} a_n b_{n+1}^{-1} b_n \ldots \rangle$$

Where the subscript n ranges over all integers. From the defining relations of N we have that the elements b_n, $n = \ldots, -1, 0, 1, \ldots$ freely generate the free nilpotent group N

modulo $\gamma_k N$ for every $k \leq 1$. If $x \in G$ and $x \notin N$ then as G/N is cyclic there is nothing to prove so we may restrict our attention to those elements $x \in G$ and $x \in N$. Since N is free $x \notin \gamma_l N$ for some l. Without loss of generality we may assume that

$$x\gamma_l N \in \langle b_o \gamma_l N, \ldots, b_{2^r-1}\gamma_l N \rangle$$

for a suitably large choice of the integer r. Now let $M = \langle c^{2r}, \gamma_l N \rangle$ then MN/M is a free nilpotent group freely generated by $b_0 M, \ldots, b_{2^r-1}M$ and G/MN is generated torsion-free nilpotent group is residually a finite 2-group. Si G/M is itself a residually a finite 2-group. Hence there is a normal subgroup L of H such that $x \in L$ and G/L is nilpotent. Therefore G is residually nilpotent. ■

Some interesting facts about parafree groups are the following.

Theorem 3.11 ([Mor21]). *Non-free surface groups are not parafree.*

Theorem 3.12 ([Bau69]). *Free products and free factors of parafree groups are parafree.*

Theorem 3.13 ([Bau69]). *The center of a parafree group is either trivial or the whole group. Therefore the only abelian parafree group is the infinite cyclic.*

We can characterize the subgroups of a parafree group with the following theorem

Theorem 3.14 ([Bau69]). *Let G be a parafree group then the abelian subgroups of G are cyclic whereas the non-abelian two-generator subgroups are free of rank two.*

Also, we can formulate a general criterion for the amalgamated product and HNN extension of parafree groups

Theorem 3.15 ([JZM21]). *Let A and B be finitely generated groups, $1 \neq a \in A$ and $1 \neq b \in B$. Consider the amalgamated free product $G = A *_{a=b} B$. Then G is parafree if and only the following conditions hold.*

1. *A and B are parafree.*
2. *The element ab^{-1} is not a proper power in the abelianization of $A * B$.*
3. *At least one of a or b is not a proper power in A or B, respectively.*

Theorem 3.16 ([JZM21]). *Let A be a finitely generated groups, $1 \neq a \in A$ and $\alpha : \langle a \rangle \to A$ a monomorphism. Consider the HNN extension $G = A*_\alpha$. Then G is parafree if and only the following conditions hold.*

1. *A is parafree.*

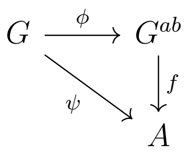
2. *The element $a\alpha(a)^{-1}$ is not a proper power in the abelianization of A.*

3. *At least one of a or $\alpha(a)$ is not a proper power in A.*

4. *The image of the element a is non-trivial in some finite nilpotent quotient of G.*

We will need some facts about the abelianization but first, we define it in terms of universal properties

Definition 3.17. *Let G be a group, then the abelianization of G, G^{ab} is an abelian group such that there exists a surjective homomorphism $f : G \to G^{\text{ab}}$ such that for every homomorphism $\phi : G \to A$, where A is an abelian group, there is a unique homomorphism $\psi : G^{\text{ab}} \to A$ such that $\phi = \phi \circ f$. That is, the following diagram commute*

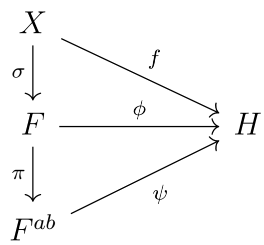


A useful fact about the abelianization of free groups is the following.

Theorem 3.18. *Let F be a free group on the set X, then F^{ab} is the abelian free group of X*

Proof. Let H be an abelian group and $f : X \to H$, by universal property of free groups there exists a unique group homomorphism $\phi : F \to H$ such that $(\phi\sigma) = f$, where $\sigma : X \to F$.

Also by the universal property of abelian free groups there exists a unique group homomorphism $\psi : F^{\text{ab}} \to H$ such that $(\psi\pi) = f$, where $\pi : F \to F^{\text{ab}}$ is the quotient map.



We have that $\psi(\pi\sigma) = f$ therefore F^{ab} is an abelian free group on the set X ∎

§4. On parafree residually free groups

First of all, we have that a parafree residually free group is a limit group.

