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A new family of aggregation functions for intervals

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Abstract
Aggregation operators are unvaluable tools when different pieces of information have to be taken into account with respect to the same object. They allow to obtain a unique outcome when different evaluations are available for the same element/object. In this contribution we assume that the opinions are not given in form of isolated values, but intervals. We depart from two "classical" aggregation functions and define a new operator for aggregating intervals based on the two original operators. We study under what circumstances this new function is well defined and we provide a general characterization for monotonicity. We also study the behaviour of this operator when the departing functions are the most common aggregation operators. We also provide an illustrative example demonstrating the practical application of the theoretical contribution to ensemble deep learning models.

Keywords Aggregation function · Intervals · Monotonicity · Injectivity

Mathematics Subject Classification 90B50 · 26B99

1 Introduction

When different assessments are available on the same element, a procedure that merges all the information and provides a unique and representative final output is necessary. This is well known, for example, in Descriptive Statistics that provides functions that allow to summarize the information retrieved from a (large) sample. Aggregation functions (Grabisch

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et al. 2009; Beliakov et al. 2007) are the formalization of the summary process described above. They allow to obtain a final unique outcome when different inputs have to be taken into account. They are therefore a cornerstone in Descriptive Statistics. Other classical examples of application can be found in classification problems (see Bustince et al. (2016); Sanz et al. (2014); Castiblanco et al. (2017) among many others), or decision making (see, for example, Pap (2015); Drygas et al. (2020)). A very common situation in decision making is that different experts provide their evaluation on an item and a final (unique) decision must be made based on those assessments. Aggregation functions are very present in Economics and Business too. One of the most interesting applications in finance is in modeling aggregate risk [8] (Belles-Sampera et al. 2017).

They have also been widely used in image processing (Beliakov et al. 2011; Galar et al. 2011; Paternain et al. 2015).

On the other hand, interval valued fuzzy sets (IVFSs, in short) (Bustince et al. 2015) express knowledge or opinions by an interval and not with an isolated value, as fuzzy (or crisp) sets do. They are widely used since they allow to capture the uncertainty inherent to real life situations in a more realistic way, by describing the membership function in a more ambiguous way. They were introduced in 1973 by Zadeh (1973) as a necessary extension of fuzzy sets. Two years later Sambuc (Sambuc 1975) used them as the mathematical basis in medical diagnosis in thyroidian pathology. Since then, IVFSs have been applied in multiple areas as image processing (Barrenechea et al. 2011), decision making (Barrenechea et al. 2014; Bentkowska et al. 2015) or medicine (Choi et al. 2012).

In the context of intervals a wide range of contributions have been devoted to the problem of aggregating these elements in the last two decades. In Yager (2004) Yager introduced OWA operators to aggregate intervals and in 2007 Xu and Chen studied the use of geometric operators to accomplish the aggregation of intervals in the context of intuitionistic fuzzy sets (Xu and Chen 2007). Deschrijver (Deschrijver 2007) introduced representable aggregation operators defined on the unit interval and generated by two aggregation functions F and G with $F \leq G$, quite often with $F = G$. In 2011 Beliakov et al. studied averaging operators (Beliakov et al. 2011) in the context of intuitionistic fuzzy sets, and a year later, they focused on the study of the mean as an aggregation operator in the same context (Beliakov et al. 2012). The results presented in da Cruz Asmus et al. (2022) are based on the representation of intervals by mid-points and half of their length. One aggregation function is applied to mid-points and another appropriate function is defined on half-lengths. In 2018 Bentkowska introduced a new type of aggregation functions in the context of interval-valued sets (Bentkowska 2018). In Bustince et al. (2020) interval-valued aggregation functions are used to measure the similarity between interval-valued fuzzy sets. Asmus et al. (2022) obtain interval-valued aggregation operators in a more general framework of fusion processes. Mesiar et al. (2015) provide an overview of the classical aggregation functions.

In this contribution we provide a new aggregation operator for intervals built from two "classical" aggregation operators. We provide necessary and sufficient conditions in order for this new operator to be well defined and monotone. We also show an example of application in the context of deep learning.

The paper is organized as follows. In Sect. 2 we recall some basic definitions and properties that will be useful in the following sections. In Sect. 3 we provide our new definition in the most general setting and in Sect. 4 we focus on the very important case of the aggregation operator being defined by the operators given by Atanassov (Atanassov 1983). In this context we characterize which of these operators lead to well defined and monotone functions when the departing aggregation operators are the most important classical functions: the minimum, the maximum and the arithmetic and geometric means. Section 5 contains an example of application and Sect. 6 draws conclusions.

2 First definitions

We first fix some basic ideas and notations.

Definition 1
- Given a partially ordered set $(D, \leq_D)$, we denote $(d_1, \ldots, d_n) \leq (d'_1, \ldots, d'_n)$ if $d_i \leq_D d'_i$ for all i. It is clear that this is a partial order on D^n.
- Given two functions $f, g : D \to I$, where $(I, \leq_I)$ is a partially ordered set, we denote $f \leq g$ if $f(d) \leq_I g(d)$ for all $d \in D$.
- Given $(D, \leq_D)$ and $(I, \leq_I)$ two partially ordered sets, we say that the function $f : D \to I$ is increasing if $f(d_1) \leq_I f(d_2)$ for all $d_1 \leq_D d_2$.

The following generalized definition of aggregation function can be found in Komorníková and Mesiar (2011). For an indepth study on aggregation functions we refer to Grabisch et al. (2009); Beliakov et al. (2007) [32] among others.

Definition 2 Given a bounded partially ordered set $(D, \leq_D)$ with minimal and maximal elements denoted as 0_D and 1_D, respectively, an n-ary aggregation function on D is an application $f : D^n \to D$ such that

- f is increasing and
- $f(0_D, \ldots, 0_D) = 0_D$ and $f(1_D, \ldots, 1_D) = 1_D$.

For the sake of simplicity of notation, we drop *n-ary* in *aggregation function on D*. And, unless otherwise stated, we assume that $n \in \mathbb{N}$ and $n > 1$ in what remains.

The weakest and strongest aggregation functions on [0, 1] are denoted A_w and A_s, respectively (see Calvo et al. (2002)), and defined as

$$\mathrm{A}_w(a_1, \ldots, a_n) = \begin{cases} 1 & \text{for } (a_1, \ldots, a_n) = (1, \ldots, 1), \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{A}_s(a_1, \ldots, a_n) = \begin{cases} 0 & \text{for } (a_1, \ldots, a_n) = (0, \ldots, 0), \\ 1 & \text{otherwise.} \end{cases}$$

Special relevance is given to aggregation functions that are idempotent. Let us recall that an application $f : D^n \to D$ is idempotent if $f(d, \ldots, d) = d$ for every $d \in D$.

As already commented, the aim of this contribution is to aggregate intervals. We denote $\mathcal{L}[0, 1]$ the set of all closed intervals in [0, 1], this is,

$$\mathcal{L}[0, 1] = \{[a, b] \mid a, b \in [0, 1],\ a \leq b\}.$$

With the usual order given by

$$[a, b] \leq [a', b'] \quad \text{if } a \leq a' \text{ and } b \leq b', \tag{1}$$

$\mathcal{L}[0, 1]$ is a bounded lattice with $\min \mathcal{L}[0, 1] = [0, 0]$ and $\max \mathcal{L}[0, 1] = [1, 1]$.We set

$$\mathcal{A}_{\mathcal{L}[0,1]} = \{\mu \colon \mathcal{L}[0, 1] \to [0, 1] \mid \mu \text{ is } \mathit{increasing}, \mu[0, 0] = 0 \text{ and } \mu[1, 1] = 1\}.$$

A particular and important family of these functions are the operators that assign to each interval a linear combination of its extreme values. They were introduced by Atanassov in 1983 Atanassov (1983) in order to associate a fuzzy set with each interval valued fuzzy set:

Definition 3 Let $\alpha \in [0, 1]$. We consider the map $k_\alpha : \mathcal{L}[0, 1] \to [0, 1]$ given by

$$k_\alpha[a, b] = (1 - \alpha)a + \alpha b$$

if $[a, b] \in \mathcal{L}[0, 1]$.

It is direct to check that k_α is increasing and $k_\alpha[0, 0] = 0$ and $k_\alpha[1, 1] = 1$, that is, $k_\alpha \in \mathcal{A}_{\mathcal{L}[0,1]}$. A direct check also allows us to see that $k_\alpha[x, x] = x$ for every $\alpha, x \in [0, 1]$.

Remark 4 Observe that k_α can also be written as

$$k_\alpha[a, b] = a + \alpha(b - a).$$

It is direct to check that $k_0[a, b] = a$ and $k_1[a, b] = b$ and $k_\alpha[0, 1] = \alpha$.

Definition 5 Given $\mu, \nu \in \mathcal{A}_{\mathcal{L}[0,1]}$ such that $\mu \leq \nu$, we call reallocation on $\mathcal{L}[0, 1]$ to any function of the type

$$(\mu, \nu) : \mathcal{L}[0, 1] \to \mathcal{L}[0, 1] \quad \text{given by}$$
$$(\mu, \nu)[a, b] = [\mu[a, b], \nu[a, b]]$$

Observe that $\mu \leq \nu$ guarantees that (μ, ν) is a well-defined map.

Proposition 6 *Every reallocation on* $\mathcal{L}[0, 1]$ *is an increasing function such that* $(\mu, \nu)[0, 0] = [0, 0]$ *and* $(\mu, \nu)[1, 1] = [1, 1]$.

Proof As μ and ν are increasing, also (μ, ν) is increasing. And $(\mu, \nu)[0, 0] = [\mu[0, 0], \nu[0, 0]] = [0, 0]$ and, analogously, $(\mu, \nu)[1, 1] = [1, 1]$. So (μ, ν) is a 1-ary aggregation function on $\mathcal{L}[0, 1]$. □

We have also immediately:

Proposition 7 *Let* $\mu \in \mathcal{A}_{\mathcal{L}[0,1]}$ *and let* F *be an aggregation function on* $[0, 1]$. *Set* $\mu^n : \mathcal{L}[0, 1]^n \to [0, 1]^n$ *for the map given by,*

$$\mu^n([a_1, b_1], \ldots, [a_n, b_n]) = (\mu[a_1, b_1], \ldots, \mu[a_n, b_n]),$$

for $([a_1, b_1], \ldots, [a_n, b_n]) \in \mathcal{L}[0, 1]^n$. *Then the composition* $\mathrm{F}\,\mu^n : \mathcal{L}[0, 1]^n \to [0, 1]$ *is increasing and* $\mathrm{F}\,\mu^n([0, 0], \ldots, [0, 0]) = 0$ *and* $\mathrm{F}\,\mu^n([1, 1], \ldots, [1, 1]) = 1$.

Proposition 8 *Let* F, G *be two aggregation functions on* $[0, 1]$ *such that* $\mathrm{F} \leq \mathrm{G}$. *If* $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in [0, 1]^n$ *satisfy* $(a_1, \ldots, a_n) \leq (b_1, \ldots, b_n)$, *then* $\mathrm{F}(a_1, \ldots, a_n) \leq \mathrm{G}(b_1, \ldots, b_n)$.

Proof For $(a_1, \ldots, a_n) \leq (b_1, \ldots, b_n)$ it holds that $\mathrm{F}(a_1, \ldots, a_n) \leq \mathrm{G}(a_1, \ldots, a_n)$, as $\mathrm{F} \leq \mathrm{G}$. Moreover, $\mathrm{G}(a_1, \ldots, a_n) \leq \mathrm{G}(b_1, \ldots, b_n)$, as G is increasing. □

Proposition 9 *Let* (μ, ν) *be a reallocation on* $\mathcal{L}[0, 1]$ *and* $\mathrm{F}, \mathrm{G} : [0, 1]^n \to [0, 1]$ *be aggregation functions on* $[0, 1]$ *such that* $\mathrm{F} \leq \mathrm{G}$. *The map*

$$(\mathrm{F}\,\mu^n, \mathrm{G}\,\nu^n) : \mathcal{L}[0, 1]^n \to \mathcal{L}[0, 1]$$

given by, if $([a_1, b_1], \ldots, [a_n, b_n]) \in \mathcal{L}[0, 1]^n$,

$$(\mathrm{F}\,\mu^n, \mathrm{G}\,\nu^n)([a_1, b_1], \ldots, [a_n, b_n]) = \tag{2}$$
$$[\mathrm{F}(\mu[a_1, b_1], \ldots, \mu[a_n, b_n]), \mathrm{G}(\nu[a_1, b_1], \ldots, \nu[a_n, b_n])] \tag{3}$$

is an aggregation function on $\mathcal{L}[0, 1]$.

Proof It is immediate. □

$$\begin{tikzcd}
 & {[0,1]^n} \arrow[r, "\mathrm{F}"] & {[0,1]} \arrow[dr, dashed, "{[*,-]}"] & & \\
\mathcal{L}[0,1]^n \arrow[ur, "\mu^n"] \arrow[dr, "\nu^n"'] \arrow[rrrr, bend right=60, "{(\mathrm{F},\mathrm{G})_{(\mu,\nu)}}"'] & & & \mathrm{im}(\mu,\nu) \arrow[r, "(\mu,\nu)^{-1}"] & \mathcal{L}[0,1] \\
 & {[0,1]^n} \arrow[r, "\mathrm{G}"] & {[0,1]} \arrow[ur, dashed, "{[-,*]}"'] & &
\end{tikzcd}$$

Fig. 1 Graphical description of the new aggregation function

3 A new way to aggregate IVFSs

Once we have introduced the necessary concepts and notation, we are ready to define the new aggregation function. Given a reallocation (μ, ν) on $\mathcal{L}[0, 1]$ and two aggregation functions F and G on $[0, 1]$, such that $\mathrm{F} \leq \mathrm{G}$, we introduce the operator

$$(\mathrm{F}, \mathrm{G})_{(\mu,\nu)} \colon \mathcal{L}[0, 1]^n \to \mathcal{L}[0, 1] \text{ given by}$$
$$(\mathrm{F}, \mathrm{G})_{(\mu,\nu)}([a_1, b_1], \ldots, [a_n, b_n]) = (\mu, \nu)^{-1}(\mathrm{F}\,\mu^n, \mathrm{G}\,\nu^n)([a_1, b_1], \ldots, [a_n, b_n]).$$

It is represented in Fig. 1.

In order for this function to be well defined it must satisfy that

i) there exists $(\mu, \nu)^{-1}$, in other words, that the reallocation (μ, ν) is injective.
ii) $(\mathrm{F}\,\mu^n, \mathrm{G}\,\nu^n)([a_1, b_1], \ldots, [a_n, b_n])$ belongs to the domain of $(\mu, \nu)^{-1}$, in other words, to the image of (μ, ν). Formally, $\mathrm{im}(\mathrm{F}\,\mu^n, \mathrm{G}\,\nu^n) \subseteq \mathrm{im}(\mu, \nu)$.

Then, the definition becomes

Definition 10 Let (μ, ν) be a reallocation on $\mathcal{L}[0, 1]$ and F, G aggregation functions on $[0, 1]$ such that $\mathrm{F} \leq \mathrm{G}$. Assume moreover that

(REQ1) $(\mu, \nu) \colon \mathcal{L}[0, 1] \to \mathcal{L}[0, 1]$ is an injective map, and
(REQ2) $\mathrm{im}(\mathrm{F}\,\mu^n, \mathrm{G}\,\nu^n) \subseteq \mathrm{im}(\mu, \nu)$.

Then we define the map $(\mathrm{F}, \mathrm{G})_{(\mu,\nu)}$ as

$$(\mathrm{F}, \mathrm{G})_{(\mu,\nu)} \colon \mathcal{L}[0, 1]^n \to \mathcal{L}[0, 1] \text{ given by}$$
$$(\mathrm{F}, \mathrm{G})_{(\mu,\nu)}([a_1, b_1], \ldots, [a_n, b_n]) = (\mu, \nu)^{-1}(\mathrm{F}\,\mu^n, \mathrm{G}\,\nu^n)([a_1, b_1], \ldots, [a_n, b_n]).$$

In the case $\mathrm{F} = \mathrm{G}$, we set $\mathrm{F}_{(\mu,\nu)} = (\mathrm{F}, \mathrm{F})_{(\mu,\nu)}$.

Assuming $F \leq G$, we denote (F, G) the operator that assigns to an element $\mathbf{u} = ([a_1, b_1], \ldots, [a_n, b_n]) \in \mathcal{L}[0, 1]^n$ the element $(\mathrm{F}(a_1, \ldots, a_n), \mathrm{G}(b_1, \ldots, b_n)) \in \mathcal{L}[0, 1]$.

Proposition 11 *Let* F *and* G *be any two aggregation operators such that $F \leq G$ then* $(\mathrm{F}, \mathrm{G})_{(k_0,k_1)} = (\mathrm{F}, \mathrm{G})$.

Proof Call $\mathbf{u} = ([a_1, b_1], \ldots, [a_n, b_n])$.

$$(\mathrm{F}, \mathrm{G})_{(k_0,k_1)}(\mathbf{u}) = (k_0, k_1)^{-1}[\mathrm{F}\,k_0^n(\mathbf{u}), \mathrm{G}\,k_1^n(\mathbf{u})]$$

As recalled in Remark 4, $k_0[a, b] = a$ and $k_1[a, b] = b$, so $(k_0, k_1)[a, b] = [a, b]$. Also, $(k_0, k_1)^{-1}[a, b] = [a, b]$.

It follows that,

$$\mathrm{F}\,k_0^n(\mathbf{u}) = \mathrm{F}(k_0[a_1, b_1], \ldots, k_0[a_n, b_n]) = \mathrm{F}(a_1, \ldots, a_n),$$
$$\mathrm{G}\,k_1^n(\mathbf{u}) = \mathrm{G}(k_1[a_1, b_1], \ldots, k_1[a_n, b_n]) = \mathrm{G}(b_1, \ldots, b_n).$$

Then,

$$\begin{aligned}(\mathrm{F}, \mathrm{G})_{(k_0,k_1)}(\mathbf{u}) &= (k_0, k_1)^{-1}[F(a_1, \ldots, a_n), G(b_1, \ldots, b_n)] \\ &= [F(a_1, \ldots, a_n), G(b_1, \ldots, b_n)] = (\mathrm{F}, \mathrm{G})(\mathbf{u}).\end{aligned}$$

□

Since aggregation operators are increasing, an immediate consequence of this result is that $(\mathrm{F}, \mathrm{F})_{(k_0,k_1)} = (\mathrm{F}, \mathrm{F})$ is the best interval representation of the operator F. Recall that the best interval representation of an operator $f : [a, b]^n \to [c, d]$ is the interval function $\hat{f} : \mathcal{L}[a, b]^n \to \mathcal{L}[c, d]$ defined by Dimuro et al. (2011):

$$\hat{f}(\mathbf{u}) = \left[\inf_{c_i \in [a_i, b_i]} \{f(c_1, \ldots, c_n)\}, \sup_{c_i \in [a_i, b_i]} \{f(c_1, \ldots, c_n)\} \right],$$

where $\mathbf{u} = ([a_i, b_1], \ldots, [a_n, b_n]) \in \mathcal{L}[a, b]^n$.

Proposition 12 *Let* $(\mathrm{F}, \mathrm{G})_{(\mu,\nu)}$ *be the operator introduced in Definition 10 where* F *and* G *are idempotent. Then*

(i) $\mathrm{im}(\mathrm{F}\,\mu^n, \mathrm{G}\,\nu^n) = \mathrm{im}(\mu, \nu)$.
(ii) $(\mathrm{F}, \mathrm{G})_{(\mu,\nu)} : \mathcal{L}[0, 1]^n \to \mathcal{L}[0, 1]$ *is an idempotent map.*

Proof (i) The content $\mathrm{im}(\mathrm{F}\,\mu^n, \mathrm{G}\,\nu^n) \subseteq \mathrm{im}(\mu, \nu)$ holds by definition. To check that also $\mathrm{im}(\mu, \nu) \subseteq \mathrm{im}(\mathrm{F}\,\mu^n, \mathrm{G}\,\nu^n)$, take $[c, d] \in \mathcal{L}[0, 1]$ such that $(\mu, \nu)[a, b] = [c, d]$ for some $[a, b] \in \mathcal{L}[0, 1]$. Then,

$$(\mathrm{F}\,\mu^n, \mathrm{G}\,\nu^n)([a, b], \overset{n}{\ldots}, [a, b]) = [\mathrm{F}(c, \overset{n}{\ldots}, c), \mathrm{G}(d, \overset{n}{\ldots}, d)] = [c, d],$$

this is, $[c, d] \in \mathrm{im}(\mathrm{F}\,\mu^n, \mathrm{G}\,\nu^n)$.

(ii) Take $[a, b] \in \mathcal{L}[0, 1]$.

$$\begin{aligned}(\mathrm{F}, \mathrm{G})_{(\mu,\nu)}([a, b], \overset{n}{\ldots}, [a, b]) &= (\mu, \nu)^{-1}[\mathrm{F}(\mu[a, b], \overset{n}{\ldots}, \mu[a, b]), \mathrm{G}(\nu[a, b], \overset{n}{\ldots}, \nu[a, b]] \\ &= (\mu, \nu)^{-1}[\mu[a, b], \nu[a, b]] = [a, b].\end{aligned}$$

□

Next result is a direct consequence of Propositions 11 and 12.

Corollary 13 *For* F, G *idempotent aggregation functions on* $[0, 1]$, $(\mathrm{F}, \mathrm{G})_{(k_0,k_1)}$ *is an idempotent aggregation function on* $\mathcal{L}[0, 1]$.

Proposition 12 does not necessarily hold if F or G are not idempotent:

Example 14 Take $\mu = \nu$ the arithmetic mean and $\mathrm{F} = \mathrm{G}$ as half of the arithmetic mean for $(a_1, \ldots, a_n) \neq (0, \ldots, 0), (1, \ldots, 1)$, this is, $\mathrm{F} = \mathrm{G} = \frac{\mathrm{am}}{2}$ defined as $\frac{\mathrm{am}}{2}(0, \ldots, 0) = 0$, $\frac{\mathrm{am}}{2}(1, \ldots, 1) = 1$ and $\frac{\mathrm{am}}{2}(a_1, \ldots, a_n) = \frac{a_1+\cdots+a_n}{2n}$ otherwise.

Then $(\mathrm{F}, \mathrm{G})_{(\mu,\nu)}$ is not idempotent. Take $a \in (0, 1)$ and recall that in this case $\mathrm{F}(a, \ldots, a) = \frac{a}{2}$. Then

$$\begin{aligned}&(\mathrm{F}, \mathrm{G})_{(\mu,\nu)}([a, a] \ldots, [a, a]) = \\ &(\mu, \nu)^{-1}([\tfrac{\mathrm{am}}{2}(a, \ldots, a), \tfrac{\mathrm{am}}{2}(a, \ldots, a)]) = \\ &(\mu, \nu)^{-1}([a/2, a/2]) = [a/2, a/2].\end{aligned}$$

Concerning the monotonicity of the compositions we are studying, we have the following partial result.

Proposition 15 *Let* $\mu, \nu \in \mathcal{A}_{\mathcal{L}[0,1]}$ *with* $\mu \leq \nu$ *and* F, G *aggregation functions on* $[0, 1]$ *such that* $\mathrm{F} \leq \mathrm{G}$. *Then* $(\mathrm{F}\,\mu^n, \mathrm{G}\,\nu^n)$ *is an increasing function.*

Proof Since they are aggregation functions, F and G are increasing. Also, μ an ν are increasing by definition, so $(\mathrm{F}\,\mu^n, \mathrm{G}\,\nu^n)$ is the composition of increasing functions. Therefore, it is increasing. □

4 The case $\mu = k_\alpha, \nu = k_\beta$

In the remainder of this contribution we focus on the particular family of functions in $\mathcal{A}_{\mathcal{L}[0,1]}$ denoted k_α for $\alpha \in [0, 1]$ and we study when $(\mathrm{F}, \mathrm{G})_{(k_\alpha, k_\beta)}$ is an aggregation function assuming that F and G are aggregation functions. We use the notation $(\mathrm{F}, \mathrm{G})_{(\alpha,\beta)}$ to refer to $(\mathrm{F}, \mathrm{G})_{(k_\alpha, k_\beta)}$ and we simply write $\mathrm{F}_{(\alpha,\beta)}$ for the case $\mathrm{F} = \mathrm{G}$, this is, $\mathrm{F}_{(\alpha,\beta)}$ stands for $(\mathrm{F}, \mathrm{F})_{(k_\alpha, k_\beta)}$. In general we have the following results.