Proposition 4.1. *Let G be a parafree group which is also residually free, then G is a limit group.*

Proof. As G is residually free, G is a subgroup of a direct product of limit groups, then by theorem 2.34 we can think of G as a subgroup of $L_1 \times \cdots \times L_n$ where every L_i is a limit group. Denote by $p_i : L_1 \to G$ the projection of L_1 into G by hypothesis these projections are surjective, then if exists one i such that p_i is also injective we have that G is isomorphic to L_i, therefore, G is a limit group.

Denote by f the embedding of G into $L_1 \times \cdots \times L_n$, and take the composition $f \circ p_i$ and suppose that all p_i are non-injective this means that the kernel of the composition is not empty and therefore there exists two subindexes j, k such that the intersections $L_j \cap f(G)$ and $L_k \cap f(G)$ are non-empty. We take K the subgroup generated by two elements g_j, g_k such that $g_j \in L_j \cap f(G)$ and $g_k \in L_k \cap f(G)$ this group is a free abelian group of rank 2 as g_j, g_k are in the kernel of the composition but this is a contradiction with the fact of all abelian subgroups of parafree groups are cyclic. ■

In [Sok21] is proven the following fact about GBS groups.

Theorem 4.2. *Let G be a non-cyclic GBS group. The following statements are equivalent.*

- *G is residually torsion-free nilpotent.*
- *G is residually free.*
- *G is isomorphic to the direct product of a free group and an infinite cyclic group.*

Theorem 4.3. *Let G be a non-cyclic GBS group then G is non-fully residually free.*

Proof. We have that a generalised Baumslag-Solitar group which is residually free is isomorphic to the direct product of a free group and an infinite cyclic group therefore it contains a subgroup isomorphic to $F_2 \times \mathbb{Z}$. ■

With this result, we have the following.

Theorem 4.4. *Non-cyclic GBS groups are not parafree.*

Proof. We have that a generalised Baumslag-Solitar is residually torsion-free nilpotent if it is residually free. Suppose that G is a parafree generalised Baumslag-Solitar then G is also fully residually free but this is a contradiction with 4.3. ■

To prove that there are no non-free parafree limit groups of rank less than three we have the following theorem

Theorem 4.5. *Free rank one extension of centralizes of free groups are non-parafree.*

Proof. Take G as a free rank one extension of centralizes of a free group of rank n, this is G has the following presentation.

$$G = \langle x_1, x_2, \dots, x_n, t \mid tvt^{-1} = v \rangle$$

With v a non-trivial element of the free group, which is not a proper power. If we take the subgroup generated by v, t we have that this is a free abelian group of rank two, but if G were parafree this will be a contradiction with all abelian subgroups being cyclic, therefore G is non-parafree. ■

In virtue of theorem 2.77 we have that

Corollary 4.6. *All parafree limit groups of rank less or equal to three are free.*

The following example is a parafree residually free group which is of rank 4.

Example 4.7. *We will take the following Baumslag Double group therefore limit and we demonstrate that is parafree.*

$$G = F \underset{a_1 w = \phi(a_1 w)}{*} \phi(F)$$

Where F is a free group generated by $a_1, \dots, a_n$, w an element in the derived group of F and ϕ an isomorphism. We use 3.15 to see that G is parafree, clearly, the first condition trivially holds.

A well known result [Sch59] is that commutators are not proper powers of a free group, therefore $a_1 w$ can not be a proper power in F. Using this result is trivial that $a_1 w \phi(a_1 w)$ is not a proper power in the abelianization of $F * \phi(F)$ which is the abelian free group generated by $a_1, \dots, a_n, \phi(a_1), \dots, \phi(a_n)$,

Finally to see that G is not necessarily free using theorem 2.72 we have that if $a_1 w$ is not a primitive element of F then G is not free.

As we have seen before the elementary free groups are limit groups containing no non-cyclic free abelian groups. One property of parafree groups is that all abelian groups are cyclic therefore parafree limit groups contain no non-cyclic free abelian groups but we have that non all parafree limit groups are elementary free. First, we have the following.