In order to study whether $(\mathrm{F}, \mathrm{G})_{(\alpha,\beta)}$ is an aggregation function we first have to check if it is well defined. As discussed above, this means that we have to check if $k_\alpha \leq k_\beta$ and Conditions REQ1 and REQ2 are satisfied. The next results will be useful to this end.

Proposition 16 *If* $\alpha, \beta \in [0, 1]$, $k_\alpha \leq k_\beta$ *if and only if* $\alpha \leq \beta$.

Proof Observe that

$$k_\beta[a, b] - k_\alpha[a, b] = (\beta - \alpha)(b - a)\,. \tag{4}$$

□

Proposition 17 *Let* $\alpha, \beta \in [0, 1]$, $\alpha \leq \beta$ *and* (k_α, k_β) *the following map:*

$$(k_\alpha, k_\beta)\colon \mathcal{L}[0, 1] \to \mathcal{L}[0, 1] \quad \textit{given by}$$
$$(k_\alpha, k_\beta)[a, b] = [k_\alpha[a, b], k_\beta[a, b]]$$

Then (k_α, k_β)*:*

a) *is an idempotent increasing function such that* $(k_\alpha, k_\beta)[0, 0] = [0, 0]$ *and* $(k_\alpha, k_\beta)[1, 1] = [1, 1]$.
b) *is injective if and only if* $\alpha < \beta$,
c) *is surjective if and only if* $\alpha = 0$ *and* $\beta = 1$,
d) *is the identity function if and only if* $\alpha = 0$ *and* $\beta = 1$, *that is,* $(k_0, k_1) = \mathrm{id}_{\mathcal{L}[0,1]}$.

Proof a) Trivial by Proposition 6.
b) Concerning injectivity,
the system $\begin{cases}(1-\alpha)a + \alpha b = c\\(1-\beta)a + \beta b = d\end{cases}$ in a, b, as $\begin{vmatrix}1-\alpha & \alpha\\1-\beta & \beta\end{vmatrix} = \beta - \alpha$, has unique solution for each c, d if and only if $\alpha \neq \beta$. Thus, if and only if $\alpha < \beta$.
c) Assume (k_α, k_β) is surjective, then $[0, 1] \in Im(k_\alpha, k_\beta)$. Equation 4 implies $1 = (\beta - \alpha)(b - a)$ for some $[a, b] \in \mathcal{L}[0, 1]$. Then, necessarily $\beta = 1$ and $\alpha = 0$.
d) It is a simple computation from Remark 4.

□

Corollary 18 *Condition REQ1 holds if and only if* $\alpha < \beta$.

Proof Follows from Proposition 17 b). □

Since Condition REQ1 is basic in order for $(\mathrm{F}, \mathrm{G})_{(\alpha,\beta)}$ to be well defined, we will assume $\alpha < \beta$ hereafter.

Concerning Condition REQ2, it must hold that

$$im(\mathrm{F}\, k_\alpha^n, \mathrm{G}\, k_\beta^n) \subseteq im(k_\alpha, k_\beta).$$

We next study the image of (k_α, k_β) (see Fig. 2).

Proposition 19 *Let* $\alpha, \beta \in [0, 1]$, $\alpha < \beta$ *and* $[c, d] \in \mathcal{L}[0, 1]$. *Then* $[c, d] \in \mathrm{im}(k_\alpha, k_\beta)$ *if and only if*

i) $\alpha d \leq \beta c$
ii) $d \leq \frac{(1-\beta)c+(\beta-\alpha)}{1-\alpha}$

Proof The element $[c, d]$ is in the image of (k_α, k_β) if and only if there exists $[a, b] \in \mathcal{L}[0, 1]$ such that $(k_\alpha, k_\beta)[a, b] = [c, d]$. Explicitly,

$$\begin{cases} (1-\alpha)a + \alpha b = c, \\ (1-\beta)a + \beta b = d. \end{cases}$$

Since $\alpha < \beta$, the system has a unique solution and the explicit expressions are

$$a = \frac{\beta c - \alpha d}{\beta - \alpha}, \quad b = \frac{(1-\alpha)d - (1-\beta)c}{\beta - \alpha}.$$

Since $[a, b] \in \mathcal{L}[0, 1]$, it must satisfy (1) $a \geq 0$, (2) $a \leq b$ and (3) $b \leq 1$.

(1) Condition $a \geq 0$ is equivalent to $\beta c \geq \alpha d$.
(2) Condition $a \leq b$ is equivalent to $\frac{\beta c-\alpha d}{\beta-\alpha} \leq \frac{(1-\alpha)d-(1-\beta)c}{\beta-\alpha}$ and this is equivalent to $c \leq d$, that holds since $[c, d] \in \mathcal{L}[0, 1]$.
(3) $b \leq 1$ is equivalent to $\frac{(1-\alpha)d-(1-\beta)c}{\beta-\alpha} \leq 1$ and equivalent to $d \leq \frac{(1-\beta)c+(\beta-\alpha)}{1-\alpha}$.

□

If we denote $\mathbf{u} = ([a_1, b_1], \ldots, [a_n, b_n])$, it holds that

$$(\mathrm{F}\, k_\alpha^n, \mathrm{G}\, k_\beta^n)(\mathbf{u}) = [\mathrm{F}(k_\alpha[a_1, b_1], \ldots, k_\alpha[a_n, b_n]), \mathrm{G}(k_\beta[a_1, b_1], \ldots, k_\beta[a_n, b_n])].$$

We denote $\mathrm{F}\, k_\alpha^n(\mathbf{u})$ and $\mathrm{G}\, k_\beta^n(\mathbf{u})$ the first and second components above, respectively:

$$\mathrm{F}\, k_\alpha^n(\mathbf{u}) := \mathrm{F}(k_\alpha[a_1, b_1], \ldots, k_\alpha[a_n, b_n])$$
$$\mathrm{G}\, k_\beta^n(\mathbf{u}) := \mathrm{G}(k_\beta[a_1, b_1], \ldots, k_\beta[a_n, b_n])$$

Then, according to Proposition 19 and in order for $(\mathrm{F}, \mathrm{G})_{(\alpha,\beta)}$ to satisfy REQ2, it must hold that

$$\alpha\, \mathrm{G}\, k_\beta^n(\mathbf{u}) \leq \beta\, \mathrm{F}\, k_\alpha^n(\mathbf{u}) \qquad \text{and} \qquad G k_\beta^n(\mathbf{u}) \leq \frac{(1-\beta) F k_\alpha^n(\mathbf{u}) + (\beta - \alpha)}{1-\alpha}.$$

We can therefore settle the conditions that $(\mathrm{F}, \mathrm{G})_{(\alpha,\beta)}$ must satisfy in order to be well defined.

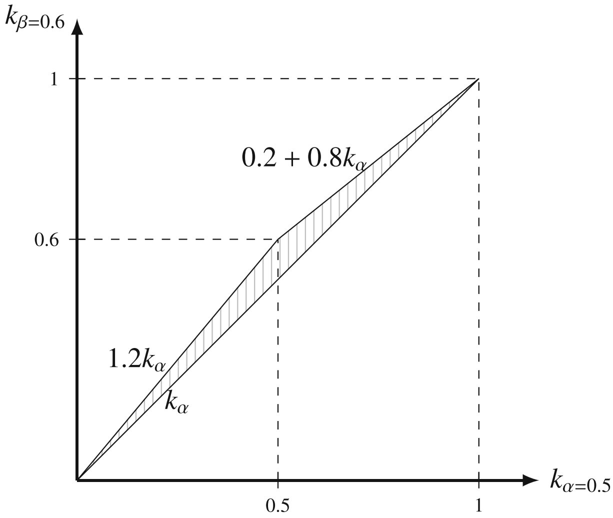

Fig. 2 A graphical representation of the image of (k_α, k_β)

Corollary 20 *Given* F *and* G *two aggregation functions on* $[0, 1]$ *and* $\alpha, \beta \in [0, 1]$*, the function* $(\mathrm{F}, \mathrm{G})_{(\alpha,\beta)}$ *is well defined if and only if:*

(WDi) $\alpha < \beta$,

(WDii) $\alpha\, \mathrm{G}\, k_\beta^n(\mathbf{u}) \leq \beta\, \mathrm{F}\, k_\alpha^n(\mathbf{u})$,

(WDiii) $\mathrm{G}\, k_\beta^n(\mathbf{u}) \leq \dfrac{(1-\beta)\, \mathrm{F}\, k_\alpha^n(\mathbf{u}) + (\beta - \alpha)}{1-\alpha}$

for any $\mathbf{u} = ([a_1, b_1], \ldots, [a_n, b_n])$ *in* $\mathcal{L}[0, 1]^n$*, where we use the notation*

$$\begin{aligned} \mathrm{F}\, k_\alpha^n(\mathbf{u}) &:= \mathrm{F}(k_\alpha[a_1, b_1], \ldots, k_\alpha[a_n, b_n]), \\ \mathrm{G}\, k_\beta^n(\mathbf{u}) &:= \mathrm{G}(k_\beta[a_1, b_1], \ldots, k_\beta[a_n, b_n]). \end{aligned}$$

Proof The operator $(\mathrm{F}, \mathrm{G})_{(\alpha,\beta)}$ is well defined if and only if Conditions REQ1 and REQ2 hold (see Definition 10). We prove that the conditions above are equivalent to Conditions REQ1 and REQ1.
(**WDi)** is equivalent to REQ1 by Proposition 17 b).
Condition REQ2 is $\mathrm{im}(\mathrm{F}\,\mu^n, \mathrm{G}\,\nu^n) \subseteq \mathrm{im}(\mu, \nu)$. We can write this as $(\mathrm{F}\,\mu^n, \mathrm{G}\,\nu^n)(\mathbf{u}) \in \mathrm{im}(\mu, \nu)$ for all $\mathbf{u} \in \mathcal{L}[0, 1]^n$. It follows from Proposition 19 that this is equivalent to **(WDii)** and **(WDiii)**. □

For the particular case $\mathrm{F} = \mathrm{G}$ and using the notation $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$, Eq. (WDii) becomes

$$\frac{\mathrm{F}(\boldsymbol{a} + \beta(\boldsymbol{b} - \boldsymbol{a}))}{\mathrm{F}(\boldsymbol{a} + \alpha(\boldsymbol{b} - \boldsymbol{a}))} \leq \frac{\beta}{\alpha}$$

for $\alpha \neq 0$, that means that function F should not show drastic increases.

Example 21 An example where all the conditions in Corollary 20 are satisfied is the following one: take $0 \leq \alpha < \beta \leq 1$

$$\mathrm{F}(a_1, \ldots, a_n) = \begin{cases} 1 & \text{, if } a_i = 1\ \forall i, \\ 0 & \text{, if } a_i = 0\ \forall i, \\ \alpha & \text{, otherwise.} \end{cases} \quad \text{and} \quad \mathrm{G}(a_1, \ldots, a_n) = \begin{cases} 1 & \text{, if } a_i = 1\ \forall i, \\ 0 & \text{, if } a_i = 0\ \forall i, \\ \beta & \text{, otherwise.} \end{cases}$$

It holds that

(WDi) $\alpha < \beta$ by definition.

(WDii) If $\alpha = 0$ the inequality holds trivially. Now, for $al > 0$, we distinguish three cases:

- If $\mathrm{G}\, k^n_\beta(\mathbf{u}) = 0$, the inequality holds trivially.
- If $0 < \mathrm{G}\, k^n_\beta(\mathbf{u}) = \beta$, then $a_i + \beta(b_i - a_i) > 0$ for some i. This inequality with $\alpha > 0$ guarantee that $a_i + \alpha(b_i - a_i) > 0$ for some i and this implies that $\mathrm{F}\, k^n_\alpha(\mathbf{u}) \geq \alpha$. The inequality follows.
- Assume $\mathrm{G}\, k^n_\beta(\mathbf{u}) = 1$. This happens if and only if $k^n_\beta(\mathbf{u}) = 1$. This implies $b_i = 1$ for all i and either $\beta = 1$ or $a_i = 1$ for all i. In both cases $k^n_\alpha(\mathbf{u}) = (1, \ldots, 1)$ and the inequality holds since it looks as $\alpha \leq \beta$.

(WDiii) In order to prove this inequality we distinguish two scenarios since the case $\mathrm{G}\, K^n_\beta(\mathbf{u}) = 0$ is trivial:

- Assume $\mathrm{G}\, K^n_\beta(\mathbf{u}) = 1$. This is equivalent to $K^n_\beta(\mathbf{u}) = 1$. This can only happen in one of the two following cases:

* If $\mathbf{u} = ([1, 1], \ldots, [1, 1])$, then $k^n_\alpha(\mathbf{u}) = (1, \ldots, 1)$ and $\mathrm{F}\, k^n_\alpha(\mathbf{u}) = 1$, whereas

$$\frac{(1-\beta)\,\mathrm{F}\, k^n_\alpha(\mathbf{u}) + (\beta - \alpha)}{1-\alpha} = 1 = \frac{(1-\beta) + (\beta-\alpha)}{1-\alpha} = 1$$

and the inequality holds.

* If $\mathbf{u} \neq ([1, 1], \ldots, [1, 1])$ then necessarily $\beta = 1$ and again

$$\frac{(1-\beta)\,\mathrm{F}\, k^n_\alpha(\mathbf{u}) + (\beta - \alpha)}{1-\alpha} = 1$$

- Assume $\mathrm{G}\, K^n_\beta(\mathbf{u}) = \beta > 0$. Assume also first that $\mathrm{F}\, k^n_\alpha(\mathbf{u}) = 0$. This equality can only hold if $\alpha = 0$ or $b_i = 0$ for all i. If $\alpha = 0$, the righ-hand side of (WDiii) becomes β, so the inequality holds. The other case ($b_i = 0$ for all i) contradicts $\mathrm{G}\, K^n_\beta(\mathbf{u}) = \beta$.
Then necessarily $\mathrm{F}\, k^n_\alpha(\mathbf{u}) > 0$. If we assume $\mathrm{F}\, k^n_\alpha(\mathbf{u}) = \alpha$, the right-hand side of (WDiii) becomes β. For $\mathrm{F}\, k^n_\alpha(\mathbf{u}) = 1$, the right-hand side of (WDiii) becomes 1. In any case, the inequality holds.

The conditions proven in Corollary 20 are quite restrictive. If we consider $\mathrm{F} = \frac{\mathrm{am}}{2}$, the function defined in Example 14, and G the arithmetic mean, Condition **(WDii)** is not satisfied for any $\alpha > 0$.

Example 22 If we take $\mathrm{F} = \frac{\mathrm{am}}{2}$, $\mathrm{G} = \mathrm{am}$, the arithmetic mean, and $0 < \alpha < \beta \leq 1$, Condition **(WDii)** does not hold. It suffices to take $\mathbf{u} = ([0, b_1], \ldots, [0, b_n])$ with $b_i > 0$ for at least some $i \in \{1, \ldots, n\}$. A simple calculus leads us to

$$\alpha\,\mathrm{G}\, k^n_\beta(\mathbf{u}) = \alpha\beta \frac{\sum_{i=1}^n b_i}{n} \not\leq \alpha\beta \frac{\sum_{i=1}^n b_i}{2n} = \beta\,\mathrm{F}\, k^n_\alpha(\mathbf{u}).$$

Proposition 23 *Let* $\mathbf{u} = ([a_1, b_1], \ldots, [a_n, b_n])$ *be any element in* $\mathcal{L}[0, 1]^n$. *Let* $\alpha < \beta$ *and* F, G *two aggregation operators on* $[0, 1]$ *and let* $(\mathrm{F}, \mathrm{G})_{(\alpha,\beta)}$ *be the function introduced in Definition 10, then* $(\mathrm{F}, \mathrm{G})_{(\alpha,\beta)}(\mathbf{u}) = [A, B]$ *with*

$$A = \frac{\beta\,\mathrm{F}\, k^n_\alpha(\mathbf{u}) - \alpha\,\mathrm{G}\, k^n_\beta(\mathbf{u})}{\beta - \alpha}, \tag{5}$$

$$B = \frac{(1-\alpha)\,\mathrm{G}\, k^n_\beta(\mathbf{u}) - (1-\beta)\,\mathrm{F}\, k^n_\alpha(\mathbf{u})}{\beta - \alpha}. \tag{6}$$

Proof

$$(\mathrm{F},\mathrm{G})_{(\alpha,\beta)}(\mathbf{u}) = (k_\alpha, k_\beta)^{-1}[\mathrm{F}\, k_\alpha^n(\mathbf{u}), \mathrm{G}\, k_\beta^n(\mathbf{u})] =: [A, B].$$

Considering the inverse function in the last equation, we obtain that

$$(k_\alpha, k_\beta)[A, B] = [k_\alpha[A, B], k_\beta[A, B]] = [\mathrm{F}\, k_\alpha^n(\mathbf{u}), \mathrm{G}\, k_\beta^n(\mathbf{u})],$$

so

$$k_\alpha[A, B] = \mathrm{F}\, k_\alpha^n(\mathbf{u}) \quad \text{and} \quad k_\beta[A, B] = \mathrm{G}\, k_\beta^n(\mathbf{u}).$$

Using the definition of k_α and k_β, we obtain the system:

$$\begin{cases} (1-\alpha)A + \alpha B = x, \\ (1-\beta)A + \beta B = y, \end{cases}$$

being $x = \mathrm{F}\, k_\alpha^n(\mathbf{u})$ and $y = \mathrm{G}\, k_\beta^n(\mathbf{u})$. Since $\alpha < \beta$, the system has a unique solution and its explicit expression is

$$A = \frac{\begin{vmatrix} x & \alpha \\ y & \beta \end{vmatrix}}{\beta - \alpha} = \frac{\beta x - \alpha y}{\beta - \alpha}, \quad B = \frac{\begin{vmatrix} 1-\alpha & x \\ 1-\beta & y \end{vmatrix}}{\beta - \alpha} = \frac{(1-\alpha)y - (1-\beta)x}{\beta - \alpha}.$$

Replacing x by $\mathrm{F}\, k_\alpha^n(\mathbf{u})$ and y by $\mathrm{G}\, k_\beta^n(\mathbf{u})$, we get the desired result. □

The objective of this contribution is to study when $(\mathrm{F},\mathrm{G})_{(\alpha,\beta)}$ is an aggregation function. A necessary condition is monotonicity. We have already proven (see Proposition 15) that $(\mathrm{F}\,\mu^n, \mathrm{G}\,\nu^n)$ is increasing. However, the following result proves that monotonicity of $(k_\alpha, k_\beta)^{-1}$ can only be warrantied for a very specific case of α and β.

Proposition 24 *Let $\alpha, \beta \in [0, 1]$, $\alpha < \beta$. Then*

$$(k_\alpha, k_\beta)^{-1}\colon\ \mathrm{im}(k_\alpha, k_\beta) \to \mathcal{L}[0, 1]$$

is increasing if and only if $\alpha = 0$ and $\beta = 1$.

Proof Since $(k_0, k_1) = \mathrm{id}_{\mathcal{L}[0,1]}$, it is clear that the function is increasing in this case. We next prove that it is not increasing for any other (α, β).

- First assume $\alpha > 0$ and take $[\alpha, \alpha], [\alpha, \beta] \in \mathrm{im}(k_\alpha, k_\beta)$. It holds that $[\alpha, \alpha] \leq [\alpha, \beta]$. However,

$$(k_\alpha, k_\beta)^{-1}[\alpha, \alpha] = [\alpha, \alpha] \not\leq [0, 1] = (k_\alpha, k_\beta)^{-1}[\alpha, \beta].$$

- For $\alpha = 0$ consider $[0, \beta], [\beta, \beta] \in \mathrm{im}(k_\alpha, k_\beta)$. We have $[0, \beta] \leq [\beta, \beta]$ but

$$(k_\alpha, k_\beta)^{-1}[0, \beta] = [0, 1] \not\leq [\beta, \beta] = (k_\alpha, k_\beta)^{-1}[\beta, \beta].$$

□

The case $\alpha = 0$ and $\beta = 1$ is a very special one since for the pair $(\alpha, \beta) = (0, 1)$ we obtain the original aggregation function.

Proposition 24 is quite disappointing since, in case $(k_\alpha, k_\beta)^{-1}$ were monotone, the composition $(\mathrm{F},\mathrm{G})_{(k_\alpha,k_\beta)}$ would be monotone too. Fortunately, the converse is not true.

The following proposition shows what inequalities must satisfy F and G in this case.

Proposition 25 (Monotonicity) *Let* $\alpha, \beta \in [0, 1]$ *with* $\alpha < \beta$. *The function* $(\mathrm{F}, \mathrm{G})_{(\alpha,\beta)}$ *is monotone if and only if:*

$$\alpha\, \mathrm{G}\, k^n_\beta(\mathbf{v}) - \beta\, \mathrm{F}\, k^n_\alpha(\mathbf{v}) \le \alpha\, \mathrm{G}\, k^n_\beta(\mathbf{u}) - \beta\, \mathrm{F}\, k^n_\alpha(\mathbf{u}), \tag{7}$$

and

$$(1-\alpha)\, \mathrm{G}\, k^n_\beta(\mathbf{u}) - (1-\beta)\, \mathrm{F}\, k^n_\alpha(\mathbf{u}) \le (1-\alpha)\, \mathrm{G}\, k^n_\beta(\mathbf{v}) - (1-\beta)\, \mathrm{F}\, k^n_\alpha(\mathbf{v}). \tag{8}$$

for every $\mathbf{u}, \mathbf{v} \in \mathcal{L}[0, 1]^n$ *such that* $\mathbf{u} \le \mathbf{v}$.

Proof In order for $(\mathrm{F}, \mathrm{G})_{(\alpha,\beta)}$ to be monotone, we must have

$$(\mathrm{F}, \mathrm{G})_{(\alpha,\beta)}(\mathbf{u}) \le (\mathrm{F}, \mathrm{G})_{(\alpha,\beta)}(\mathbf{v})$$

whenever $\mathbf{u} \le \mathbf{v}$. Using Eqs. 5 and 6 the previous inequality is equivalent to

$$\frac{\beta\, \mathrm{F}\, k^n_\alpha(\mathbf{u}) - \alpha\, \mathrm{G}\, k^n_\beta(\mathbf{u})}{\beta - \alpha} \le \frac{\beta\, \mathrm{F}\, k^n_\alpha(\mathbf{v}) - \alpha\, \mathrm{G}\, k^n_\beta(\mathbf{v})}{\beta - \alpha}$$

and

$$\frac{(1-\alpha)\, \mathrm{G}\, k^n_\beta(\mathbf{u}) - (1-\beta)\, \mathrm{F}\, k^n_\alpha(\mathbf{u})}{\beta - \alpha} \le \frac{(1-\alpha)\, \mathrm{G}\, k^n_\beta(\mathbf{v}) - (1-\beta)\, \mathrm{F}\, k^n_\alpha(\mathbf{v})}{\beta - \alpha}.$$

And Eqs. 7 and 8 are simplified versions of these two inequalities. □

Example 26 The functions included in Example 21 satisfy this condition. The checking is tedious because different cases have to be considered, but straightforward.

4.1 Some particular cases. The case F = G

In this subsection we study some particular cases and we explore if we obtain an aggregation operator for intervals when both F and G are replaced by the same aggregation function.

We first present a general result concerning aggregation functions that are linear. We say that an aggregation function F is linear if for any $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in [0, 1]^n$ and $\beta \in [0, 1]$ it holds that

$$\mathrm{F}(a_1 + \beta b_1, \ldots, a_n + \beta b_n) = \mathrm{F}(a_1, \ldots, a_n) + \beta\, \mathrm{F}(b_1, \ldots, b_n).$$

Proposition 27 *Let* $\alpha, \beta \in [0, 1]$, $\alpha < \beta$ *and* F *a linear aggregation function. Then* $\mathrm{F}_{(\alpha,\beta)} = (\mathrm{F}, \mathrm{F})$.