Theorem 4.8. *Baumslag doubles are non elementary free.*

Proof. We know that Baumslag doubles are non-free and in [Sel06] Sela proved an elementary free group contains a QH vertex in the JSJ decomposition but in this case both vertexes are rigid. ■

Proposition 4.9. *Let G be a non-abelian parafree limit group, then G is not necessarily elementary free.*

Proof. We will use the fact that doubles are non elementary free, the group

$$G = F_2 \underset{w=\phi(w)}{*} \phi(F_2)$$

Where F_2 is the rank 2 free group,ϕ is an isomorphism and w is an element in the derived group of F is a parafree group and a limit group but non-elementary free. ■

4.1 Constructing parafree limit groups.

Now we aim to find a way to construct these groups, to this purpose we use the JSJ decomposition of limit groups. As we know non-free surface groups are non-parafree and the only abelian parafree group is the infinite cyclic therefore parafree limit groups admit a non-trivial cyclic JSJ decomposition. We use the following theorem that describes how is the graph of groups of a parafree group.

Corollary 4.10 ([JZM21])**.** *Let (G, X) be a graph of groups and $\pi_1(G, Y, T)$ be its fundamental group. Assume that all vertex subgroups are finitely generated and all edge subgroups are cyclic, then $\pi_1(G, Y, T)$ is parafree if and only if the following conditions hold:*

1. *All the vertex subgroups are parafree.*
2. *The abelianization of $\pi_1(G, Y, T)$ is torsion-free.*
3. *All centralizers of non-trivial elements $\pi_1(G, Y, T)$ are cyclic*
4. *For each non-trivial edge subgroup there is a finite nilpotent quotient of $\pi_1(G, Y, T)$ where the image of this edge subgroup is non-trivial*

Proposition 4.11. *Let G be a non-abelian parafree group, if G is elementary free then G is a free group.*

Proof. As before we use the fact that a non-free group elementary equivalent to a free group contains a QH-vertex in his JSJ decomposition but by the last theorem all vertexes have to be parafree and QH-vertexes are surface groups hence non-parafree. ■

Now we take G as a limit group and reduce the JSJ decomposition to the two simple cases: amalgamated product and HNN extensions. Then we see that all the vertexes must be rigid, in fact, parafree, if we want to construct non-free limit parafree groups as they can't be QH vertexes and for the abelian vertexes we have the following cases

If $G = G_1 *_C G_2$ is the non-trivial JSJ decomposition of G we have two cases:

1. G_1 and G_2 are abelian-vertexes, therefore, both are cyclic but G will be a GBS group therefore non-parafree.

2. G_1 (or G_2) is abelian-vertex therefore is cyclic then it is a rigid vertex.

In the case of $G = G_1 *_C$, G_1 has to be rigid as if it is abelian then it is cyclic and G will be a GBS therefore non-parafree

With this, we define the constructive class of parafree limit groups.

Definition 4.12. *The class of* ***constructible parafree limit groups*** *is defined inductively as follows*

1. *Level 0 of the class are finitely generated free groups.*

2. *A group G is of level n if and only if either*

   (a) $G = G_1 * G_2$ *with* G_1 *and* G_1 *groups of level lower than* n*.*

   (b) *The abelianization of G is torsion-free and there exists a homomorphism* $\rho : G \to G'$ *with* G' *of level lower than* n *and G has a generalized cyclic decomposition such that*

      - ρ *is injective on each edge group* G_e *and at least one of the images of* G_e *in a vertex group of the one-edged splitting induced by* G_e *is a maximal abelian subgroup.*

      - *Every vertex B is parafree and,* ρ *is injective on B.*

With this definition, we have that example 4.7 and group constructed in the proof of proposition 4.9 are constructible parafree limit groups of level 2.

Theorem 4.13. *Constructible parafree limit groups are parafree and limit groups.*

Proof. Case 1. and 2.(a) are trivial therefore we will demonstrate 2.(b). We use induction over the level of the group and corollary 4.10 to see that the generalized cyclic decomposition holds all the conditions. Trivially every group described above is a limit group.

Let G be a level zero constructive parafree limit group, it is free hence parafree and limit group. Suppose now that G is a level n constructive parafree limit group, take his generalized cyclic decomposition and ρ a homomorphism between G and a constructive parafree limit group of level less than n G' that holds the conditions described. Now we compare this decomposition to the one in 4.10.

By definition, all vertexes are parafree therefore condition one holds. Now we study the abelianization of G, the condition of G being a limit group is not enough as there are limit groups with abelianization non-torsion free [WG16]. So we need the condition of the abelianization of G being torsion-free.

To study the centralizers we have that they are abelian as G is a limit group and as every abelian subgroup can be conjugated into a vertex group we have that every centralizer is an abelian subgroup of a vertex group which are parafree therefore it is cyclic.

In the last case, we study the finite nilpotent quotients, as limit groups are residually torsion-free nilpotent we have that the last condition holds.

■

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