Proof Call $\mathbf{u} = ([a_1, b_1], \ldots, [a_n, b_n])$.

$$(\mathrm{F}, \mathrm{F})_{(\alpha,\beta)}(\mathbf{u}) = (k_\alpha, k_\beta)^{-1}[\mathrm{F}\, k^n_\alpha(\mathbf{u}), \mathrm{G}\, k^n_\beta(\mathbf{u})]$$

On the other hand,

$$\mathrm{F}\, k^n_\alpha(\mathbf{u}) = \mathrm{F}(a_1 + \beta(b_1 - a_1), \ldots, a_n + \beta(b_n - a_n))$$
$$\mathrm{F}\, k^n_\beta(\mathbf{u}) = \mathrm{F}(a_1 + \beta(b_1 - a_1), \ldots, a_n + \beta(b_n - a_n))$$

Since F is linear,

$$\mathrm{F}\, k^n_\alpha(\mathbf{u}) = \mathrm{F}(a_1, \ldots, a_n) + \alpha\, \mathrm{F}((b_1 - a_1), \ldots, (b_n - a_n))$$
$$\mathrm{F}\, k^n_\beta(\mathbf{u}) = \mathrm{F}(a_1, \ldots, a_n) + \beta\, \mathrm{F}((b_1 - a_1), \ldots, (b_n - a_n))$$

Now call $c := \mathrm{F}(a_1, \ldots, a_n)$ and $d := \mathrm{F}(((b_1 - a_1), \ldots, (b_n - a_n))$ then

$$(\mathrm{F}, \mathrm{F})_{(\alpha,\beta)}(\mathbf{u}) = (k_\alpha, k_\beta)^{-1}([c + \alpha d, c + \beta d]) = [c, c + d].$$

Finally, since F is linear,

$$\begin{aligned} c + d &= \mathrm{F}(a_1, \ldots, a_n) + \mathrm{F}((b_1 - a_1), \ldots, (b_n - a_n)) \\ &= \mathrm{F}(a_1 + (b_1 - a_1), \ldots, a_n + (b_n - a_n)) = \mathrm{F}(b_1, \ldots, b_n). \end{aligned}$$

So,

$$(\mathrm{F}, \mathrm{F})_{(\alpha,\beta)}(\mathbf{u}) = [\mathrm{F}(a_1, \ldots, a_n), \mathrm{F}(b_1, \ldots, b_n)],$$

equivalently, $(\mathrm{F}, \mathrm{F})_{(\alpha,\beta)} = (\mathrm{F}, \mathrm{F})$. □

A relevant particular case is the arithmetic mean.

Corollary 28 *Let* $[a_1, b_1], \ldots, [a_n, b_n] \in \mathcal{L}[0, 1]$ *and* $\alpha, \beta \in [0, 1]$, $\alpha < \beta$. *Then we have, for the arithmetic mean* am,

$$\mathrm{am}_{(\alpha,\beta)}([a_1, b_1], \ldots, [a_n, b_n]) = [\mathrm{am}(a_1, \ldots, a_n), \mathrm{am}(b_1, \ldots, b_n)].$$

Proof Follows from Proposition 27 and the fact that am is a linear aggregation function. □

The remaining of this subsection concerns the most classical aggregation functions. We first consider the weakest and strongest aggregation operators (Calvo et al. 2002) but we later focus on the cases most used in practical situations, those that are idempotent: the minimum, the maximum and the arithmetic and geometric means.

Proposition 29 *Let* $\alpha, \beta \in [0, 1]$, $\alpha < \beta$. *Let* A_w *be the weakest aggregation function. Then* $\mathrm{A}_{w\,(\alpha,\beta)} : \mathcal{L}[0, 1]^n \to \mathcal{L}[0, 1]$ *is well defined for* $\beta < 1$ *and if* $\beta = 1$ *for* $\alpha = 0$. *It is increasing whenever it is well defined.*

Proof Consider the case $\beta < 1$. We first prove that in this case, for any $\alpha < \beta$ we have that:

$$k^n_\beta(\mathbf{u}) = (1, \ldots, 1) \quad \text{if and only if} \quad \mathrm{k}^n_\alpha(\mathbf{u}) = (1, \ldots, 1). \tag{9}$$

Observe that if $\mathbf{u} = ([1, 1], \ldots, [1, 1])$, then $k^n_\beta(\mathbf{u}) = k^n_\alpha(\mathbf{u}) = (1, \ldots, 1)$.

In case $\mathbf{u} = ([a_1, b_1], \ldots, [a_n, b_n]) \neq ([1, 1], \ldots, [1, 1])$, there exists $a_i < 1$. Since both $\alpha, \beta < 1$, it holds that both $k_\beta([a_i, b_i]) < 1$ and $k_\alpha([a_i, b_i]) < 1$, whereas both $k^n_\beta(\mathbf{u}) \neq (1, \ldots, 1)$ and $k^n_\alpha(\mathbf{u}) \neq (1, \ldots, 1)$.

It follows from Eq. 9 that for $\beta < 1$, then

$$\mathrm{G}\, k^n_\beta(\mathbf{u}) = 1 \quad \Leftrightarrow \quad \mathrm{F}\, k^n_\alpha(\mathbf{u}) = 1, \qquad \mathrm{G}\, k^n_\beta(\mathbf{u}) = 0 \quad \Leftrightarrow \quad \mathrm{F}\, k^n_\alpha(\mathbf{u}) = 0, \tag{10}$$

since in our case $\mathrm{G} = \mathrm{F} = \mathrm{A}_w$.

Once proved the previous equivalences, it is easy to check that Conditions (WDii) and (WDiii) hold:

Condition (WDii) becomes $\alpha \leq \beta$ if $u = ([1, 1], \ldots, [1, 1])$. And $0 \leq 0$ for $u \neq ([1, 1], \ldots, [1, 1])$. The inequality holds in any case.

Condition (WDiii) becomes $1 \leq \dfrac{(1 - \beta) + (\beta - \alpha)}{1 - \alpha} = \dfrac{1 - \alpha}{1 - \alpha}$ if $u = ([1, 1], \ldots, [1, 1])$. And $0 \leq \dfrac{\beta - \alpha}{1 - \alpha}$ for $u \neq ([1, 1], \ldots, [1, 1])$. The condition holds in both cases too. So the operator is well defined.

Let us recall that in order to prove monotonicity, we have to check Eqs. 7 and 8. We distinguish three situations:

- If $u = ([1, 1], \ldots, [1, 1])$, then $v = ([1, 1], \ldots, [1, 1])$ and
$$\mathrm{G}\, k_\beta^n(\mathbf{u}) = 1 = \mathrm{G}\, k_\beta^n(\mathbf{v}) = \mathrm{F}\, k_\alpha^n(\mathbf{u}) = \mathrm{F}\, k_\alpha^n(\mathbf{v}).$$
Eqs. 7 and 8 become $\alpha - \beta \leq \alpha - \beta$ and $(1-\alpha) - (1-\beta) \leq (1-\alpha) - (1-\beta)$, respectively, so both hold.
- If $u \neq ([1, 1], \ldots, [1, 1])$ and $v = ([1, 1], \ldots, [1, 1])$, then
$$\mathrm{G}\, k_\beta^n(\mathbf{u}) = 0 = \mathrm{F}\, k_\alpha^n(\mathbf{u}) \quad \text{and} \quad \mathrm{G}\, k_\beta^n(\mathbf{v}) = 1 = \mathrm{F}\, k_\alpha^n(\mathbf{v}).$$
Eqs. 7 and 8 become $\alpha - \beta \leq 0$ and $0 \leq (1-\alpha) - (1-\beta)$, so both hold.
- Finally, if $u \leq v \neq ([1, 1], \ldots, [1, 1])$, then
$$\mathrm{G}\, k_\beta^n(\mathbf{u}) = 0 = \mathrm{G}\, k_\beta^n(\mathbf{v}) = \mathrm{F}\, k_\alpha^n(\mathbf{u}) = \mathrm{F}\, k_\alpha^n(\mathbf{v}),$$
and both Eqs. 7 and 8 become $0 \leq 0$, so they are satisfied.

Let us assume now $\beta = 1$. We first prove that for $\alpha > 0$ the operator $\mathrm{A}_{w(\alpha,\beta)}$ is not well defined.

Take $u = ([a_1, 1], \ldots, [a_n, 1])$ with $a_i < 1$, then $k_\beta^n(\mathbf{u}) = (1, \ldots, 1)$ but $k_\alpha^n(\mathbf{u}) \neq (1, \ldots, 1)$ since $k_\alpha([a_i, b_i]) < 1$.

It follows that
$$\mathrm{G}\, k_\beta^n(\mathbf{u}) = 1 \quad \text{and} \quad \mathrm{F}\, k_\alpha^n(\mathbf{u}) = 0$$
and Condition (WDii) becomes $\alpha \leq 0$, that only holds for $\alpha = 0$.

For the case $\alpha = 0$ and $\beta = 1$ it follows from Proposition 11 that $\mathrm{A}_{w(\alpha,\beta)} = \mathrm{A}_w$ and therefore it is well defined and monotone. □

Proposition 30 *Let $\alpha, \beta \in [0, 1]$, $\alpha < \beta$. Let A_s be the strongest aggregation function. Then $\mathrm{A}_{s(\alpha,\beta)} \colon \mathcal{L}[0, 1]^n \to \mathcal{L}[0, 1]$ is well defined whenever $\alpha > 0$. For the case $\alpha = 0$ it is only well defined if $\beta = 1$. Moreover, it is increasing whenever it is well defined.*

Proof It is analogous to the previous one. □

Proposition 31 *Let $\alpha, \beta \in [0, 1]$, $\alpha < \beta$. Let* min *be the aggregation function which returns the smallest value. Then $\min_{(\alpha,\beta)} \colon \mathcal{L}[0, 1]^n \to \mathcal{L}[0, 1]$ is well defined. It is increasing if and only if $\alpha = 0$ and $\beta = 1$.*

Proof Let us see that $\min_{(\alpha,\beta)}$ is well defined. Since Condition (WDi) holds trivially, it suffices to check Conditions (WDii) and (WDiii), that is, it suffices to check that
$$\alpha \min k_\beta^n(\boldsymbol{u}) \leq \beta \min k_\alpha^n(\boldsymbol{u}) \quad \text{and} \quad \min k_\beta^n(\boldsymbol{u}) \leq \frac{(1-\beta) \min k_\alpha^n(\boldsymbol{u}) + (\beta - \alpha)}{1-\alpha},$$
for every $\boldsymbol{u} = ([a_1, b_1], \ldots, [a_n, b_n]) \in \mathcal{L}[0, 1]$.
Assume that $[a_i, b_i]$ and $[a_j, b_j]$ satisfy that
$$\begin{aligned}(1-\alpha)a_i + \alpha b_i &= \min((1-\alpha)a_1 + \alpha b_1, \ldots, (1-\alpha)a_n + \alpha b_n) = \min k_\alpha^n(\boldsymbol{u}),\\ (1-\beta)a_j + \beta b_j &= \min((1-\beta)a_1 + \beta b_1, \ldots, (1-\beta)a_n + \beta b_n) = \min k_\beta^n(\boldsymbol{u}).\end{aligned}$$
In order to prove Condition (WDii), that reads as,
$$\alpha[(1-\beta)a_j + \beta b_j] \leq \beta[(1-\alpha)a_i + \alpha b_i],$$
and since $(1-\beta)a_j + \beta b_j = \min_k((1-\beta)a_k + \beta b_k) \leq (1-\beta)a_i + \beta b_i$, it suffices to prove
$$\alpha[(1-\beta)a_i + \beta b_i] \leq \beta[(1-\alpha)a_i + \alpha b_i].$$

Operating in this last expression we can see that it is equivalent to $\alpha a_i \leq \beta a_i$, so it holds.

We next prove that Condition (WDiii) holds. It reads as

$$(1-\alpha)[(1-\beta)a_j + \beta b_j] \leq (1-\beta)[(1-\alpha)a_i + \alpha b_i] + (\beta - \alpha).$$

Then, since $(1-\beta)a_j + \beta b_j \leq (1-\beta)a_i + \beta b_i$ it suffices to prove that

$$(1-\alpha)[(1-\beta)a_i + \beta b_i] \leq (1-\beta)[(1-\alpha)a_i + \alpha b_i] + (\beta - \alpha).$$

This is equivalent to prove that

$$\cancel{(1-\alpha)(1-\beta)a_i} + \beta b_i - \cancel{\alpha\beta b_i} \leq \cancel{(1-\alpha)(1-\beta)a_i} + \alpha b_i - \cancel{\alpha\beta b_i} + (\beta - \alpha)$$

and equivalent to $(\beta - \alpha)b_i \leq \beta - \alpha$. So it holds.

We now see when $\min_{(\alpha,\beta)}$ is increasing.
The case $\alpha = 0$ and $\beta = 1$ was studied in Proposition 11. Assume then, that $\alpha \neq 0$ or $\beta \neq 1$.

(1) If $\beta = 1$ but $0 < \alpha < 1$, then for any $t \in (0, 1)$ we have that $0 < \alpha t < t$ and therefore, there exists $s \in (0, 1)$ such that $\alpha t < s < t$. Thus, if we consider

$$\begin{aligned}
\mathbf{u} &= ([0,t],[s,s],[1,1],\ldots,[1,1])\,,\\
\mathbf{v} &= ([0,t],[s,t],[1,1],\ldots,[1,1])\,.
\end{aligned}$$

then $\mathbf{u} < \mathbf{v}$ and

$$\begin{aligned}
\min k_\alpha^n(\mathbf{u}) &= \min(\alpha t, s, 1, \ldots, 1) = \alpha t\,,\\
\min k_1^n(\mathbf{u}) &= \min(t, s, 1, \ldots, 1) = s\,,\\
\min k_\alpha^n(\mathbf{v}) &= \min(\alpha t, (1-\alpha)s + \alpha t, 1, \ldots, 1) = \alpha t\,,\\
\min k_1^n(\mathbf{v}) &= \min(t, t, 1, \ldots, 1) = t\,.
\end{aligned}$$

Then Eq. (7) is not satisfied because it becomes

$$\alpha t - \alpha t \leq \alpha s - \alpha t.$$

But $s - t < 0$, whereas $\min_{(\alpha,1)}$ is not increasing.

(2) Assume now that $\alpha = 0$ and $\beta \neq 1$, then for any $t \in (0, 1)$, then $\beta t > 0$ and therefore, there exists $s \in (0, \beta t)$, that is, $0 < s < \beta t$ and

$$\begin{aligned}
\mathbf{u} &= ([0,t],[s,s],[1,1],\ldots,[1,1])\,,\\
\mathbf{w} &= ([t,t],[s,s],[1,1],\ldots,[1,1])\,.
\end{aligned}$$

So $\mathbf{u} < \mathbf{w}$ and

$$\begin{aligned}
\min k_0^n(\mathbf{u}) &= \min(0, s, 1) = 0\,,\\
\min k_\beta^n(\mathbf{u}) &= \min(\beta t, s, 1) = s\,,\\
\min k_0^n(\mathbf{w}) &= \min(t, s, 1) = s\,,\\
\min k_\beta^n(\mathbf{w}) &= \min(t, s, 1) = s\,.
\end{aligned}$$

hence Eq. (8) is not satisfied since $\beta < 1$:

$$(1-\alpha)s - (1-\beta)0 > (1-\alpha)s - (1-\beta)s.$$

And therefore the operator $\min_{(0,\beta)}$ is not increasing.

(3) Let now $\alpha \neq 0$, $\beta \neq 1$. Take

$$\mathbf{x} = ([0,1],[\alpha,\beta],[1,1],\dots,[1,1]),$$
$$\mathbf{y} = ([0,1],[\beta,\beta],[1,1],\dots,[1,1])$$

We have $\mathbf{x} < \mathbf{y}$ and

$$\begin{aligned}
\min k_\alpha^n(\mathbf{x}) &= \min(\alpha, (1-\alpha)\alpha + \alpha\beta, 1) = \alpha\,,\\
\min k_\beta^n(\mathbf{x}) &= \min(\beta, (1-\beta)\alpha + \beta^2, 1) = (1-\beta)\alpha + \beta^2\,,\\
\min k_\alpha^n(\mathbf{y}) &= \min(\alpha, \beta, 1) = \alpha\,,\\
\min k_\beta^n(\mathbf{y}) &= \min(\beta, \beta, 1) = \beta\,.
\end{aligned}$$

The last equality in the first row follows from $(1-\alpha)\alpha + \alpha\beta = \alpha + \alpha(\beta - \alpha) > \alpha$ and the last equality in row two from $(1-\beta)\alpha + \beta^2 = \beta - (1-\beta)(\beta - \alpha) < \beta$. Then Eq. (7) is not satisfied:

$$\alpha\beta - \beta\alpha \not\leq \alpha[(1-\beta)\alpha + \beta^2] - \beta\alpha,$$

since, once simplified, this inequality becomes $\beta \leq (1-\beta)\alpha + \beta^2$ and we have just proven that this one does not hold.

□

Proposition 32 *Let $\alpha, \beta \in [0,1]$, $\alpha < \beta$. Let* gm *be the geometric mean. Then* $\mathrm{gm}_{(\alpha,\beta)}$ *is well defined if and only if $\beta = 1$. It is also increasing if and only if, in addition, $\alpha = 0$.*

Proof Let us see that $\mathrm{gm}_{(\alpha,1)}$ is well defined. According to Corollary 20 we have to check Conditions (WDi), (WDii) and (WDiii).

(WDi) holds by hypothesis.

(WDii) the second condition that must be satisfied becomes $\alpha\,\mathrm{gm}\,k_1(\mathbf{u}) \leq \mathrm{gm}\,k_\alpha(\mathbf{u})$. Call $\mathbf{u} = ([a_1,b_1],\dots,[a_n,b_n])$, then $\mathrm{gm}\,k_1(\mathbf{u})) = \sqrt[n]{\prod_{i=1}^n b_i}$ and $\mathrm{gm}\,k_\alpha(\mathbf{u}) = \sqrt[n]{\prod_{i=1}^n((1-\alpha)a_i + \alpha b_i)}$ and therefore,

$$\begin{aligned}
&\alpha\,\mathrm{gm}\,k_1(\mathbf{u}) \leq \mathrm{gm}\,k_\alpha(\mathbf{u})\\
\Leftrightarrow\ &\alpha \sqrt[n]{\prod_{i=1}^n b_i} \leq \sqrt[n]{\prod_{i=1}^n((1-\alpha)a_i + \alpha b_i)}\\
\Leftrightarrow\ &\alpha^n \prod_{i=1}^n b_i = \prod_{i=1}^n \alpha b_i \leq \prod_{i=1}^n((1-\alpha)a_i + \alpha b_i)\,.
\end{aligned}$$

And this inequality holds since $\alpha b_i \leq (1-\alpha)a_i + \alpha b_i$ for all i.

(WDiii) holds trivially since for $\beta = 1$ it becomes

$$(1-\alpha)\,\mathrm{gm}\,k_1(\mathbf{u}) \leq 1 - \alpha.$$

So $\mathrm{gm}_{(\alpha,1)}$ is well defined.
We now prove that for $\beta < 1$ it is not. In particular, Condition (WDiii) fails for any α, β such that $\alpha < \beta < 1$.

Set $\mathbf{u} = ([0,1],[1,1],\overset{n-1}{\ldots},[1,1])$.

For this element, Condition (WDiii) becomes

$$(1-\alpha)\beta^{1/n} \leq (1-\beta)\alpha^{1/n} + (\beta - \alpha).$$

This inequality is equivalent to $(1-\beta)\alpha^{1/n} + (\beta-\alpha) - (1-\alpha)\beta^{1/n} \geq 0$ and equivalent to

$$(1-\alpha)\left[1-\beta^{1/n}\right] - (1-\beta)\left[1-\alpha^{1/n}\right] \geq 0$$

Set $s = \alpha^{1/n}$, $t = \beta^{1/n}$, so that $\alpha = s^n$, $\beta = t^n$, where $s, t \in [0, 1]$, $s < t$ (hence $s < 1$). Thus

$$\frac{1-\beta}{1-\alpha} = \frac{1-t^n}{1-s^n} = \frac{(1-t)(1+t+\cdots+t^{n-1})}{(1-s)(1+s+\cdots+s^{n-1})} > \frac{1-t}{1-s} = \frac{1-\beta^{1/n}}{1-\alpha^{1/n}},$$

whereas

$$(1-\alpha)\left[1-\beta^{1/n}\right] - (1-\beta)\left[1-\alpha^{1/n}\right] \ngeq 0.$$

Let us see the assertion on the monotonicity. For it we show that if $\alpha \neq 0$, then $\mathrm{gm}_{(\alpha,1)}$ is not increasing. Take

$$\mathbf{v} = ([\alpha, \alpha], [0, 1], \ldots, [0, 1]), \quad \mathbf{w} = ([\alpha, 1], [0, 1], \ldots, [0, 1]).$$

Then $\mathbf{u} < \mathbf{v}$ and

$$\begin{aligned}
&\mathrm{gm}\, k_\alpha^n(\mathbf{v}) = \mathrm{gm}(\alpha, \ldots, \alpha) = \alpha,\\
&\mathrm{gm}\, k_1^n(\mathbf{v}) = \mathrm{gm}(\alpha, 1, \ldots, 1) = \alpha^{1/n},\\
&\mathrm{gm}\, k_\alpha^n(\mathbf{w}) = \mathrm{gm}(\alpha(2-\alpha), \alpha, \ldots, \alpha) = \alpha(2-\alpha)^{1/n},\\
&\mathrm{gm}\, k_1^n(\mathbf{w}) = \mathrm{gm}(1, \ldots, 1) = 1.
\end{aligned}$$

Thus, Eq. (7) does not hold: it becomes

$$\alpha - \alpha(2-\alpha)^{1/n} \leq \alpha\alpha^{1/n} - \alpha.$$

And this is equivalent to

$$2 - \alpha^{1/n} \leq (2-\alpha)^{1/n}.$$

However, this inequality does not hold: Take $x := \alpha^{1/n}$. In order for Eq. (7) to hold, we should have $(2-x)^n \leq 2 - x^n$. However, the function $f(x) = 2 - x^n - (2-x)^n$ is continuous and it verifies that $f'(x) = n[(2-x)^{n-1} - x^{n-1}] > 0$ for any $x \in [0, 1]$, this is, f is increasing in $[0, 1]$ and $f(1) = 0$, whereas $f(x) \leq 0$ for every $x < 1$. In particular, $2 - \alpha^{1/n} > (2-\alpha)^{1/n}$ for every $\alpha < 1$. □

The study of $\mathrm{am}_{(\alpha,\beta)}$ operator was carried out in Corollary 28. It was proven there that $\mathrm{am}_{(\alpha,\beta)}$ is well defined for every $\alpha < \beta$ and in particular, that $\mathrm{am}_{(\alpha,\beta)} = (\mathrm{am}, \mathrm{am})$ for all $\alpha < \beta$.

Proposition 33 *Let $\alpha, \beta \in [0, 1]$, $\alpha < \beta$. Let* max *be the aggregation function which returns the greatest value. Then $\max_{(\alpha,\beta)} : \mathcal{L}[0, 1]^n \to \mathcal{L}[0, 1]$ is well defined. It is increasing if and only if $\alpha = 0$ and $\beta = 1$.*

Proof (A) Let us see that it is well defined.

Set $\mathbf{u} = ([a_1, b_1], \ldots, [a_n, b_n]) \in \mathcal{L}[0, 1]^n$ and assume that $[a_i, b_i]$ and $[a_j, b_j]$ satisfy that

$$k_\alpha[a_i, b_i] = \max k_\alpha^n \mathbf{u}, \quad k_\beta[a_j, b_j] = \max k_\beta^n \mathbf{u}.$$

Then $k_\alpha[a_j, b_j] = (1-\alpha)a_j + \alpha b_j \leq (1-\alpha)a_i + \alpha b_i = k_\alpha[a_i, b_i]$.

(WDi) holds by hypothesis.

(WDii) With the notation above, $\alpha \max k^n_\beta \mathbf{u} \leq \beta \max k^n_\alpha \mathbf{u}$ can be written as:

$$\alpha((1-\beta)a_j + \beta b_j) \leq \beta((1-\alpha)a_i + \alpha b_i).$$

Since $(1-\alpha)a_j + \alpha b_j \leq (1-\alpha)a_i + \alpha b_i$, it suffices to prove that $\alpha((1-\beta)a_j + \beta b_j) \leq \beta((1-\alpha)a_j + \alpha b_j)$. Equivalently,

$$\alpha(1-\beta)a_j + \alpha\beta b_j \leq \beta(1-\alpha)a_j + \alpha\beta b_j.$$

And this becomes equivalent to $\alpha a_j \leq \beta a_j$. So Condition (WDii) holds for every $a_j \in [0,1]$.

(WDiii) Using the convention above, Condition (WDiii) becomes

$$(1-\alpha)\left[(1-\beta)a_j + \beta b_j\right] \leq (1-\beta)\left[(1-\alpha)a_i + \alpha b_i\right] + (\beta - \alpha).$$

Since $(1-\alpha)a_j + \alpha b_j \leq (1-\alpha)a_i + \alpha b_i$, it suffices to prove

$$(1-\alpha)\left[(1-\beta)a_j + \beta b_j\right] \leq (1-\beta)\left[(1-\alpha)a_j + \alpha b_j\right] + (\beta - \alpha).$$

Equivalently,

$$(1-\alpha)(1-\beta)a_j + (1-\alpha)\beta b_j \leq (1-\alpha)(1-\beta)a_j + (1-\beta)\alpha b_j + \beta - \alpha$$

This is also equivalent to

$$\beta b_j - \alpha\beta b_j \leq \alpha b_j - \alpha\beta b_j + \beta - \alpha \Leftrightarrow (\beta-\alpha)b_j \leq \beta - \alpha.$$

That holds for every $b_j \in [0,1]$.

(B) We next prove that $\max_{(\alpha,\beta)}$ is increasing if and only if $\alpha = 0$ and $\beta = 1$. It follows from Proposition 11 that $\max_{(0,1)}$ is increasing. Assume now that $\alpha \neq 0$ or $\beta \neq 1$.

(1) If $\alpha \neq 0$, let $0 < a, b, e \in [0,1]$ such that $\alpha b < a < \beta b$ and $b + e < 1$ (which implies $a + e < 1$). Take

$$\mathbf{u} = ([a,a], [0,0], \overset{n-1}{\ldots}, [0,0]),$$
$$\mathbf{v} = ([a, a+e], [0, b+e], [0,0], \overset{n-2}{\ldots}, [0,0]).$$

It is $\mathbf{u} < \mathbf{v}$ and

$$\max k^n_\alpha(\mathbf{u}) = \max(a, 0) = a,$$
$$\max k^n_\beta(\mathbf{u}) = \max(a, 0) = a,$$
$$\max k^n_\alpha(\mathbf{v}) = \max(a + \alpha e, \alpha(b+e), 0) = a + \alpha e,$$
$$\max k^n_\beta(\mathbf{v}) = \max(a + \beta e, \beta(b+e), 0) = \beta(b+e).$$

where the equality in the third row follows from $\alpha b < a$ and the equality in the fourth one follows from $a < \beta b$.

Equation (7) does not hold in this case. We should have

$$\alpha\beta(b+e) - \beta(a + \alpha e) \leq \alpha a - \beta a.$$

Since $\alpha > 0$, this is equivalent to $\beta b \leq a$, but we have chosen a and b satisfying $a < \beta b$.

(2) Let us finally assume $\alpha = 0$ and $\beta \neq 1$. Take $a, b, c, e \in [0,1]$ such that $0 < a < b < c < a + \beta e \leq 1$ and let

$$\mathbf{u} = ([b,b], [a, a+e], [0,0], \overset{n-2}{\ldots}, [0,0]),$$

$$\mathbf{v} = ([c, c], [a, a+e], [0, 0], \overset{n-2}{\ldots}, [0, 0])\,.$$

Then $\mathbf{u} < \mathbf{v}$ and

$$\begin{aligned}
\max k_0^n(\mathbf{u}) &= \max(b, a, 0) = b\,,\\
\max k_\beta^n(\mathbf{u}) &= \max(b, a+\beta e, 0) = a+\beta e\,,\\
\max k_0^n(\mathbf{v}) &= \max(c, a, 0) = c\,,\\
\max k_\beta^n(\mathbf{v}) &= \max(c, a+\beta e, 0) = a+\beta e\,.
\end{aligned}$$

Then Eq. (8) does not hold. We should have $a+\beta e-(1-\beta)b \leq a+\beta e-(1-\beta)c$ which is equivalent to $(1-\beta)c \leq (1-\beta)b$ and this is not true since $\beta < 1$ and $c > b$. □

4.2 Some particular cases. The case F < G

Proposition 34 *Let* $\mathrm{F}, \mathrm{G} \in \{\min, \mathrm{gm}, \mathrm{am}, \max\}$, $\mathrm{F} < \mathrm{G}$ *and* $\alpha, \beta \in [0, 1]$ *with* $\alpha < \beta$. *Then* $(\mathrm{F}, \mathrm{G})_{(\alpha,\beta)}$ *is well defined if and only if* $\alpha = 0$ *and* $\beta = 1$.

Proof It follows from Proposition 11 that $(\mathrm{F}, \mathrm{G})_{(0,1)}$ is well defined and is increasing for any pair of aggregation functions F and G, in particular, for $\mathrm{F}, \mathrm{G} \in \{\min, \mathrm{gm}, \mathrm{am}, \max\}$.

Consider $\alpha, \beta \in [0, 1]$ such that $(\alpha, \beta) \neq (0, 1)$. We will prove that $(\mathrm{F}, \mathrm{G})_{(\alpha,\beta)}$ is not well defined.

According to Corollary 20, the operator $(\mathrm{F}, \mathrm{G})_{(\alpha,\beta)}$ is well defined if and only if Conditions (WDi), (WDii) and (WDiii) hold. We then assume $\alpha < \beta$.

The following table shows some elements in $\mathcal{L}[0, 1]^n$ and their images by k_γ^n for different γ. They are used in different sections of this proof.

Elements in $\mathcal{L}[0, 1]^n$	$k_\gamma^n, \gamma \in [0, 1]$
$\mathbf{u} = ([0, 1], [1, 1], \overset{n-1}{\ldots}, [1, 1])$	$(\gamma, 1, \overset{n-1}{\ldots}, 1)$
$\mathbf{v} = ([0, b], [0, 1], \overset{n-1}{\ldots}, [0, 1])$	$(\gamma b, \gamma, \overset{n-1}{\ldots}, \gamma)$
$\mathbf{w} = ([0, 0], [1, 1], \overset{n-1}{\ldots}, [1, 1])$	$(0, 1, \overset{n-1}{\ldots}, 1)$
$\mathbf{x} = ([0, b], [1, 1], \overset{n-1}{\ldots}, [1, 1])$	$(\gamma b, 1, \overset{n-1}{\ldots}, 1)$
$\mathbf{y} = ([0, 0], \overset{n-1}{\ldots}, [0, 0], [0, 1])$	$(0, \overset{n-1}{\ldots}, 0, \gamma)$

We consider different situations.

(1) $\mathrm{F} = \min$, $\mathrm{G} = \mathrm{gm}$.
Take $\mathbf{u} = ([0, 1], [1, 1], \overset{n-1}{\ldots}, [1, 1])$,

$$\begin{aligned}
\mathrm{F}\, k_\alpha^n(\mathbf{u}) &= \min(\alpha, 1) = \alpha\,,\\
\mathrm{G}\, k_\beta^n(\mathbf{u}) &= \mathrm{gm}(\beta, 1, \overset{n-1}{\ldots}, 1) = \beta^{1/n}\,.
\end{aligned}$$

In order for Condition (WDii) to be satisfied, we should have

$$\alpha\beta^{1/n} \leq \beta\alpha.$$

That does not hold unless $\alpha = 0$ or $\beta = 1$.

- Assume $\alpha = 0$ then in order for Condition (WDiii) to be satisfied we should have $\beta^{1/n} \leq \beta$. But this only holds if $\beta = 1$.
  So both (WDii) and (WDiii) can only hold if $\alpha = 0$ and $\beta = 1$.

- Assume now $\beta = 1$ and take $\mathbf{v} = ([0, b], [0, 1], \overset{n-1}{\ldots}, [0, 1])$, where $b \in [0, 1]$ with $0 < b < 1$. Then

$$\mathrm{F}\, k^n_\alpha(\mathbf{v}) = \min(\alpha b, \alpha, \overset{n-1}{\ldots}, \alpha) = \alpha b\,,$$
$$\mathrm{G}\, k^n_1(\mathbf{v}) = \mathrm{gm}(b, 1, \overset{n-1}{\ldots}, 1) = b^{1/n}\,.$$

Then Condition (WDii) becomes $\alpha b^{1/n} \not\leq \alpha b$ and it only holds if $\alpha = 0$.

(2) $\mathrm{F} = \min$, $\mathrm{G} = \mathrm{am}$.
In this case

$$\mathrm{F}\, k^n_\alpha(\mathbf{u}) = \min(\alpha, 1) = \alpha\,,$$
$$\mathrm{G}\, k^n_\beta(\mathbf{u}) = \mathrm{am}(\beta, 1, \overset{n-1}{\ldots}, 1) = \beta + \frac{(n-1)(1-\beta)}{n}\,.$$

Condition (WDii) becomes

$$\alpha\left(\beta + \frac{(n-1)(1-\beta)}{n}\right) \leq \alpha\beta,$$

that only holds if $\alpha = 0$ or $\beta = 1$.

- Assume $\alpha = 0$ then Condition (WDiii) becomes

$$\beta + \frac{(n-1)(1-\beta)}{n} \leq \beta,$$

that implies $\beta = 1$.
- Assume $\beta = 1$ and $\mathbf{w} = ([0, 0], [1, 1], \overset{n-1}{\ldots}, [1, 1])$. Then

$$\mathrm{F}\, k^n_\alpha(\mathbf{w}) = \min(0, \alpha) = 0\,,$$
$$\mathrm{G}\, k^n_1(\mathbf{w}) = \mathrm{am}(0, 1, \overset{n-1}{\ldots}, 1) = \frac{n-1}{n}\,.$$

In this case Condition (WDii) becomes $\alpha\frac{n-1}{n} \leq 0$ and it holds only if $\alpha = 0$ as we wanted to prove.

(3) $\mathrm{F} = \min$, $\mathrm{G} = \max$.
Let $b \in [0, 1]$, $0 < b < 1$, and $\mathbf{x} = ([0, b], [1, 1], \overset{n-1}{\ldots}, [1, 1])$.
Then

$$\mathrm{F}\, k^n_\alpha(\mathbf{x}) = \min(\alpha b, 1) = \alpha b\,,$$
$$\mathrm{G}\, k^n_\beta(\mathbf{x}) = \max(\beta b, 1) = 1\,.$$

Then Condition (WDii) becomes $\alpha \leq \beta\alpha b$ and it can only hold if $\alpha = 0$. With $\alpha = 0$ Condition (WDiii) becomes $1 \leq \beta$, whereas $\beta = 1$.

(4) $\mathrm{F} = \mathrm{gm}$, $\mathrm{G} = \mathrm{am}$.
Take $\mathbf{y} = ([0, 0], \overset{n-1}{\ldots}, [0, 0], [0, 1])$. Then

$$\mathrm{F}\, k^n_\alpha(\mathbf{y}) = \mathrm{gm}(0, \overset{n-1}{\ldots}, 0, 1) = 0\,,$$
$$\mathrm{G}\, k^n_\beta(\mathbf{y}) = \mathrm{am}(0, \overset{n-1}{\ldots}, 0, \beta) = \frac{\beta}{n}\,.$$

So Condition (WDiii) becomes $\alpha\frac{\beta}{n} \leq 0$ which implies $\alpha = 0$ or $\beta = 0$ but this contradicts $\alpha < \beta$.

On the other hand, considering $\alpha = 0$, for the element $\mathbf{u}$ it holds that

$$\mathrm{F}\,k_0^n(\mathbf{u}) = \mathrm{gm}(0, 1, \overset{n-1}{\ldots}, 1) = 0\,,$$
$$\mathrm{G}\,k_\beta^n(\mathbf{u}) = \mathrm{am}(\beta, 1, \overset{n-1}{\ldots}, 1) = \beta + \frac{(n-1)(1-\beta)}{n}\,.$$

and Condition (WDiii), that reads $\beta + \frac{(n-1)(1-\beta)}{n} \leq \beta$ only holds if $\beta = 1$.

(5) $\mathrm{F} = \mathrm{gm}$, $\mathrm{G} = \max$.
One has

$$\mathrm{F}\,k_\alpha^n(\mathbf{y}) = \mathrm{gm}(0, \overset{n-1}{\ldots}, 0, \alpha) = 0\,,$$
$$\mathrm{G}\,k_\beta^n(\mathbf{y}) = \max(0, \overset{n-1}{\ldots}, 0, \beta) = \beta\,.$$

and Condition (WDiii) becomes $\beta \leq \beta - \alpha$, whereas $\alpha = 0$.
Now, assuming $\alpha = 0$,

$$\mathrm{F}\,k_0^n(\mathbf{u}) = \mathrm{gm}(0, 1, \overset{n-1}{\ldots}, 1) = 0\,,$$
$$\mathrm{G}\,k_\beta^n(\mathbf{u}) = \max(\beta, 1, \overset{n-1}{\ldots}, 1) = 1\,.$$

And Condition (WDiii), that reads as $1 \leq \beta$ implies $\beta = 1$.

(6) $\mathrm{F} = \mathrm{am}$, $\mathrm{G} = \max$
We have that

$$\mathrm{F}\,k_\alpha^n(\mathbf{y}) = \mathrm{am}(0, \overset{n-1}{\ldots}, 0, \alpha) = \frac{\alpha}{n}\,,$$
$$\mathrm{G}\,k_\beta^n(\mathbf{y}) = \max(0, \overset{n-1}{\ldots}, 0, \beta) = \beta\,.$$

And Condition (WDii) becomes $\alpha\beta \leq \beta\frac{\alpha}{n}$, implying $\alpha = 0$ or $\beta = 0$, but this last option contradicts $\alpha < \beta$. On the other hand, assuming $\alpha = 0$, for the element $\mathbf{u}$

$$\mathrm{F}\,k_0^n(\mathbf{u}) = \mathrm{am}(0, 1, \overset{n-1}{\ldots}, 1) = \frac{n-1}{n}\,,$$
$$\mathrm{G}\,k_\beta^n(\mathbf{u}) = \max(\beta, 1, \overset{n-1}{\ldots}, 1) = 1\,,$$

and Condition (WDiii) becomes $1 \leq (1-\beta)\frac{n-1}{n} + \beta$ and this implies $\beta = 1$. □

The following table summarizes the cases and functions that lead to an aggregation function.

Conditions on α and β	Conditions on F and G
$\alpha < \beta$ $\alpha = 0, \beta = 1$	$\mathrm{F} = \mathrm{G}$ linear (Prop. 27) $\mathrm{F} \leq \mathrm{G}$ (Prop. 11)
$\alpha < \beta < 1$ or $\alpha = 0, \beta = 1$	$\mathrm{F} = \mathrm{G} = \mathrm{A}_w$ (Prop. 29)
$0 < \alpha < \beta$ or $\alpha = 0, \beta = 1$	$\mathrm{F} = \mathrm{G} = \mathrm{A}_s$ (Prop. 30)
$\alpha = 0, \beta = 1$	$\mathrm{F} = \mathrm{G} = \min$ (Prop. 31) $\mathrm{F} = \mathrm{G} = \mathrm{gm}$ (Prop. 32) $\mathrm{F} = \mathrm{G} = \max$ (Prop. 33)
$\alpha = 0, \beta = 1$	$\mathrm{F}, \mathrm{G} \in \{\min, \mathrm{gm}, \mathrm{am}, \max\}$ $\mathrm{F} < \mathrm{G}$ (Prop. 34)

5 Application

In order to apply the theoretical concepts introduced in the previous sections, we propose as an illustrative example to build an ensemble of deep learning models. The aim is to stabilize classification performance, by combining the contribution of different neural network classifiers.

To begin with, we select r different deep learning architectures. For each architecture $\mathcal{A}_i \in \{\mathcal{A}_1, \ldots, \mathcal{A}_r\}$, we train m different models $\mathcal{M}^{\mathcal{A}_i}$ using various parameter initializations. These models are trained to fit the training data. Once trained, they are used to predict the classes of the objects in the testing dataset. Specifically, for a single object in the testing dataset, each model is used to compute the probability of the object belonging to each of the n possible classes $\{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$. From now on we focus on the procedure followed by a single object of the testing dataset, as the aim is to classify the object using an ensemble of models. Let ζ_k^j be the probability of the object belonging to class $\mathcal{C}_j$ according to model $\mathcal{M}_k^{\mathcal{A}_i}$ that uses the architecture $\mathcal{A}_i$. The results for the object, considering all the models trained with architecture $\mathcal{A}_i$, can be expressed as

$$\begin{array}{llll}
 & \mathcal{C}_1 & \mathcal{C}_2 & \ldots\ \mathcal{C}_n \\
\mathcal{M}_1^{\mathcal{A}_i} & \zeta_1^1 & \zeta_1^2 & \ldots\ \zeta_1^n \\
\mathcal{M}_2^{\mathcal{A}_i} & \zeta_2^1 & \zeta_2^2 & \ldots\ \zeta_2^n \\
\ldots & & & \\
\mathcal{M}_m^{\mathcal{A}_i} & \zeta_m^1 & \zeta_m^2 & \ldots\ \zeta_m^n
\end{array}$$

having that for any of the models $\mathcal{M}_k^{\mathcal{A}_i}$ occurs

$$\zeta_k^j \in [0, 1] \quad \text{and} \quad \sum_{j=1}^{n} \zeta_k^j = 1,$$

which means that the sum of all the predicted probabilities by the model $\mathcal{M}_k^{\mathcal{A}_i}$ for each possible class $\mathcal{C}_j$ should add up to 1, indicating that the model has assigned a probability for the object to belong to each class and that these probabilities are normalized.

Each model selects the class that maximizes the predicted probability for classifying the object. Since different models may predict different classes for the same object, we wish to combine the predictions of all models available to obtain a final classification. We aim not only to combine the outputs of these different initializations of the same model, but also to incorporate the outputs produced by the rest of model architectures.

First of all, we start by combining the models based on the same architecture, which would generate a matrix such as the one presented above. The information provided by the different models is combined into a single probability interval for each class. To obtain the interval for each class, we select the minimum and maximum probabilities among the considered models. Referring to the previous matrix, we combine the values of ζ by column, resulting in a set of intervals. For each class, the aim is to have a single probability interval that encompasses the probabilities given by all models. This way, we adopt an epistemic point of view (Dubois and Prade 2012): we do not know the actual probability value and the interval represents all the possible values. The left endpoint of the interval is the minimum value in the column corresponding to the class $\mathcal{C}_j$, and the right endpoint is the maximum value in the same column. More specifically, the probability interval $[\underline{\gamma_j}, \overline{\gamma_j}]_i$ for an object to belong to class $\mathcal{C}_j$, generated by the m models of the same architecture $\mathcal{A}_i$, is:

$$[\underline{\gamma_j}, \overline{\gamma_j}]_i = [\min\{\zeta_1^j, \zeta_2^j, \ldots, \zeta_m^j\}, \max\{\zeta_1^j, \zeta_2^j, \ldots, \zeta_m^j\}].$$

Applying this to all the r different architectures, we end up with:

$$\begin{array}{llll} & \mathcal{C}_1 & \mathcal{C}_2 & \ldots\ \mathcal{C}_n \\ \mathcal{A}_1 & [\underline{\gamma_1}, \overline{\gamma_1}]_1 & [\underline{\gamma_2}, \overline{\gamma_2}]_1 & \ldots\ [\underline{\gamma_n}, \overline{\gamma_n}]_1 \\ \mathcal{A}_2 & [\underline{\gamma_1}, \overline{\gamma_1}]_2 & [\underline{\gamma_2}, \overline{\gamma_2}]_2 & \ldots\ [\underline{\gamma_n}, \overline{\gamma_n}]_2 \\ \ldots & & & \\ \mathcal{A}_r & [\underline{\gamma_1}, \overline{\gamma_1}]_r & [\underline{\gamma_2}, \overline{\gamma_2}]_r & \ldots\ [\underline{\gamma_n}, \overline{\gamma_n}]_r \end{array}$$

At this point, we apply the theoretical concepts introduced in this paper. We use the intervals obtained for each architecture to build a consensus opinion using Eq. 2, as described in Fig. 1. To do this, we consider as individual inputs the probability intervals of the object belonging to each class, with the aim to create a single interval by class that summarizes those given by all the architectures. The interval is defined by considering all models trained with different parameter initializations of the same architecture. The values of these models impact the width of the resulting interval. On one hand, if the models perform similarly, the interval will be narrow, signifying greater agreement. On the other hand, in situations with greater uncertainty, a wider interval is obtained, reflecting increased disagreement. Subsequently, the width of the interval influences the decision-making process, which explains why we do not employ a direct aggregation function on the numbers. To be clear, the process shown in Fig. 1 is repeated a total of n times, combining each time the probability intervals corresponding to class $\mathcal{C}_j$ for all models. After the n repetitions a set of n intervals is obtained, each one corresponding to a different class.

$$\begin{array}{lll} \mathcal{C}_1 & \mathcal{C}_2 & \ldots\ \mathcal{C}_n \\ [\underline{\Gamma_1}, \overline{\Gamma_1}] & [\underline{\Gamma_2}, \overline{\Gamma_2}] & \ldots\ [\underline{\Gamma_n}, \overline{\Gamma_n}] \end{array}$$

We then use this information to classify the object. In classical deep learning models, obtaining the predicted class for an object is a straight-forward process of selecting the class associated with the highest predicted probability. However, since the intervals have different widths and values, a simple maximum probability approach cannot be used to determine the final class, since some of these intervals may not be comparable with one another. Instead, we propose to use a function that takes into account both the width and values of the intervals to determine the final class. By doing so, the proposed function for determining the final class from the probability intervals is based on two criteria. The first criterion gives more weight to intervals with higher k_α values, since higher values could be interpreted to higher confidence in the classification. The second criterion is inversely proportional to the width of the interval, which rewards smaller intervals, as we understand that these represent less uncertainty among the models building the ensemble. The weight parameter w can be adjusted to give more weight to one criterion over the other. Overall, this approach provides a single value Δ_j for each class $\mathcal{C}_j$ which allows to model a trade-off between confidence and uncertainty and can be expressed as

$$\Delta_j = w \cdot k_\alpha([\underline{\Gamma_j}, \overline{\Gamma_j}]) + (1 - w) \cdot (1 - (\overline{\Gamma_j} - \underline{\Gamma_j}))^p.$$

In the ensemble process, the class $\mathcal{C}_j$ with highest Δ_j is selected as the one to which the object belongs. The procedure described is applied to all the objects in the testing dataset.

To demonstrate the effectiveness of our approach, we apply it to the CIFAR10 dataset [36] that contains images of real world objects, such as animals and vehicles, which can belong to 10 possible classes. The training data consist of 50000 objects (5000 from each class) and the testing data has 10000 objects for the evaluation of the models.

Table 1 Results of the accuracy obtained in the experiments

Architecture	Models					Ensemble
	1	2	3	4	5	
1) `final_lenet_max`	0.7392	0.7412	0.7402	0.7414	0.7343	0.9174
2) `final_lenet_avg`	0.7539	0.7560	0.7551	0.7530	0.7534	
3) `final_vgg16_small_CIFAR__pool_max`	0.8597	0.8653	0.8701	0.8570	0.8000	
4) `remaining_vgg16_small_CIFAR__pool_avg`	0.8536	0.8559	0.8513	0.8521	0.8517	
5) `dense_pool_avg`	0.9219	0.9223	0.9214	0.9232	0.9222	
6) `dense_pool_max`	0.9103	0.9108	0.9060	0.9136	0.9140	

In particular, in our application 6 different architectures have been considered. For each one, 5 different models have been trained using different parameter initializations. The accuracy obtained for each model is shown in Table 1 as well as the accuracy resulting from the ensemble process described in this section. We have set the mean function as functions F and G, as well as k_α operators for the μ and ν functions ($\mu = k_0$, $\nu = k_1$) and have used a value of $w = 0.5$ giving equal importance to the k_α and width of the interval. We will be happy to share the code for result replication with anyone who requests it. In the case of the proposed technique, the results obtained are more robust than individual models because the opinion of many predictive models is taken into account.

To sum up, our approach aims to provide a good consensus by combining the predictions of multiple models. While it may not result in the best possible classification performance, it shows that the final results are not worse than the worst-performing model. Additionally, the inclusion of multiple models helps to minimize the impact of any bad models, leading to a more robust and reliable classification system. Overall, our approach strikes a balance between performance and robustness, providing a useful tool for real-world applications where accuracy and reliability are both crucial.

6 Conclusions

Aggregation functions allow to merge the information provided by different sources (judges/decision makers) and obtain a unique final outcome that is assumed to be a consensus answer. When the inputs are intervals, the result should be a new interval. We have proposed a way to obtain new aggregation operators for fusing intervals based on two "classical" aggregation functions. We have proven under which conditions the new operator is well defined and we have also studied the necessary and sufficient conditions in order to guarantee that this operator is monotone and therefore an aggregation function. In addition to the general characterization, we have provided particular expressions for the most relevant aggregation operators handled in practice: the maximum and the minimum and the arithmetic and geometric means. The practical value of the theoretical results has been demonstrated with an illustrative example, specifically in constructing ensemble deep learning models from various initializations and architectures trained on the same dataset. This application reveals that the suggested procedure effectively stabilizes the individual outcomes, thereby confirming their utility in real-world scenarios for information fusion.

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